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Abstract

In this work cylindrical Wiener processes on Banach spaces are defined by means of cylindrical stochastic processes, which are a well considered mathematical object. This approach allows a definition which is a simple straightforward extension of the real-valued situation. We apply this definition to introduce a stochastic integral with respect to cylindrical Wiener processes. Again, this definition is a straightforward extension of the real-valued situation which results now in simple conditions on the integrand. In particular, we do not have to put any geometric constraints on the Banach space under consideration. Finally, we relate this integral to well-known stochastic integrals in literature.

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1 Introduction

Cylindrical Wiener processes appear in a huge variety of models in infinite dimensional spaces as a source of random noise or random perturbation. Almost in the same amount as models with cylindrical Wiener processes one can find different definitions of cylindrical Wiener processes in literature. Most of these definitions suffer from the fact that they do not generalise comprehensibly the real-valued definition to the infinite dimensional situation.

In this note cylindrical Wiener processes on a Banach space are introduced by virtue of the core mathematical object which underlies all these definitions but which is most often not mentioned: a *cylindrical stochastic process*. A cylindrical stochastic process is a generalised stochastic process whose distribution at a fixed time defines only a finite countably additive set function on the Banach space. These finite countably additive set functions are called *cylindrical measures*. We give a very transparent definition of a weakly cylindrical Wiener process as a *cylindrical stochastic process which is Wiener*. Our approach has the side-effect that the appearance of the word *cylindrical* is given a reason.

This definition of a weakly cylindrical Wiener process is a straightforward extension of the real-valued situation but it is immediately seen to be too general in order to be analytically tractable. An obvious request is that the covariance operator of the associated Gaussian cylindrical measures exists and has the analogue properties as in the case of ordinary Gaussian measures on infinite-dimensional spaces. This leads to a second definition of a strongly cylindrical Wiener process.

For strongly cylindrical Wiener processes we derive a representation by a series with independent real-valued Wiener processes. On the other hand, we see, that by such a series a strongly cylindrical Wiener process can be constructed.

The obvious question when is a cylindrical Wiener process actually a Wiener process in the ordinary sense can be answered easily thanks to our approach by the self-suggesting answer: if and only if the underlying cylindrical measure extends to an infinite countably additive set function, i.e. a measure.

Utilising furthermore the approach by cylindrical measures we define a stochastic integral with respect to cylindrical Wiener processes. Again, this definition is a straightforward extension of the real-valued situation which results now in simple conditions on the integrand. In particular, we do not have to put any geometric constraints on the Banach space under consideration. The cylindrical approach yields that the distribution of the integral is a cylindrical measure. We finish with two corollaries giving conditions such that the cylindrical distribution of the stochastic integral extends to a probability measure. These results relate our integral to other well-known integrals in literature.

To summarise, this article introduces two major ideas:

- A cylindrical Wiener process is defined by a straightforward extension of the real-valued situation and the requirement of having a nice covariance operator. It can be seen that most of the existing definitions in literature have the same purpose of

guaranteeing the existence of an analytically tractable covariance operator. Thus, our definition unifies the existing definitions and respects the core mathematical object underlying the idea of a cylindrical Wiener process.

- Describing a random dynamic in an infinite dimensional space by an ordinary stochastic process requires the knowledge that it is a *real* infinite dimensional phenomena, i.e. that there exits a probability measure on the state space. Whilst describing the dynamic by a cylindrical stochastic process it is sufficient to know only the finite dimensional dynamic under the application of all linear bounded functionals. Our introduced stochastic integral allows the development of such a theory of cylindrical stochastic dynamical systems and has the advantage that no constraints are put on the underlying space.

We do not claim that we accomplish very new mathematics in this work. But the innovation might be seen by relating several mathematical objects which results in a straightforward definition of a cylindrical Wiener process and its integral. Even these relations might be well known to some mathematicians but they do not seem to be accessible in a written form.

Our work relies on several ingredients from the theory of cylindrical and ordinary measures on infinite dimensional spaces. Based on the monographs Bogachev [2] and Vakhaniya et al [8] we give an introduction to this subject. The section on γ-radonifying operators is based on the notes by Jan van Neerven [9]. Cylindrical Wiener processes in Banach or Hilbert spaces and their integral are treated for example in the monographs Da Prato and Zabcyzk [3], Kallianpur [5] and Metivier and Pellaumail [6]. In van Gaans [4] the series representation of the cylindrical Wiener process is used to define a stochastic integral in Hilbert spaces and in Berman and Root [1] an approach similar to ours is introduced. The fundamental observation in this work that not every Gaussian cylindrical measure has a nice covariance operator was pointed out to me the first time by Dave Applebaum.

2 Preliminaries

Throughout this notes let U be a separable Banach space with dual U^*. The dual pairing is denoted by $\langle u, u^* \rangle$ for $u \in U$ and $u^* \in U^*$. If V is another Banach space then $L(U, V)$ is the space of all linear, bounded operators from U to V equipped with the operator norm $\|\cdot\|_{U \to V}$.

The Borel σ-algebra is denoted by $\mathcal{B}(U)$. Let Γ be a subset of U^*. Sets of the form

$$\mathcal{Z}(u_1^*, \ldots, u_n^*, B) := \{ u \in U : (\langle u, u_1^* \rangle, \cdots, \langle u, u_n^* \rangle) \in B \},$$

where $u_1^*, \ldots, u_n^* \in \Gamma$ and $B \in \mathcal{B}(\mathbb{R}^n)$ are called *cylindrical sets* or *cylinder with respect to* (U, Γ). The set of all cylindrical sets is denoted by $Z(U, \Gamma)$, which turns out to be an algebra. The generated σ-algebra is denoted by $\mathcal{C}(U, \Gamma)$ and it is called *cylindrical σ-algebra with respect to* (U, Γ). If $\Gamma = U^*$ we write $\mathcal{C}(U) := \mathcal{C}(U, \Gamma)$. If U is separable then both the Borel $\mathcal{B}(U)$ and the cylindrical σ-algebra $\mathcal{C}(U)$ coincide.

A function $\mu : \mathcal{C}(U) \to [0, \infty]$ is called *cylindrical measure on* $\mathcal{C}(U)$, if for each finite subset $\Gamma \subseteq U^*$ the restriction of μ on the σ-algebra $\mathcal{C}(U, \Gamma)$ is a measure. A cylindrical measure is called finite if $\mu(U) < \infty$.

For every function $f : U \to \mathbb{R}$ which is measurable with respect to $\mathcal{C}(U, \Gamma)$ for a finite subset $\Gamma \subseteq U^*$ the integral $\int f(u) \, \mu(du)$ is well defined as a real-valued Lebesgue integral if it exists. In particular, the characteristic function $\varphi_\mu : U^* \to \mathbb{C}$ of a finite cylindrical measure μ is defined by

$$\varphi_\mu(u^*) := \int e^{i\langle u, u^* \rangle} \, \mu(du) \qquad \text{for all } u^* \in U^*.$$

In contrast to measures on infinite dimensional spaces there is an analogue of Bochner's Theorem for cylindrical measures:

Theorem 2.1. *A function* $\varphi : U^* \to \mathbb{C}$ *is a characteristic function of a cylindrical measure on* U *if and only if*

(a) $\varphi(0) = 0$;

(b) φ *is positive definite;*

(c) *the restriction of* φ *to every finite dimensional subset* $\Gamma \subseteq U^*$ *is continuous with respect to the norm topology.*

For a finite set $\{u_1^*, \dots, u_n^*\} \subseteq U^*$ a cylindrical measure μ defines by

$$\mu_{u_1^*, \dots, u_n^*} : \mathcal{B}(\mathbb{R}^n) \to [0, \infty], \qquad \mu_{u_1^*, \dots, u_n^*}(B) := \mu\big(\{u \in U : (\langle u, u_1^* \rangle, \dots, \langle u, u_n^* \rangle) \in B\}\big)$$

a measure on $\mathcal{B}(\mathbb{R}^n)$. We call $\mu_{u_1^*, \dots, u_n^*}$ *the image of the measure* μ *under the mapping* $u \mapsto (\langle u, u_1^* \rangle, \dots, \langle u, u_n^* \rangle)$. Consequently, we have for the characteristic function $\varphi_{\mu_{u_1^*, \dots, u_n^*}}$ of $\mu_{u_1^*, \dots, u_n^*}$ that

$$\varphi_{\mu_{u_1^*, \dots, u_n^*}}(\beta_1, \dots, \beta_n) = \varphi_\mu(\beta_1 u_1^* + \cdots + \beta_n u_n^*)$$

for all $\beta_1, \dots, \beta_n \in \mathbb{R}$.

