Pairs of compatible associative algebras, classical Yang-Baxter and quiver representations

Odesskii, Alexander and Sokolov, Vladimir

2007

MIMS EPrint: **2007.191**

ISSN 1749-9097

Commun. Math. Phys.
Digital Object Identifier (DOI) 10.1007/s00220-007-0361-9

Communications in
**Mathematical
Physics**

Pairs of Compatible Associative Algebras, Classical Yang-Baxter Equation and Quiver Representations

Alexander Odesskii[1,2], **Vladimir Sokolov**[1]

[1] Landau Institute for Theoretical Physics, Kosygina 2, 119334, Moscow, Russia.
E-mail: odesskii@itp.ac.ru; sokolov@itp.ac.ru
[2] School of Mathematics, The University of Manchester, Manchester, UK

Received: 17 November 2006 / Accepted: 23 April 2007

Abstract: Given an associative multiplication in matrix algebra compatible with the usual one or, in other words, a linear deformation of the matrix algebra, we construct a solution to the classical Yang-Baxter equation. We also develop a theory of such deformations and construct numerous examples. It turns out that these deformations are in one-to-one correspondence with representations of certain algebraic structures, which we call M-structures. We also describe an important class of M-structures related to the affine Dynkin diagrams of A, D, E-type. These M-structures and their representations are described in terms of quiver representations.

Introduction

Two associative algebras with multiplications $(a, b) \to ab$ and $(a, b) \to a \circ b$ defined on the same finite dimensional vector space are said to be compatible if the multiplication

$$a \bullet b = ab + \lambda\, a \circ b \tag{0.1}$$

is associative for any constant λ. The multiplication $\bullet$ can be regarded as a deformation of the multiplication $(a, b) \to ab$ linear in the parameter λ.

In [1] we have studied multiplications compatible with the matrix product or, in other words, linear deformations of matrix multiplication. It turns out that these deformations of the matrix algebra are in one-to-one correspondence with representations of certain algebraic structures, which we call M-structures. The case of a direct sum of several matrix algebras corresponds to representations of the so-called PM-structures (see [1]).

Given a pair of compatible associative products, one can construct a hierarchy of integrable systems of ODEs via the Lenard-Magri scheme [2]. The Lax representations for these systems are described in [3]. If one of the multiplications is the usual matrix product, the integrable systems are Hamiltonian $gl(N)$-models with quadratic Hamiltonians [4]. These systems can be regarded as a generalization of the matrix equations considered in [5]. Their skew-symmetric reductions give rise to new integrable quadratic $so(n)$-Hamiltonians.

The main ingredient of the M-structure is a pair of associative algebras $\mathcal{A}$ and $\mathcal{B}$ of the same dimension. The simplest version of a structure of this kind can be regarded as an associative analog of the Lie bi-algebra [6].

We define an *infinitesimal bi-algebra* (see [20]) as a pair of associative algebras $\mathcal{A}$ and $\mathcal{B}$ with a non-degenerated pairing and a $\mathcal{B} \otimes \mathcal{A}^{op}$-module structure on the space $\mathcal{L} = \mathcal{A} \oplus \mathcal{B}$ such that the algebra $\mathcal{A}$ acts on $\mathcal{A} \subset \mathcal{L}$ by right multiplications, the algebra $\mathcal{B}$ acts on $\mathcal{B} \subset \mathcal{L}$ by left multiplications and the pairing is invariant with respect to this action (that is $(bb', a) = (b, b'a)$ and $(b, aa') = (ba, a')$ for $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$). Here $\mathcal{A}^{op}$ stands for the algebra opposite to $\mathcal{A}$. Given an infinitesimal bi-algebra, one has the structure of associative algebra on the space $\mathcal{A} \oplus \mathcal{B} \oplus \mathcal{A} \otimes \mathcal{B}$ (this is an analog of the Drinfeld double).

In this paper we introduce the notion of *associative r-matrices*, which is a particular case of the usual classical r-matrices. It turns out that the constant associative r-matrices can be classified in terms of infinitesimal bi-algebras. Moreover, one can introduce spectral parameters into the definition of infinitesimal bi-algebras and obtain a classification of non-constant associative r-matrices.

