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2001

MIMS EPrint: **2006.99**

ISSN 1749-9097

Hyperbolicity of the invariant set for the logistic map with $\mu > 4$

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Classic results due to Guckenheimer and Misiurewicz imply that the invariant set of the logistic map with $\mu \in (4, 2+\sqrt{5}]$ is hyperbolic. This is well known, but the only obvious reference in the literature uses relatively sophisticated ideas from complex variable theory. This pedagogical note provides a brief, self-contained account of this result using only elementary real analysis. The method also gives a good estimate of the expansion rate on the invariant set.

1. INTRODUCTION

Almost every introduction to chaos and dynamical systems contains an account of the dynamics of the logistic map

$$F_\mu(x) = \mu x(1-x). \tag{1}$$

If $\mu \in (0,4]$ then the interval $[0,1]$ is invariant under the map, and 'interesting' phenomena (period-doubling, chaos etc.) can be observed as μ is varied. If $\mu > 4$ then there is no invariant interval, but if we define

$$\Omega_n = \{x \in [0,1] \mid F_\mu^r(x) \in [0,1], \quad r = 0, \dots, n-1\} \tag{2}$$

then $\Omega := \cap_1^\infty \Omega_n$ is a closed invariant set on which the dynamics is topologically conjugate to a one-sided shift on two symbols. Such an unstable set is said to be *hyperbolic* if there exists $\lambda > 1$ (the *expansion rate*) such that for all $x \in \Omega$ and all $n \in \mathbf{Z}^+$

$$|DF_\mu^n(x)| \geq c\lambda^n \tag{3}$$

where DF^n denotes the derivative of F^n (the n^{th} iterate of F) and c is a constant. It is a standard graduate exercise (see [2], Ch. 1.7, Theorem 7.5 and Ex. 2b for example) to show that Ω is hyperbolic if $\mu > 2+\sqrt{5}$, which follows immediately from a direct calculation: $|DF_\mu(x)| > 1$ for all $x \in \Omega_2$ if $\mu > 2+\sqrt{5}$. This leaves an uncomfortable gap in the parameter space: $4 < \mu \leq 2+\sqrt{5}$. It is well known that Ω is hyperbolic for these parameters, and a standard statement (see e.g. [8]) is that it can be proved using the techniques developed by Guckenheimer [3] and Misiurewicz [7] (see also [5,6,9]). Robinson [8] gives a shorter proof using ideas from complex variable theory, but the proof is by no

*This article was motivated by a question posed by Mrs Das after my talk in the session chaired by Prof. Sengupta at WCNA2000, Catania, Sicily.

means simple. In most graduate courses it is probably enough to assure students that Ω is indeed hyperbolic for these parameter values, but that the proof is more complicated. The more persistent student might like more detail, and it is this curiosity which I hope to satisfy below. So this note provides a new and elementary proof of the following result:

Theorem 1.1 *If $\mu > 4$ then Λ is hyperbolic.*

The proof uses ideas which can be found in [4,6], see, for example, Lemma 4.1 of [4] where the technique is used to study expansion in logistic maps with $\mu < 4$. This method uses the standard topological conjugacy between the logistic map with $\mu = 4$ and the tent map with slope two. It is shown below that this conjugacy (or more precisely, the congugacy for a related quadratic map) can be used to conjugate the quadratic map to a map which has a slope with modulus greater than one for an appropriate range of values – this makes it possible to obtain estimates of the derivative of the original map and hence prove hyperbolicity. The proof also provides good estimates of the expansion rate λ which I have not seen elsewhere in the literature.

2. QUADRATIC MAPS

The logistic map, (1), is just one of several equivalent parametrizations of a quadratic map of the interval to itself. It turns out (for reasons alluded to in the next section, but which only really reveal themselves if the relevant calculations are attempted) that the algebraic manipulations needed for the proof are greatly simplfied if the family

$$f_r(x) = 1 - rx^2, \qquad r > 2 \tag{4}$$

is used instead of the standard logistic map. If $r > 0$ then each member of the family (f_r) is topologically conjugate to the member of the logistic family (F_μ) with μ and r related by the formula

$$r = \frac{1}{4}\mu(\mu - 2) \tag{5}$$

and, in particular, $\mu = 4$ gives $r = 2$, and $\mu > 4$ corresponds to $r > 2$. The interesting dynamics of (f_r) lies in an interval $I_r = [-a, a]$, where $x = -a$ is the fixed point of f_r with

$$a = \frac{1 + \sqrt{1 + 4r}}{2r}. \tag{6}$$

Note that if $r < 2$ then $I_r \subset [-1, 1]$. The conjugating function, p, such that $F_\mu = p^{-1} \circ f_r \circ p$, is affine (i.e. $p(x) = Ax + B$); the precise values of the coefficients A and B are unimportant and left as an exercise. Note that p maps the interval $[0, 1]$ onto I_r, and Ω and Ω_n onto the corresponding sets for f_r, which will be denoted Λ and Λ_n. For the remainder of this note we concentrate on (4) rather than (1). The results of the next three sections establish the following result.

