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LOCALLY POLYNOMIALLY BOUNDED STRUCTURES

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Abstract

We prove a theorem which provides a method for constructing points on varieties defined by certain smooth functions. We require that the functions are definable in a definably complete expansion of a real closed field and are locally definable in a fixed o-minimal and polynomially bounded reduct.

As an application we show that in certain o-minimal structures definable functions are piecewise implicitly defined over the basic functions in the in the language.

1. *Introduction*

In his proof of the model completeness of the real exponential field ([**7**]), the second author develops a theory of Noetherian differential rings of definable functions, and studies varieties defined by these functions. One of the main results of this theory is Theorem 5.1 which provides a method for constructing points on such varieties.

Our aim in this paper is to prove a version of this theorem without the Noetherianity assumption. Instead we suppose that the functions considered are definable in an expansion of a real closed field, $\mathcal{M}$ say, which is definably complete (see [**5**],[**6**]) and further, that the functions are what we call *locally tame*. We will give precise definitions later, but the idea is that certain restrictions (to bounded boxes) of the (total) functions considered, are definable in a fixed o-minimal polynomially bounded reduct of $\mathcal{M}$. Then we can use Miller's results ([**3**],[**4**]) to bound orders of vanishing and it is this that makes up for the lack of Noetherianity.

After proving the main result, we specialize to the o-minimal situation. We call an o-minimal structure $\mathcal{M}$ with model complete theory *locally polynomially bounded* if the reduct generated by all restrictions of the basic functions to bounded open boxes is polynomially bounded. We show that being locally polynomially bounded is preserved under elementary equivalence. Combining this with model completeness and the main theorem, we show that definable functions are piecewise implicitly defined over the basic functions in the language. This implies that these structures have smooth cell decomposition. Under a further assumption on these basic functions, this gives uniform control over the derivatives of definable functions.

2. *Locally tame functions*

Let $\bar{M} = \langle M, <, +, \cdot, 0, 1\rangle$ be a fixed real closed field and let $\mathcal{M} = \langle \bar{M}, \dots\rangle$ be an arbitrary, but fixed, expansion of $\mathcal{M}$. We also fix an o-minimal, polynomially

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bounded reduct $\mathcal{M}_0$ of $\mathcal{M}$ such that $\mathcal{M}_0$ is also an expansion of $\bar{M}$. We use *definable* to mean definable with parameters and *0-definable* to mean definable without parameters, and unless we specifically mention another structure, we are referring to definability in $\mathcal{M}$.

DEFINITION 1. Suppose that $f : U \to M$ is a definable function on some open $U \subseteq M^n$. We say that f is *locally tame* if f is smooth (i.e. infinitely differentiable on U in the sense of the usual $\varepsilon - \delta$ definition formulated in $\mathcal{M}$) and, for every open box $B \subseteq M^n$ having sides of length ≤ 1 and satisfying $\overline{B} \subseteq U$, we have that $f|_B$ is definable in $\mathcal{M}_0$.

EXAMPLE 1. Suppose that $\mathcal{M} = \langle \bar{\mathbb{R}}, \exp \rangle$ and $\mathcal{M}_0 = \langle \bar{\mathbb{R}}, \exp |[0,1] \rangle$. Then exp is locally tame. Now consider the function

$$\begin{aligned} g : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto \begin{cases} \exp(-1/t^2) & t \neq 0 \\ 0 & t = 0. \end{cases} \end{aligned}$$

This function is smooth and definable, but, by the following result, it is not locally tame.

PROPOSITION 2.1. *Let $f : U \to M$ be a locally tame function. Then the set of flat points of f (i.e. the points at which all derivatives of f of all orders vanish) is definable and is both open and closed in U. Further, if B is any open box having sides of length at most 1, satisfying $\overline{B} \subseteq U$, and B contains a flat point of f, then f vanishes throughout B.*

Proof. Let X be the set of all flat points of f and let

$$Y = \{\bar{x} \in U : \text{ there is an open box around } \bar{x} \text{ on which } f \text{ vanishes}\}.$$

Clearly we have $Y \subseteq X$. Suppose that $\bar{a} \in X$ and let B be a box containing $\bar{a}$ with sides of length ≤ 1 such that $\overline{B} \subseteq U$. Since f is locally tame, the restriction $f|_B$ is definable in $\mathcal{M}_0$. This structure is o-minimal and polynomially bounded so a result of Miller's ([**3**]) shows that $\bar{a} \in Y$. So $Y = X$ and X is definable and since Y is open, so is X.