In [1] we have discovered an important class of M and PM-structures. These structures are related to the Cartan matrices of affine Dynkin diagrams of the $\tilde{A}_{2k-1}$, $\tilde{D}_k$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$-type. In this paper we describe these M-structures and their representations in terms of quiver representations.

The paper is organized as follows. In Sect. 1, we consider an associative analog of the classical Yang-Baxter equation. Since semi-simple associative algebras are more rigid algebraic structures than semi-simple Lie algebras, it turns out to be possible to construct a developed theory of the associative Yang-Baxter equation in the semi-simple case. This theory is suitable for constructing a wide class of solutions to the Yang-Baxter equation. We are planning to write a separate paper devoted to systematic search for solutions.

In Sect. 2, we give an explicit construction of a solution to the Yang-Baxter equation by each pair of compatible Lie brackets provided that the first bracket is rigid. The corresponding r-matrices are not unitary and therefore they are not included in the classification by A. Belavin and V. Drinfeld [7]. In particular, compatible associative products give rise to solutions of the associative Yang-Baxter equation. This gives us a way to construct r-matrices related to M-structures.

In Sect. 3 we recall the notion of M-structure and formulate the main results describing the relationship between associative multiplications in matrix algebra compatible with the usual matrix product and M-structures.

In Sect. 4 we describe all M-structures with semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$. It turns out that such M-structures are related to the Cartan matrices of affine Dynkin diagrams of the $\tilde{A}_{2k-1}$, $\tilde{D}_k$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$-type. We describe these M-structures and their representations in terms of representations of affine quivers [10–12].

In the Appendix we give explicit formulas for these M-structures of A and D types, their representations and for corresponding solutions to the classical Yang-Baxter equation.[1]

[1] The explicit formulas for these M-structures of E type can be found in the preprint version of this article.

1. Classical Yang-Baxter Equation

Let $\mathfrak{g}$ be a Lie algebra. Let $r(u, v)$ be a meromorphic function of two complex variables with values in $End(\mathfrak{g})$. For each $u \in \mathbb{C}$ we denote by $\mathfrak{g}_u$ a vector space canonically isomorphic to $\mathfrak{g}$. Let $\tilde{\mathfrak{g}} = \oplus_u \mathfrak{g}_u$. We define a bracket on the space $\tilde{\mathfrak{g}}$ by the formula

$$[x_u, y_v] = ([x, r(u, v)y])_u + ([r(v, u)x, y])_v. \tag{1.2}$$

Lemma 1.1. *The bracket* (1.2) *defines a structure of a Lie algebra on* $\tilde{\mathfrak{g}}$ *iff* $r(u, v)$ *satisfies the following equation*

$$[r(u, w)x, r(u, v)y] - r(u, v)[r(v, w)x, y] - r(u, w)[x, r(w, v)y] \in Cent(\mathfrak{g}), \tag{1.3}$$

where x, y *are arbitrary elements of* $\mathfrak{g}$ *and* $Cent(\mathfrak{g})$ *stands for the center of* $\mathfrak{g}$.

Proof. of the lemma is straightforward.

Remark 1. Here and in the sequel by Lie algebra we mean partial Lie algebra. Namely, the bracket (1.2) is defined iff the functions $r(u, v)$ and $r(v, u)$ are defined at the point (u, v). The anti-commutativity condition and the Yacobi identity hold whenever the left hand side is defined.

Definition. *The operator relation*

$$[r(u, w)x, r(u, v)y] - r(u, v)[r(v, w)x, y] - r(u, w)[x, r(w, v)y] = 0 \tag{1.4}$$

is called the **classical Yang-Baxter equation**. *A solution* $r(u, v)$ *to the classical Yang-Baxter equation is called the* **classical r-matrix**. *Arguments of* $r(u, v)$ *are called* **spectral parameters**.

Note that the arguments u, v of r could be also elements of $\mathbb{C}^n$ for $n > 1$ or elements of some complex manifold called the manifold of spectral parameters.

Suppose $\mathfrak{g}$ possesses a non-degenerate invariant scalar product $(\cdot, \cdot)$. An r-matrix is called unitary if $(x, r(u, v)y) = -(r(v, u)x, y)$.

