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ON THE PROBLEM OF STOCHASTIC INTEGRAL REPRESENTATIONS OF FUNCTIONALS OF THE BROWNIAN MOTION. II*

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(*Translated by A. A. Sergeev*)

Abstract. In the first part of this paper [A. N. Shiryaev and M. Yor, *Theory Probab. Appl.*, 48 (2004), pp. 304–313], a method of obtaining stochastic integral representations of functionals $S(\omega)$ of Brownian motion $B = (B_t)_{t \geqq 0}$ was stated. Functionals $\max_{t \leqq T} B_t$ and $\max_{t \leqq T_{-a}} B_t$, where $T_{-a} = \inf\{t\colon B_t = -a\}$, $a > 0$, were considered as an illustration. In the present paper we state another derivation of representations for these functionals and two proofs of representation for functional $\max_{t \leqq g_T} B_t$, where (non-Markov time) $g_T = \sup\{0 \leqq t \leqq T\colon B_t = 0\}$ are given.

Key words. Brownian motion, Itô integral, max-functionals, stochastic integral representation

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2′. The second derivation of the representation for $S_T = \max_{t \leqq T} B_t$.

2.1. According to relation (4) of the first part of the paper,

$$S_T = \mathbf{E}S_T + 2\int_0^T \left[1 - \Phi\left(\frac{S_t - B_t}{\sqrt{T-t}}\right)\right] dB_t$$

or, equivalently,

$$S_T = \mathbf{E}S_T + \int_0^T \Psi\left(\frac{S_t - B_t}{\sqrt{T-t}}\right) dB_t, \tag{45}$$

where $S_t = \max_{u \leqq t} B_u$, $\mathbf{E}S_t = \sqrt{2T/\pi}$, and $\Psi(x) = 2[1 - \Phi(x)]$ $(= 2\mathbf{P}\{\mathcal{N}(0,1) > x\}$, $\mathcal{N}(0,1)$ having the standard Gaussian distribution).

2.2. Let us demonstrate that for all $t \geqq 0$ the following relation holds:

$$\mathbf{E}(S_T \mid \mathcal{F}_t) = \sqrt{\frac{2}{\pi}T} + \int_0^{T \wedge t} \Psi\left(\frac{S_u - B_u}{\sqrt{T-u}}\right) dB_u, \tag{46}$$

which implies, obviously, formula (45) too. (Recall that $\mathcal{F}_t = \sigma(B_s, s \leqq t)$ is the σ-algebra, generated by the Brownian motion and completed with sets of $\mathbf{P}$-probability zero from σ-algebra $\mathcal{F}$ of the original complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$.)

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Fix $0 \leqq t < T$. Then, according to (5),

$$\begin{aligned}\mathbf{E}[S_T \mid \mathcal{F}_t] &= \mathbf{E}\Big(\int_0^\infty \mathbf{E}\big[I(a < S_t) \mid \mathcal{F}_t\big]\, da\Big) = \int_0^\infty \mathbf{E}\big[I(T_a < T) \mid \mathcal{F}_t\big]\, da \\ &= \int_0^\infty \Big(I(T_a \leqq t) + \mathbf{E}\big[I(t < T_a < T) \mid \mathcal{F}_t\big]\Big)\, da \\ &= \int_0^\infty I(T_a \leqq t)\, da + \int_0^\infty \mathbf{P}(t < T_a < T \mid \mathcal{F}_t)\, da \\ &= S_t + \int_0^\infty \mathbf{P}(t < T_a < T \mid \mathcal{F}_t)\, da. \qquad (47)\end{aligned}$$

On the set $\{t < T_a\}$, according to the Markov property and relations (14), (15), we have

$$\begin{aligned}\mathbf{P}(t < T_a < T \mid \mathcal{F}_t) &= \mathbf{P}(\exists\, s \in (t,T)\colon\ B_s > a \mid \mathcal{F}_t) \\ &= \mathbf{P}_{B_t}\big\{\exists\, s \in (0, T-t)\colon\ B_s > a\big\} = \mathbf{P}_{B_t}\{T_a < T - t\} \\ &= \int_0^{T-t} \frac{a - B_t}{\sqrt{2\pi s^3}} \exp\Big\{-\frac{(a - B_t)^2}{2s}\Big\}\, ds, \qquad (48)\end{aligned}$$

where $\mathbf{P}_x(\,\cdot\,)$ is the distribution of the Brownian motion starting at point x.

From (47) and (48) we find that

$$\begin{aligned}\mathbf{E}[S_T \mid \mathcal{F}_t] &= S_t + \int_{S_t}^\infty \int_0^{T-t} \frac{a - B_t}{\sqrt{2\pi s^3}} \exp\Big\{-\frac{(a - B_t)^2}{2s}\Big\}\, ds\, da \\ &= S_t + \int_0^{T-t} \frac{1}{\sqrt{2\pi s}} \Big(\int_{S_t}^\infty \frac{a - B_t}{\sqrt{2\pi s^2}} \exp\Big\{-\frac{(a - B_t)^2}{2s}\Big\}\, da\Big)\, ds \\ &= S_t + \int_0^{T-t} \frac{1}{\sqrt{2\pi s}} \exp\Big\{-\frac{(S_t - B_t)^2}{2s}\Big\}\, ds = S_t + H(S_t - B_t, t), \qquad (49)\end{aligned}$$

where

$$H(x,t) = \int_0^{T-t} \frac{1}{\sqrt{2\pi s}}\, e^{-x^2/(2s)}\, ds, \qquad x \in \mathbf{R}, \quad 0 \leqq t < T.$$

It is obvious that

$$H(0,0) = \int_0^T \frac{1}{\sqrt{2\pi s}}\, ds = \sqrt{\frac{2T}{\pi}} \qquad (50)$$

and for $x > 0$ and $0 < t < T$

$$\frac{\partial}{\partial x} H(x,t) = -\int_0^{T-t} \frac{x}{\sqrt{2\pi s}}\, e^{-x^2/(2s)}\, ds = -\mathbf{P}\{T_x < T - t\} = -\Psi\Big(\frac{x}{\sqrt{T-t}}\Big). \qquad (51)$$