Now suppose for a contradiction that X is not closed in U. Then there is some $\bar{b} \in U$ such that $\bar{b} \in \mathrm{fr}X$. Fix $\alpha \in \mathbb{N}^n$. There are points arbitrarily close to $\bar{b}$ at which f is flat. At such a point, $\bar{x}$ say, $D^\alpha f(\bar{x}) = 0$. Hence $D^\alpha f(\bar{b}) = 0$. Since $\alpha \in \mathbb{N}^n$ was arbitrary, it follows that f is flat at $\bar{b}$. So $\bar{b} \in X$ which is a contradiction.

Finally, if B is an open box having sides of length at most 1, satisfying $\overline{B} \subseteq U$ and containing a flat point of f, then we may apply the above argument in the o-minimal, polynomially bounded structure $\mathcal{M}_0$ to the function $f|B$ and use the fact that B is definably connected. □

We now suppose that we have, for each $n \geq 1$, a $\mathbb{Q}$-algebra R_n of locally tame functions $f : M^n \to M$, which is closed under partial differentiation. We will also assume that $R_n \subseteq R_{n+1}$ (in the obvious sense) and that

$$\mathbb{Q}[X_1, \ldots, X_n] \subseteq R_n.$$

Before giving our main result, we recall some notation from [**7**]. Let $f \in R_n$. We define $\nabla f : M^n \to M^n$ by

$$\nabla f(\bar{a}) := \langle \frac{\partial f}{\partial x_1}(\bar{a}), \ldots, \frac{\partial f}{\partial x_n}(\bar{a}) \rangle \qquad \text{for } \bar{a} \in M^n.$$

Note that $\nabla f \in R_n^n$. For $p \geq 1$ and $f_1, \ldots, f_p \in R_n$ we let

$$V_n(f_1, \ldots, f_p) := \{\bar{x} \in M^n : f_1(\bar{x}) = \ldots = f_p(\bar{x}) = 0\}$$

and

$$V_n^{reg}(f_1, \ldots, f_p) := \{\bar{x} \in V_n(f_1, \ldots, f_p) : \nabla f_1(\bar{x}), \ldots, \nabla f_p(\bar{x}) \text{ are linearly independent}\}.$$

Here, linear independence is in the $\bar{M}$ vector space M^n. The *Jacobian matrix* of $f_1, \ldots, f_p$ is the matrix

$$J_n(f_1, \ldots, f_p) := \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_p \end{pmatrix}.$$

The rows of $J_n(f_1, \ldots, f_p)$ are linearly independent when evaluated at $\bar{a} \in M^n$ if and only if $p \leq n$ and there is a $p \times p$ submatrix whose determinant is non-zero when evaluated at $\bar{a}$. So, if we let $Q = Q_{n,f_1,\ldots,f_p} \in R_n$ be the sum of squares of all such determinants we have

$$\text{for all } \bar{a} \in M^n, \bar{a} \in V_n^{reg}(f_1, \ldots, f_p) \leftrightarrow \bar{a} \in V_n(f_1, \ldots, f_p) \text{ and } Q(\bar{a}) > 0. \qquad (2.1)$$

LEMMA 2.2. *If we regard $f_1, \ldots, f_p$ as elements of R_{n+1}, then there is a function $f_{p+1} \in R_{n+1}$ such that*

$$V_{n+1}(f_1, \ldots, f_{p+1}) = V_{n+1}^{reg}(f_1, \ldots, f_{p+1}),$$

and $V_{n+1}(f_1, \ldots, f_{p+1})$ projects onto $V_n^{reg}(f_1, \ldots, f_p)$. In particular, $V_{n+1}^{reg}(f_1, \ldots, f_{p+1})$ is closed in M^{n+1}.

Proof. Let $f_{p+1}(x_1, \ldots, x_{n+1}) = x_{n+1} \cdot Q(x_1, \ldots, x_n) - 1$ where Q is the function defined before the Lemma. Then by (2.1),

$$\langle \bar{a}, a_{n+1} \rangle \in V_{n+1}^{reg}(f_1, \ldots, f_{p+1}) \leftrightarrow \bar{a} \in V_n^{reg}(f_1, \ldots, f_p) \text{ and } a_{n+1} = Q(\bar{a})^{-1}.$$

An easy calculation shows that for such $\langle \bar{a}, a_{n+1} \rangle$, we have $Q_0(\bar{a}, a_{n+1}) \geq Q(\bar{a})^3$, where $Q_0 := Q_{n+1,f_1,\ldots,f_{p+1}}$. So

$$V_{n+1}(f_1, \ldots, f_{p+1}) = V_{n+1}^{reg}(f_1, \ldots, f_{p+1}).$$

as required. □

THEOREM 2.3. *Assume that $\mathcal{M}$ is definably complete (i.e. every definable subset of M with an upper bound has a least upper bound). Suppose that $n \geq 1$ and that $f \in R_n$ is such that $V(f)$ is nonempty. Then there exist $m \geq 0$ and $f_1, \ldots, f_{n+m} \in R_{n+m}$ such that*

$$V_{n+m}^{reg}(f_1, \ldots, f_{n+m}) \cap V_{n+m}(f) \neq \emptyset.$$

Here we regard f as an element of R_{n+m}, so $V_{n+m}(f) = V_n(f) \times M^m$.