Cylindrical measures are described uniquely by their characteristic functions and therefore by their one-dimensional distributions μ_{u^*} for $u^* \in U^*$.

3   Gaussian cylindrical measures

A measure μ on $\mathcal{B}(\mathbb{R})$ is called Gaussian with mean $m \in \mathbb{R}$ and variance $\sigma^2 \geqslant 0$ if either $\mu = \delta_m$ and $\sigma^2 = 0$ or it has the density

$$f : \mathbb{R} \to \mathbb{R}_+, \qquad f(s) = \tfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\tfrac{1}{2\sigma^2}(s - m)^2\right).$$

In case of a multidimensional or an infinite dimensional space U a measure μ on $\mathcal{B}(U)$ is called Gaussian if the image measures μ_{u^*} are Gaussian for all $u^* \in U^*$. Gaussian cylindrical measures are defined analogously but due to some reasons explained below we have to distinguish between two cases: weakly and strongly Gaussian.

Definition 3.1. *A cylindrical measure μ on $\mathcal{C}(U)$ is called* weakly Gaussian *if μ_{u^*} is Gaussian on $\mathcal{B}(\mathbb{R})$ for every $u^* \in U^*$.*

Because of well-known properties of Gaussian measures in finite dimensional Euclidean spaces a cylindrical measure μ is weakly Gaussian if and only if $\mu_{u_1^*, \ldots, u_n^*}$ is a Gaussian measure on $\mathcal{B}(\mathbb{R}^n)$ for all $u_1^*, \ldots, u_n^* \in U^*$ and all $n \in \mathbb{N}$.

Theorem 3.2. *Let μ be a weakly Gaussian cylindrical measure on $\mathcal{C}(U)$. Then its characteristic function φ_μ is of the form*

$$\varphi_\mu : U^* \to \mathbb{C}, \qquad \varphi_\mu(u^*) = \exp\left(im(u^*) - \tfrac{1}{2}s(u^*)\right), \qquad (3.1)$$

where the functions $m : U^* \to \mathbb{R}$ and $s : U^* \to \mathbb{R}_+$ are given by

$$m(u^*) = \int_U \langle u, u^* \rangle \, \mu(du), \qquad s(u^*) = \int_U \langle u, u^* \rangle^2 \mu(du) - (m(u^*))^2.$$

Conversely, if μ is a cylindrical measure with characteristic function of the form

$$\varphi_\mu : U^* \to \mathbb{C}, \qquad \varphi_\mu(u^*) = \exp\left(im(u^*) - \tfrac{1}{2}s(u^*)\right),$$

for a linear functional $m : U^* \to \mathbb{R}$ and a quadratic form $s : U^* \to \mathbb{R}_+$, then μ is a weakly Gaussian cylindrical measure.

Proof. Follows from [8, Prop.IV.2.7], see also [8, p.393]. $\square$

Example 3.3. Let H be a separable Hilbert space. Then the function

$$\varphi : H \to \mathbb{C}, \qquad \varphi(u) = \exp(-\tfrac{1}{2}\|u\|_H^2)$$

satisfies the condition of Theorem 3.2 and therefore there exists a weakly Gaussian cylindrical measure γ with characteristic function φ. We call this cylindrical measure *standard Gaussian cylindrical measure on H*. If H is infinite dimensional the cylindrical measure γ is not a measure, see [2, Cor.2.3.2].

Note, that this example might be not applicable for a Banach space U because then $x \mapsto \|x\|_U^2$ need not to be a quadratic form.

For a weakly Gaussian cylindrical measure μ one defines for $u^*, v^* \in U^*$:

$$r(u^*, v^*) := \int_U \langle u, u^* \rangle \langle u, v^* \rangle \, \mu(du) - \int_U \langle u, u^* \rangle \, \mu(du) \int_U \langle u, v^* \rangle \, \mu(du).$$

These integrals exist as μ is a Gaussian measure on the cylindrical σ-algebra generated by u^* and v^*. One defines the *covariance operator Q of μ* by

$$Q : U^* \to (U^*)', \qquad (Qu^*)v^* := r(u^*, v^*) \qquad \text{for all } v^* \in U^*,$$

where $(U^*)'$ denotes the algebraic dual of U^*, i.e. all linear but not necessarily continuous functionals on U^*. Hence, the characteristic function φ_μ of μ can be written as

$$\varphi_\mu : U^* \to \mathbb{C}, \qquad \varphi_\mu(u^*) = \exp\left(im(u^*) - (Qu^*)u^*\right).$$

The cylindrical measure μ is called *centered* if $m(u^*) = 0$ for all $u^* \in U^*$.
If μ is a Gaussian measure or more general, a measure of weak order 2, i.e.

$$\int_U |\langle u, u^* \rangle|^2 \, \mu(du) < \infty \qquad \text{for all } u^* \in U^*,$$

then the covariance operator Q is defined in the same way as above. However, in this case it turns out that Qu^* is not only continuous and thus in U^{**} but even in U considered as a subspace of U^{**}, see [8, Thm.III.2.1]. This is basically due to properties of the Pettis integral in Banach spaces. For cylindrical measures we have to distinguish this property and define:

Definition 3.4. *A centred weakly Gaussian cylindrical measure μ on $\mathcal{C}(U)$ is called strongly Gaussian if the covariance operator $Q : U^* \to (U^*)'$ is U-valued.*

Below Example 3.6 gives an example of a weakly Gaussian cylindrical measure which is not strongly. This example can be constructed in every infinite dimensional space in particular in every Hilbert space.
Strongly Gaussian cylindrical measures exhibit an other very important property:

Theorem 3.5. *For a cylindrical measure μ on $\mathcal{C}(U)$ the following are equivalent:*

(a) *μ is a continuous linear image of the standard Gaussian cylindrical measure on a Hilbert space;*

(b) *there exists a symmetric positive operator $Q : U^* \to U$ such that*

$$\varphi_\mu(u^*) = \exp\left(-\tfrac{1}{2}\langle Qu^*, u^* \rangle\right) \qquad \text{for all } u^* \in U^*.$$

Proof. See [8, Prop.VI.3.3]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 3.5 provides an example of a weakly Gaussian cylindrical measure which is not strongly Gaussian:

Example 3.6. For a discontinuous linear functional $f : U^* \to \mathbb{R}$ define

$$\varphi : U^* \to \mathbb{C}, \qquad \varphi(u^*) = \exp\left(-\frac{1}{2}(f(u^*))^2\right).$$

Then φ is the characteristic function of a weakly Gaussian cylindrical measure due to Theorem 3.2 but this measure can not be strongly Gaussian by Theorem 3.5 because every symmetric positive operator $Q : U^* \to U$ is continuous.