Denoting $X_t = S_t - B_t$ and applying Itô's formula to $H(X_t, t)$, we find that (in differential form)

$$dH(X_t,t) = \frac{\partial}{\partial t} H(X_t,t)\, dt + \frac{\partial}{\partial x} H(X_t,t)\, dX_t + \frac{1}{2}\frac{\partial^2}{\partial x^2} H(X_t,t)\, d[X]_t,$$

where $([X_t])_{t\leqq T}$ is the square variation of the process $(X_t)_{t\leqq T}$. It is obvious that in the case in question $[X]_t = [B]_t = t$. So (in the integral form)

$$H(S_t - B_t, t) = H(0,0) + \Bigg[\int_0^t \frac{\partial}{\partial u} H(X_u, u)\, du + \int_0^t \frac{\partial}{\partial x} H(X_u, u)\, dS_u + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} H(X_u, u)\, du\Bigg] - \int_0^t \frac{\partial}{\partial x} H(X_u, u)\, dB_u.$$

Denoting the expression in the square brackets A_t, we find from (49) that

$$\mathbf{E}(S_T \mid \mathcal{F}_t) = H(0,0) + (A_t + B_t) - \int_0^t \frac{\partial}{\partial x} H(X_u, u)\, dB_u.$$

Since the processes $(\mathbf{E}(S_T \mid \mathcal{F}_t))_{t\leqq T}$ and $(\int_0^t \frac{\partial}{\partial x} H(X_u, u)\, dB_u)_{t\leqq T}$ are continuous martingales, the process of bounded variation $(A_t + B_t)_{t\leqq T}$ with $A_0 + B_0 = 0$ is also a martingale. Therefore, this process is identically (to stochastic indistinguishability) equal to zero, and, hence,

$$\mathbf{E}(S_T \mid \mathcal{F}_t) = H(0,0) - \int_0^t \frac{\partial}{\partial x} H(X_u, u)\, dB_u,$$

which leads, taking formulas (50) and (51) into account, to the required representation (46).

3′. The second derivation of the representation for $S_{T_{-a}} = \max_{t\leqq T_{-a}} B_t$.

3.1. According to relation (44) from the first part of the paper,

$$S_{T_{-a}} = \int_0^{T_{-a}} \log \frac{a}{a + S_u}\, dB_u. \tag{52}$$

Let us demonstrate that for every $M > 0$ and $t \geqq 0$ the following representation holds:

$$\mathbf{E}\big[S_{T_{-a}} \wedge M \mid \mathcal{F}_t\big] = a \log \frac{a+M}{a} + \int_0^{t\wedge T_{-a}} \log \frac{a+M}{a + M \wedge S_u}\, dB_u, \tag{53}$$

whence (52) is derived by transition to the limit, as shown in what follows.

For fixed $M > 0$

$$\begin{aligned}
\mathbf{E}\big[S_{T_{-a}} \wedge M \mid \mathcal{F}_t\big] &= \mathbf{E}\Bigg[\int_0^M I(\alpha < S_{T_{-a}} \wedge M)\, d\alpha \,\Big|\, \mathcal{F}_t\Bigg] \\
&= \int_0^M \mathbf{E}\big[I(\alpha < S_{T_{-a}\wedge M}) \mid \mathcal{F}_t\big]\, d\alpha = \int_0^M \mathbf{E}\big[I(T_\alpha < T_{-a}) \mid \mathcal{F}_t\big]\, d\alpha \\
&= \int_0^M I(T_\alpha < T_{-a} \leqq t)\, d\alpha \\
&\quad + \int_0^M \Big(I(T_\alpha < T_{-a}) + \mathbf{E}\big[I(t < T_\alpha < T_{-a}) \mid \mathcal{F}_t\big]\Big)\, d\alpha.
\end{aligned} \tag{54}$$

Due to the Markov property of the Brownian motion we find that on the set $\{t < T_{-a}\}$

$$\begin{aligned}\mathbf{E}\big[S_{T_{-a}} \wedge M \mid \mathcal{F}_t\big] &= \int_0^M I(T_\alpha \leqq t)\, d\alpha + \int_0^M \mathbf{E}\big[I(t < T_\alpha < T_{-a}) \mid \mathcal{F}_t\big]\, d\alpha \\ &= \int_0^M I(\alpha < S_t)\, d\alpha \\ &\quad + \int_0^M \mathbf{E}\big[I(t < T_\alpha)\, I(0 < T_\alpha \circ \theta_t < T_{-a} \circ \theta_t) \mid \mathcal{F}_t\big]\, d\alpha \\ &= S_t \wedge M + \int_0^M \mathbf{P}_{B_t}\{T_\alpha < T_{-a}\}\, I(t < T_\alpha)\, d\alpha, \end{aligned} \tag{55}$$

where θ_t is the shift operator. Yet, on the set $\{T_{-a} \leqq t\}$

$$\mathbf{E}\big[S_{T_{-a}} \wedge M \mid \mathcal{F}_t\big] = \int_0^M I(T_\alpha < T_{-a} < t)\, d\alpha = S_{T_{-a}} \wedge M. \tag{56}$$

Using the well-known relation

$$\mathbf{P}_x\{T_\alpha < T_{-a}\} = \frac{x+a}{\alpha+a}, \qquad -a < x < \alpha,$$

we find from (55) that on the set $\{t < T_{-a}\}$

$$\begin{aligned}\mathbf{E}\big[S_{T_{-a}} \wedge M \mid \mathcal{F}_t\big] &= S_t \wedge M + \int_0^M \frac{B_t + a}{\alpha + a}\, I(t < T_\alpha)\, d\alpha \\ &= S_t \wedge M + B_t \int_0^M \frac{1}{\alpha+a}\, I(S_t < \alpha)\, d\alpha + \int_0^M \frac{a}{\alpha+a}\, I(S_t < \alpha)\, d\alpha \\ &= S_t \wedge M + (B_t + a)\log(M+a) - B_t \log(S_t \wedge M + a) \\ &\quad - a\log(S_t \wedge M + a) = A_t + \int_0^t \log\frac{M+a}{S_u \wedge M + a}\, dB_u, \end{aligned} \tag{57}$$

where $(A_t)_{t \geqq 0}$ is the continuous process of bounded variation specified by the relation

(58)

$$A_t = a\log(M+a) + S_t \wedge M - a\log(S_t \wedge M + a) - \int_0^t \frac{B_\alpha}{S_u \wedge M + a}\, I(S_u \leqq M)\, dS_u.$$

(The last equality in (57) was obtained with the use of Itô's formula applied to $B_t \log(S_t \wedge M + a)$.)