Proof. If f vanishes identically then we let $m = 0$ and $f_i(x_1, \dots, x_n) = x_i$ so that we have $V_n^{reg}(f_1, \dots, f_n) \cap V_n(f) = \{\bar{0}\}$. So we may suppose that f is not identically zero.

We will show by induction on p, for $1 \leq p \leq n$, that

$$\begin{array}{c}\text{there exist } m \geq 0 \text{ and } f_1, \dots, f_{p+m} \in R_{p+m} \text{ such that} \\ V_{n+m}^{reg}(f_1, \dots, f_{p+m}) \cap V_{n+m}(f) \neq \emptyset.\end{array} \tag{2.2}$$

Suppose first that $p = 1$. We choose any point $\bar{a} \in V_n(f)$. Since $\mathcal{M}$ is definably complete, the set M is definably connected (see [**5**]) and a simple argument shows that M^n is also definably connected. Hence since f is locally tame and is not identically zero, Proposition 2.1 gives an $\alpha \in \mathbb{N}^n$ such that, with $f_1 = D^\alpha f$, we have $\bar{a} \in V_n^{reg}(f_1)$. This proves (2.2) for $p = 1$, with $m = 0$.

Now suppose that p is such that $1 \leq p < n$ and that (2.2) holds for p. Then we have $m \geq 0$ and $f_1, \dots, f_{p+m} \in R_{n+m}$ such that

$$V_{n+m}^{reg}(f_1, \dots, f_{p+m}) \cap V_{n+m}(f) \neq \emptyset.$$

Case 1. There is some $\bar{a} \in V_{n+m}^{reg}(f_1, \dots, f_{p+m}) \cap V_{n+m}(f)$ such that f is not identically zero on $B \cap V_{n+m}^{reg}(f_1, \dots, f_{p+m})$ for any open box $B \subseteq M^{n+m}$ with $\bar{a} \in B$.

Since $\bar{a} \in V_{n+m}^{reg}(f_1, \dots, f_{p+m})$, there is some $(p+m) \times (p+m)$ submatrix of $J_{n+m}(f_1, \dots, f_{p+m})$ whose determinant is non-zero at $\bar{a}$. We will assume that this submatrix consists of the last $(p+m)$ columns, and write Δ for its determinant. Note that Δ is a function in R_{n+m}. For $\bar{y} = \langle y_1, \dots, y_{n+m} \rangle \in M^{n+m}$, we let $\tilde{y} := \langle y_1, \dots, y_{n-p} \rangle$. Since the functions $f_1, \dots, f_{p+m}$ are locally tame, there is an open box $B_0 \subseteq M^{n+m}$ such that $\bar{a} \in B_0$ and $f_1|_{B_0}, \dots, f_{p+m}|_{B_0}$ are definable in $\mathcal{M}_0$. By the implicit function theorem, applied in the o-minimal structure $\mathcal{M}_0$ (see [**1**], Chapter 7) there is an open box $U \subseteq M^{n-p}$ with $\tilde{a} \in U$ and a smooth map $\phi : U \to M^{p+m}$, definable in $\mathcal{M}_0$, such that

(i) $\phi(\tilde{a}) = \langle a_{n-p+1}, \dots, a_{n+m} \rangle$,
(ii) $\{\langle \tilde{y}, \phi(\tilde{y}) \rangle : \tilde{y} \in U\} = B \cap V_{n+m}^{reg}(f_1, \dots, f_{p+m})$

for some open box $B \subseteq M^{n+m}$ with $\bar{a} \in B$. We may suppose that Δ has no zeroes in B. Since f is locally tame, $f|_B$ is definable in $\mathcal{M}_0$ and hence so is the function

$$\begin{aligned} g : U &\to M \\ \tilde{y} &\mapsto f(\tilde{y}, \phi(\tilde{y})). \end{aligned}$$

Now, by the hypothesis of case 1 and (i) and (ii) above, g is not identically zero on U, and as $\mathcal{M}_0$ is polynomially bounded, there is some $\alpha \in \mathbb{N}^{n-p}$ such that $g^* := D^\alpha g$ vanishes at $\tilde{a}$ but, for some $j = 1, \dots, n-p$, $\frac{\partial g^*}{\partial y_j}$ does not.

