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ON FINITE SOLUBLE GROUPS WITH ALMOST FIXED-POINT-FREE AUTOMORPHISMS OF NON-COPRIME ORDER

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Abstract. It is proved that if a finite p-soluble group G admits an automorphism φ of order p^n having at most m fixed points on every φ-invariant elementary abelian p'-section of G, then the p-length of G is bounded above in terms of p^n and m; if in addition the group G is soluble, then the Fitting height of G is bounded above in terms of p^n and m. It is also proved that if a finite soluble group G admits an automorphism ψ of order $p^a q^b$ for some primes p, q, then the Fitting height of G is bounded above in terms of $|\psi|$ and $|C_G(\psi)|$.

to Yurii Leonidovich Ershov on the occasion of his 75-th birthday

1. Introduction

Studying groups with "almost fixed-point-free" automorphisms means obtaining restrictions on the structure of groups depending on their automorphisms and certain restrictions imposed on the fixed-point subgroups. In this paper we consider questions of bounding the p-length and Fitting height of finite p-soluble and soluble groups admitting almost fixed-point-free automorphisms of non-coprime order.

Let $\varphi \in \operatorname{Aut} G$ be an automorphism of a finite group G. Studying the structure of the group G depending on φ and the fixed-point subgroup $C_G(\varphi)$ is one of the most important and fruitful avenues in finite group theory. The celebrated Brauer–Fowler theorem [1] (bounding the index of the soluble radical in terms of the order of $|C_G(\varphi)|$ when $|\varphi| = 2$) and Thompson's theorem [2] (giving the nilpotency of G when φ is of prime order and acts fixed-point-freely, that is, $C_G(\varphi) = 1$) lie in the foundations of the classification of finite simple groups. The classification was used for obtaining further results on solubility of G, or of a suitable "large" subgroup. For example, using the classification Hartley [3] generalized the Brauer–Fowler theorem to any order of φ: the group G has a soluble subgroup of index bounded in terms of $|\varphi|$ and $|C_G(\varphi)|$.

Now suppose that the group G is soluble. Further information on the structure of G is sought first of all in the form of bounds for the Fitting height (nilpotent length). A bound for the Fitting height naturally reduces further studies to the case of nilpotent groups with (almost) fixed-point-free automorphisms, for which, in turn, problems arise of bounding the derived length, or the nilpotency class of the group or of a suitable "large" subgroup. Such bounds for nilpotent groups so far have been obtained in the cases of φ being of prime order or of order 4 in [4, 5, 6, 7, 8, 9]. In addition, definitive general results have been obtained in the study of almost fixed-point-free p-automorphisms of finite p-groups [10, 11, 12, 13, 14, 15].

On bounding the Fitting height, especially strong results have been obtained in the case of soluble groups of automorphisms $A \leqslant \operatorname{Aut} G$ of coprime order. Thompson [16] proved that if both groups G and A are soluble and have coprime orders, then the Fitting height

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of G is bounded in terms of the Fitting height of $C_G(A)$ and the number $\alpha(A)$ of prime factors of $|A|$ with account for multiplicities. Later the bounds in Thompson's theorem were improved in numerous papers, with definitive results obtained by Turull [17] and Hartley and Isaacs [18] with linear bounds in terms of $\alpha(A)$ for the Fitting height of the group or of a "large subgroup".

The case of non-coprime orders of G and $A \leqslant \operatorname{Aut} G$ is more difficult. Bell and Hartley [19] constructed examples showing that for any non-nilpotent finite group A there are soluble groups G of arbitrarily high Fitting height admitting A as a fixed-point-free group of automorphisms. But if A is nilpotent and $C_G(A) = 1$, then the Fitting height of G is bounded in terms of $\alpha(A)$ by a special case of Dade's theorem [20]. Unlike the aforementioned "linear" results in the coprime case, the bound in Dade's theorem is exponential. Improving this bound to a linear one is a difficult problem; it was tackled in some special cases by Ercan and Güloğlu [21, 22, 23].

In the almost fixed-point-free situation, even for a cyclic group of automorphisms $\langle\varphi\rangle \leqslant \operatorname{Aut} G$ it is still an open problem to obtain a bound for the Fitting height of a finite soluble group G in terms of $|\varphi|$ and $|C_G(\varphi)|$ (this question is equivalent to the one recorded by Belyaev in Kourovka Notebook [24] as Hartley's Problem 13.8(a)). Beyond the fixed-point-free case of Dade's theorem, so far the only cases where an affirmative solution is known are the cases of automorphisms of primary order p^n (Hartley and Turau [25]) and of biprimary order $p^a q^b$ (which is discussed in the present paper).