Remark 2. There are several algebraic interpretations of the Yang-Baxter equation ([7–9]). For our purposes the interpretation from Lemma 1.1 is the most convenient. All definitions lead to the same equation for $r(u, v)$ provided that the r-matrix is unitary. In particular, it is easy to see [8] that Eq. (1.4) is equivalent to the classical Yang-Baxter equation written in the tensor form. The unitary r-matrices were classified in [7]. The case of the non-unitary r-matrix was considered in ([8, 9]). There is not any classification of r-matrices in the general case.

It turns out that a theory of (non-unitary) r-matrices can be developed in the special case of associative algebras. Let A be an associative algebra. Let $r(u, v)$ be a meromorphic function in two complex variables with values in $End(A)$. For each $u \in \mathbb{C}$ we denote by A_u a vector space canonically isomorphic to A. Let $\tilde{A} = \oplus_u A_u$. We define a product on the space $\tilde{A}$ by the formula

$$x_u y_v = (x(r(u, v)y))_u + ((r(v, u)x)y)_v. \tag{1.5}$$

Lemma 1.2. *The product* (1.5) *defines a structure of an associative algebra on* $\tilde{A}$ *iff* $r(u,v)$ *satisfies the following equation:*

$$(r(u,w)x)(r(u,v)y) - r(u,v)((r(v,w)x)y) - r(u,w)(x(r(w,v)y)) \in Null(A), \tag{1.6}$$

where $Null(A)$ *is the set of* $z \in A$ *such that* $zt = tz = 0$ *for all* $t \in A$.

Proof. of the lemma is straightforward.

Definition. *The relation*

$$(r(u,w)x)(r(u,v)y) - r(u,v)((r(v,w)x)y) - r(u,w)(x(r(w,v)y)) = 0 \tag{1.7}$$

is called the **associative Yang-Baxter equation**.

Lemma 1.3. *Let* $\mathfrak{g}$ *be a Lie algebra with the brackets* $[x,y] = xy - yx$. *Then any solution of* (1.7) *is a solution of* (1.4).

Proof. of the lemma is straightforward.

Let $A = Mat_n$. It is easy to see that any operator from $End(A)$ to $End(A)$ has the form $x \to a_1\, x\, b^1 + \cdots + a_p\, x\, b^p$ for some matrices $a_1, \ldots, a_p, b^1, \ldots, b^p$. Moreover, p is the smallest possible for such a representation iff the sets matrices $\{a_1, \ldots, a_p\}$ and $\{b^1, \ldots, b^p\}$ are both linear independent.

Theorem 1.1. *Let*

$$r(u,v)x = a_1(u,v)\, x\, b^1(v,u) + \cdots + a_p(u,v)\, x\, b^p(v,u),$$

where $a_1(u,v), \ldots, b^p(u,v)$ *are meromorphic functions with values in* Mat_n *such that* $\{a_1(u,v), \ldots, a_p(u,v)\}$ *are linear independent over the field of meromorphic functions in* u, v *as well as* $\{b^1(u,v), \ldots, b^p(u,v)\}$. *Then* $r(u,v)$ *satisfies* (1.7) *iff there exist meromorphic functions* $\phi^k_{i,j}(u,v,w)$ *and* $\psi^k_{i,j}(u,v,w)$ *such that*

$$\begin{aligned} a_i(u,v)a_j(v,w) &= \phi^k_{i,j}(u,v,w)a_k(u,w), \\ b^i(u,v)b^j(v,w) &= \psi^{i,j}_k(u,v,w)b^k(u,w), \\ b^i(u,v)a_j(v,w) &= \phi^i_{j,k}(v,w,u)b^k(u,w) + \psi^{k,i}_j(w,u,v)a_k(u,w). \end{aligned} \tag{1.8}$$

The tensors $\phi^k_{i,j}(u,v,w)$ *and* $\psi^k_{i,j}(u,v,w)$ *satisfy the following equations:*

$$\begin{aligned} \phi^s_{i,j}(u,v,w)\phi^l_{s,k}(u,w,t) &= \phi^l_{i,s}(u,v,t)\phi^s_{j,k}(v,w,t), \\ \psi^{i,j}_s(u,v,w)\psi^{s,k}_l(u,w,t) &= \psi^{i,s}_l(u,v,t)\psi^{j,k}_s(v,w,t), \\ \phi^s_{j,k}(v,w,t)\psi^{l,i}_s(t,u,v) &= \phi^l_{s,k}(u,w,t)\psi^{s,i}_j(w,u,v) + \phi^i_{j,s}(v,w,u)\psi^{l,s}_k(t,u,w). \end{aligned} \tag{1.9}$$

Proof. of the theorem is similar to the proof of Theorem 3.1 from [1].