Now we have

$$\begin{aligned} f_i(\tilde{y}, \phi(\tilde{y})) = 0 \text{ for } i = 1, \dots, m+p \text{ and } \tilde{y} \in U, \\ g(\tilde{y}) = f(\tilde{y}, \phi(\tilde{y})) \text{ for } \tilde{y} \in U, \end{aligned}$$

and by differentiating these relations, we obtain a function $F \in R_{n+m}$ such that

$$g^*(\tilde{y}) = \frac{F(\tilde{y}, \phi(\tilde{y}))}{\Delta(\tilde{y}, \phi(\tilde{y}))^d} \text{ for all } \tilde{y} \in U$$

for some d. We also have that $F(\tilde{a}, \phi(\tilde{a})) = F(\bar{a}) = 0$, and since $\frac{\partial g^*}{\partial y_j}(\tilde{a}) \neq 0$, it follows from Lemma 4.7 in [**7**] that $\nabla f_1(\bar{a}), \dots, \nabla f_p(\bar{a}), \nabla F(\bar{a})$ are linearly independent. So we obtain (2.2) for $p+1$ by taking $f_{p+m+1} = F$ and not changing m.

Case 2. Not case 1.

By Lemma 2.2 we may suppose (after increasing m) that $V^{reg}_{n+m}(f_1, \dots, f_{p+m}) = V_{n+m}(f_1, \dots, f_{p+m})$. Let $\mathcal{C} = V^{reg}_{n+m}(f_1, \dots, f_{p+m}) \cap V_{n+m}(f)$. Then $\mathcal{C}$ is nonempty (by (2)) and closed in M^{n+m}. Now, if we can find some $h \in R_{n+m}$ which has a zero in $\mathcal{C}$ but is not identically zero on $B \cap V^{reg}_{n+m}(f_1, \dots, f_{p+m})$ for any open box B containing this zero, then we can apply the method of Case 1 to h and we will be done.

To find such an h we proceed as in the proof of Theorem 5.1 in [**7**]. Let $\bar{\eta} = \langle \eta_1, \dots, \eta_{n+m} \rangle \in \mathbb{Q}^{n+m}$. Then, since $\mathcal{C}$ is closed, there is a point $\bar{b} \in \mathcal{C}$ at minimum distance from $\bar{\eta}$. (This follows easily from the definable completeness of $\mathcal{M}$.) Let $H_{\bar{\eta}}(\bar{x}) := \Sigma(x_i - \eta_i)^2$. Then $H_{\bar{\eta}} \in R_{n+m}$ and the function $H_{\bar{\eta}}|\mathcal{C}$ has a minimum at $\bar{b}$. However, by the hypotheses of Case 2, $\mathcal{C}$ coincides with $V^{reg}_{n+m}(f_1, \dots, f_{p+m})$ on some open box in M^{n+m} containing the point $\bar{b}$ and hence, by the method of Lagrange multipliers (see 4.10 in [**7**]; we should also remark that we may work in the o-minimal structure $\mathcal{M}_0$ at this point), the vectors $\nabla f_1(\bar{b}), \dots, \nabla f_{p+m}(\bar{b}), \nabla H_{\bar{\eta}}(\bar{b})$ are linearly dependent. Now, by (1), this is equivalent to the vanishing at $\bar{b}$ of the function $Q_{\bar{\eta}} := Q_{n+m, f_1, \dots, f_{p+m}, H_{\bar{\eta}}} \in R_{n+m}$. Now consider the function $\tilde{f} := Q^2_{\bar{\eta}} + f^2$. Either it will serve as the required function h, or else it too satisfies the same hypothesis of Case 2 as did f (including the fact that $V^{reg}_{n+m}(f_1, \dots, f_{p+m}) \cap V_{n+m}(\tilde{f}) \neq \emptyset$).

So by successive repetition of this argument we either succeed in finding a suitable h, or else for any positive integer r and any sequence of points $\bar{\eta}_1, \dots \bar{\eta}_r \in \mathbb{Q}^{n+m}$, we find a point $\bar{c} \in \mathcal{C}$ such that for each $i = 1, \dots, r$, the vector $\nabla H_{\bar{\eta}_i}(\bar{c})$ lies in the vector space spanned by $\nabla f_1(\bar{c}), \dots, \nabla f_{p+m}(\bar{c})$. However, for $\bar{\eta} \in \mathbb{Q}^{n+m}$ one calculates that $\nabla H_{\bar{\eta}}(\bar{c}) = \langle 2(c_1 - \eta_1), \dots, 2(c_{n+m} - \eta_{n+m}) \rangle$. Thus, if we take $r = n+m+1$, $\bar{\eta}_1 = \bar{0}$ and $\bar{\eta}_2, \dots, \bar{\eta}_{n+m+1}$ to be any basis for $\mathbb{Q}^{n+m}$ we see that (for any $\bar{c} \in M^{n+m}$), the set $\{\nabla H_{\bar{\eta}_1}(\bar{c}), \dots, \nabla H_{\bar{\eta}_{n+m+1}}(\bar{c})\}$ spans M^{n+m}, contradicting the fact that $p + m < n + m$. Thus we will find a suitable h and this completes the proof of Theorem 2.3. □

Examples of definably complete structures include any structure elementarily equivalent to an expansion of $\bar{\mathbb{R}}$, and any o-minimal structure. See [**5**] and [**6**] for discussions of definably complete structures.