As in the end of section 2′, we make use of the fact that a continuous martingale, which is at the same time a process of bounded variation, is constant. Then we find from (57) that the processes

$$\big(\mathbf{E}[S_{T_{-a}} \wedge M \mid \mathcal{F}_{t \wedge T_{-a}}]\big)_{t \geqq 0} \quad \text{and} \quad \left(A_0 + \int_0^{t \wedge T_{-a}} \log\frac{M+a}{S_u \wedge M + a}\, dB_u\right)_{t \geqq 0}$$

are indistinguishable.

From (58) we obtain equality $A_0 = a\log[(M+a)/a]$, and, therefore, for every $t \geqq 0$

$$\mathbf{E}(S_{T_{-a}} \wedge M \mid \mathcal{F}_t) = a\log\frac{M+a}{a} + \int_0^{t\wedge T_{-a}} \log\frac{M+a}{S_u \wedge M + a}\, dB_u \quad (\mathbf{P}\text{-a.s.}),$$

which is just the required relation (53).

Note that according to this formula

$$\mathbf{E}(S_{T_{-a}} \wedge M) = a\log\frac{M+a}{a}.$$

This latter formula one can also find directly:

$$\begin{aligned}\mathbf{E}(S_{T_{-a}} \wedge M) &= \int_0^M \mathbf{P}\{S_{T_{-a}} \wedge M > \alpha\}\, d\alpha = \int_0^M \mathbf{P}\{T_\alpha < T_{-a}\}\, d\alpha \\ &= \int_0^M \frac{a}{\alpha + a}\, d\alpha = a\log\frac{M+a}{a}.\end{aligned}$$

If one assumes $t = T_{-a}$ in (53), then one will find that for every $M > 0$

$$\begin{aligned}S_{T_{-a}} \wedge M &= a\log\frac{M+a}{a} + \int_0^{T_{-a}} \big[\log(M+a) - \log(S_u \wedge M + a)\big]\, dB_u \\ &= a\log\frac{M+a}{a} - a\log(M+a) - \int_0^{T_{-a}} \log(S_u \wedge M + a)\, dB_u \\ &= \int_0^{T_{-a}} \big[\log a - \log(S_u \wedge M + a)\big]\, dB_u = \int_0^{T_{-a}} \log\frac{a}{S_u \wedge M + a}\, dB_u.\end{aligned}$$

Assuming here $M \to \infty$ and using the continuity of the integral with respect to M, we get

$$S_{T_{-a}} = \int_0^{T_{-a}} \log\frac{a}{a + S_u}\, dB_u,$$

which is just the required relation (52).

3.2. Let $T^b_{-a} = T_b \wedge T_{-a}$, $a, b > 0$. In other words, let $T^b_{-a} = \inf\{t > 0\colon B_t \notin (-a, b)\}$. The reasoning, similar to the adduced one, allows us to validate the following representations:

$$S_{T^b_{-a}} = a\log\frac{a+b}{a} + \int_0^{T^b_{-a}} \log\frac{b+a}{S_u + a}\, dB_u \tag{59}$$

and

$$\mathbf{E}(S_{T^b_{-a}} \mid \mathcal{F}_t) = a\log\frac{a+b}{a} + \int_0^{t\wedge T^b_{-a}} \log\frac{b+a}{S_u + a}\, dB_u. \tag{60}$$

Indeed, since $\mathbf{P}\{S_{T^b_{-a}} \leqq b\} = 1$ and

$$\mathbf{P}\{S_{T^b_{-a}} > u\} = \mathbf{P}\{T_u < T_{-a}\} = \frac{a}{u+a}, \qquad 0 < u \leqq b,$$

one can see that

$$\mathbf{E}S_{T^b_{-a}} = a\log\frac{b+a}{a}.$$

Fix $t>0$. As in section 1 (see (54)), we have

$$\begin{aligned}\mathbf{E}[S_{T^b_{-a}} \mid \mathcal{F}_t] &= \int_0^\infty \mathbf{E}\big[I(u<S_{T^b_{-a}}) \mid \mathcal{F}_t\big]\,du = \int_0^\infty \mathbf{E}\big[I(T_u<T^b_{-a}) \mid \mathcal{F}_t\big]\,du\\ &= \int_0^\infty \Big(I(T_u<T^b_{-a}\leqq t) + I(T_u\leqq t<T^b_{-a})\\ &\qquad + \mathbf{E}\big[I(t<T_u<T^b_{-a}) \mid \mathcal{F}_t\big]\Big)\,du.\end{aligned}$$

Using the Markov property of the Brownian motion, we find that

$$\begin{aligned}\mathbf{E}\big[I(t<T_u<T^b_{-a}) \mid \mathcal{F}_t\big] &= \mathbf{P}\big(\exists\, s\in(t,T^b_{-a})\colon\ B_s>u \mid \mathcal{F}_t\big)\, I(t<T^b_{-a})\\ &= \mathbf{P}_{B_t}\big\{\exists\, s\in(0,T^b_{-a})\colon\ B_s>u\big\}\, I(t<T^b_{-a})\\ &= \psi(B_t,u)\, I(t<T^b_{-a}),\end{aligned}$$

where

$$\psi(x,u) = \begin{cases} 1, & x\geqq u,\ u\in(0,b),\\ \dfrac{x+a}{u+a}, & -a<x<u,\ u\in(0,b),\\ 0, & \text{otherwise.}\end{cases}$$