Another generalization of fixed-point-free automorphisms in the non-coprime case is Thompson's problem on bounding the p-length of a finite p-soluble group G admitting a p-group of automorphisms P that acts fixed-point-freely on every P-invariant p'-section of G. Rae [26] and Hartley and Rae [27] solved this problem in the affirmative for $p \neq 2$, as well as for cyclic P for any p. A special case of this problem is when a p-soluble group G admits a so-called p^n-splitting automorphism φ, which means that $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p^n-1}} = 1$ for all $x \in G$ (this also implies $\varphi^{p^n} = 1$); then of course φ automatically acts fixed-point-freely on φ-invariant p'-sections. This case was actually considered earlier by Kurzweil [28] who obtained bounds for the Fitting height of a soluble group G, and these bounds were improved to linear ones by Meixner [29]. If it is only known that φ induces a p^n-splitting automorphism on a φ-invariant Sylow p-subgroup of G, then there is already a bound in terms of n for the p-length of G: for $p \neq 2$ such a bound was obtained by Wilson [30], and for all primes p in [31] even under a weaker assumption.

In this paper we consider the natural generalization of Thompson's problem for a p-soluble group G admitting an automorphism φ of order p^n in which the condition that φ acts fixed-point-freely on φ-invariant p'-sections is replaced by that φ acts almost fixed-point-freely on these sections. It is actually sufficient to impose the restriction on the number of fixed points of φ only on elementary abelian φ-invariant p'-sections.

Theorem 1.1. *If a finite p-soluble group G admits an automorphism φ of order p^n such that φ has at most m fixed points on every φ-invariant elementary abelian p'-section of G, then the p-length of G is bounded above in terms of p^n and m.*

It would be interesting to obtain a bound of the p-length in terms of n (or at least in terms of p^n) for some subgroup of index bounded in terms of p^n and m.

Remark 1.2. There is a certain similarity with the situation for a p^n-splitting automorphism described above. Namely, if, for a p-soluble group G with an automorphism φ of order p^n, instead of a restriction on the number of fixed points on p'-sections, we have a restriction $|C_P(\varphi)| = p^m$ on the number of fixed points of φ in a φ-invariant Sylow p-subgroup P, then we also obtain a bound for the p-length of G. Indeed, then the derived length of P is bounded in terms of p, n, and m by Shalev's theorem [12], so the bound for the p-length immediately follows from the Hall–Higman theorems [32] for $p \neq 2$, and the theorems of Hoare [33], Berger and Gross [34], and Bryukhanova [35]. Moreover, by [13] the group P even has a (normal) subgroup of index bounded in terms of p, n, and m that has p^n-bounded derived length. Therefore by the Hall–Higman–Hartley Theorem 2.3 (see below) there is a characteristic subgroup H of G such that the p-length of H is p^n-bounded and a Sylow p-subgroup of the quotient G/H has order bounded in terms of p, n, and m.

For soluble groups, Theorem 1.1 can be combined with known results to give a bound for the Fitting height.

Corollary 1.3. *If a finite soluble group G admits an automorphism φ of order p^n such that φ has at most m fixed points on every φ-invariant elementary abelian p'-section of G, then the Fitting height of G is bounded above in terms of p^n and m.*

The technique used in the proof of Theorem 1.1 is also applied in the proof of the soluble case of the following theorem on almost fixed-point-free automorphism of biprimary order; the reduction to the soluble case is given by Hartley' theorem [3] (based on the classification of finite simple groups).

Theorem 1.4. *If a finite group G admits an automorphism φ of order $p^a q^b$ for some primes p, q and nonnegative integers a, b, then G has a soluble subgroup whose index and Fitting height are bounded above in terms of $p^a q^b$ and $|C_G(\varphi)|$.*

Standard inverse limit arguments yield the following corollary for locally finite groups.

Corollary 1.5. *If a locally finite group G contains an element g of order $p^a q^b$ for some primes p, q and nonnegative integers a, b with finite centralizer $C_G(g)$, then G has a subgroup of finite index that has a finite normal series with locally nilpotent factors.*

Another corollary is of more technical nature but it may be useful in further studies.

Corollary 1.6. *If a finite group G admits an automorphism φ such that there are at most two primes dividing both $|\varphi|$ and $|G|$, then G has a soluble subgroup whose index and Fitting height are bounded above in terms of $|\varphi|$ and $|C_G(\varphi)|$.*

Remark 1.7. After this paper was prepared for publication, the author became aware of an unpublished manuscript of Brian Hartley, which contains the result of Theorem 1.4; the author together with A. Borovik and P. Shumyatsky published this manuscript as [36] on the web-site of the University of Manchester.

2. Preliminaries

Induced automorphisms of invariant sections are denoted by the same letters. The following lemma is well known.

Lemma 2.1. *If φ is an automorphism of a finite group G and N is a normal φ-invariant subgroup, then $|C_{G/N}(\varphi)| \leqslant |C_G(\varphi)|$.*

The next lemma is also a well-known consequence of considering the Jordan normal form of a linear transformation of order p^k in characteristic p.

Lemma 2.2. *If an elementary abelian p-group P admits an automorphism φ of order p^k such that $|C_P(\varphi)| = p^m$, then the rank of P is bounded in terms of p^k and m.*

We shall use the following consequence of the Hall–Higman–type theorems in Hartley's paper [37].