3. *Locally polynomially bounded structures*

From now on in this paper, we fix an o-minimal structure $\mathcal{M} = \langle \bar{M}, \mathcal{F} \rangle$ with model complete theory, where $\mathcal{F}$ is a collection of smooth functions $f : M^n \to M$ for various n. Let $\mathcal{F}^{res}$ denote the collection of all functions of the form $f|_B$, for $f \in \mathcal{F}$ and B an open box in M^n. We say $\mathcal{M}$ is *locally polynomially bounded* if the structure $\mathcal{M}_0 := \langle \bar{M}, \mathcal{F}^{res} \rangle$ is polynomially bounded.

So, for example, suppose that $\tilde{\mathbb{R}}$ is a polynomially bounded o-minimal expansion of $\bar{\mathbb{R}}$ and that $\tilde{\mathbb{R}}$ has smooth cell decomposition. Then the structure $\langle \bar{\mathbb{R}}, \mathcal{F} \rangle$, where $\mathcal{F}$ is the collection of all smooth $\tilde{\mathbb{R}}$-definable functions, is LPB, and this structure and $\tilde{\mathbb{R}}$ have the same definable sets. Another example of an LPB structure is the real exponential field discussed in the previous section. These examples can be combined: suppose that $\tilde{\mathbb{R}}$ is a polynomially bounded o-minimal expansion of $\bar{\mathbb{R}}$ with smooth cell decomposition and that the restricted exponential function, $\exp|_{[0,1]}$, is definable in $\tilde{\mathbb{R}}$. Let $\mathcal{F}$ denote the collection of all total smooth definable functions.

By a Theorem of van den Dries and Speissegger (Theorem B in [**2**]) the structure $\langle\bar{\mathbb{R}}, \mathcal{F}, \exp\rangle$ is model complete and hence LPB.

Now, let $\mathcal{M}$ be an LPB structure with $\mathcal{M}_0$ and $\mathcal{F}$ as described above. Let $\mathcal{N} = \langle\bar{N}, \mathcal{G}\rangle$ be a structure for the same language (with $\bar{N}$ a real closed field) and form $\mathcal{G}^{res}$ and $\mathcal{N}_0$ in the analogous way.

Theorem 3.1. *If $\mathcal{M} \equiv \mathcal{N}$ then $\mathcal{N}$ is also LPB.*

Proof. We must show that $\mathcal{N}_0$ is polynomially bounded. We will first show it is power bounded, in the sense of [**4**], so suppose that it is not. Then by Miller's Dichotomy Theorem ([**4**]), there is an exponential function $E : N \to N$ which is 0-definable in $\mathcal{N}_0$. This means that there is some formula, $\Phi(F_1, \dots, F_n, x, y)$ say, in the language of ordered rings together with n function variables (of various arities) but only first order quantifiers, functions $g_1, \dots, g_n \in \mathcal{G}$ (of the corresponding arities) and bounded open boxes $B_1, \dots, B_n$ (in the corresponding spaces) such that

$$\text{for all } a, b \in N, E(a) = b \text{ if and only if } \mathcal{N} \models \Phi(g_1|_{B_1}, \dots, g_n|_{B_n}, a, b).$$

We now write $\Phi(x, y)$ for $\Phi(F_1, \dots, F_n, x, y)$ and let $\Psi(F_1, \dots, F_n)$ be the formula

$$\forall x \exists! y \Phi(x, y) \wedge \Phi(0, 1) \wedge \forall x, x', y, y' \big[(x < x' \wedge \Phi(x, y) \wedge \Phi(x', y')) \to (y < y' \wedge \Phi(x + x', y \cdot y'))\big].$$

Then

$$\mathcal{N} \models \exists B_1, \dots, B_n \Psi(g_1|_{B_1}, \dots, g_n|_{B_n}).$$

Now quantification over boxes is first order, as we can quantify over the corners. So, by the elementary equivalence of $\mathcal{M}$ and $\mathcal{N}$, we have

$$\mathcal{M} \models \exists B_1, \dots, B_n \Psi(f_1|_{B_1}, \dots, f_n|_{B_n})$$

where the $f_1, ..., f_n \in \mathcal{F}$ correspond to $g_1, ..., g_n \in \mathcal{G}$. Hence an exponential function is definable in $\mathcal{M}_0$, contradicting the fact that $\mathcal{M}$ is locally polynomially bounded.

