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CONJUGATE CONNECTIONS AND DIFFERENTIAL EQUATIONS ON INFINITE DIMENSIONAL MANIFOLDS

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Abstract. On a smooth manifold M, the vector bundle structures of the second order tangent bundle, T^2M bijectively correspond to linear connections. In this paper we classify such structures for those Fréchet manifolds which can be considered as projective limits of Banach manifolds. We investigate also the relation between ordinary differential equations on Fréchet spaces and the linear connections on their trivial bundle. Such equations arise in theoretical physics.

Keywords Banach manifold, Frechet manifold, connection, conjugate, ordinary differential equation

Introduction

The structure of T^2M *ie* the bundle of accelerations for a smooth manifold M of finite dimension was studied by Dodson and Radivoiovici (see [3]). They proved that T^2M admits a vector bundle structure over M if and only if M is endowed with a linear connection.

In [1] Dodson and Galanis have established the structure of T^2M for Banach manifolds and also for those Fréchet manifolds which are projective limits of Banach manifolds. They proved that existence of a vector bundle structure on T^2M is equivalent to the existence of a linear connection (in the sense of [9]) on M. By this means, vector bundle structures of T^2M were classified by Dodson, Galanis and Vassiliou for the Banach case (see [2]).

In this paper we extend this classification to a large class of Fréchet manifolds. Also, we investigate some relations between connections and ordinary differential equations on Fréchet spaces which generalize a result of Vassiliou ([8]) in the Banach case. As Galanis and Vassiliou have pointed out in [5], there is no specific method to solve a given differential equation on Fréchet spaces. Here we introduce a method for solving such problems and we give also a relation between these equations and the induced connections.

This method can solve any ordinary differential equation on Fréchet spaces because every Fréchet space can be considered as a projective limit of Banach spaces. Furthermore this extends to solve differential equations on those Fréchet manifolds which are obtained as projective limits of Banach manifolds.

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1. Preliminaries

Let M be a smooth manifold modelled on the Banach space $\mathbb{E}$ with the corresponding atlas $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in I}$. For each $x \in M$ we define $C_x = \{f : (-\epsilon,\epsilon) \longrightarrow M$; f is smooth and f(0)=x$\}$. For $f, g \in C_x$, we define $f \sim_x g$ *iff* $f'(0) = g'(0)$, so $T_xM = c_x/\sim_x$ and $TM = \bigcup_{x\in M} T_xM$. It is easy to check that TM is a smooth Banach manifold modelled on $\mathbb{E}\times\mathbb{E}$. Moreover it is a vector bundle over M by the projection $\pi_M : TM \longrightarrow M$. Consider the trivialization $\{(\pi_M^{-1}(U_\alpha),\Psi_\alpha)\}_{\alpha\in I}$ for TM and similarly the trivialization $\{(\pi_{TM}^{-1}(\pi_M^{-1}(U_\alpha)),\tilde{\Psi}_\alpha)\}_{\alpha\in I}$ for T(TM).

Following eg Vilms [9], a connection on M is a vector bundle morphism $\nabla : T(TM) \longrightarrow TM$ with the local forms $\omega_\alpha : \psi_\alpha(U_\alpha)\times\mathbb{E} \longrightarrow L(\mathbb{E}\times\mathbb{E})$. Local representation of ∇ is as follows:

$$\nabla_\alpha : \psi_\alpha(U_\alpha)\times\mathbb{E}\times\mathbb{E}\times\mathbb{E} \longrightarrow \psi_\alpha(U_\alpha)\times\mathbb{E}$$

with $\nabla_\alpha = \Psi_\alpha o \nabla o \tilde{\Psi}_\alpha^{-1}$ for $\alpha\in I$, and the relation

$$\nabla_\alpha(y,u,v,w) = (y, w + \omega_\alpha(y,u).v)$$

is satisfied. Furthermore ∇ is a linear connection iff $\{\omega_\alpha\}_{\alpha\in I}$ are linear with respect to their second variables. This connection ∇ is completely determined by its Christoffel symbols $\{\Gamma_\alpha\}_{\alpha\in I}$ which are smooth and:

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \longrightarrow L(\mathbb{E}, L(\mathbb{E},\mathbb{E}))$$

defined by $\Gamma_\alpha(y)[u] = \omega_\alpha(y,u)$ for each $(y,u)\in\psi_\alpha(U_\alpha)\times\mathbb{E}$.