Theorem 2.3 (Hall–Higman–Hartley). *Let P be a Sylow p-subgroup of a p-soluble group G. If R is a normal subgroup of P and the derived length of R is d, then $R \leqslant O_{p',p,p',\dots,p',p}(G)$, where p occurs on the right-hand side d times if $p > 3$, $2d$ times if $p = 3$, and $3d$ times if $p = 2$.*

Proof. As a refinement of some of the Hall–Higman theorems [32], Hartley [37] proved that if A is an abelian normal subgroup of a Sylow p-subgroup of G, then

$$A \leqslant O_{p',p}(G) \qquad \text{if} \quad p > 3,$$

$$A \leqslant O_{3',3,3',3}(G) \qquad \text{if} \quad p = 3,$$

and

$$A \leqslant O_{2',2,2',2,2'2}(G) \qquad \text{if} \quad p = 2.$$

The result follows from these inclusions for $A = R^{(d-1)}$ by a straightforward induction on the derived length d. □

We now recall some definitions and notation from representation theory. If V is a kG-module for a field k and a group G, we use the right operator notation vg for $v \in V$ and $g \in G$. We use the centralizer notation for fixed points, like $C_V(g) = \{v \in V \mid vg = v\}$. We also use the commutator notation $[v, g] = -v + vg$ for $v \in V$ and $g \in G$. The commutator subspaces are defined accordingly: if $B \leqslant G$, then $[V, B]$ is the span of all commutators $[v, b]$, where $v \in V$ and $b \in B$. The subspace $[V, B]$ coincides with the commutator subgroup $[V, B]$ in the natural semidirect product VG when V is regarded as the additive group acted upon by G. In particular, $[V, B]$ is B-invariant, and thus can be regarded as a kB-submodule.

For a group G and a field k, a free kG-module of rank n is a direct sum of n copies of the group algebra kG each of which is regarded as a vector space over k of dimension $|G|$ with a basis $\{b_g \mid g \in G\}$ labelled by elements of G on which G acts in a regular permutation representation: $b_g h = b_{gh}$. In other words, a free kG-module $V = \bigoplus_{g\in G} V_g$ is a direct sum of subspaces that are regularly permuted by G so that $V_g h = V_{gh}$.

The following lemma is known in the literature (see, for example, [25, Lemma 4.5]), but we give a proof for completeness.

Lemma 2.4. *Suppose that an abelian p-group M is acted upon by a cyclic group $\langle\varphi\rangle$ of order p^n and V is a $kM\langle\varphi\rangle$-module for a field k of characteristic different from p. If the subgroup $[M, \varphi^{p^{n-1}}]$ acts non-trivially on V, then the subspace $[V, [M, \varphi^{p^{n-1}}]]$ is a free $k\langle\varphi\rangle$-module.*

Here, of course, $\varphi^{p^{n-1}} = \varphi$ if $n = 1$.

Proof. The subspace $[V, [M, \varphi^{p^{n-1}}]]$ is clearly $M\langle\varphi\rangle$-invariant, so is an $kM\langle\varphi\rangle$-module. We extend the ground field to its algebraic closure $\bar{k}$ and denote by $W = V \otimes_k \bar{k}$ the resulting $\bar{k}M\langle\varphi\rangle$-module. Then $[W, [M, \varphi^{p^{n-1}}]]$ is a $\bar{k}M\langle\varphi\rangle$-module obtained from $[V, [M, \varphi^{p^{n-1}}]]$ by the field extension.

Since the characteristic of the ground field is coprime to $|M\langle\varphi\rangle|$, by Maschke's theorem

$$W = C_W([M, \varphi^{p^{n-1}}]) \oplus [W, [M, \varphi^{p^{n-1}}]]$$

is a completely reducible $\bar{k}M\langle\varphi\rangle$-module. Let U be an irreducible $\bar{k}M\langle\varphi\rangle$-submodule of $[W, [M, \varphi^{p^{n-1}}]]$ on which $[M, \varphi^{p^{n-1}}]$ acts non-trivially.

By Clifford's theorem, $U = U_1 \oplus \cdots \oplus U_m$ decomposes into homogeneous $\bar{k}M$-submodules U_i (Wedderburn components). The group $\langle\varphi\rangle$ transitively permutes the U_i. If the kernel of this permutational action was non-trivial, then $\varphi^{p^{n-1}}$ would stabilize all the U_i. But the abelian group M acts by scalar transformations on each homogeneous component U_i. Hence $[M, \varphi^{p^{n-1}}]$ would act trivially on each U_i and therefore on U, contrary to our assumption. Thus, U is a free $\bar{k}\langle\varphi\rangle$-module.