Remark 3. It is easy to give an invariant description of the corresponding algebraic structure. In the case of a constant r-matrix this leads to the infinitesimal bi-algebras [20] described in the Introduction.

Remark 4. A similar statement holds in the case of a semi-simple algebra A.

Example 1. Let $A = Mat_n$ and $r(u, v)x = \frac{1}{u-v}e(u, v)xf(v, u)$, where

$$
\begin{aligned}
e(u, v)e(v, w) &= e(u, w), \quad f(u, v)f(v, w) = f(u, w), \\
e(u, v)f(v, w) &= \frac{u-v}{u-w}e(u, w) + \frac{v-w}{u-w}f(u, w).
\end{aligned} \tag{1.10}
$$

Then $r(u, v)$ is an associative r-matrix. These equations hold if we assume, for example, that $e(u, v) = 1,\ f(u, v) = (u+C)(v+C)^{-1}$, where C is an arbitrary constant matrix.

Example 2. Let $A = \mathbb{C}^p$. The algebra A has a basis $\{e_i, i = 1, \ldots, p\}$ such that $e_i e_j = \delta_{i,j} e_i$. The formula

$$
r(u, v)e_i = \sum_{1 \le j \le p} \frac{\psi_i(v)}{\phi_j(u) - \phi_i(v)} e_j
$$

gives an associative r-matrix for any functions $\phi_1, \ldots, \phi_p, \psi_1, \ldots, \psi_p$ of one variable, where $\phi_1, \ldots, \phi_p$ are not constant. This r-matrix can be written in the form

$$
r(\vec{u}, \vec{v})e_i = \sum_{1 \le j \le p} \frac{\psi_i(\vec{v})}{u_j - v_i} e_j,
$$

where $\vec{u} = (u_1, \ldots, u_p)$, $\vec{v} = (v_1, \ldots, v_p)$, $\psi_i(\vec{v})$ are functions of p variables. In this case the manifold of spectral parameters is $\mathbb{C}^p$.

2. Compatible Products and Solutions to the Classical Yang-Baxter Equation

Two Lie brackets $[\cdot, \cdot]$ and $[\cdot, \cdot]_1$ defined on the same vector space $\mathfrak{g}$ are said to be compatible if $[\cdot, \cdot]_\lambda = [\cdot, \cdot] + \lambda[\cdot, \cdot]_1$ is a Lie bracket for any λ. In the papers [13–16] different applications of the notion of compatible Lie brackets to the integrability theory have been considered.

Suppose that the bracket $[\cdot, \cdot]$ is rigid, i.e. $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ with respect to $[\cdot, \cdot]$. In this case the Lie algebras with the brackets $[\cdot, \cdot]_\lambda$ are isomorphic to the Lie algebra with the bracket $[\cdot, \cdot]$ for almost all values of the parameter λ. This means that there exists a meromorphic function $\lambda \to S_\lambda$ with values in $End(\mathfrak{g})$ such that $S_0 = Id$ and

$$
[S_\lambda(x), S_\lambda(y)] = S_\lambda([x, y] + \lambda[x, y]_1). \tag{2.11}
$$

Theorem 2.1. *The formula*

$$
r(u, v) = \frac{1}{u-v} S_u S_v^{-1} \tag{2.12}
$$

defines a solution to the classical Yang-Baxter equation (1.4).