4 Reproducing kernel Hilbert space

According to Theorem 3.5 a centred strongly Gaussian cylindrical measure is the image of the standard Gaussian cylindrical measure on a Hilbert space H under an operator $F \in L(H, U)$. In this section we introduce a possible construction of this Hilbert space H and the operator F.

For this purpose we start with a bounded linear operator $Q : U^* \to U$, which is positive,

$$\langle Qu^*, u^* \rangle \geqslant 0 \qquad \text{for all } u^* \in U^*,$$

and symmetric,

$$\langle Qu^*, v^* \rangle = \langle Qv^*, u^* \rangle \qquad \text{for all } u^*, v^* \in U^*.$$

On the range of Q we define a bilinear form by

$$[Qu^*, Qv^*]_{H_Q} := \langle Qu^*, v^* \rangle.$$

It can be seen easily that this defines an inner product $[\cdot, \cdot]_{H_Q}$. Thus, the range of Q is a pre-Hilbert space and we denote by H_Q the real Hilbert space obtained by its completion with respect to $[\cdot, \cdot]_{H_Q}$. This space will be called the *reproducing kernel Hilbert space associated with Q*.

In the following we collect some properties of the reproducing kernel Hilbert space and its embedding:

(a) The inclusion mapping from the range of Q into U is continuous with respect to the inner product $[\cdot, \cdot]_{H_Q}$. For, we have

$$\|Qu^*\|_{H_Q}^2 = |\langle Qu^*, u^* \rangle| \leqslant \|Q\|_{U^* \to U} \|u^*\|^2,$$

which allows us to conclude

$$|\langle Qu^*, v^* \rangle| = \left| [Qu^*, Qv^*]_{H_Q} \right| \leqslant \|Qu^*\|_{H_Q} \|Qv^*\|_{H_Q} \leqslant \|Qu^*\|_{H_Q} \|Q\|_{U^* \to H_Q} \|v^*\|.$$

Therefore, we end up with

$$\|Qu^*\| = \sup_{\|v^*\| \leqslant 1} |\langle Qu^*, v^* \rangle| \leqslant \|Q\|_{U^* \to H_Q} \|Qu^*\|_{H_Q}.$$

Thus the inclusion mapping is continuous on the range of Q and it extends to a bounded linear operator i_Q from H_Q into U.

(b) The operator Q enjoys the decomposition

$$Q = i_Q i_Q^*.$$

For the proof we define $h_{u^*} := Qu^*$ for all $u^* \in U^*$. Then we have $i_Q(h_{u^*}) = Qu^*$ and

$$[h_{u^*}, h_{v^*}]_{H_Q} = \langle Qu^*, v^* \rangle = \langle i_Q(h_{u^*}), v^* \rangle = [h_{u^*}, i_Q^* v^*]_{H_Q}.$$

Because the range of Q is dense in H_Q we arrive at

$$h_{v^*} = i_Q^* v^* \qquad \text{for all } v^* \in U^* \tag{4.2}$$

which finally leads to

$$Q v^* = i_Q(h_{v^*}) = i_Q(i_Q^* v^*) \qquad \text{for all } v^* \in U^*.$$

(c) By (4.2) it follows immediately that the range of i_Q^* is dense in H_Q.

(d) the inclusion mapping i_Q is injective. For, if $i_Q h = 0$ for some $h \in H_Q$ it follows that

$$[h, i_Q^* u^*]_{H_Q} = \langle i_Q h, u^* \rangle = 0 \qquad \text{for all } u^* \in U^*,$$

which results in $h = 0$ because of (c).

(e) If U is separable then H_Q is also separable.

Remark 4.1. Let μ be a centred strongly Gaussian cylindrical measure with covariance operator $Q : U^* \to U$. Because Q is positive and symmetric we can associate with Q the reproducing kernel Hilbert space H_Q with the inclusion mapping i_Q as constructed above. For the image $\gamma \circ i_Q^{-1}$ of the standard cylindrical measure γ on H_Q we calculate

$$\begin{aligned}
\varphi_{\gamma \circ i_Q^{-1}}(u^*) &= \int_U e^{i\langle u, u^* \rangle} (\gamma \circ i_Q^{-1})(du) \\
&= \int_{H_Q} e^{i\langle h, i_Q^* u^* \rangle} \gamma(dh) \\
&= \exp\left(-\tfrac{1}{2} \left\| i_Q^* u^* \right\|_{H_Q}^2 \right) \\
&= \exp\left(-\tfrac{1}{2} \langle Q u^*, u^* \rangle \right).
\end{aligned}$$

Thus, $\mu = \gamma \circ i_Q^{-1}$ and we have found one possible Hilbert space and operator satisfying the condition in Theorem 3.5.

But note, that there might exist other Hilbert spaces exhibiting this feature. But the reproducing kernel Hilbert space is characterised among them by a certain "minimal property", see [2].

5 γ-radonifying operators

This section follows the notes [9].

Let $Q : U^* \to U$ be a positive symmetric operator and H the reproducing kernel Hilbert space with the inclusion mapping $i_Q : H \to U$. If U is a Hilbert space then it is a well known result by Mourier ([8, Thm. IV.2.4]) that Q is the covariance operator

of a Gaussian measure on U if and only if Q is nuclear or equivalently if i_Q is Hilbert-Schmidt. By Remark 4.1 it follows that the cylindrical measure $\gamma \circ i_Q^{-1}$ extends to a Gaussian measure on $\mathcal{B}(U)$ and Q is the covariance operator of this Gaussian measure. The following definition generalises this property of $i_Q : H \to U$ to define by $Q := i_Q i_Q^*$ a covariance operator to the case when U is a Banach space:

Definition 5.1. *Let γ be the standard Gaussian cylindrical measure on a separable Hilbert space H. A linear bounded operator $F : H \to U$ is called γ-radonifying if the cylindrical measure $\gamma \circ F^{-1}$ extends to a Gaussian measure on $\mathcal{B}(U)$.*

Theorem 5.2. *Let γ be the standard Gaussian cylindrical measure on a separable Hilbert space H with orthonormal basis $(e_n)_{n \in \mathbb{N}}$ and let $(G_n)_{n \in \mathbb{N}}$ be a sequence of independent standard real normal random variables. For $F \in L(H, U)$ the following are equivalent:*

(a) *F is γ-radonifying;*

(b) *the operator $FF^* : U^* \to U$ is the covariance operator of a Gaussian measure μ on $\mathcal{B}(U)$;*

(c) *the series $\displaystyle\sum_{k=1}^{\infty} G_k F e_k$ converges a.s. in U.*

(d) *the series $\displaystyle\sum_{k=1}^{\infty} G_k F e_k$ converges in $L^p(\Omega; U)$ for some $p \in [1, \infty)$.*

(e) *the series $\displaystyle\sum_{k=1}^{\infty} G_k F e_k$ converges in $L^p(\Omega; U)$ for all $p \in [1, \infty)$.*

In this situation we have for every $p \in [1, \infty)$:

$$\int_U \|u\|^p \, \mu(du) = E \left\| \sum_{k=1}^{\infty} G_k F e_k \right\|^p .$$

Proof. As in Remark 4.1 we obtain for the characteristic function of $\nu := \gamma \circ F^{-1}$:

$$\varphi_\nu(u^*) = \exp\left(-\tfrac{1}{2}\langle FF^* u^*, u^* \rangle\right) \qquad \text{for all } u^* \in U^*.$$

This establishes the first equivalence between (a) and (b). The proofs of the remaining part can be found in [9, Prop.4.2]. $\qquad\square$

To show that γ-radonifying generalise Hilbert-Schmidt operators to Banach spaces we prove the result by Mourier mentioned already above. Other proofs only relying on Hilbert theory can be found in the literature.