Taking account of this designation, we find that on the set $\{t<T^b_{-a}\}$

$$\mathbf{E}\big[S_{T^b_{-a}} \mid \mathcal{F}_t\big] = S_t + \int_{S_t}^b \psi(B_t,u)\,du = S_t + \int_{S_t}^b \frac{B_t+a}{u+a}\,du = S_t + (B_t+a)\,\log\frac{b+a}{S_t+a}.$$

Applying Itô's formula to the right-hand side of this relation and again, as in section 1, ignoring members with bounded variation, we arrive at the following relation:

$$\mathbf{E}\big[S_{T^b_{-a}} \mid \mathcal{F}_t\big] = a\log\frac{a+b}{a} + \int_0^{t\wedge T^b_{-a}} \log\frac{b+a}{S_u+a}\,dB_u,$$

which is just the required relation (60), which obviously implies (59).

4. The case $S_{g_T} = \max_{t\leqq g_T} B_t$.

4.1. Let $g_T = \sup\{0<t\leqq T\colon\ B_t=0\}$ be the time of the last reaching of zero by the Brownian motion on $(0,T]$. If $B_t\neq 0$ for all $0<t\leqq T$, then assume $g_T=0$.

THEOREM 3. *For S_{g_T} the following stochastic integral representation is true*:

$$S_{g_T} = \frac{1}{2}\,\mathbf{E}S_T + \int_0^T \left[1-\Psi\left(\frac{2S_u-B_u}{\sqrt{T-u}}\right) - Z_u(B_u, S_u-S_{g_u})\right] dB_u, \tag{61}$$

where $\mathbf{E}S_t = \sqrt{2T/\pi}$, $\Psi(x) = 2[1-\Phi(x)]$, *and*

$$Z_u(B_u, S_u-S_{g_u}) = (S_u-S_{g_u})\,\varphi_{T-u}(B_u) \qquad \textit{with} \quad g_u = \sup\{0<t\leqq u\colon\ B_t=0\},$$

or, equivalently,

$$S_{g_T} = \frac{1}{2}\,\mathbf{E}S_T + \int_0^T \left[\frac{1}{2}\,\Psi\left(\frac{2S_u - B_u}{\sqrt{T-u}}\right) - Z_u(B_u, S_u - S_{g_u})\right] dB_u. \tag{62}$$

We give two different proofs, each of them of independent interest in view of the techniques used.

4.2. *First proof.* We have

$$S_{g_T} = \int_0^\infty I(a < S_{g_T})\,da = \int_0^\infty I(g_T > T_a)\,da = \int_0^\infty I(d_{T_a} < T)\,da, \tag{63}$$

where for $K > 0$

$$d_K = \inf\{t > K\colon\ B_t = 0\}.$$

By analogy with the scheme of the proof of Theorem 1 it is natural first to obtain a stochastic integral representation for $I(d_{T_a} < T)$ (cf. Lemma 1, which provides the representation for $I(T_a < T)$).

LEMMA 4. *For any $a > 0$ and any $T > 0$*

$$\begin{aligned} I(d_{T_a} < T) &= \mathbf{P}\{T_{2a} < T\} + 2\int_{T_a\wedge T}^{d_{T_a}\wedge T} \varphi_{T-s}(B_s)\,dB_s \\ &\quad - 2\int_0^{T_a\wedge T} \varphi_{T-s}(B_s - 2a)\,dB_s \quad (\mathbf{P}\text{-}a.s.). \end{aligned} \tag{64}$$

Proof. It is obvious that

$$\begin{aligned} d_{T_a} &= \inf\{t > T_a\colon\ B_t = 0\} = T_a + \inf\{u \geqq 0\colon\ B_{T_a+u} = 0\} \\ &= T_a + \inf\{u \geqq 0\colon\ B_{T_a+u} - a = -a\} = T_a + \inf\{u \geqq 0\colon\ \widehat{B}_u = -a\}, \end{aligned} \tag{65}$$

where $\widehat{B} = (\widehat{B}_u)_{u\geqq 0}$ with $\widehat{B}_u = B_{T_a+u} - a$ is the Brownian motion, independent of σ-algebra $\mathcal{F}_{T_a} = \sigma\{A \in \mathcal{F}\colon\ A \cap \{T_a \leqq t\} \in \mathcal{F}_t,\ t > 0\}$ with $\mathcal{F} = \bigvee_{t>0}\mathcal{F}_t$.

Denote $\widehat{T}_{-a} = \inf\{u \geqq 0\colon\ \widehat{B}_u = -a\}$. Then from (65) we have $d_{T_a} = T_a + \widehat{T}_{-a}$, and, hence,

$$I(d_{T_a} < T) = I(\widehat{T}_{-a} < T - T_a). \tag{66}$$

Let us find a representation for $I(\widehat{T}_{-a} < b)$. If one denotes $\widehat{T}_{-a} = \widehat{T}_{-a}(\widehat{B})$, then one can see that $\widehat{T}_{-a}(\widehat{B}) = \widehat{T}_a(-\widehat{B})$. According to Lemma 1,

$$\begin{aligned} I(\widehat{T}_a(-\widehat{B}) < b) &= \mathbf{P}\{\widehat{T}_a(-\widehat{B}) < b\} + 2\int_0^{\widehat{T}_a(-\widehat{B})\wedge b} \varphi_{b-u}(-\widehat{B}_u - a)\,d(-\widehat{B}_u) \\ &= \mathbf{P}\{T_a < b\} - 2\int_0^{\widehat{T}_{-a}\wedge b} \varphi_{b-u}(\widehat{B}_u + a)\,d\widehat{B}_u. \end{aligned}$$