The necessary condition for ∇ to be well defined on overlaps of M is that the Christoffel symbols satisfy the following compatibility condition;

$$\begin{aligned}\Gamma_\alpha(\sigma_{\alpha\beta}(y))(d\sigma_{\alpha\beta}(y)(u))[d\sigma_{\alpha\beta}(y)(v)] &+ (d^2\sigma_{\alpha\beta}(y)(v))(u)\\ &= d\sigma_{\alpha\beta}(y)((\Gamma_\beta(y)(u))(v))\end{aligned}$$

for all $(y,u,v)\in\psi_\alpha(U_\alpha)\times\mathbb{E}\times\mathbb{E}$. Here $\sigma_{\alpha\beta} = \psi_\alpha o \psi_\beta^{-1}$ and d and d^2 denote the first and the second order differentials respectively.

Recalling our above definition of C_x, we define the equivalence relation $\approx_x$ as follows, for $f, g \in C_x$,

$$f \approx_x g \Longleftrightarrow f'(0) = g'(0) \text{ and } f''(0) = g''(0).$$

Then $T^2M_x = C_x/\approx_x$ and $T^2M = \bigcup_{x\in M} T_x^2M$. Here we see that T_x^2M is a topological vector space isomorphic to $\mathbb{E}\times\mathbb{E}$ with the mapping:

$$\begin{aligned}\phi_x : T_x^2M &\longrightarrow \mathbb{E}\times\mathbb{E}\\ [f,x]_2 &\longmapsto (\psi_\alpha o f)'(0), (\psi_\alpha o f)''(0)).\end{aligned}$$

Locally, a trivialization for T^2M is provided as follows:

$$\begin{aligned}\Phi_\alpha : \pi_2^{-1}(U_\alpha) &\longrightarrow U_\alpha\times\mathbb{E}\times\mathbb{E}\\ [f,x]_2 &\longmapsto (x, (\psi_\alpha o f)'(0), (\psi_\alpha o f)''(0) + \Gamma_\alpha(\psi_\alpha(x))((\psi_\alpha o f)'(0)\\ &, \quad (\psi_\alpha o f)'(0))),\end{aligned}$$

where $\pi_2 : T^2M \longrightarrow M$ sending $[f,x]_2$ to x. In this way we see that T^2M becomes a vector bundle over M with fibres of type $\mathbb{E}\times\mathbb{E}$ and the structure group $GL(\mathbb{E}\times\mathbb{E})$.

Let $\Phi_{\alpha,x}$ be the restriction of Φ_α to the fibres T^2_xM. Then its transition functions will be:

$$\begin{array}{rcl} T_{\alpha\beta} : U_\alpha \cap U_\beta & \longrightarrow & L(\mathbb{E}\times\mathbb{E}, \mathbb{E}\times\mathbb{E}) \\ x & \longmapsto & \Phi_{\alpha,x} o {\Phi_{\beta,x}}^{-1} \end{array}$$

More precisely they have the form $T_{\alpha\beta} = (d(\sigma_{\alpha\beta} o \phi_\beta), d(\sigma_{\alpha\beta} o \phi_\beta))$, for more details see [1].

Next we turn to a class of Fréchet manifolds that are obtained as projective limits of Banach manifolds. Let $\{M^i, \varphi^{ji}\}$ be a projective system of Banach manifolds modelled on the Banach spaces $\{\mathbb{E}^i\}$ respectively; we require the model spaces also to form a projective system. Suppose that for $x = (x^i) \in M = \varprojlim M^i$ there exists a projective system of charts $\{(U_i, \psi_i)\}_{i\in\mathbb{N}}$ such that $x^i \in U_i$. Then the projective limit $M = \varprojlim M_i$ has a Fréchet manifold structure modelled on $\mathbb{F} = \varprojlim E_i$ with the atlas $\{(\varprojlim U_i, \varprojlim \psi_i)\}$. Let $\{M^i, \phi^{ji}\}_{i,j\in\mathbb{N}}$ and $\{N^i, \phi'^{ji}\}_{i,j\in\mathbb{N}}$ be two projective systems of manifolds, with smooth maps $g^i : M^i \longrightarrow N^i$ such that $\varprojlim g^i = g$ exists.