So $\mathcal{N}_0$ is power bounded. We now need to show that it is polynomially bounded.

Claim. *Suppose that for any formula $\Phi(F_1, \dots, F_n, x, y)$ (in the language of ordered rings, together with n function variables but only first order quantifiers) and any collection of functions $f_1, \dots, f_n \in \mathcal{F}$, the formula $\Phi(f_1|_{B_1}, \dots, f_n|_{B_n}, x, y)$ defines in $\mathcal{M}$ the graphs of only finitely many power functions as the boxes $B_1, \dots, B_n$ vary. Then $\mathcal{N}_0$ is polynomially bounded.*

Proof. Note that it suffices to show that there is no non-polynomially bounded power function definable without parameters in $\mathcal{N}_0$. So, suppose that $g_1 \dots, g_n \in \mathcal{G}$ and $B_1, \dots, B_n$ are open boxes such that the formula $\Phi(g_1|_{B_1}, \dots, g_n|_{B_n}, x, y)$ defines a power function, x^α say, in $\mathcal{N}_0$. By the hypothesis of the Claim and the fact that $\mathcal{M}_0$ is polynomially bounded, there is a $k \in \mathbb{N}$ such that the sentence

$$\forall B_1, \dots, B_n (\text{if } \Phi(f_1|_{B_1}, \dots, f_n|_{B_n}, x, y) \text{ defines a power function then this function is bounded by } x^k)$$

holds in $\mathcal{M}$, where the $f_1, ..., f_n \in \mathcal{F}$ correspond to $g_1, ..., g_n \in \mathcal{G}$. (To see that the set of boxes for which $\Phi(f_1|_{B_1}, \dots, f_n|_{B_n}, x, y)$ defines a power function is definable, write out a formula analogous to the formula Ψ above.) Hence this sentence is true in $\mathcal{N}$ and so $\alpha \leq k$ and $\mathcal{N}_0$ is polynomially bounded. $\square$

We will now establish the hypothesis of the Claim, so fix a formula $\Phi(F_1, \ldots, F_n, x, y)$ and functions $f_1, \ldots, f_n \in \mathcal{F}$. Let K be the (definable) set of exponents of power functions defined by $\Phi(f_1|_{B_1}, \ldots, f_n|_{B_n}, x, y)$ as the boxes $B_1, \ldots, B_n$ vary and suppose for a contradiction that K contains a nonempty open interval, J say. Using definable choice and monotonicity (in the o-minimal structure $\mathcal{M}$) there is a bounded subinterval J_0 say, with $\overline{J}_0 \subseteq J$ and a continuous definable function G on $\overline{J}_0$, whose values are n-tuples of boxes, such that

$$\begin{array}{l}\text{for all } \alpha \in \overline{J}_0, G(\alpha) = \langle B_1^\alpha, \ldots, B_n^\alpha \rangle \text{ is such that} \\ \Phi(f_1|_{B_1^\alpha}, \ldots, f_n|_{B_n^\alpha}, x, y) \text{ defines } y = x^\alpha \text{ for } x > 0.\end{array} \tag{3.1}$$

Since $\overline{J}_0$ is a closed bounded interval, G is bounded and we may take bounded open boxes $D_1, \ldots, D_n$ such that $B_i^\alpha \subseteq D_i$, for all $\alpha \in \overline{J_0}$ and $i = 1, \ldots, n$. Now, if we repeat the above argument with the structure $\langle \bar{M}, f_1|_{D_1}, \ldots, f_n|_{D_n} \rangle$ in place of $\mathcal{M}$, we obtain an interval J_1 and a function G_1 defined in the structure $\langle \bar{M}, f_1|_{D_1}, \ldots, f_n|_{D_n} \rangle$ such that (3.1) holds with J_1, G_1 in place of J_0, G. Hence the function $\langle x, y \rangle \mapsto x^y$ with $x > 0$ and $y \in J_1$ is definable in $\langle \bar{M}, f_1|_{D_1}, \ldots, f_n|_{D_n} \rangle$ and hence in $\mathcal{M}_0$. But this is impossible, by the proof of 4.2 in [**4**], as $\mathcal{M}_0$ is polynomially bounded. $\square$

4. *Consequences of model completeness*

For the remainder of the paper, we fix an LPB structure, $\mathcal{M} = \langle \bar{M}, \mathcal{F} \rangle$. Let $\widetilde{\mathcal{F}}$ be the smallest collection of functions containing $\mathcal{F}$ and all polynomials over $\mathbb{Q}$ and closed under the $\mathbb{Q}$-algebra operations and under partial differentiation. For each $n \geq 1$, let R_n be the $\mathbb{Q}$-algebra consisting of all n-ary functions in $\widetilde{\mathcal{F}}$. Then each R_n is closed under partial differentiation and consists of locally tame functions, so the results of the first section apply. These results also apply to the rings $R_n^{\bar{a}}$, for $\bar{a} \in M^p$, consisting of all functions of the form $\bar{x} \mapsto f(\bar{a}, \bar{x})$ for some $f \in R_{p+n}$.