By (14) and (15)

$$\mathbf{P}\{T_a < b\} = \int_0^\infty I(t < b)\,\gamma_a(t)\,dt \qquad \text{with} \quad \gamma_a(t) = \frac{a}{\sqrt{2\pi t^3}}\,e^{-a^2/(2t)}.$$

Therefore,

$$I(\widehat{T}_{-a} < b) = \int_0^\infty I(t < b)\,\gamma_a(t)\,dt - 2\int_0^{\widehat{T}_{-a}\wedge b} \varphi_{b-u}(\widehat{B}_u + a)\,d\widehat{B}_u. \tag{67}$$

Using the independence of $\widehat{T}_{-a}$ from σ-algebra $\mathcal{F}_{T_a}$, we find from (67) that

$$\begin{aligned}\int_0^{\widehat{T}_{-a}\wedge(T-T_a)} \varphi_{T-T_a-u}(\widehat{B}_u + a)\,d\widehat{B}_u &= \int_0^{\widehat{T}_{-a}\wedge(T-T_a)} \varphi_{T-T_a-u}(B_{T_a+u})\,dB_{T_a+u}\\ &= \int_{T_a}^{(\widehat{T}_{-a}+T_a)\wedge T} \varphi_{T-s}(B_s)\,dB_s = \int_{T_a\wedge T}^{d_{T_{-a}}\wedge T} \varphi_{T-s}(B_s)\,dB_s.\end{aligned}$$

From this, (67), (66), and (6) we obtain

$$\begin{aligned}I(d_{T_a} < T) &= \int_0^T I(T_a < T-t)\,\gamma_a(t)\,dt - 2\int_{T_a\wedge T}^{d_{T_a}\wedge T} \varphi_{T-s}(B_s)\,dB_s\\ &= \int_0^T \mathbf{P}\{T_a < T-t\}\,\gamma_a(t)\,dt\\ &\quad + 2\int_0^T \left[\int_0^{T_a\wedge(T-t)} \varphi_{T-t-s}(B_s - a)\,dB_s\right]\gamma_a(t)\,dt\\ &\quad - 2\int_{T_a\wedge T}^{d_{T_a}\wedge T} \varphi_{T-s}(B_s)\,dB_s\\ &= \mathbf{P}\{T_{2a} < T\} + 2\int_0^{T_a\wedge T}\left[\int_0^{T-s} \gamma_a(t)\varphi_{T-t-s}(B_s - a)\,dt\right]dB_s\\ &\quad - 2\int_{T_a\wedge T}^{d_{T_a}\wedge T} \varphi_{T-s}(B_s)\,dB_s.\end{aligned} \tag{68}$$

According to Lemma 5, mentioned below in section 3, for all $s < T_a$ such that $B_s < a$, the following equality holds:

$$\int_0^{T-s} \gamma_a(t)\,\varphi_{T-t-s}(B_s - a)\,dt = \varphi_{T-s}(B_s - 2a). \tag{69}$$

Hence, it follows from (68) that

$$\begin{aligned}I(d_{T_a} < T) = \mathbf{P}\{T_{2a} < T\} &+ 2\int_0^{T_a\wedge T} \varphi_{T-s}(B_s - 2a)\,dB_s\\ &- 2\int_{T_a\wedge T}^{d_{T_a}\wedge T} \varphi_{T-s}(B_s)\,dB_s,\end{aligned}$$

which is just the required relation (64). This proves Lemma 4.

Now we turn to proving representation (61).

By virtue of (63) and (64)

$$\begin{aligned}S_{g_T} &= \int_0^\infty I(d_{T_a} < T)\,da\\ &= \int_0^\infty \mathbf{P}\{T_{2a} < T\}\,da + 2\int_0^\infty \left[\int_0^{T_a\wedge T} \varphi_{T-s}(B_s - 2a)\,dB_s\right]da\\ &\quad - 2\int_0^\infty \left[\int_{T_a\wedge T}^{d_{T_a}\wedge T} \varphi_{T-s}(B_s)\,dB_s\right]da.\end{aligned} \tag{70}$$

Here

$$\int_0^\infty \mathbf{P}\{T_{2a} < T\}\,da = \frac{1}{2}\int_0^\infty \mathbf{P}\{T_b < T\}\,db = \frac{1}{2}\int_0^\infty \mathbf{P}\{S_T > b\}\,db = \frac{1}{2}\,\mathbf{E}S_T \tag{71}$$

and

$$\begin{aligned}
&\int_0^\infty \left[\int_0^{T_a\wedge T} \varphi_{T-s}(B_s - 2a)\,dB_s\right] da \\
&\quad = \int_0^T \left[\int_0^\infty \varphi_{T-u}(B_u - 2a)\,I(S_u < a)\,da\right] dB_u \\
&\quad = \int_0^T \left[\int_{S_u}^\infty \varphi_{T-u}(B_u - 2a)\,da\right] dB_u = \frac{1}{2}\int_0^T \left[\int_{2S_u}^\infty \varphi_{T-u}(B_u - b)\,db\right] dB_u \\
&\quad = \frac{1}{2}\int_0^T \left[1 - \Phi\left(\frac{2S_u - B_u}{\sqrt{T-u}}\right)\right] dB_u. \qquad (72)
\end{aligned}$$

Finally, let us transform the last expression in the right-hand side of (70).

We have

$$\begin{aligned}
\int_0^\infty \left[\int_{T_a\wedge T}^{d_{T_a}\wedge T} \varphi_{T-s}(B_s)\,dB_s\right] da &= \int_0^\infty\int_0^\infty I(T_a \wedge T < s < d_{T_a} \wedge T)\,\varphi_{T-s}(B_s)\,dB_s\,da \\
&= \int_0^T \left[\int_0^\infty I(T_a < s < d_{T_a} \wedge T)\,da\right] \varphi_{T-s}(B_s)\,dB_s \\
&= \int_0^T \left[\int_0^\infty I(S_{g_s} < a < S_s)\,da\right] \varphi_{T-s}(B_s)\,dB_s \\
&= \int_0^T (S_u - S_{g_u})\,\varphi_{T-u}(B_u)\,dB_u. \qquad (73)
\end{aligned}$$

Thus, it follows from (70)–(73) that

$$S_{g_T} = \frac{1}{2}\,\mathbf{E}S_T + \int_0^T \left[1 - \Phi\left(\frac{2S_u - B_u}{\sqrt{T-u}}\right)\right] dB_u + 2\int_0^T (S_u - S_{g_u})\,\varphi_{T-u}(B_u)\,dB_u.$$

Thereby (61) and (62) are proved.