Since $[W, [M, \varphi^{p^{n-1}}]]$ is the direct sum of such U, we obtain that $[W, [M, \varphi^{p^{n-1}}]]$ is also a free $\bar{k}\langle\varphi\rangle$-module. Then $[V, [M, \varphi^{p^{n-1}}]]$ is a free $k\langle\varphi\rangle$-module. Indeed, by the Deuring–Noether theorem [38, Theorem 29.7] two representations over a smaller field are equivalent if they are equivalent over a larger field. Being a free $\bar{k}\langle\varphi\rangle$-module, or a free $k\langle\varphi\rangle$-module, means having a basis, as of a vector space over the corresponding field, elements of which are permuted by φ so that all orbits are regular. In such a basis $\langle\varphi\rangle$ is represented by the corresponding permutational matrices, all of which are defined over k. □

3. Automorphism of order p^n

First we state separately the following proposition, which will also be used in the next section in a different situation.

Proposition 3.1. *Suppose that a cyclic group $\langle\varphi\rangle$ of order p^n acts by automorphisms on a finite p-group P, and V is a faithful $\mathbb{F}_q P\langle\varphi\rangle$-module, where $\mathbb{F}_q$ is a prime field of order $q \neq p$. Then the derived length of $[P, \varphi^{p^{n-1}}]$ is bounded in terms of $|C_V(\varphi)|$ and p^n.*

Proof. Let M be a maximal abelian normal subgroup of the semidirect product $P\langle\varphi\rangle$. If $[M, \varphi^{p^{n-1}}] \neq 1$, then by Lemma 2.4, $[V, [M, \varphi^{p^{n-1}}]]$ is a free $\mathbb{F}_q\langle\varphi\rangle$-module. Obviously, in a free $\mathbb{F}_q\langle\varphi\rangle$-module the fixed points of φ are exactly the "diagonal" elements. Hence the order of $[V, [M, \varphi^{p^{n-1}}]]$ is equal to

$$|C_{[V,[M,\varphi^{p^{n-1}}]]}(\varphi)|^{|\varphi|} = |C_{[V,[M,\varphi^{p^{n-1}}]]}(\varphi)|^{p^n}$$

and therefore is bounded in terms of $|C_V(\varphi)|$ and p^n. The group $[M, \varphi^{p^{n-1}}]$ acts faithfully on V; therefore by Maschke's theorem it also acts faithfully on $[V, [M, \varphi^{p^{n-1}}]]$. Hence the order of $[M, \varphi^{p^{n-1}}]$ is bounded in terms of $|C_V(\varphi)|$ and p^n. The same of course holds if $[M, \varphi^{p^{n-1}}] = 1$.

It follows that the index $|M : C_M(\varphi^{p^{n-1}})|$ is bounded in terms of $|C_V(\varphi)|$ and p^n, since this index is equal to the number of different commutators $[m, \varphi^{p^{n-1}}]$ for $m \in M$.

Consider a central series of $P\langle\varphi\rangle$ connecting 1 and M. Since $|M : C_M(\varphi^{p^{n-1}})|$ is bounded in terms of $|C_V(\varphi)|$ and p^n, the number of factors of this series that are not covered by $C_M(\varphi^{p^{n-1}})$ is bounded in terms of $|C_V(\varphi)|$ and p^n. Therefore there is a normal series of bounded length connecting 1 and M each factor of which is either central in $P\langle\varphi\rangle$ or is covered by $C_M(\varphi^{p^{n-1}})$. Obviously, then $\varphi^{p^{n-1}}$ acts trivially on each factor of this series, and therefore so does $[P, \varphi^{p^{n-1}}]$. By Kaluzhnin's theorem, the automorphism group induced by the action of $[P, \varphi^{p^{a-1}}]$ on M is nilpotent of bounded class. Since M contains its centralizer in $P\langle\varphi\rangle$, it follows that $[P, \varphi^{p^{n-1}}]$ is soluble of bounded derived length, since by the above $\gamma_s([P, \varphi^{p^{n-1}}]) \leqslant M \cap [P, \varphi^{p^{n-1}}]$ for some number s bounded in terms of $|C_V(\varphi)|$ and p^n. □

Proof of Theorem 1.1. Recall that G is a finite p-soluble group admitting an automorphism φ of order p^n such that φ has at most m fixed points in every φ-invariant elementary abelian p'-section of G. We need to bound the p-length of G in terms of p^n and m. Henceforth in this section, saying for brevity that a certain parameter is simply "bounded" we mean that this parameter is bounded above in terms of p^n and m.