Theorem 2.1 *If $r > 2$ then for all $x \in \Lambda$ and $n \in \mathbf{Z}^+$*

$$|Df_r^n(x)| \geq C_r \lambda_r^n \tag{7}$$

where $\lambda_r = \sqrt{2r}$ and $C_r > 0$ with

$$C_r^2 = \frac{2r^2 - 2r - 1 - \sqrt{1+4r}}{2r^2 - 2r + 1 + \sqrt{1+4r}}. \tag{8}$$

Note that as $r \downarrow 2$, $C_r \downarrow 0$ and as $r \to \infty$, $C_r \uparrow 1$.

3. A CHANGE OF VARIABLE

If $r = 2$ then the change of variable $h : [-1,1] \to [-1,1]$ defined by

$$h(\theta) = \cos \frac{1}{2}\pi(\theta - 1), \qquad -1 \leq \theta \leq 1, \tag{9}$$

conjugates the quadratic map f_2 to the tent map $g_2 : [-1,1] \to [-1,1]$, where

$$g_2(x) = \begin{cases} 1 + 2\theta & \text{if } \theta \in [-1,0] \\ 1 - 2\theta & \text{if } \theta \in [0,1] \end{cases} \tag{10}$$

i.e.

$$h^{-1} \circ f_2 \circ h(\theta) = g_2(\theta). \tag{11}$$

The main idea behind the elementary proof of Theorem 2.1 given below is to use the function h above to conjugate f_r $(r > 2)$ to a new map g_r which has $|Dg_r(\theta)| > 1$ for all θ in $h^{-1}(\Lambda)$. Once this has been established, the proof is simple.

Thus, for $r > 2$ define a new map g_r by

$$g_r(\theta) = h^{-1} \circ f_r \circ h(\theta). \tag{12}$$

Note that the invariant set Λ of f_r is contained in the set $\Lambda_2 = I_r \backslash \Delta$ where $\Delta = (-b, b)$ with

$$b^2 = \frac{2r - 1 - \sqrt{1+4r}}{2r^2} \tag{13}$$

and so we are only interested in g_r for $\theta \in h^{-1}(\Lambda_2)$. (The value of b is found by computing the set of points near $x = 0$ for which $f_r(x) = a$.) Differentiating (12) and rearranging a little gives

$$|Df_r^n(x)| = \left| \frac{Dh(g_r^n(y))}{Dh(y)} \right| . |Dg_r^n(y)|, \qquad y = h^{-1}(x). \tag{14}$$

The aim of the next two sections is to control the two terms making up the right hand side of (14) for $y \in h^{-1}(\Lambda)$.

Equation (14) also explains, at least implicitly, why we have chosen to work with f_r instead of the logistic map F_μ: as noted earlier, if $r > 2$ then the invariant set of f_r is contained *inside* the interval $[-1,1]$ on which the conjugacy (9) was defined for f_2, and h is a diffeomorphism on any closed interval contained in $(-1,1)$. Had the logistic map been used this would no longer have been the case, and rather than use the conjugating function it would have been necessary to try families of rescaled versions of the conjugating function, thus adding an extra level of complexity to the manipulations.

4. BOUNDING THE FIRST TERM OF (14)

In the previous section it was established that if $r > 2$ then the invariant set of f_r is contained in $\Lambda_2 = [-a, -b] \cup [b, a]$. Thus, for all $y \in h^{-1}(\Lambda)$,

$$\left| \frac{Dh(g_r^n(y))}{Dh(y)} \right| \geq C_r \tag{15}$$

where

$$C_r = \min_{\theta, \theta' \in h^{-1}(\Lambda_2)} \left| \frac{Dh(\theta)}{Dh(\theta')} \right| = \min_{\theta, \theta' \in h^{-1}(\Lambda_2)} \frac{|\sin \frac{1}{2}\pi(\theta - 1)|}{|\sin \frac{1}{2}\pi(\theta' - 1)|}. \tag{16}$$