Proof. For $r(u, v)$ given by (2.12), Eq. (1.4) is equivalent to

$$
\begin{aligned}
&\frac{1}{(u-v)(u-w)}[S_u S_w^{-1}(x), S_u S_v^{-1}(y)] - \frac{1}{(u-v)(v-w)} S_u S_v^{-1}([S_v S_w^{-1}(x), y]) \\
&\quad - \frac{1}{(u-w)(w-v)} S_u S_w^{-1}([x, S_w S_v^{-1}(y)]) = 0.
\end{aligned} \tag{2.13}
$$

Using (2.11), we get

$$
\begin{aligned}
[S_u S_w^{-1}(x), S_u S_v^{-1}(y)] &= S_u([S_w^{-1}(x), S_v^{-1}(y)] + u[S_w^{-1}(x), S_v^{-1}(y)]_1),\\
S_u S_v^{-1}([S_v S_w^{-1}(x), y]) &= S_u([S_w^{-1}(x), S_v^{-1}(y)] + v[S_w^{-1}(x), S_v^{-1}(y)]_1),\\
S_u S_w^{-1}([x, S_w S_v^{-1}(y)]) &= S_u([S_w^{-1}(x), S_v^{-1}(y)] + w[S_w^{-1}(x), S_v^{-1}(y)]_1).
\end{aligned}
$$

Substituting these expressions into the left hand side of (2.13), we obtain the statement.

Remark 1. It is clear that the r-matrix (2.12) is unitary with respect to an invariant form $(\cdot, \cdot)$ if the operator S_λ is orthogonal. In this case formula (2.11) implies that the form $(\cdot, \cdot)$ is invariant with respect to the second bracket.

Two associative algebras with multiplications $(x, y) \to xy$ and $(x, y) \to x \circ y$ defined on the same finite dimensional vector space A are said to be *compatible* if the multiplication (0.1) is associative for any constant λ. Suppose $H^2(A, A) = 0$ with respect to the first multiplication; then there exists a meromorphic function $\lambda \to S_\lambda$ with values in $End(A)$ such that $S_0 = Id$ and

$$
S_\lambda(x) S_\lambda(y) = S_\lambda(xy + \lambda x \circ y). \tag{2.14}
$$

The Taylor decomposition of S_λ at $\lambda = 0$ has the following form:

$$
S_\lambda = 1 + R\,\lambda + T\,\lambda^2 + \cdots, \tag{2.15}
$$

where $R, T, \ldots$ are some linear operators on A. Substituting this decomposition into (2.14) and equating the coefficients of λ, we obtain the formula

$$
x \circ y = R(x)y + xR(y) - R(xy), \tag{2.16}
$$

where R is defined by (2.15). It is clear that for any $a \in A$ the transformation

$$
R \longrightarrow R + ad_a, \tag{2.17}
$$

where ad_a is a linear operator $v \to av - va$, does not change the multiplication $\circ$.

Definition. *Operators R and R' are said to be equivalent if $R - R' = ad_a$ for some $a \in A$.*

The following analog of Theorem 2.1 can be proved similarly.

Theorem 2.2. *Suppose that S_λ satisfies* (2.14)*, then formula* (2.12) *defines a solution to the associative Yang-Baxter equation* (1.7).

Remark 2. In the important particular case $S_\lambda = 1 + \lambda R$ the r-matrix (2.12) is equivalent to

$$
r(u, v) = \frac{1}{u - v} + (v + R)^{-1}. \tag{2.18}
$$

Let $A = Mat_N$. Consider the following classification problem: describe all possible associative multiplications $\circ$ compatible with the usual matrix product in A. Since $H^2(A, A) = 0$ for any semi-simple associative algebra A, an operator-valued meromorphic function S_λ with the properties $S_0 = Id$ and (2.14) exists for any such multiplication and the multiplication is given by formula (2.16).

Example. Let $a \in Mat_N$ be an arbitrary matrix and R be the operator of left multiplication by a. Then (2.16) yields the multiplication $x \circ y = xay$, which is associative and compatible with the standard one. It is clear that S_λ can be chosen in the form $S_\lambda(x) = (1 + \lambda a)x$. In this case we have

$$r(u, v) = \frac{1}{u - v} + (v + a)^{-1}.$$

Any linear operator R on the space Mat_N may be written in the form $R(x) = a_1 x b^1 + \ldots + a_l x b^l$ for some matrices $a_1, \ldots, a_l, b^1, \ldots, b^l$. Indeed, the operators $x \to e_{i,j} x e_{i_1,j_1}$ form a basis in the vector space of linear operators on Mat_N.