We use induction on n. It is convenient to consider the case of $n = 0$ as the basis of induction, when $|\varphi| = p^0 = 1$, that is, φ acts trivially on G. Then the hypothesis means that every elementary abelian p'-section of G has bounded order. We claim that the nilpotency class of a Sylow p-subgroup P of $\hat{G} = G/O_p(G)$ is bounded. Indeed, since the order of P is coprime to $|O_{p'}(\hat{G})|$, for every prime q dividing $|O_{p'}(\hat{G})|$ there is a P-invariant Sylow q-subgroup Q of $O_{p'}(\hat{G})$. The quotient $P/C_P(Q)$ acts faithfully on the Frattini quotient $Q/\Phi(Q)$, which has order at most m by the assumption. Hence $P/C_P(Q)$ has bounded order and therefore bounded nilpotency class. Since P acts faithfully on $O_{p'}(\hat{G})$, we have $\bigcap C_P(Q_i) = 1$, where Q_i runs over all P-invariant Sylow subgroups of $O_{p'}(\hat{G})$. Hence P is a subdirect product of groups of bounded nilpotency class and therefore has bounded nilpotency class itself. We now obtain that the p-length of $\hat{G} = G/O_p(G)$ is bounded by the Hall–Higman theorem [32]. As a result, the p-length of G is bounded.

From now on we assume that $n \geqslant 1$.

Let $\hat{G} = G/O_p(G)$. Consider a Sylow p-subgroup of the semidirect product $\hat{G}\langle\varphi\rangle$ containing $\langle\varphi\rangle$ and let P be its intersection with $\hat{G}$, so that P is a φ-invariant Sylow p-subgroup of $\hat{G}$. Since the order of the p-group $P\langle\varphi\rangle$ is coprime to $|O_{p'}(\hat{G})|$, for every prime q dividing $|O_{p'}(\hat{G})|$ there is a $P\langle\varphi\rangle$-invariant Sylow q-subgroup Q of $O_{p'}(\hat{G})$.

The quotient $\bar{P} = P/C_P(Q)$ acts faithfully on the Frattini quotient $V = Q/\Phi(Q)$, which we regard as an $\mathbb{F}_qP\langle\varphi\rangle$-module. By hypothesis, $|C_V(\varphi)| \leqslant m$, so by Proposition 3.1 the derived length of $[\bar{P}, \varphi^{p^{n-1}}]$ is bounded. In other words, $[P, \varphi^{p^{n-1}}]^{(s)} \leqslant C_P(Q)$ for some bounded number s. Since P acts faithfully on $O_{p'}(\hat{G})$, we have $\bigcap C_P(Q_i) = 1$, where Q_i runs over all $P\langle\varphi\rangle$-invariant Sylow subgroups of $O_{p'}(\hat{G})$. Hence, $[P, \varphi^{p^{n-1}}]^{(s)} = 1$.

By the Hall–Higman–Hartley Theorem 2.3 we now obtain that the normal subgroup $[P, \varphi^{p^{n-1}}]$ of the Sylow p-subgroup P is contained in $H = O_{p',p,p',\dots,p',p}(\hat{G})$, where p occurs boundedly many times.

Consider the action of φ on the quotient $\tilde{G} = \hat{G}/H$. Since $[P, \varphi^{p^{n-1}}] \leqslant H$, it follows that $\varphi^{p^{n-1}}$ acts trivially on the image of P, which is a Sylow p-subgroup of $\tilde{G}$. In particular, $\varphi^{p^{n-1}}$ acts trivially on $O_{p',p}(\tilde{G})/O_{p'}(\tilde{G})$, and therefore so does $[\tilde{G}, \varphi^{p^{n-1}}]$. Since $O_{p',p}(\tilde{G})/O_{p'}(\tilde{G})$ contains its centralizer in $\tilde{G}/O_{p'}(\tilde{G})$, we obtain that $[\tilde{G}, \varphi^{p^{n-1}}] \leqslant O_{p',p}(\tilde{G})$. In other words, $\varphi^{p^{n-1}}$ acts trivially on the quotient $\tilde{G}/O_{p',p}(\tilde{G})$. Therefore the order of the automorphism induced by φ on $\tilde{G}/O_{p',p}(\tilde{G})$ is at most p^{n-1}. By the induction hypothesis the p-length of this quotient is bounded. Then the p-length of $G/O_{p,p'}(G)$ is bounded, and therefore the p-length of G is bounded, as required. □

Proof of Corollary 1.3. Here, G is a finite soluble group admitting an automorphism φ of order p^n such that φ has at most m fixed points in every φ-invariant elementary abelian p'-section of G. By Theorem 1.1 the p-length of G is bounded. It remains to obtain a bound for the Fitting height of every p'-factor T of the upper p-series consisting of the subgroups $O_{p',p,p',p,\dots}$. It is known that the rank of a finite group is bounded in terms of the ranks of its elementary abelian sections. Here, by definition, the rank of a group is the minimum number r such that every subgroup can be generated by r elements. Of course every elementary abelian section of $C_T(\varphi)$ is a φ-invariant p'-section of G and therefore has bounded order by hypothesis. It is also known that the Fitting height of a soluble finite group is bounded in terms of its rank. Thus $C_T(\varphi)$ has bounded Fitting height and therefore so does G by Thompson's theorem [16]. □

Remark 3.2. If we could obtain in Theorem 1.1 a "strong" bound for the p-length, in terms of $\alpha(\langle\varphi\rangle)$ only, for a subgroup of bounded index, then a similar strong bound could be obtained in Corollary 1.3 for the Fitting height of a subgroup of bounded index. This would follow from a rank analogue of the Hartley–Isaacs theorem proved in [39], which states that if a finite soluble group K admits a soluble group of automorphisms L of coprime order, then K has a normal subgroup N of Fitting height at most $5(4^{\alpha(L)}-1)/3$ such that the order of K/N is bounded in terms of $|L|$ and the rank of $C_K(L)$.