4.3. In the above-mentioned proof, integral relation (69) linking the densities

$$\varphi_t(a) = \frac{1}{\sqrt{2\pi t}}\,e^{-a^2/(2t)} \quad \text{and} \quad \gamma_a(t) = \frac{a}{\sqrt{2\pi t^3}}\,e^{-a^2/(2t)} \quad \left(= -\frac{\partial}{\partial a}\,\varphi_t(a)\right)$$

was used. It follows from the following lemma.

LEMMA 5. *For all $a > 0$ and $\theta > 0$*

$$\int_0^\theta \gamma_a(t)\,\varphi_{\theta-t}(x-a)\,dt = \begin{cases} \varphi_\theta(x), & x > a, \\ \varphi_\theta(x-2a), & x \leqq a. \end{cases} \tag{74}$$

Proof. Let

$$I(a,x) = \frac{1}{\varphi_\theta(x)}\int_0^\theta \gamma_a(t)\,\varphi_{\theta-t}(x-a)\,dt.$$

Using the above-mentioned explicit form of the functions $\varphi_\theta(a)$ and $\gamma_a(t)$ and making the change of variables $u = \sqrt{\theta/t - 1}$, we find that

$$I(a,x) = \frac{2a}{\sqrt{2\pi t}}\, e^{a(x-a)/\theta} \int_0^\infty e^{-\alpha u^2 - \beta/u^2}\, du$$

with $\alpha = a^2/(2\theta)$ and $\beta = (x-a)^2/(2\theta)$.

By formula 3.325 of [2]

$$\int_0^\infty e^{-\alpha u^2 - \beta/u^2}\, du = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\, e^{-2\sqrt{\alpha\beta}} = \frac{\sqrt{2\pi\theta}}{2a}\, e^{-a|x-a|/\theta}.$$

Therefore,

$$I(a,x) = e^{a(x-a)/\theta}\, e^{-a|x-a|/\theta} = \begin{cases} 1, & x > a, \\ e^{-2a(a-x)/\theta}, & x \leqq a, \end{cases}$$

which proves (74).

Along with the adduced "analytic" proof of (74) the following "probabilistic" proof of this relation is not without interest.

Let $f = f(x)$ be a measurable bounded function. Then

$$\mathbf{E} f(B_\theta) = \mathbf{E}\big[f(B_\theta)\, I(T_a < \theta)\big] + \mathbf{E}\big[f(B_\theta)\, I(T_a \geqq \theta)\big]. \tag{75}$$

Here

$$\begin{aligned} \mathbf{E}\big[f(B_\theta)\, I(T_a < \theta)\big] &= \mathbf{E}\big[f(B_{T_a + (\theta - T_a)})\, I(T_a < \theta)\big] \\ &= \int_0^\theta \gamma_a(t) \left[\int_{-\infty}^\infty f(x)\, \varphi_{\theta-t}(x-a)\, dx\right] dt \end{aligned} \tag{76}$$

and

$$\begin{aligned} \mathbf{E}\big[f(B_\theta)\, I(T_a \geqq \theta)\big] &= \int_{-\infty}^\infty \mathbf{E}\big[f(B_\theta)\, I(\theta \leqq T_a) \mid B_\theta = x\big] \varphi_\theta(x)\, dx \\ &= \int_{-\infty}^\infty f(x) \mathbf{P}(T_a \geqq \theta \mid B_\theta = x)\, \varphi_\theta(x)\, dx. \end{aligned} \tag{77}$$

From (75)–(77) we obtain

$$\begin{aligned} \mathbf{E} f(B_\theta) &= \int_{-\infty}^\infty f(x)\, \varphi_\theta(x)\, dx \\ &= \int_{-\infty}^\infty f(x) \left[\int_0^\theta \varphi_{\theta-t}(x-a)\, \gamma_a(t)\, dt + \mathbf{P}(T_a \geqq \theta \mid B_\theta = x)\, \varphi_\theta(x)\right] dx. \end{aligned}$$

From this in view of the arbitrariness of the function $f(x)$ we have

$$\int_0^\theta \gamma_a(t)\, \varphi_{\theta-t}(x-a)\, dt = \varphi_\theta(x)[1 - \mathbf{P}(T_a \geqq \theta \mid B_\theta = x)]. \tag{78}$$

If $x > a$, then $\mathbf{P}(T_a \geqq \theta \,|\, B_\theta = x) = 0$, and (78) gives (74).

Now let $x \leqq a$; then

$$\begin{aligned} 1 - \mathbf{P}(T_a \geqq \theta \mid B_\theta = x) &= \mathbf{P}(T_a < \theta \mid B_\theta = x) = \mathbf{P}\Big(\max_{u \leqq \theta} B_u > a \,\Big|\, B_\theta = x\Big) \\ &= \mathbf{P}(S_\theta > a \mid B_\theta = x) \end{aligned}$$

with $S_\theta = \max_{u \leqq \theta} B_u$.

Probability $\mathbf{P}(S_\theta > a \,|\, B_\theta = x)$ can be found, using, for example, the Seshadri result (see [3]) that random variables $S_\theta(S_\theta - B_\theta)$ and B_θ are independent and

$$S_\theta(S_\theta - B_\theta) \overset{\text{law}}{=} \frac{\theta}{2}\,\mathcal{E},$$

where $\mathcal{E}$ is the standard exponentially distributed random variable ($\mathbf{P}\{\mathcal{E} > t\} = e^{-t}$, $t > 0$).