2. Classification for vector bundle structures of T^2M

Suppose that for each $i \in \mathbb{N}$; M^i and N^i are endowed with linear connections ∇_{M^i} and ∇_{N^i}. Let $\nabla_M = \varprojlim \nabla_{M^i}$ and $\nabla_N = \varprojlim \nabla_{N^i}$ be two linear connections over M and N respectively.

Theorem 2.1. *Let M^i and N^i be g^i-conjugate for each $i \in \mathbb{N}$ then M and N are g-conjugate.*

Proof. We have to show that

$$\nabla_N \circ T(Tg) = Tg \circ \nabla_M.$$

First we prove that $\varprojlim \nabla_{N^i} \circ T(Tg^i)$ exists and hence:

$$\nabla_N \circ T(Tg) = \varprojlim \nabla_{N^i} \circ T(Tg^i).$$

Let $i \leq j$, then:

$$\begin{array}{rcl} T\phi'^{ji}(\nabla_{N^j} \circ T(Tg^j)) & = & (\nabla_{N^i} \circ T(T\phi'^{ji})) \circ (T(Tg^i)) \\ & = & \nabla_{N^i} \circ [(T(Tg^i)) \circ T(T\phi^{ji})] \\ & = & (\nabla_{N^i} \circ T(Tg^i)) \circ T(T\phi^{ji}), \end{array}$$

hence $\nabla_N \circ T(Tg) = \varprojlim \nabla_{N^i} \circ T(Tg^i)$ exists. Now we show that $\{Tg^i \circ \nabla_{M^i}\}_{i\in\mathbb{N}}$ is a projective system of maps. Let $i \leq j$, then

$$\begin{array}{rcl} T\phi'^{ji} \circ (Tg^j \circ \nabla_{M^j}) & = & (Tg^i \circ T\phi^{ji}) \circ \nabla_{M^j} \\ & = & Tg^i \circ (\nabla_{M^i} \circ T(T\phi^{ji})) \\ & = & (Tg^i \circ \nabla_{M_i}) \circ T(T\phi^{ji}). \end{array}$$

Hence $\varprojlim \nabla_{N^i} \circ T(Tg^i)$ exists and $\nabla_N \circ T(Tg) = \varprojlim \nabla_{N^i} \circ T(Tg^i)$. One can check easily that $M = \varprojlim M^i$ and $N = \varprojlim N^i$ are $g = \varprojlim g^i$-conjugate. □

Theorem 2.2. *If* $\nabla_M = \varprojlim \nabla_{M^i}$ *and* $\nabla_N = \varprojlim \nabla_{N^i}$ *and* ∇_{M^i} *and* ∇_{N^i} *are* g^i*-conjugate, then* $T^2g : T^2M \longrightarrow T^2N$ *is linear on the fibres.*

Proof. According to [2] since ∇_{M^i} and ∇_{N^i} are g^i-conjugate then $T^2_{x^i}g^i$ is linear for each $x^i \in M^i$. Considering $T^2_x g = \varprojlim T^2_{x^i} g^i$ the result follows. □

Theorem 2.3. *Let* $g^i : M^i \longrightarrow N^i$ *be smooth maps and* ∇_{M^i} *and* ∇_{N^i} *be* g^i*-conjugate for each* $i \in \mathbb{N}$*; then* $T^2g : T^2M \longrightarrow T^2N$ *is a vector bundle morphism.*

Sketch of proof. As proved in [2], each $T^2g^i : T^2M^i \longrightarrow T^2N^i$ is a vector bundle morphism; then $T^2g = \varprojlim T^2g^i$ implies that $T^2g : T^2M \longrightarrow T^2N$ is a vector bundle morphism.

Considering the above discussion, we deduce the following result:

Theorem 2.4. *Let* $g^i : M^i \longrightarrow N^i$ *be a diffeomorphism and* ∇^i *and* ∇'^i *be two* g^i*-conjugate linear connections on* M^i*, for each* $i \in \mathbb{N}$*. If* $\nabla = \varprojlim \nabla^i$ *and* $\nabla' = \varprojlim \nabla'^i$ *then the vector bundle structures on* T^2M *induced by* ∇ *and* ∇' *are isomorphic.*

Let (M, ∇) denote the vector bundle structure of T^2M induced by ∇. For a diffeomorphism $g : M \longrightarrow M$ we define the equivalence relation $\sim_g$ as follows:

$$(M, \nabla) \sim_g (M, \nabla') \Longleftrightarrow \nabla \text{ and } \nabla' \text{ are } g - \text{conjugate.}$$

Hence if (M, ∇) and (M, ∇') are in the same g-conjugate class $[(M, \nabla)]_g$, then their induced vector bundle structures on T^2M are isomorphic.