4. Automorphism of order $p^a q^b$

Proof of Theorem 1.4. Recall that G is a finite group admitting an automorphism φ of order $p^a q^b$. By Hartley's theorem [3] (based on the classification of finite simple groups), G has a soluble subgroup of index bounded in terms of $p^a q^b$ and $|C_G(\varphi)|$. Therefore we can assume from the outset that G is soluble, so that we need to bound the Fitting height of G in terms of $p^a q^b$ and $|C_G(\varphi)|$. Throughout this section we say for brevity that a certain parameter is "bounded" meaning that this parameter is bounded above in terms of $p^a q^b$ and $|C_G(\varphi)|$. We use without special references the fact that the number of fixed points of φ in every φ-invariant section of G is at most $|C_G(\varphi)|$ by Lemma 2.1.

We use induction on $a+b$. As a basis of induction we consider the case when either $a=0$ or $b=0$. Then $|\varphi|$ is a prime-power, and by the Hartley–Turau theorem [25] the group G has a subgroup of bounded index that has Fitting height at most $\alpha(\varphi)$. (Actually, for our 'weak' bound a simpler argument would suffice: if, say, $|\varphi|=p^a$, then the rank of the Frattini quotient of $O_{p',p}(G)/O_{p'}(G)$ is bounded by Lemma 2.2, which implies a bound for the Fitting height of $G/O_{p'}(G)$, and the Fitting height of $O_{p'}(G)$ is bounded in terms of a by Thompson's theorem [16].) Moreover, the following proposition holds, which apparently was noted by Hartley but may have remained unpublished. We state this proposition in a more general form, without assuming that the automorphism has biprimary order.

Proposition 4.1. *If a finite soluble group G admits an automorphism ψ such that there is at most one prime dividing both $|\psi|$ and $|G|$, then the Fitting height of G is bounded above in terms of $|\psi|$ and $|C_G(\psi)|$.*

Proof. If $(|\psi|,|G|)=1$, then the result follows from the stronger theorem of Thompson [16]. Now let $\langle\psi\rangle=\langle\psi_r\rangle\times\langle\psi_{r'}\rangle$, where $\langle\psi_r\rangle$ is the Sylow r-subgroup of $\langle\psi\rangle$ and r is the only common prime divisor of $|G|$ and $|\psi|$. The centralizer $C_G(\psi_{r'})$ admits the automorphism ψ_r of prime-power order whose centralizer $C_{C_G(\psi_{r'})}(\psi_r)$ is equal to $C_G(\psi)$. By the Hartley–Turau theorem, the Fitting height of $C_G(\psi_{r'})$ is bounded. We now apply Thompson's theorem to the automorphism $\psi_{r'}$ of G of coprime order to obtain that the Fitting height of G is bounded as required. □

We return to the proof of Theorem 1.4. Let $a\geqslant 1$ and $b\geqslant 1$. Let $\varphi_p=\varphi^{q^b}$ and $\varphi_q=\varphi^{p^a}$, so that $|\varphi_p|=p^a$ and $|\varphi_q|=q^b$, while $\langle\varphi\rangle=\langle\varphi_p\rangle\times\langle\varphi_q\rangle$. The subgroup $O_{q'}(G)$ admits the automorphism φ whose order has at most one prime divisor p in common with $|O_{q'}(G)|$. By Proposition 4.1 the Fitting height of $O_{q'}(G)$ is bounded.

Therefore we can assume that $O_{q'}(G)=1$. Then the quotient $\bar G=G/O_q(G)$ acts faithfully on the Frattini quotient $V=O_q(G)/\Phi(O_q(G))$, which we regard as an $\mathbb{F}_q\bar G\langle\varphi\rangle$-module. The fixed-point subspace $C_V(\varphi_p)$ has bounded order. This follows from Lemma 2.2 applied to the action of the linear transformation φ_q of order q^b on $C_V(\varphi_p)$, since the fixed points of φ_q in $C_V(\varphi_p)$ are contained in the fixed-point subspace $C_V(\varphi)$ of bounded order.

Choose a Sylow p-subgroup of $\bar G\langle\varphi_p\rangle$ containing $\langle\varphi_p\rangle$, and let P be its intersection with $\bar G$, so that P is a φ_p-invariant Sylow p-subgroup of $\bar G$. By Proposition 3.1, the subgroup $[P,\varphi_p^{p^{a-1}}]$ has bounded derived length.