Corollary 5.3. *If H and U are separable Hilbert spaces then the following are equivalent for $F \in L(H, U)$:*

(a) *F is γ-radonifying;*

(b) *F is Hilbert-Schmidt.*

Proof. Let $(e_k)_{k \in \mathbb{N}}$ be an orthonormal basis of H. The equivalence follows immediately from

$$E \left\| \sum_{k=m}^{n} G_k F e_k \right\|^2 = \sum_{k=m}^{n} \|F e_k\|^2$$

for every family $(G_k)_{k \in \mathbb{N}}$ of independent standard normal random variables. $\square$

In general, the property of being γ-radonifying is not so easily accessible as Hilbert-Schmidt operators in case of Hilbert spaces. However, for some specific Banach spaces, as L^p or l^p spaces, the set of all covariance operators of Gaussian measures can be also described more precisely, see [8, Thm.V.5.5 and Thm.V.5.6].
It turns out that the set of all γ-radonifying operators can be equipped with a norm such that it is a Banach space, see [9, Thm. 4.14].

6   Cylindrical stochastic processes

Let $(\Omega, \mathcal{A}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$.
Similarly to the correspondence between measures and random variables there is an analogue random object associated to cylindrical measures:

Definition 6.1. *A* cylindrical random variable X in U *is a linear map*

$$X : U^* \to L^0(\Omega).$$

A cylindrical process X in U *is a family* $(X(t) : t \geqslant 0)$ *of cylindrical random variables in* U.

The characteristic function of a cylindrical random variable X is defined by

$$\varphi_X : U^* \to \mathbb{C}, \qquad \varphi_X(u^*) = E[\exp(iXu^*)].$$

The concepts of cylindrical measures and cylindrical random variables match perfectly. Because the characteristic function of a cylindrical random variable is positive-definite and continuous on finite subspaces there exists a cylindrical measure μ with the same characteristic function. We call μ the *cylindrical distribution of* X. Vice versa, for every cylindrical measure μ on $\mathcal{C}(U)$ there exists a probability space $(\Omega, \mathcal{A}, P)$ and a cylindrical random variable $X : U^* \to L^0(\Omega)$ such that μ is the cylindrical distribution of X, see [8, VI.3.2].

Example 6.2. A cylindrical random variable $X : U^* \to L^0(\Omega)$ is called weakly Gaussian, if Xu^* is Gaussian for all $u^* \in U^*$. Thus, X defines a weakly Gaussian cylindrical

measure μ on $\mathcal{C}(U)$. The characteristic function of X coincide with the one of μ and is of the form

$$\varphi_X(u^*) = \exp(im(u^*) - \tfrac{1}{2}s(u^*))$$

with $m : U^* \to \mathbb{R}$ linear and $s : U^* \to \mathbb{R}_+$ a quadratic form. If X is strongly Gaussian there exists a covariance operator $Q : U^* \to U$ such that

$$\varphi_X(u^*) = \exp(im(u^*) - \tfrac{1}{2}\langle Qu^*, u^* \rangle).$$

Because $\varphi_X(u^*) = \varphi_{Xu^*}(1)$ it follows

$$E[Xu^*] = m(u^*) \qquad \text{and} \qquad \text{Var}[Xu^*] = \langle Qu^*, u^* \rangle.$$

In the same way by comparing the characteristic function

$$\varphi_{Xu^*, Xv^*}(\beta_1, \beta_2) = E\left[\exp\left(i(\beta_1 Xu^* + \beta_2 Xv^*)\right)\right] = E\left[\exp\left(i(X(\beta_1 u^* + \beta_2 v^*))\right)\right]$$

for $\beta_1, \beta_2 \in \mathbb{R}$ with the characteristic function of the two-dimensional Gaussian vector (Xu^*, Xv^*) we may conclude

$$\text{Cov}[Xu^*, Xv^*] = \langle Qu^*, v^* \rangle.$$

Let H_Q denote the reproducing kernel Hilbert space of the covariance operator Q. Then we obtain

$$E\left|Xu^* - m(u^*)\right|^2 = \text{Var}[Xu^*] = \langle Qu^*, u^* \rangle = \left\|i_Q^* u^*\right\|_{H_Q}^2.$$

The cylindrical process $X = (X(t) : t \geqslant 0)$ is called *adapted to a given filtration* $\{\mathcal{F}_t\}_{t \geqslant 0}$, if $X(t)u^*$ is $\mathcal{F}_t$-measurable for all $t \geqslant 0$ and all $u^* \in U^*$. The cylindrical process X has weakly independent increments if for all $0 \leqslant t_0 < t_1 < \cdots < t_n$ and all $u_1^*, \ldots, u_n^* \in U^*$ the random variables

$$(X(t_1) - X(t_0))u_1^*, \ldots, (X(t_n) - X(t_{n-1}))u_n^*$$

are independent.

Remark 6.3. Our definition of cylindrical processes is based on the definitions in [1] and [8]. In [6] and [7] cylindrical random variables are considered which have values in $L^p(\Omega)$ for $p > 0$. They assume in addition that a cylindrical random variable is continuous. The continuity of a cylindrical variable is reflected by continuity properties of its characteristic function, see [8, Prop.IV. 3.4]. The notion of weakly independent increments origins from [1].

Example 6.4. Let $Y = (Y(t) : t \geqslant 0)$ be a stochastic process with values in a separable Banach space U. Then $\hat{Y}(t)u^* := \langle Y(t), u^* \rangle$ for $u^* \in U^*$ defines a cylindrical process $\hat{Y} = (\hat{Y}(t) : t \geqslant 0)$. The cylindrical process $\hat{Y}$ is adapted if and only if Y is also adapted and $\hat{Y}$ has weakly independent increments if and only if Y has also independent increments. Both statements are due to the fact that the Borel and the cylindrical σ-algebras coincide for separable Banach spaces due to Pettis' measurability theorem.

An $\mathbb{R}^n$-*valued Wiener process* $B = (B(t) : t \geqslant 0)$ is an adapted stochastic process with independent, stationary increments $B(t) - B(s)$ which are normally distributed with expectation $E[B(t) - B(s)] = 0$ and covariance $\text{Cov}[B(t) - B(s), B(t) - B(s)] = |t - s| \, C$ for a non-negative definite symmetric matrix C. If $C = \text{Id}$ we call B a *standard* Wiener process.

Definition 6.5. *An adapted cylindrical process* $W = (W(t) : t \geqslant 0)$ *in* U *is a* weakly cylindrical Wiener process, *if*

(a) *for all* $u_1^*, \ldots, u_n^* \in U^*$ *and* $n \in \mathbb{N}$ *the* $\mathbb{R}^n$-*valued stochastic process*

$$\left((W(t)u_1^*, \ldots, W(t)u_n^*) : t \geqslant 0 \right)$$

is a Wiener process.

Our definition of a weakly cylindrical Wiener process is an obvious extension of the definition of a finite-dimensional Wiener processes and is exactly in the spirit of cylindrical processes. The multidimensional formulation in Definition 6.5 would be already necessary to define a finite-dimensional Wiener process by this approach and it allows to conclude that a weakly cylindrical Wiener process has weakly independent increments. The latter property is exactly what is needed in addition to an one-dimensional formulation:

Lemma 6.6. *For an adapted cylindrical process* $W = (W(t) : t \geqslant 0)$ *the following are equivalent:*

(a) W *is a weakly cylindrical Wiener process;*

(b) W *satisfies*

    (i) W *has weakly independent increments;*

    (ii) $(W(t)u^* : t \geqslant 0)$ *is a Wiener process for all* $u^* \in U^*$.