Indeed, from the stated assertions we find that for $x \leqq a$, $a \geqq 0$, and $b = a(a-x)$

$$\begin{aligned}\mathbf{P}(S_\theta > a \mid B_\theta = x) &= \mathbf{P}(S_\theta(S_\theta - B_\theta) > b \mid B_\theta = x) = \mathbf{P}\left\{\frac{\theta}{2}\mathcal{E} > b\right\}\\ &= \mathbf{P}\left\{\mathcal{E} > \frac{2b}{\theta}\right\} = e^{-2b/\theta} = e^{-2a(a-x)/\theta}.\end{aligned}$$

Thereby, expression $1 - \mathbf{P}(T_a \geqq \theta \,|\, B_\theta = x)$, being a part of (78), equals

$$\mathbf{P}(S_\theta > a \mid B_\theta = x) = e^{-2a(a-x)/\theta}$$

and, hence, for $x \leqq a$,

$$\int_0^\theta \gamma_a(t)\,\varphi_{\theta-t}(x-a)\,dt = \varphi_\theta(x)\,e^{-2a(a-x)/\theta},$$

which is claimed in (74).

4.4. Second proof. By analogy with the distributions used in sections $2'$ and $3'$ let us demonstrate that for every $t > 0$ the following equality is true ($\mathbf{P}$-a.s.):

$$\mathbf{E}[S_{g_T} \mid \mathcal{F}_t] = \frac{1}{2}\,\mathbf{E}S_T + \int_0^{t\wedge T}\left[\frac{1}{2}\,\Psi\left(\frac{2S_u - B_u}{\sqrt{T-u}}\right) - Z_u(B_u, S_u - S_{g_u})\right] dB_u. \tag{79}$$

(Formula (61), of course, follows from this representation.)

Fix $0 \leqq t < T$. Then

$$\begin{aligned}\mathbf{E}[S_{g_T} \mid \mathcal{F}_t] &= \int_0^\infty \mathbf{E}\big[I(a < S_{g_T}) \mid \mathcal{F}_t\big]\,da = \int_0^\infty \mathbf{E}\big[I(T_a < g_T) \mid \mathcal{F}_t\big]\,da\\ &= \int_0^\infty \mathbf{E}\big[I(T_a < g_T \leqq t) + I(T_a \leqq t < g_T) + I(t < T_a < g_T) \mid \mathcal{F}_t\big]\,da. \qquad (80)\end{aligned}$$

Using the Markov property of the Brownian motion, we obtain the following relations:

$$\begin{aligned}\text{(a)}\quad &\mathbf{E}\big[I(T_a < g_T \leqq t) \mid \mathcal{F}_t\big] = \mathbf{E}\big[I(T_a < g_T < t) \mid \mathcal{F}_t\big]\\ &= \mathbf{E}\big[I(T_a < g_T < t) \mid \mathcal{F}_t\big]\,I(T_a < t)\\ &= \mathbf{P}\big(\exists\, s_1 < s_2 < t\colon\ B_{s_1} > a, B_{s_2} = 0;\ B_s \neq 0 \text{ for } s \in (t,T) \mid \mathcal{F}_t\big)\,I(T_a < t)\\ &= \mathbf{P}\big(T_a + T_0 \circ \theta_{T_a} < t \text{ and } B_s \neq 0 \text{ for } s \in (t,T) \mid \mathcal{F}_t\big)\,I(T_a < t)\\ &= \mathbf{P}_{B_t}\big\{B_s \neq 0 \text{ for } s \in (0, T-t)\big\}\,I(T_a + T_0 \circ \theta_{T_a} < t)\\ &= \mathbf{P}\big\{T_{|B_t|} > 1 - t\big\}\,I(T_a + T_0 \circ \theta_{T_a} < t);\end{aligned}$$

$$
\begin{aligned}
\text{(b)}\quad & \mathbf{E}\big[I(T_a \leqq t \leqq g_T) \mid \mathcal{F}_t\big] = \mathbf{E}\big[I(T_a < t < g_T) \mid \mathcal{F}_t\big] \\
&= \mathbf{E}\big[I(T_a < t < g_T) \mid \mathcal{F}_t\big]\, I(t > T_a) \\
&= \mathbf{P}\big(\exists\, s < t\colon\ B_s > a \text{ and } \exists\, s_1 \in (t,T)\colon\ B_{s_1} = 0 \mid \mathcal{F}_t\big)\, I(t > T_a) \\
&= \mathbf{P}_{B_t}\big\{\exists\, s_1 \in (0, T-t)\colon\ B_{s_1} = 0\big\}\, I(t > T_a) \\
&= \mathbf{P}\big\{T_{|B_t|} > T - t\big\}\, I(t > T_a); \\
\text{(c)}\quad & \mathbf{E}\big[I(t < T_a < g_T) \mid \mathcal{F}_t\big] \\
&= \mathbf{P}\big(\exists\, s_1, s_2\colon\ t < s_1 < s_2 < T,\ B_{s_1} > a, B_{s_2} = 0 \mid \mathcal{F}_t\big)\, I(t < T_a) \\
&= \mathbf{P}_{B_t}\big(\exists\, s_1, s_2\colon\ 0 < s_1 < s_2 < T - t, B_{s_1} > a, B_{s_2} = 0\big)\, I(t < T_a) \\
&= \mathbf{P}_{B_t}\big\{\exists\, s \in (0, T-t)\colon\ B_s = 2a\big\}\, I(t < T_a) \\
&= \mathbf{P}\{T_{2a - B_t} < T - t\}\, I(t < T_a).
\end{aligned}
$$