Now, since $\Lambda_2 = [-a, -b] \cup [b, a]$ with $0 < b < a < 1$, $h^{-1}(\Lambda_2)$ is also a union of two intervals, $[-\alpha, -\beta] \cup [\beta, \alpha]$ with $0 < \beta < \alpha < 1$, $\cos \frac{1}{2}\pi(\alpha - 1) = a$ and $\cos \frac{1}{2}\pi(\beta - 1) = b$. Hence

$$C_r = \frac{|\sin \frac{1}{2}\pi(\alpha - 1)|}{|\sin \frac{1}{2}\pi(\beta - 1)|}. \tag{17}$$

Since a and b are known from equations (6) and (13), this implies that

$$C_r^2 = \frac{1 - a^2}{1 - b^2} = \frac{2r^2 - 2r - 1 - \sqrt{1 + 4r}}{2r^2 - 2r + 1 + \sqrt{1 + 4r}}. \tag{18}$$

5. BOUNDING THE SECOND TERM OF (14)

Using (9) and (11), g_r may be written explicitly as

$$g_r(\theta) = 1 + \frac{2}{\pi} \cos^{-1} \left(1 - r \cos^2 \frac{1}{2}\pi(\theta - 1) \right) \tag{19}$$

and so by the chain rule

$$\begin{aligned} Dg_r(\theta) &= 2r \frac{\cos \frac{1}{2}\pi(\theta - 1) \sin \frac{1}{2}\pi(\theta - 1)}{\left(1 - (1 - r \cos^2 \frac{1}{2}\pi(\theta - 1))^2\right)^{\frac{1}{2}}} \\ &= \sqrt{2r} \frac{\sin \frac{1}{2}\pi(\theta - 1)}{\left(1 - \frac{1}{2} r \cos^2 \frac{1}{2}\pi(\theta - 1)\right)^{\frac{1}{2}}}. \end{aligned}$$

Set $w = |\cos^2 \frac{1}{2}\pi(\theta - 1)| = |h(\theta)|^2$, so if $\theta \in h^{-1}(\Lambda)$ then $b^2 \leq w \leq a^2$ and

$$|Dg_r(\theta)| = \sqrt{2r} \left(\frac{1 - w}{1 - \frac{1}{2} r w} \right)^{\frac{1}{2}}. \tag{20}$$

If $r > 2$ then, provided $1 - \frac{1}{2} r w > 0$, this implies that

$$|Dg_r(\theta)| \geq \sqrt{2r}. \tag{21}$$

For w in the interval $[b^2, a^2]$, $1 - \frac{1}{2} r w \geq 1 - \frac{1}{2} r a^2$, and by looking at the derivative of the expression $1 - \frac{1}{2} r a^2$ with respect to r it is straightforward to show that for if $r > 2$ then $1 - \frac{1}{2} r a^2 > 0$. Hence $1 - \frac{1}{2} r w > 0$ if $r > 2$ and so (21) does indeed hold for all $r > 2$.

6. COMPLETING THE PROOF

To complete the proof of Theorem 2.1 it only remains to substitute the bounds obtained in the previous two sections into (14) to obtain

$$|Df_r^n(x)| \geq C_r|2r|^{\frac{n}{2}} \qquad \text{for all } \ n > 0 \ \text{ and } \ x \in \Lambda \tag{22}$$

if $r > 2$. In fact, if more care is taken with the minimization of the right hand side of (20) it is possible to obtain the slightly better estimate, $|Df_r^n(x)| \geq C_r\lambda_r^n$ with

$$\lambda_r = 2\left(\frac{2r^2 - 2r + 1 + \sqrt{1+4r}}{2r + 1 + \sqrt{1+4r}}\right)^{\frac{1}{2}}. \tag{23}$$

However, this expression is not quite as transparent as the expression derived in the sections above!

Note that throughout this proof the only mathematical techniques used have been from elementary calculus: differentiating functions of functions (the chain rule) and finding maxima and minima. From dynamical systems, the proof exploits the idea of (smooth) topological conjugacy. The proof of Robinson [8] uses a more advanced definition of distance together with the Schwarz Lemma from complex variable theory, and does not appear to give good estimates of the expansion rate without more work. I have never seen a complete proof of this result using the earlier ideas of Guckenheimer [3] and Misiurewicz [7]. It is not hard to see how this would work, but such a proof would involve considerably longer arguments than the reasoning above (although it would be more general).

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