Definition 2. Let $\bar{a} \in M^p$ and $b \in M$. We say that b is *$\mathcal{F}$-defined over $\bar{a}$* if there exist $m \geq 1, f_1, \ldots, f_m \in R_m^{\bar{a}}$ and $b_1, \ldots, b_m \in M$ with $b = b_i$ for some i, such that

$$\bar{b} \in V_m^{reg}(f_1, \ldots, f_m).$$

The following is an easy consequence of the model completeness of $\langle \bar{M}, \mathcal{F} \rangle$ and 2.3, together with a standard trick on representing definable sets as projections of zero sets.

Theorem 4.1. *Let $\bar{a} \in M^p, b \in M$. Then b is in the definable closure of $\bar{a}$ if and only if b is $\mathcal{F}$-defined over $\bar{a}$. In particular, "$\mathcal{F}$-defined over" is a pregeometry.*

Definition 3. We say that a 0-definable function $f : U \to M$, where $U \subseteq M^n$ is open, is *implicitly $\mathcal{F}$-defined* if there exist $m \geq 1$, functions $g_1, \ldots, g_m \in R_{n+m}$ and 0-definable functions $\phi_1, \ldots, \phi_m : U \to M$ such that

(1) $f = \phi_i$, for some $i = 1, \ldots, m$,

(2) $\langle \phi_1(\bar{x}), \ldots, \phi_m(\bar{x}) \rangle \in V_n^{reg}(g_1(\bar{x}, \cdot), \ldots, g_m(\bar{x}, \cdot))$, for all $\bar{x} \in U$.

COROLLARY 4.2. *If $\bar{a} \in M^n$ is generic (for the pregeometry given by definable closure) and $f : U \to M$ is a 0-definable function on a neighbourhood U of $\bar{a}$, then there is an open 0-definable $V \subseteq U$ with $\bar{a} \in V$, such that $f|_V$ is implicitly $\mathcal{F}$-defined.*

Proof. Since $f(\bar{a})$ is in the definable closure of $\bar{a}$, it follows from the previous Theorem that there exist $m \geq 1$, functions $g_1, \ldots, g_m \in R_{n+m}$ and a tuple $\langle b_1, \ldots, b_m \rangle \in M^m$ such that $f(\bar{a}) \in \{b_1, \ldots, b_m\}$ and

$$\bar{b} \in V_m^{reg}(g_1(\bar{a}, \cdot), \ldots, g_m(\bar{a}, \cdot)).$$

Consider the 0-definable set

$$X := \{\langle \bar{x}, \bar{y} \rangle \in M^{n+m} : \bar{y} \in V_m^{reg}(g_1(\bar{x}, \cdot), \ldots, g_m(\bar{x}, \cdot))\}.$$

For each $\bar{x}$ there are at most finitely many $\bar{y}$ such that $\langle \bar{x}, \bar{y} \rangle \in X$. Hence by cell decomposition and the fact that $\bar{a}$ is generic there is an open cell, C say, containing $\bar{a}$ and 0-definable functions $\phi_1, \ldots, \phi_m : C \to M$ such that

$$\langle \phi_1(\bar{x}), \ldots, \phi_m(\bar{x}) \rangle \in V_m^{reg}(g_1(\bar{x}, \cdot), \ldots, g_m(\bar{x}, \cdot))$$

and $\phi_i(\bar{a}) = f(\bar{a})$ for some i. Then as $\bar{a}$ is generic, ϕ_i and f agree on some open neighbourhood V of $\bar{a}$ and so $f|_V$ is implicitly $\mathcal{F}$- defined. □

Using this Corollary and Theorem 3.1, a standard compactness argument yields the following:

COROLLARY 4.3. *Suppose that $f : U \to M$ is a 0-definable function on an open set $U \subseteq M^n$. Then there are 0-definable open sets $U_1, \ldots, U_k \subseteq U$ with $\dim(U \setminus \bigcup_{i=1}^k U_i) < n$ such that $f|_{U_i}$ is implicitly $\mathcal{F}$-defined, for each i.*

Now the implicit function theorem implies that functions which are implicitly $\mathcal{F}$-defined are smooth, and so we have:

COROLLARY 4.4. *Locally polynomially bounded structures have smooth cell decomposition.*

5. *Controlling the derivatives*

DEFINITION 4. A smooth definable function $f : U \to M$ on an open set $U \subseteq M^n$ is said to have *controlled derivatives* if there exists a definable continuous function $\omega : U \to M_{\geq 0}$ and $C_i \in M, E_i \in \mathbb{N}$, for each $i \in \mathbb{N}$ such that

$$|D^\alpha f(\bar{x})| \leq C_{|\alpha|} \cdot \omega(\bar{x})^{E_{|\alpha|}} \text{ for all } \alpha \in \mathbb{N}^n \text{ and } \bar{x} \in U.$$

We say that such an ω is a *control function* for f and that $\{\omega, C_i, E_i\}$ is *control data* for f.

We now suppose that each of the functions $f \in \mathcal{F}$ has controlled derivatives. It follows that, in the notation of the previous section, the functions in $\widetilde{\mathcal{F}}$ (and hence in R_n) also have controlled derivatives. Note that, because of the presence of exp, this assumption holds for the examples of LPB structures given in Section 3.

PROPOSITION 5.1. *Suppose that $f : U \to M$ is implicitly $\mathcal{F}$-defined. Then f has controlled derivatives.*

Proof. Let $g_1, \dots, g_m \in R_{n+m}$ and $\phi_1, \dots, \phi_m : U \to M$ witness the fact that f is implicitly defined. Since $g_1, \dots, g_m$ have controlled derivatives, there is a continuous definable function $\omega : M^n \to M$ and $C_i \in M, E_i \in \mathbb{N}$ such that for each $i = 1, \dots, m$ and all $\alpha \in N^n$,

$$|D^\alpha g_i(\bar{x}, \bar{y})| \leq C_{|\alpha|} \cdot \omega(\bar{x}, \bar{y})^{E_{|\alpha|}} \text{ for all } \langle \bar{x}, \bar{y} \rangle \in M^{n+m}.$$

Let Δ be the determinant of the matrix

$$\begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial y_1} & \cdots & \frac{\partial g_m}{\partial y_m} \end{pmatrix}.$$

We will show by induction on $|\alpha|$ that there are $C'_{|\alpha|} \in M, E'_{|\alpha|} \in \mathbb{N}$ such that for each i and all $\bar{x} \in U$,

$$|D^\alpha \phi_i(\bar{x})| \leq C'_{|\alpha|} \left(\frac{\omega(\bar{x}, \phi_1(\bar{x}), \dots, \phi_m(\bar{x}))}{\Delta(\bar{x}, \phi_1(\bar{x}), \dots, \phi_m(\bar{x}))} \right)^{E'_{|\alpha|}},$$

which suffices as f is one of the ϕ_i.

Suppose first that $|\alpha| = 1$. We write $\bar{\phi}(\bar{x}) := \langle \phi_1(\bar{x}), \dots, \phi_m(\bar{x}) \rangle$. Since the derivative $\frac{\partial \phi_i}{\partial y_j}(\bar{x})$ has the form

$$\frac{\text{polynomial in } \frac{\partial g_l}{\partial y_k} \text{ evaluated at } \langle \bar{x}, \bar{\phi}(\bar{x}) \rangle, \text{ for various } k, l}{\Delta(\bar{x}, \bar{\phi}(\bar{x}))},$$

the required C'_1, E'_1 clearly exist.

Now suppose that $|\alpha| > 1$. By the chain rule, $D^\alpha \phi_i(\bar{x})$ has the form

$$\frac{\begin{array}{c}\text{polynomial in } D^\beta g_j \text{ evaluated at } \langle \bar{x}, \bar{\phi}(\bar{x}) \rangle \text{ and } D^{\beta'} \phi_k(\bar{x}), \\ \text{for various } j, k, \beta, \beta' \text{ with } |\beta| \leq |\alpha|, |\beta'| < |\alpha| \end{array}}{\Delta(\bar{x}, \bar{\phi}(\bar{x}))^d},$$

and by the induction hypothesis, we can find suitable $C'_{|\alpha|}, E'_{|\alpha|}$. □

Combining this with Corollary 4.2, we obtain

COROLLARY 5.2. *Suppose that $f : U \to M$ is a smooth definable function. Then there are definable open sets $U_1, \dots, U_k \subseteq U$ with $\dim(U \setminus \bigcup U_i) < n$ such that for each $i = 1, \dots, k$, $f|_{U_i}$ has controlled derivatives.*

REMARK 1. In polynomially bounded structures, all smooth functions have controlled derivatives. It seems feasible that a more careful analysis of the derivatives of implicit functions may show that exponents of the form $|\alpha|$ are preserved. This could lead to new results in the polynomially bounded case.

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