Proof. We have only to show that (b) implies (a). By linearity we have

$$\beta_1(W(t) - W(s))u_1^* + \cdots + \beta_n(W(t) - W(s))u_n^* = (W(t) - W(s)) \left(\sum_{i=1}^n \beta_i u_i^* \right),$$

for all $\beta_i \in \mathbb{R}$ and $u_i^* \in U^*$ which shows that the increments of $((W(t)u_1^*, \ldots, W(t)u_n^*) : t \geqslant 0)$ are normally distributed and stationary. The independence of the increments follows by (i). $\qquad\square$

Because $W(1)$ is a centred weakly Gaussian cylindrical random variable there exists a weakly Gaussian cylindrical measure μ such that

$$\varphi_{W(1)}(u^*) = E[\exp(iW(1)u^*)] = \varphi_\mu(u^*) = \exp\left(-\tfrac{1}{2}s(u^*)\right)$$

for a quadratic form $s : U^* \to \mathbb{R}_+$. Therefore, one obtains

$$\varphi_{W(t)}(u^*) = E[\exp(iW(t)u^*)] = E[\exp(iW(1)(tu^*))] = \exp\left(-\tfrac{1}{2}t^2 s(u^*)\right)$$

for all $t \geqslant 0$. Thus, the cylindrical distributions of $W(t)$ for all $t \geqslant 0$ are only determined by the cylindrical distribution of $W(1)$.

Definition 6.7. *A weakly cylindrical Wiener process $(W(t) : t \geqslant 0)$ is called* strongly cylindrical Wiener process*, if*

(b) *the cylindrical distribution μ of $W(1)$ is strongly Gaussian.*

The additional condition on a weakly cylindrical Wiener process to be strongly requests the existence of an U-valued covariance operator for the Gaussian cylindrical measure. To our knowledge weakly cylindrical Wiener processes are not defined in the literature and (strongly) cylindrical Wiener processes are defined by means of other conditions. Often, these definition are formulated by assuming the existence of the reproducing kernel Hilbert space. But this implies the existence of the covariance operator. Another popular way for defining cylindrical Wiener processes is by means of a series. We will see in the next chapter that this is also equivalent to our definition.
Later, we will compare a strongly cylindrical Wiener process with an U-valued Wiener process. Also the latter is defined as a direct generalisation of a real-valued Wiener process:

Definition 6.8. *An adapted U-valued stochastic process $(W(t) : t \geqslant 0)$ is called a* Wiener process *if*

(a) $W(0) = 0$ *P-a.s.;*

(b) W *has independent, stationary increments;*

(c) *there exists a Gaussian covariance operator $Q : U^* \to U$ such that*

$$W(t) - W(s) \overset{d}{=} N(0, (t-s)Q) \qquad \textit{for all } 0 \leqslant s \leqslant t.$$

If U is finite dimensional then Q can be any symmetric, positive semi-definite matrix. In case that U is a Hilbert space we know already that Q has to be nuclear. For the general case of a Banach space U we can describe the possible Gaussian covariance operator by Theorem 5.2.
It is obvious that every U-valued Wiener process W defines a strongly cylindrical Wiener process $(\hat{W}(s) : t \geqslant 0)$ in U by $\hat{W}(s)u^* := \langle W(s), u^* \rangle$. For the converse question, if a cylindrical Wiener process can be represented in such a way by an U-valued Wiener process we will derive later necessary and sufficient conditions.

7 Representations of cylindrical Wiener processes

In this section we derive representations of cylindrical Wiener processes and U-valued Wiener processes in terms of some series. In addition, these representations can also serve as a construction of these processes, see Remark 7.5.

Theorem 7.1. *For an adapted cylindrical process $W := (W(t) : t \geqslant 0)$ the following are equivalent:*

(a) *W is a strongly cylindrical Wiener process;*

(b) *there exist a Hilbert space H with an orthonormal basis $(e_n)_{n \in \mathbb{N}}$, $F \in L(H, U)$ and independent real-valued standard Wiener processes $(B_n)_{n \in \mathbb{N}}$ such that*

$$W(t)u^* = \sum_{k=1}^{\infty} \langle Fe_k, u^* \rangle B_k(t) \qquad in \ L^2(\Omega) \ for \ all \ u^* \in U^*.$$

Proof. (b) $\Rightarrow$ (a) By Doob's inequality we obtain for any $m, n \in \mathbb{N}$

$$E\left[\sup_{t \in [0,T]} \left| \sum_{k=n}^{n+m} \langle Fe_k, u^* \rangle B_k(t) \right|^2 \right] \leqslant 4E \left| \sum_{k=n}^{n+m} \langle Fe_k, u^* \rangle B_k(T) \right|^2$$

$$= 4T \sum_{k=n}^{n+m} \langle e_k, F^* u^* \rangle^2$$

$$\to 0 \qquad for \ m, n \to \infty.$$

Thus, for every $u^* \in U^*$ the random variables $W(t)u^*$ are well defined and form a cylindrical process $(W(t) : t \geqslant 0)$. For any $0 = t_0 < t_1 < \cdots < t_m$ and $\beta_k \in \mathbb{R}$ we calculate

$$E\left[\exp\left(i \sum_{k=0}^{m-1} \beta_k (W(t_{k+1})u^* - W(t_k)u^*) \right) \right]$$

$$= \lim_{n \to \infty} E\left[\exp\left(i \sum_{k=0}^{m-1} \beta_k \sum_{l=1}^{n} \langle Fe_l, u^* \rangle (B_l(t_{k+1}) - B_l(t_k)) \right) \right]$$

$$= \lim_{n \to \infty} \prod_{k=0}^{m-1} \prod_{l=1}^{n} E\left[\exp\left(i\beta_k \langle Fe_l, u^* \rangle (B_l(t_{k+1}) - B_l(t_k)) \right) \right]$$

$$= \lim_{n \to \infty} \prod_{k=0}^{m-1} \prod_{l=1}^{n} \exp\left(-\tfrac{1}{2} \beta_k^2 \langle Fe_l, u^* \rangle^2 (t_{k+1} - t_k) \right)$$

$$= \prod_{k=0}^{m-1} \exp\left(-\tfrac{1}{2} \beta_k^2 \|F^* u^*\|_H^2 (t_{k+1} - t_k) \right),$$

which shows that $(W(t)u^* : t \geqslant 0)$ has independent, stationary Gaussian increments. Because the partial sums converge uniformly on every finite interval the process $(W(t)u^* : t \geqslant 0)$ has a.s. continuous paths and is therefore established as a real-valued Wiener process.

The calculation above of the characteristic function yields

$$E\left[\exp(iW(1)u^*)\right] = \exp\left(-\tfrac{1}{2}\left\|F^*u^*\right\|_H^2\right) = \exp\left(-\tfrac{1}{2}\langle FF^*u^*, u^*\rangle^2\right).$$

Hence, the process W is a strongly cylindrical Wiener process with covariance operator $Q := FF^*$.