From (a), (b), and (c) we get

$$
\begin{aligned}
\text{(a}^*\text{)}\quad & \int_0^\infty \mathbf{E}\big[I(T_a < g_T \leqq t) \mid \mathcal{F}_t\big]\, da \\
&= \mathbf{P}\{T_{|B_t|} > T - t\} \int_0^\infty I(T_a + T_0 \circ \theta_{T_a} < t)\, da \\
&= \mathbf{P}\{T_{|B_t|} > T - t\} S_{g_t} = \big[1 - \mathbf{P}\{T_{|B_t|} < T - t\}\big] S_{g_t}; \\
\text{(b}^*\text{)}\quad & \int_0^\infty \mathbf{E}\big[I(T_a < t < g_t) \mid \mathcal{F}_t\big]\, da = \mathbf{P}\{T_{|B_t|} < T - t\} S_t; \\
\text{(c}^*\text{)}\quad & \int_0^\infty \mathbf{E}\big[I(t < T_a < g_T) \mid \mathcal{F}_t\big]\, da \\
&= \int_0^\infty \mathbf{P}\{T_{2a - B_t} < T - t\}\, I(t < T_a)\, da \\
&= \int_{S_t}^\infty \int_0^{T-t} \frac{2a - B_t}{\sqrt{2\pi s^3}}\, \exp\bigg\{-\frac{(2a - B_t)^2}{2s}\bigg\}\, ds\, da \\
&= \int_0^{T-t} \int_{S_t}^\infty \frac{2a - B_t}{\sqrt{2\pi s^3}}\, \exp\bigg\{-\frac{(2a - B_t)^2}{2s}\bigg\}\, da\, ds \\
&= \frac{1}{2} \int_0^{T-t} \frac{1}{\sqrt{2\pi s}}\, \exp\bigg\{-\frac{(2S_t - B_t)^2}{2s}\bigg\}\, ds \\
&= \frac{1}{2} \int_0^{T-t} \varphi_s(2S_t - B_t)\, ds = \frac{1}{2}\, H(2S_t - B_t, t),
\end{aligned}
$$

where $H(x,t) = \int_0^{T-t} \varphi_s(x)\, ds$ and $0 \leqq t < T$.

Gathering relations (a*), (b*), and (c*), we obtain

$$
\begin{aligned}
\mathbf{E}(S_{g_T} \mid \mathcal{F}_t) &= \frac{1}{2}\, H(2S_t - B_t, t) + \mathbf{P}\{T_{|B_t|} < T - t\}(S_t - S_{g_t}) + S_{g_t} \\
&= \frac{1}{2}\, H(2S_t - B_t, t) + \int_0^{T-t} \frac{|B_t|}{\sqrt{2\pi s^3}}\, \exp\bigg\{-\frac{|B_t|^2}{2s}\bigg\}\, ds \cdot (S_t - S_{g_t}) + S_{g_t}. \qquad (81)
\end{aligned}
$$

Applying the Itô formula to $H(X_t, t)$ with $X_t = 2S_t - B_t$ (Bessel process of order 3),

we find that for $t < T$

$$H(2S_t - B_t, t) = H(0,0) - \int_0^t \frac{\partial}{\partial x} H(X_u, u)\, dB_u + A_t, \tag{82}$$

where $(A_t)_{t<T}$ is a process of bounded variation. From this by virtue of (50) and (51) we have

$$H(2S_t - B_t, t) = \sqrt{\frac{2T}{\pi}} + \int_0^t \Psi\left(\frac{2S_u - B_u}{\sqrt{T-u}}\right) dB_u + A_t, \tag{83}$$

where $(A_t)_{t<T}$ is a continuous process of bounded variation.

Let

$$\widetilde{H}(x,t) = \int_0^{T-t} \frac{x}{\sqrt{2\pi s^3}}\, e^{-x^2/(2s)}\, ds \quad \left(= \Psi\left(\frac{x}{\sqrt{T-t}}\right)\right). \tag{84}$$

Applying the Itô–Tanaka formula to $\widetilde{H}(|B_t|, t)$, we find that

$$\begin{aligned} d\widetilde{H}(|B_t|,t) &= \Psi\left(\frac{|B_t|}{\sqrt{T-t}}\right) = -2\varphi\left(\frac{|B_t|}{\sqrt{T-t}}\right) \frac{1}{\sqrt{T-t}}\, \operatorname{sign} B_t\, dB_t + d\widetilde{A}_t \\ &= -2\varphi_{T-t}(B_t) \operatorname{sign} B_t\, dB_t + d\widetilde{A}_t, \end{aligned} \tag{85}$$

where $(\widetilde{A}_t)_{t<T}$ is a process of bounded variation.

Thus, from (81), (83)–(85) we find, neglecting members with bounded variation (cf. the reasoning in the end of section 2′), that for $t < T$,

$$\begin{aligned} \mathbf{E}[S_{g_T} \mid \mathcal{F}_t] = \sqrt{\frac{T}{2\pi}} &+ \frac{1}{2} \int_0^{t\wedge T} \Psi\left(\frac{2S_u - B_u}{\sqrt{T-t}}\right) dB_u \\ &- 2\int_0^{t\wedge T} \varphi_{T-u}(B_u)\, \operatorname{sign} B_u \cdot (S_u - S_{g_u})\, dB_u. \end{aligned} \tag{86}$$

Note that one can omit $\operatorname{sign} B_u$ here, since if $\operatorname{sign} B_u = -1$, then $S_u - S_{g_u} = 0$. Hence,

$$\int_0^{t\wedge T} \varphi_{T-u}(B_u) \operatorname{sign} B_u \cdot (S_u - S_{g_u})\, dB_u = \int_0^{t\wedge T} \varphi_{T-u}(B_u)(S_u - S_{g_u})\, dB_u. \tag{87}$$

The required relation (79) follows for $t < T$ from (86) and (87). In the general case (when $t \geqq 0$) it is sufficient to note that $\lim_{t\uparrow T} \mathbf{E}[S_{g_T} \mid \mathcal{F}_t] = S_{g_T}$ and the limits $\lim_{t\uparrow T}$ of the integrals $\int_0^{t\wedge T} (\,\cdot\,)\, dB_u$ in (86) are equal to the integrals $\int_0^T (\,\cdot\,)\, dB_u$.

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