It is convenient to represent the operator R from formula (2.16) in the form

$$R(x) = a_1\, x\, b^1 + \cdots + a_p\, x\, b^p + c\, x \tag{2.19}$$

with p being the smallest possible in the class of equivalence of R. This means that the matrices $\{a_1, \ldots, a_p, 1\}$ are linear independent as well as the matrices $\{b^1, \ldots, b^p, 1\}$. According to (2.16), the second product has the following form:

$$x \circ y = \sum_i (a_i\, x\, b^i\, y + x\, a_i\, y\, b^i - a_i\, x y\, b^i) + x\, c\, y. \tag{2.20}$$

It turns out that the matrices $\{a_1, \ldots, a_p, b^1, \ldots, b^p, c\}$ form a representation of a certain algebraic structure. We describe this structure in the next section.

3. *M*-Structures and the Corresponding Associative Algebras

In this section we formulate the results of the paper [1] and their simple consequences we will use below.

Definition. *By weak M-structure on a linear space $\mathcal{L}$ we mean the following data:*

- *Two subspaces $\mathcal{A}$ and $\mathcal{B}$ and a distinguished element* $1 \in \mathcal{A} \cap \mathcal{B} \subset \mathcal{L}$.
- *A non-degenerate symmetric scalar product* $(\cdot, \cdot)$ *on the space $\mathcal{L}$.*
- *Associative products $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$ with unity* 1.
- *A left action $\mathcal{B} \times \mathcal{L} \to \mathcal{L}$ of the algebra $\mathcal{B}$ and a right action $\mathcal{L} \times \mathcal{A} \to \mathcal{L}$ of the algebra $\mathcal{A}$ on the space $\mathcal{L}$ that commute to each other.*

These data should satisfy the following properties:

1. $\dim \mathcal{A} \cap \mathcal{B} = \dim \mathcal{L}/(\mathcal{A} + \mathcal{B}) = 1$.
2. The restriction of the action $\mathcal{B} \times \mathcal{L} \to \mathcal{L}$ to the subspace $\mathcal{B} \subset \mathcal{L}$ is the product in $\mathcal{B}$. The restriction of the action $\mathcal{L} \times \mathcal{A} \to \mathcal{L}$ to the subspace $\mathcal{A} \subset \mathcal{L}$ is the product in $\mathcal{A}$.
3. $(a_1, a_2) = (b_1, b_2) = 0$ and $(b_1 b_2, v) = (b_1, b_2 v)$, $(v, a_1 a_2) = (v a_1, a_2)$ for any $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $v \in \mathcal{L}$.

It follows from these properties that $(\cdot, \cdot)$ defines a non - degenerate pairing between $\mathcal{A}/\mathbb{C}1$ and $\mathcal{B}/\mathbb{C}1$. Therefore $\dim \mathcal{A} = \dim \mathcal{B}$ and $\dim \mathcal{L} = 2 \dim \mathcal{A}$.

Given a weak M-structure $\mathcal{L}$, we define an associative algebra $U(\mathcal{L})$ generated by $\mathcal{L}$ and satisfying natural compatibility and universality conditions.

Definition. *By weak M-algebra associated with a weak M-structure $\mathcal{L}$ we mean an associative algebra $U(\mathcal{L})$ with a linear mapping $j : \mathcal{L} \to U(\mathcal{L})$ such that the following conditions are satisfied:*

1. $j(b)j(x) = j(bx)$ *and* $j(x)j(a) = j(xa)$ *for* $a \in \mathcal{A}$, $b \in \mathcal{B}$ *and* $x \in \mathcal{L}$.
2. *For any algebra X with a linear mapping $j' : \mathcal{L} \to X$ satisfying property 1 there exists a unique homomorphism of algebras $f : U(\mathcal{L}) \to X$ such that $f \circ j = j'$.*

It is easy to see that $U(\mathcal{L})$ exists and is unique for given $\mathcal{L}$.