Hence by the Hall–Higman–Hartley Theorem 2.3 the normal subgroup $[P, \varphi_p^{p^{a-1}}]$ of the Sylow p-subgroup P is contained in $H = O_{p',p,p',\dots,p',p}(\bar{G})$, where p occurs boundedly many times.

Consider the action of φ on the quotient $\tilde{G} = \bar{G}/H$. Since $[P, \varphi_p^{p^{a-1}}] \leqslant H$, it follows that $\varphi_p^{p^{a-1}}$ acts trivially on the image of P, which is a Sylow p-subgroup of $\tilde{G}$. In particular, $\varphi_p^{p^{a-1}}$ acts trivially on $O_{p',p}(\tilde{G})/O_{p'}(\tilde{G})$, and therefore so does $[\tilde{G}, \varphi_p^{p^{a-1}}]$. Since $O_{p',p}(\tilde{G})/O_{p'}(\tilde{G})$ contains its centralizer in $\tilde{G}/O_{p'}(\tilde{G})$, we obtain that $[\tilde{G}, \varphi_p^{p^{a-1}}] \leqslant O_{p',p}(\tilde{G})$. In other words, $\varphi_p^{p^{a-1}}$ acts trivially on the quotient $\tilde{G}/O_{p',p}(\tilde{G})$. Therefore the order of the automorphism induced by φ on $\tilde{G}/O_{p',p}(\tilde{G})$ divides $p^{a-1}q^b$. By induction, the Fitting height of this quotient is bounded.

It remains to obtain a bound for the Fitting height of each of the boundedly many φ-invariant normal p'-sections that appear in the upper p-series of the groups H and $O_{p',p}(\tilde{G})$. Such a bound follows from Proposition 4.1. □

Proof of Corollary 1.5. This corollary for locally finite groups follows from Theorem 1.4 by the standard inverse limit argument. □

Proof of Corollary 1.6. Here, a finite group G admits an automorphism φ such that there are at most two primes dividing both $|\varphi|$ and $|G|$. Again, by Hartley's theorem [3] we can assume from the outset that G is soluble. If $(|\varphi|, |G|)$ is 1 or a prime power, then the result follows from Proposition 4.1. Now let $\langle\varphi\rangle = \langle\varphi_{pq}\rangle \times \langle\psi\rangle$, where $\langle\varphi_{pq}\rangle$ is the Hall $\{p, q\}$-subgroup of $\langle\varphi\rangle$ and p, q are the only common prime divisors of $|G|$ and $|\varphi|$. The centralizer $C_G(\psi)$ admits the automorphism φ_{pq} of biprimary order whose centralizer $C_{C_G(\psi)}(\varphi_{pq})$ is equal to $C_G(\varphi)$. By Theorem 1.4, the Fitting height of $C_G(\psi)$ is bounded in terms of $|\varphi_{pq}|$ and $|C_G(\varphi)|$. We now apply Thompson's theorem [16] to the automorphism ψ of G of coprime order to obtain that the Fitting height of G is bounded in terms of $|\varphi|$ and $|C_G(\varphi)|$. □

References

[1] R. Brauer and K. A. Fowler, On groups of even order, *Ann. Math.* (2) **62** (1955), 565–583.

[2] J. Thompson, Finite groups with fixed-point-free automorphosms of prime order, *Proc. Nat. Acad. Sci. U.S.A.*, **45** (1959), 578–581.

[3] B. Hartley A general Brauer–Fowler theorem and centralizers in locally finite groups, *Pacific J. Math.* **152** (1992), 101–117.

[4] G. Higman, Groups and rings which have automorphisms without non-trivial fixed elements, *J. London Math. Soc.* (2) **32** (1957), 321–334.

[5] V. A. Kreknin, The solubility of Lie algebras with regular automorphisms of finite period, *Dokl. Akad. Nauk SSSR* **150** (1963), 467–469; English transl., *Math. USSR Doklady* **4** (1963), 683–685.

[6] V. A. Kreknin and A. I. Kostrikin, Lie algebras with regular automorphisms, *Dokl. Akad. Nauk SSSR* **149** (1963), 249–251 (Russian); English transl., *Math. USSR Doklady* **4**, 355–358.

[7] E. I. Khukhro, Groups and Lie rings admitting an almost regular automorphism of prime order, *Mat. Sbornik* **181**, no. 9 (1990), 1207–1219; English transl., *Math. USSR Sbornik* **71**, no. 9 (1992), 51–63.

[8] L. G. Kovács, Groups with regular automorphisms of order four, *Math. Z.* **75** (1960/61), 277–294.

[9] N. Yu. Makarenko and E. I. Khukhro, Finite groups with an almost regular automorphism of order four, *Algebra Logika* **45** (2006), 575–602; English transl., *Algebra Logic* **45** (2006), 326–343.

[10] J. L. Alperin, Automorphisms of solvable groups, *Proc. Amer. Math. Soc.* **13** (1962), 175–180.