Corollary 2.5. *All the elements of the class* $[(M, \nabla)]_g$ *have isomorphic induced vector bundle structures on* T^2M*.*

3. Connections And Ordinary Differential Equations

Let $\mathbb{E}$ and $\mathbb{B}$ be Banach spaces and $L = (\mathbb{B} \times \mathbb{E}, \mathbb{B}, pr_1)$ be the trivial bundle over $\mathbb{B}$ with fibres of type $\mathbb{E}$. In [8] it is stated that we can correspond an ordinary differential equation to a connection over the trivial bundle with the solution ξ being the horizontal global section of that connection. Furthermore it is shown that connections ∇ and ∇' over L are $(\phi, id_{\mathbb{B}})$-related iff the corresponding differential equations $dx/dt = A(t)x$ and $dx/dt = B(t)y$ are equivalent.

Here we extend these concepts to those Fréchet spaces which can be considered as projective limits of Banach spaces. Let $\mathbb{F}$ and $\mathbb{H}$ be Fréchet spaces whith $\mathbb{F} = \varprojlim \mathbb{E}^i$ and $\mathbb{H} = \varprojlim \mathbb{B}^i$, for two inverse systems $\{\mathbb{E}^i, \rho^{ji}, \mathbb{N}\}$ and $\{\mathbb{B}^i, \tau^{ji}, \mathbb{N}\}$ of Banach spaces. (ρ^{ji} and τ^{ji} are linear maps.)

Consider the trivial bundle $L = (\mathbb{H} \times \mathbb{F}, \mathbb{H}, pr_1)$ with respect to the usual atlas $\mathcal{A}$ for $\mathbb{H}$ which contains $(\mathbb{H}, id_{\mathbb{H}})$ and the open covering $\mathcal{C}$ of $\mathbb{H}$ that can be realized as a projective limit of its components in each $\mathbb{B}^i$.

Assume that ∇^i is a linear connection over $L^i = (\prod_{j=1}^i \mathbb{B}^i \times \mathbb{E}^i, \prod_{j=1}^i \mathbb{B}^i, pr_1^i)$ where $pr_1^i = (pr_1^1, ..., pr_1^i)$ and, $\nabla = \varprojlim \nabla^i$ a linear connection on $L = \varprojlim L^i$. According to the above atlas $\mathcal{A}$ the Christoffel symbols of ∇ are :

$$\Gamma_\alpha : \phi_\alpha(U_\alpha) \longrightarrow L^2(\mathbb{H} \times \mathbb{F}, \mathbb{F}).$$

In particular take Γ_1 as the symbol defined over $(\mathbb{H}, id_{\mathbb{H}})$. Let $A_\alpha(t) = \Gamma_\alpha(t)(.,1)$ where 1 is the unit of $\mathbb{H}$, especially;

$$A = A_1 : \mathbb{H} \longrightarrow L(\mathbb{F})$$

.

Theorem 3.1. *The linear connections of the form* $\nabla = \varprojlim \nabla^i$ *are in one-to-one correspondence with the ordinary differential equations* $dx/dt = A(t)x$. *Moreover, for each* $t_0 \in \mathbb{H}$ *there exists a unique horizontal global section*

$$\xi : \mathbb{H} \longrightarrow \mathbb{H} \times \mathbb{F}$$

with $\xi_p(t_0) = f_0$, *where* $\xi_p : \mathbb{H} \longrightarrow \mathbb{H}$ *is the principal part of* ξ.

Proof. We know that $\nabla = \varprojlim \nabla^i$ such that each ∇^i is a linear connection over L^i. As stated in [8], each ∇^i corresponds bijectively to an ordinary differential equation $dx^i/dt = A^i(t)x^i$. Furthermore the solution of $dx^i/dt = A^i(t)x^i$ is the principal part of of the horizontal global section of ∇^i, which we call ξ^i.