Definition. *A weak M-structure $\mathcal{L}$ is called M-structure if there exists a central element $K \in U(\mathcal{L})$ of the algebra $U(\mathcal{L})$ quadratic with respect to $\mathcal{L}$.*

Theorem 3.1. *Let $\mathcal{L}$ be an M-structure. Then there exists a basis $\{1, A_1, \dots, A_p, B^1, \dots, B^p, C\}$ in $\mathcal{L}$ such that $\{1, A_1, \dots, A_p\}$ is a basis in $\mathcal{A}$, $\{1, B^1, \dots, B^p\}$ is a basis in $\mathcal{B}$, and*

$$K = A_1 B^1 + \cdots + A_p B^p + C.$$

Theorem 3.2. *Let $R \in End(U(\mathcal{L}))$ be given by the formula*

$$R(x) = A_1 x B^1 + \cdots + A_p x B^p + Cx,$$

and $\circ$ be defined by (2.16). Then $\circ$ is associative and compatible with the usual product in $U(\mathcal{L})$.

Notice that $K = R(1)$.

Theorem 3.3. *Let $\circ$ be an associative product in the space Mat_N compatible with the usual one and written in the form* (2.16)*, where R is given by* (2.19) *with p being smallest possible in the class of equivalence of R. Then there exists an M-structure $\mathcal{L}$ with representation $U(\mathcal{L}) \to Mat_N$ such that* $\dim \mathcal{A} = \dim \mathcal{B} = p + 1$*, the image of $\mathcal{A}$ has the basis $\{1, a_1, \dots, a_p\}$, and the image of $\mathcal{B}$ has the basis $\{1, b^1, \dots, b^p\}$.*

Definition. *A representation of $U(\mathcal{L})$ is called non-degenerate if its restrictions on the algebras $\mathcal{A}$ and $\mathcal{B}$ are exact.*

Theorem 3.4. *There is one-to-one correspondence between N- dimensional non-degenerate representations of algebras $U(\mathcal{L})$ corresponding to M-structures and associative products in Mat_N compatible with the usual matrix product.*

The structure of the algebra $U(\mathcal{L})$ for an M-structure $\mathcal{L}$ can be described as follows.

Theorem 3.5. *The algebra $U(\mathcal{L})$ is spanned by the elements of the form $a\, b\, K^s$, where $a \in \mathcal{A}, b \in \mathcal{B},\ s \in \mathbb{Z}_+$.*

We need also the following

Definition. *Let $\mathcal{L}$ be a weak M-structure. By the opposite weak M-structure $\mathcal{L}^{op}$ we mean the M-structure with the same linear space $\mathcal{L}$, the same scalar product and algebras $\mathcal{A}$, $\mathcal{B}$ replaced by the opposite algebras $\mathcal{B}^{op}$, $\mathcal{A}^{op}$, correspondingly.*

It is easy to see that if $\mathcal{L}$ is an M-structure, then $\mathcal{L}^{op}$ is an M-structure as well.

4. M-Structures with Semi-Simple Algebras $\mathcal{A}$ and $\mathcal{B}$ and Quiver Representations

4.1. Matrix of multiplicities. By V^l we denote the direct sum of l copies of a linear space V. By definition, we put $V^0 = \{0\}$. Recall [17] that any semi-simple associative algebra over $\mathbb{C}$ has the form $\oplus_{1\le i\le r} End(V_i)$, any left $End(V)$-module has the form V^l, and any right $End(V)$-module has the form $(V^\star)^l$ for some r and l.

Lemma 4.1. *Let $\mathcal{L}$ be a weak M-structure. Suppose $\mathcal{A} = \oplus_{1\le i\le r} End(V_i)$, where* $\dim V_i = m_i$. *Then $\mathcal{L}$ as a right $\mathcal{A}$-module is isomorphic to $\oplus_{1\le i\le r}(V_i^\star)^{2m_i}$.*