(a) $\Rightarrow$ (b): Let $Q : U^* \to U$ be the covariance operator of $W(1)$ and H its reproducing kernel Hilbert space with the inclusion mapping $i_Q : H \to U$. Because the range of i_Q^* is dense in H and H is separable there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}} \subseteq \mathrm{range}(i_Q^*)$ of H. We choose $u_n^* \in U^*$ such that $i_Q^* u_n^* = e_n$ for all $n \in \mathbb{N}$ and define $B_n(t) := W(t)u_n^*$. Then we obtain

$$E\left|\sum_{k=1}^{n}\langle i_Q e_k, u^*\rangle B_k(t) - W(t)u^*\right|^2 = E\left[W(t)\left(\sum_{k=1}^{n}\langle i_Q e_k, u^*\rangle u_k^* - u^*\right)\right]^2$$

$$= t\left\|i_Q^*\left(\sum_{k=1}^{n}\langle i_Q e_k, u^*\rangle u_k^* - u^*\right)\right\|_H^2$$

$$= t\left\|\sum_{k=1}^{n}[e_k, i_Q^* u^*]_{H_Q} e_k - i_Q^* u^*\right\|_H^2$$

$$\to 0 \qquad \text{for } n \to \infty.$$

Thus, W has the required representation and it remains to establish that the Wiener processes $B_n := (B_n(t) : t \geqslant 0)$ are independent. Because of the Gaussian distribution it is sufficient to establish that $B_n(s)$ and $B_m(t)$ for any $s \leqslant t$ and $m, n \in \mathbb{N}$ are independent:

$$E[B_n(s)B_m(t)] = E[W(s)u_n^* W(t)u_m^*]$$
$$= E[W(s)u_n^*(W(t)u_m^* - W(s)u_m^*)] + E[W(s)u_n^* W(s)u_m^*].$$

The first term is zero by Theorem 6.6 and for the second term we obtain

$$E[W(s)u_n^* W(s)u_m^*] = s\langle Qu_n^*, u_m^*\rangle = s[i_Q^* u_n^*, i_Q^* u_m^*]_{H_Q} = s[e_n, e_m]_{H_Q} = s\delta_{n,m}.$$

Hence, $B_n(s)$ and $B_m(t)$ are uncorrelated and therefore independent. $\qquad\square$

Remark 7.2. The proof has shown that the Hilbert space H in part (b) can be chosen as the reproducing kernel Hilbert space associated to the Gaussian cylindrical distribution of $W(1)$. In this case the function $F : H \to U$ is the inclusion mapping i_Q.

Remark 7.3. Let H be a separable Hilbert space with orthonormal basis $(e_k)_{k \in \mathbb{N}}$ and $(B_k(t) : t \geqslant 0)$ be independent real-valued Wiener processes. By setting $U = H$ and $F = \mathrm{Id}$ Theorem 7.1 yields that a strongly cylindrical Wiener process $(W_H(t) : t \geqslant 0)$ is defined by

$$W_H(t)h = \sum_{k=1}^{\infty} \langle e_k, h \rangle B_k(t) \qquad \text{for all } h \in H.$$

The covariance operator of W_H is $\mathrm{Id} : H \to H$. This is the approach how a cylindrical Wiener process is defined for example in [2] and [10].

If in addition U is a separable Banach space and $F \in L(H, U)$ we obtain by defining

$$W(t)u^* := W_H(t)(F^* u^*) \qquad \text{for all } u^* \in U^*,$$

a strongly cylindrical Wiener process $(W(t) : t \geqslant 0)$ with covariance operator $Q := FF^*$ according to our Definition 6.7.

Theorem 7.4. *For an adapted U-valued process $W := (W(t) : t \geqslant 0)$ the following are equivalent:*

(a) *W is an U-valued Wiener process;*

(b) *there exist a Hilbert space H with an orthonormal basis $(e_n)_{n \in \mathbb{N}}$, a γ-radonifying operator $F \in L(H, U)$ and independent real-valued standard Wiener processes $(B_n)_{n \in \mathbb{N}}$ such that*

$$W(t) = \sum_{k=1}^{\infty} F e_k B_k(t) \qquad in \ L^2(\Omega; U).$$

Proof. (b) $\Rightarrow$ (a): As in the proof of Theorem 7.1 we obtain by Doob's Theorem (but here for infinite-dimensional spaces) for any $m, n \in \mathbb{N}$

$$E \left[\sup_{t \in [0,T]} \left\| \sum_{k=n}^{n+m} F e_k B_k(t) \right\|^2 \right] \leqslant 4E \left\| \sum_{k=n}^{n+m} F e_k B_k(T) \right\|^2$$
$$\to 0 \qquad \text{for } m, n \to \infty,$$

where the convergence follows by Theorem 5.2 because F is γ-radonifying. Thus, the random variables $W(t)$ are well defined and form an U-valued stochastic process $W := (W(t) : t \geqslant 0)$. As in the proof of Theorem 7.1 we can proceed to establish that W is an U-valued Wiener process.

(a) $\Rightarrow$ (b): By Theorem 7.1 there exist a Hilbert space H with an orthonormal basis $(e_n)_{n \in \mathbb{N}}$, $F \in L(H, U)$ and independent real-valued standard Wiener processes $(B_n)_{n \in \mathbb{N}}$ such that

$$\langle W(t), u^* \rangle = \sum_{k=1}^{\infty} \langle F e_k, u^* \rangle B_k(t) \qquad in \ L^2(\Omega) \text{ for all } u^* \in U^*.$$

The Itô-Nisio Theorem [8, Thm.V.2.4] implies

$$W(t) = \sum_{k=1}^{\infty} F e_k B_k(t) \qquad P\text{-a.s. for all } u^* \in U^*$$

and a result by Hoffmann-Jorgensen [8, Cor.2 in V.3.3] yields the convergence in $L^2(\Omega; U)$. Theorem 5.2 verifies F as γ-radonifying. $\qquad\square$

Remark 7.5. In the proofs of the implication from (b) to (a) we established in both Theorems 7.1 and 7.4 even more than required: we established the convergence of the series in the specified sense without assuming the existence of the limit process, respectively. This means, that we can read these results also as a construction principle of cylindrical or U-valued Wiener processes without assuming the existence of the considered process a priori.

The construction of these random objects differs significantly in the required conditions on the involved operator F. For a cylindrical Wiener process no conditions are required, however, for an U-valued Wiener process we have to guarantee $Q = FF^*$ to be a covariance operator of a Gaussian measure by assuming F to be γ-radonifying.

8   When is a cylindrical Wiener process U-valued ?

In this section we give equivalent conditions for a strongly cylindrical Wiener process to be an U-valued Wiener process. To be more precise a cylindrical random variable $X : U^* \to L^0(\Omega)$ is called *induced by a random variable* $Z : \Omega \to U$, if P-a.s.

$$Xu^* = \langle Z, u^* \rangle \qquad \text{for all } u^* \in U^*.$$

This definition generalises in an obvious way to cylindrical processes.

Because of the correspondence to cylindrical measures the question whether a cylindrical random variable is induced by an U-valued random variable is reduced to the question whether the cylindrical measure extends to a Radon measure ([8, Thm. Vi.3.1]). There is a classical answer by Prokhorov ([8, Thm. VI.3.2]) to this question in terms of tightness. A cylindrical measure μ on $\mathcal{C}(U)$ is called *tight* if for each $\varepsilon > 0$ there exists a compact subset $K \subseteq U$ such that

$$\mu(K) \geqslant 1 - \varepsilon.$$

In case of non-separable Banach spaces U one has to be more careful because then compact sets are not necessarily admissible arguments of a cylindrical measure.