In the first step we notice that $A^i(t^i) = \Gamma^i(t^i)(.,1)$ where Γ^i is the Christoffel symbol of ∇^i over L^i assigned to the chart $(\mathbb{E}^i, id_{\mathbb{E}^i})$. Since $\nabla = \varprojlim \nabla^i$ we get $\Gamma(t)(.,1) = \varprojlim \Gamma^i(t^i)(.,1)$, where $t = (t^i)$. Hence $A(t) = \varprojlim A^i(t^i)$ is well defined and consequently $dx/dt = A(t)x$ is an ordinary differential equation on $\mathbb{B}$. (For more details see [4].)

In the second step we claim that $\{\xi^i\}_{i\in\mathbb{N}}$ is a projective system of maps and $\xi_p = \varprojlim \xi_p^i$ is the solution of $dx/dt = A(t)x$. It is enough to prove that:

$$\rho^{ji} o {\xi_p}^j = {\xi_p}^i$$

for $i \leq j$ and ${\xi_p}^i : \mathbb{B}^i \longrightarrow \mathbb{E}^i$. We start our proof by showing that $\rho^{ji} o {\xi_p}^j$ satisfies

$$\dot{\xi}_p^i(t) = [A^i(t)](\xi_p^i(t)).$$

In fact:

$$\begin{aligned}
(\rho^{ji} o \xi_p^j)'(t) &= \rho^{ji} o (\xi_p^j)'(t) = \rho^{ji} o [A^j(t)](\xi_p^j(t)) \\
&= [\rho^{ji} o A^j(t)](\xi_p^j(t)) \\
&= [A^i(t) o \rho^{ji}](\xi_p^j(t)) \\
&= [A^i(t)](\xi_p^i(t))
\end{aligned}$$

Next, $\xi_p^i(t_0^i) = f_0^i$ and $(\rho^{ji} o \xi_p^j)(t_0) = \rho^{ji}(f_0^j) = f_0^j$. Hence ξ_p^i (or equivalently ξ^i) is unique, so $\rho^{ji} o \xi_p^j = \xi_p^i$. This implies that $\xi = \varprojlim \xi^i$ is the horizontal global section of L and it is also the solution of the given differential equation. At last we see that:

$$\begin{aligned}
\xi_p'(t) = ({\xi_p^i}'(t^i))_{i\in\mathbb{N}} &= (A^i(t^i)({\xi_p^i}'(t^i)))_{i\in\mathbb{N}} \\
&= A(t)(\xi_p(t)).
\end{aligned}$$

□

Let $\nabla = \varprojlim \nabla^i$ and $\nabla' = \varprojlim \nabla'^i$ be two linear connection over L such that for each $i \in \mathbb{N}$, ∇^i and ∇'^i are g^i-related connections on L^i and $g = \varprojlim g^i$.

Theorem 3.2. *Using the previous notations, let* $\nabla = \varprojlim \nabla^i$ *and* $\nabla' = \varprojlim \nabla'^i$ *be two linear connections over* L. *Then* ∇ *and* ∇' *are* $(g, id_{\mathbb{B}})$*-related iff their corresponding differential equations given by* $dx/dt = A(t)$ *and* $dy/dt = C(t)y$, *are equivalent, ie there exists a smooth transformation* $Q : \mathbb{H} \longrightarrow \mathcal{H}_0(\mathbb{F})$ *such that* $x(t) = Q(t)y(t)$.

Proof. By [8] ∇^i and ∇'^i are g^i-related connections over L^i *iff* $dx^i/dt = A^i(t)x^i$ and $dy^i/dt = C^i(t)y^i$ are equivalent *ie* $x^i(t) = Q^i(t)y^i(t)$ where:

$$Q^i \in \mathcal{H}_0^i(\mathbb{F}) = \{(l^i, ..., l^i) \in \prod_{j=i}^{i} Lis(\mathbb{E}^j) : \rho^{ji} o l^j = l^k o \rho^{jk};\ \text{for } k \le j \le i\}.$$

It can be shown that $x(t) = Q(t)y(t)$ where $Q(t) = \varprojlim Q^i(t) \in \mathcal{H}_0(\mathbb{F})$. □

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