[11] E. I. Khukhro, Finite p-groups admitting an automorphism of order p with a small number of fixed points, *Mat. Zametki* **38** (1985), 652–657 (Russian); English transl., *Math. Notes* **38** (1986), 867–870.

[12] Shalev A. On almost fixed point free automorphisms, *J. Algebra*, **157**, 271–282.

[13] Khukhro E. I. Finite p-groups admitting p-automorphisms with few fixed points, *Mat. Sbornik*, **184** (1993), 53–64; English transl., *Russian Acad. Sci. Sbornik Math.*, **80** (1995), 435–444.

[14] Yu. Medvedev p-Divided Lie rings and p-groups, *J. London Math. Soc.* (2), **59** (1999), 787–798.
[15] A. Jaikin-Zapirain, On almost regular automorphisms of finite p-groups, *Adv. Math.* **153** (2000), 391–402.
[16] J. Thompson, Automorphisms of solvable groups, *J. Algebra* **1** (1964), 259–267.
[17] A. Turull, Fitting height of groups and of fixed points, *J. Algebra* **86** (1984), 555–566.
[18] B. Hartley and I. M. Isaacs, On characters and fixed points of coprime operator groups, *J. Algebra* **131** (1990), 342–358.
[19] S. D. Bell, B. Hartley, A note on fixed-point-free actions of finite groups, *Quart. J. Math. Oxford Ser.* (2), **41** (1990), N 162, 127–130.
[20] E. C. Dade, Carter subgroups and Fitting heights of finite solvable groups, *Illinois J. Math.* **13** (1969), 449–514.
[21] G. Ercan, On a Fitting length conjecture without the coprimeness condition, *Monatsh. Math.* **167**, no. 2 (2012), 175–187.
[22] G. Ercan and İ. Güloğlu, On finite groups admitting a fixed point free automorphism of order pqr, *J. Group Theory* **7**, no. 4 (2004), 437–446.
[23] G. Ercan and İ. Güloğlu, Fixed point free action on groups of odd order, *J. Algebra* **320** (2008), 426–436.
[24] *Unsolved Problems in Group Theory. The Kourovka Notebook*, no. 18, Institute of Mathematics, Novosibirsk, 2014.
[25] B. Hartley and V. Turau, Finite soluble groups admitting an automorphism of prime power order with few fixed points, *Math. Proc. Cambridge Philos. Soc.*, **102** (1987), 431–441.
[26] Rae, A. Sylow p-subgroups of finite p-soluble groups. J. London Math. Soc. (2) 7 (1973), 117–123.
[27] Hartley, B.; Rae, A. Finite p-groups acting on p-soluble groups. Bull. London Math. Soc. 5 (1973), 197–198.
[28] Kurzweil, Hans. Eine Verallgemeinerung von fixpunktfreien Automorphismen endlicher Gruppen. (German) Arch. Math. (Basel) 22 (1971), 136–145.
[29] Meixner, Thomas. The Fitting length of solvable H_{p^n}-groups. Israel J. Math. 51 (1985), no. 1-2, 68–78
[30] Wilson, John S. On the structure of compact torsion groups. Monatsh. Math. 96 (1983), no. 1, 57–66.
[31] Khukhro, E. I.; Shumyatsky, P. Words and pronilpotent subgroups in profinite groups. J. Aust. Math. Soc. 97 (2014), no. 3, 343–364.
[32] P. Hall and G. Higman, The p-length of a p-soluble group and reduction theorems for Burnside's problem, *Proc. London Math. Soc.* (3), **6** (1956), 1–42.
[33] A. H. M. Hoare, A note on 2-soluble groups, *J. London Math. Soc.* **35** 1960, 193–199.
[34] T. R. Berger and F. Gross, 2-length and the derived length of a Sylow 2-subgroup, *Proc. London Math. Soc.* (3) **34** (1977), 520–534.
[35] E. G. Bryukhanova, The relation between 2-length and derived length of a Sylow 2-subgroup of a finite soluble group, *Mat. Zametki* **29**, no. 2 (1981), 161–170; English transl., *Math. Notes* **29**, no. 1–2 (1981), 85–90.
[36] B. Hartley, Automorphisms of finite soluble groups. Preliminary version, MIMS EPrint: 2014.52 `http://eprints.ma.man.ac.uk/2188/01/covered/MIMS_ep2014_52.pdf`
[37] B. Hartley, Some theorems of Hall–Higman type for small primes, Proc. London Math. Soc. (3) **41** (1980), 340–362.
[38] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Interscience, New York–London, 1962.
[39] E. Khukhro and V. Mazurov, Automorphisms with centralizers of small rank, *Groups St. Andrews* 2005. *Vol. II. Selected papers of the conference*, *St. Andrews*, *UK*, *July* 30*–August* 6, 2005, London Math. Soc. Lecture Note Ser., vol. 340, Cambridge Univ. Press, Cambrisge, 2007, 564–585.

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