Theorem 8.1. *For a strongly cylindrical Wiener process $W := (W(t) : t \geqslant 0)$ with covariance operator $Q = i_Q i_Q^*$ the following are equivalent:*

(a) *W is induced by an U-valued Wiener process;*

(b) *i_Q is γ-radonifying;*

(c) *the cylindrical distribution of $W(1)$ is tight;*

(d) *the cylindrical distribution of $W(1)$ extends to a measure.*

Proof. (a) $\Rightarrow$ (b) If there exists an U-valued Wiener process $(\tilde{W}(t) : t \geqslant 0)$ with $W(t)u^* = \langle \tilde{W}(t), u^* \rangle$ for all $u^* \in U^*$, then $\tilde{W}(1)$ has a Gaussian distribution with covariance operator Q. Thus, i_Q is γ-radonifying by Theorem 5.2.
(b)$\Leftrightarrow$ (c) $\Leftrightarrow$ (d) This is Prokhorov's Theorem on cylindrical measures.
(b)$\Rightarrow$(a) Due to Theorem 7.1 there exist an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of the reproducing kernel Hilbert space of Q and independent standard real-valued Wiener process $(B_k(t) : t \geqslant 0)$ such that

$$W(t)u^* = \sum_{k=1}^{\infty} \langle i_Q e_k, u^* \rangle B_k(t) \qquad \text{for all } u^* \in U^*.$$

On the other hand, because i_Q is γ-radonifying Theorem 7.4 yields that

$$\tilde{W}(t) = \sum_{k=1}^{\infty} i_Q e_k B_k(t)$$

defines an U-valued Wiener process $(\tilde{W}(t) : t \geqslant 0)$. Obviously, we have $W(t)u^* = \langle \tilde{W}(t), u^* \rangle$ for all u^*. $\qquad \square$

If U is a separable Hilbert space we can replace the condition (b) by

 (b$'$) i_Q is Hilbert-Schmidt

because of Theorem 5.3.

9   Integration

In this section we introduce an integral with respect to a strongly cylindrical Wiener process $W = (W(t) : t \geqslant 0)$ in U. The integrand is a stochastic process with values in $L(U, V)$, the set of bounded linear operators from U to V, where V denotes a separable Banach space. For that purpose we assume for W the representation according to Theorem 7.1:

$$W(t)u^* = \sum_{k=1}^{\infty} \langle i_Q e_k, u^* \rangle B_k(t) \qquad \text{in } L^2(\Omega) \text{ for all } u^* \in U^*,$$

where H is the reproducing kernel Hilbert space of the covariance operator Q with the inclusion mapping $i_Q : H \to U$ and an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of H. The real-valued standard Wiener processes $(B_k(t) : t \geqslant 0)$ are defined by $B_k(t) = W(t)u_k^*$ for some $u_k^* \in U^*$ with $i_Q^* u_k^* = e_k$.

Definition 9.1. *The set $M_T(U, V)$ contains all random variables $\Phi : [0, T] \times \Omega \to L(U, V)$ such that:*

(a) $(t, \omega) \mapsto \Phi^*(t, \omega)v^*$ is $\mathcal{B}[0, T] \otimes \mathcal{A}$ measurable for all $v^* \in V^*$;

(b) $\omega \mapsto \Phi^*(t, \omega)v^*$ is $\mathcal{F}_t$-measurable for all $v^* \in V^*$ and $t \in [0, T]$;

(c) $\int_0^T E \|\Phi^*(s, \cdot)v^*\|_{U^*}^2 \, ds < \infty$ for all $v^* \in V^*$.

As usual we neglect the dependence of $\Phi \in M_T(U, V)$ on ω and write $\Phi(s)$ for $\Phi(s, \cdot)$ as well as for the dual operator $\Phi^*(s) := \Phi^*(s, \cdot)$ where $\Phi^*(s, \omega)$ denotes the dual operator of $\Phi(s, \omega) \in L(U, V)$.

We define the candidate for a stochastic integral:

Definition 9.2. For $\Phi \in M_T(U, V)$ we define

$$I_t(\Phi)v^* := \sum_{k=1}^\infty \int_0^t \langle \Phi(s)i_Q e_k, v^* \rangle \, dB_k(s) \qquad in \ L^2(\Omega)$$

for all $v^* \in V^*$ and $t \in [0, T]$.

The stochastic integrals appearing in Definition 9.2 are the known real-valued Itô integrals and they are well defined thanks to our assumption on Φ. In the next Lemma we establish that the asserted limit exists:

Lemma 9.3. $I_t(\Phi) : V^* \to L^2(\Omega)$ is a well-defined cylindrical random variable in V which is independent of the representation of W, i.e. of $(e_n)_{n \in \mathbb{N}}$ and $(u_n^*)_{n \in \mathbb{N}}$.

Proof. We begin to establish the convergence in $L^2(\Omega)$. For that, let $m, n \in \mathbb{N}$ and we define for simplicity $h(s) := i_Q^* \Phi^*(s)v^*$. Doob's theorem implies

$$E \left| \sup_{0 \leqslant t \leqslant T} \sum_{k=m+1}^n \int_0^t \langle \Phi(s)i_Q e_k, v^* \rangle \, dB_k(s) \right|^2$$

$$\leqslant 4 \sum_{k=m+1}^n \int_0^T E \left[e_k, h(s)\right]_H^2 \, ds$$

$$\leqslant 4 \sum_{k=m+1}^\infty \int_0^T E \left[[e_k, h(s)]_H e_k, h(s)\right]_H \, ds$$

$$= 4 \sum_{k=m+1}^\infty \sum_{l=m+1}^\infty \int_0^T E \left[[e_k, h(s)]_H e_k, [e_l, h(s)]_H e_l\right]_H \, ds$$

$$= 4 \int_0^T E \|(\mathrm{Id} - \pi_m)h(s)\|_H^2 \, ds,$$

where $\pi_m : H \to H$ denotes the projection onto the span of $\{e_1, \ldots, e_m\}$. Because $\|(\mathrm{Id} - \pi_m)h(s)\|_H^2 \to 0$ for $m \to \infty$ and

$$\int_0^T E \|(\mathrm{Id} - \pi_m)h(s)\|_H^2 \, ds \leqslant \|i_Q^*\|_{U^* \to H}^2 \int_0^T E \|\Phi^*(s, \cdot)v^*\|_{U^*}^2 \, ds < \infty$$

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we obtain by Lebesgue's theorem the convergence in $L^2(\Omega)$.

To prove the independence on the chosen representation of W let $(f_l)_{l \in \mathbb{N}}$ be an other orthonormal basis of H and $w_l^* \in U^*$ such that $i_Q^* w_l^* = f_l$ and $(C_l(t) : t \geqslant 0)$ independent Wiener processes defined by $C_l(t) = W(t) w_l^*$. As before we define in $L^2(\Omega)$:

$$\tilde{I}_t(\Phi) v^* := \sum_{l=1}^{\infty} \int_0^t \langle \Phi(s) i_Q f_l, v^* \rangle \, dC_l(s) \qquad \text{for all } v^* \in V^*.$$

The relation $\mathrm{Cov}(B_k(t), C_l(t)) = t \left[i_Q^* u_k^*, i_Q^* w_l^* \right]_H = t \, [e_k, f_l]_H$ enables us to calculate

$$
\begin{aligned}
E &\left| I_t(\Phi) v^* - \tilde{I}_t(\Phi) v^* \right|^2 \\
&= E \left| I_t(\Phi) v^* \right|^2 + E \left| \tilde{I}_t(\Phi) v^* \right|^2 - 2E \left[\left(I_t(\Phi) v^* \right) \left(\tilde{I}_t(\Phi) v^* \right) \right] \\
&= \sum_{k=1}^{\infty} \int_0^t E \langle \Phi(s) i_Q e_k, v^* \rangle^2 \, ds + \sum_{l=1}^{\infty} \int_0^t E \langle \Phi(s) i_Q f_l, v^* \rangle^2 \, ds \\
&\quad - 2 \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \int_0^t E \left[\langle \Phi(s) i_Q e_k, v^* \rangle \langle \Phi(s) i_Q f_l, v^* \rangle \left[i_Q^* u_k^*, i_Q^* w_l^* \right]_H \right] ds \\
&= 2 \int_0^t E \left\| i_Q^* \Phi^*(s) v^* \right\|_H^2 \, ds - 2 \int_0^t E \left\| i_Q^* \Phi^*(s) v^* \right\|_H^2 \, ds \\
&= 0,
\end{aligned}
$$

which proves the independence of $I_t(\Phi)$ on $(e_k)_{k \in \mathbb{N}}$ and $(u_k^*)_{k \in \mathbb{N}}$.

The linearity of $I_t(\Phi)$ is obvious and hence the proof is complete. $\qquad\square$

Our next definition is not very surprising:

Definition 9.4. *For $\Phi \in M_T(U, V)$ we call the cylindrical random variable*

$$\int_0^t \Phi(s) \, dW(s) := I_t(\Phi)$$

cylindrical stochastic integral with respect to W.

Because the cylindrical stochastic integral is strongly based on the well-known real-valued Itô integral many features can be derived easily. We collect the martingale property and Itô's isometry in the following theorem.

Theorem 9.5. *Let Φ be in $M_T(U, V)$. Then we have*

(a) *for every $v^* \in V^*$ the family*

$$\left(\left(\int_0^t \Phi(s) \, dW(s) \right) v^* : t \in [0, T] \right)$$

forms a continuous square-integrable martingale.

(b) *the Itô's isometry:*

$$E\left|\left(\int_0^t \Phi(s)\,dW(s)\right)v^*\right|^2 = \int_0^t E\left\|i_Q^*\Phi^*(s)v^*\right\|_H^2\,ds.$$

Proof. (a) In Lemma 9.3 we have identified $I_t(\Phi)v^*$ as the limit of

$$M_n(t) := \sum_{k=1}^n \int_0^t \langle\Phi(s)i_Q e_k, v^*\rangle\,dB_k(s),$$

where the convergence takes place in $L^2(\Omega)$ uniformly on the interval $[0,T]$. As $(M_n(t) : t \in [0,T])$ are continuous martingales the assertion follows.

(b) Using Itô's isometry for real-valued stochastic integrals we obtain

$$\begin{aligned}
E\left|\left(\int_0^t \Phi(s)\,dW(s)\right)v^*\right|^2 &= \sum_{k=1}^\infty E\left[\int_0^T \langle\Phi(s)i_Q e_k, v^*\rangle\,dB_k(s)\right]^2 \\
&= \sum_{k=1}^\infty \int_0^T E\left[e_k, i_Q^*\Phi^*(s)v^*\right]_H^2\,ds \\
&= \int_0^T E\left\|i_Q^*\Phi^*(s)v^*\right\|_H^2\,ds.
\end{aligned}$$

$\square$

An obvious question is under which conditions the cylindrical integral is induced by a V-valued random variable. The answer to this question will also allow us to relate the cylindrical integral with other known definitions of stochastic integrals in infinite dimensional spaces.

From our point of view the following corollary is an obvious consequence. We call an stochastic process $\Phi \in M_T(U,V)$ non-random if it does not depend on $\omega \in \Omega$.

Corollary 9.6. *For non-random $\Phi \in M_T(U,V)$ the following are equivalent:*

(a) $\int_0^T \Phi(s)\,dW(s)$ *is induced by a V-valued random variable;*

(b) *there exists a Gaussian measure μ on V with covariance operator R such that:*

$$\int_0^T \left\|i_Q^*\Phi^*(s)v^*\right\|_H^2\,ds = \langle Rv^*, v^*\rangle \qquad \text{for all } v^* \in V^*.$$

Proof. (a) $\Rightarrow$ (b): If the integral $I_T(\Phi)$ is induced by a V-valued random variable then the random variable is centred Gaussian, say with a covariance operator R. Then Itô's isometry yields

$$\langle Rv^*, v^*\rangle = E\left|I_T(\Phi)v^*\right|^2 = \int_0^T \left\|i_Q^*\Phi^*(s)v^*\right\|_H^2\,ds.$$

(b)$\Rightarrow$(a): Again Itô's isometry shows that the weakly Gausian cylindrical distribution of $I_T(\Phi)$ has covariance operator R and thus, extends to a Gaussian measure on V. $\square$

The condition (b) of Corollary 9.6 is derived in van Neerven and Weis [10] as a sufficient and necessary condition for the existence of the stochastic Pettis integral introduced in this work. Consequently, it is easy to see that under the equivalent conditions (a) or (b) the cylindrical integral coincides with this stochastic Pettis integral. Further relation of condition (b) to γ-radonifying properties of the integrand Φ can also be found in [10].

Our next result relates the cylindrical integral to the stochastic integral in Hilbert spaces as introduced in Da Prato and Zabczyk [3]. For that purpose, we assume that U and V are separable Hilbert spaces. Let W be a strongly cylindrical Wiener process in U and let the inclusion mapping $i_Q : H_Q \to U$ be Hilbert-Schmidt. Then there exist an orthonormal basis $(f_k)_{k \in \mathbb{N}}$ in U and real numbers $\lambda_k \geqslant 0$ such that $Q f_k = \lambda_k f_k$ for all $k \in \mathbb{N}$. For the following we can assume that $\lambda_k \neq 0$ for all $k \in \mathbb{N}$. By defining $e_k := \sqrt{\lambda_k}\, f_k$ for all $k \in \mathbb{N}$ we obtain an orthonormal basis of H_Q and W can be represented as usual as a sum with respect to this orthonormal basis.

Our assumption on i_Q to be Hilbert-Schmidt is not a restriction because in general the integral with respect to a strongly cylindrical Wiener process is defined in [3] by extending U such that i_Q becomes Hilbert-Schmidt.

Corollary 9.7. *Let W be a strongly cylindrical Wiener process in a separable Hilbert space U with $i_Q : H_Q \to U$ Hilbert-Schmidt. If V is a separable Hilbert space and $\Phi \in M_T(U,V)$ is such that*

$$\sum_{k=1}^{\infty} \lambda_k \int_0^T E \left\| \Phi(s) i_Q e_k \right\|_V^2 \, ds < \infty,$$

then the cylindrical integral

$$\int_0^T \Phi(s) \, dW(s)$$

is induced by a V-valued random variable. This random variable is the standard stochastic integral in Hilbert spaces of Φ with respect to W.

Proof. By Theorem 8.1 the cylindrical Wiener process W is induced by an U-valued Wiener process Y. We define U-valued Wiener processes $(Y_N(t) : t \in [0,T])$ by

$$Y_N(t) = \sum_{k=1}^{N} i_Q e_k B_k(t).$$

Theorem 7.4 implies that $Y_N(t)$ converges to Y in $L^2(\Omega; U)$. By our assumption on Φ the stochastic integrals $\Phi \circ Y_N(T)$ in the sense of Da Prate and Zabczyk [3] exist and converge to the stochastic integral $\Phi \circ Y(T)$ in $L^2(\Omega; V)$, see [3, Ch.4.3.2].

On the other hand, by first considering simple functions Φ and then extending to the general case we obtain

$$\langle \Phi \circ Y_N(T), v^* \rangle = \sum_{k=1}^{N} \int_0^t \langle \Phi(s) i_Q e_k, v^* \rangle \, dB_k(s)$$

for all $v^* \in V^*$. By Definition 9.2 the right hand side converges in $L^2(\Omega)$ to

$$\left(\int_0^T \Phi(s) \, dW(s) \right) v^*,$$

whereas at least a subsequence of $(\langle \Phi \circ Y_N(T), v^* \rangle)_{N \in \mathbb{N}}$ converges to $\langle \Phi \circ Y(T), v^* \rangle$ P-a.s.. $\qquad\qquad\square$

Based on the cylindrical integral one can build up a whole theory of *cylindrical stochastic differential equations*. Of course, a solution will be in general a cylindrical process but there is no need to put geometric constrains on the state space under consideration. If one is interested in classical stochastic processes as solutions for some reasons one can tackle this problem as in our two last results by deriving sufficient conditions guaranteeing that the cylindrical solution is induced by a V-valued random process.

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