Proof. Since any right $\mathcal{A}$-module has the form $\oplus_{1\le i\le r}(V_i^\star)^{l_i}$ for some $l_1,\dots,l_r \ge 0$, we have $\mathcal{L} = \oplus_{1\le i\le r}\mathcal{L}_i$, where $\mathcal{L}_i = (V_i^\star)^{l_i}$. Note that $\mathcal{A} \subset \mathcal{L}$ and, moreover, $End(V_i) \subset \mathcal{L}_i$ for $i = 1,\dots,r$. Besides, $End(V_i)\perp\mathcal{L}_j$ for $i \ne j$. Indeed, we have $(v, a) = (v, Id_i a) = (v Id_i, a) = 0$ for $v \in \mathcal{L}_j$, $a \in End(V_i)$, where Id_i is the unity of the subalgebra $End(V_i)$. Since $(\cdot,\cdot)$ is non-degenerate and $End(V_i)\perp End(V_i)$ by property 3 of the weak M-structure, we have $\dim\mathcal{L}_i \ge 2\dim End(V_i)$. But $\sum_i \dim\mathcal{L}_i = \dim\mathcal{L} = 2\dim\mathcal{A} = \sum_i 2\dim End(V_i)$ and we obtain $\dim\mathcal{L}_i = 2\dim End(V_i)$ for each $i = 1,\dots,r$, which is equivalent to the statement of Lemma 4.1.

Lemma 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be semi-simple associative algebras:*

$$\mathcal{A} = \oplus_{1\le i\le r} End(V_i), \qquad \mathcal{B} = \oplus_{1\le j\le s} End(W_j), \qquad \dim V_i = m_i, \quad \dim W_j = n_j. \tag{4.21}$$

Then $\mathcal{L}$ as the $\mathcal{A}^{op}\otimes\mathcal{B}$-module is given by the formula

$$\mathcal{L} = \oplus_{1\le i\le r, 1\le j\le s}(V_i^\star\otimes W_j)^{a_{i,j}}, \tag{4.22}$$

where $a_{i,j} \ge 0$ and

$$\sum_{j=1}^{s} a_{i,j}n_j = 2m_i, \qquad \sum_{i=1}^{r} a_{i,j}m_i = 2n_j. \tag{4.23}$$

Proof. It is known that any $\mathcal{A}^{op}\otimes\mathcal{B}$-module has the form $\oplus_{1\le i\le r,1\le j\le s}(V_i^\star\otimes W_j)^{a_{i,j}}$, where $a_{i,j}\ge 0$. Applying Lemma 4.1, we obtain $\dim\mathcal{L}_i = 2m_i^2$, where $\mathcal{L}_i = \oplus_{1\le j\le s}(V_i^\star\otimes W_j)^{a_{i,j}}$. This gives the first equation from (4.23). The second equation can be obtained similarly.

Definition. *The $r\times s$-matrix $(a_{i,j})$ from Lemma 4.2 is called the matrix of multiplicities of the weak M-structure $\mathcal{L}$.*

Definition. *The $r\times s$-matrix $(a_{i,j})$ is called decomposable if there exist partitions $\{1,\dots,r\} = I\sqcup I'$ and $\{1,\dots,s\} = J\sqcup J'$ such that $a_{i,j} = 0$ for $(i,j)\in I\times J'\sqcup I'\times J$.*

Lemma 4.3. *The matrix of multiplicities is indecomposable.*

Proof. Suppose $(a_{i,j})$ is decomposable. We have $\mathcal{A} = \mathcal{A}'\oplus\mathcal{A}''$, $\mathcal{B} = \mathcal{B}'\oplus\mathcal{B}''$ and $\mathcal{L} = \mathcal{L}'\oplus\mathcal{L}''$, where

$$\begin{aligned}&\mathcal{A}' = \oplus_{i\in I} End(V_i), \quad \mathcal{A}'' = \oplus_{i\in I'} End(V_i), \quad \mathcal{B}' = \oplus_{j\in J} End(W_j),\\ &\mathcal{B}'' = \oplus_{j\in J'} End(W_j), \quad \mathcal{L}' = \oplus_{(i,j)\in I\times J}(V_i^\star\otimes W_j)^{a_{i,j}},\\ &\qquad \mathcal{L}'' = \oplus_{(i,j)\in I'\times J'}(V_i^\star\otimes W_j)^{a_{i,j}}.\end{aligned}$$

Let $1 = e_1 + e_2$, where $e_1\in\mathcal{L}'$ and $e_2\in\mathcal{L}''$. It is clear that $e_1, e_2\in\mathcal{A}\cap\mathcal{B}$. Therefore, $\dim\mathcal{A}\cap\mathcal{B} > 1$, which contradicts property 1 of the weak M-structure.

Note that if A is the matrix of multiplicities of a weak M structure with semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$, then A^t is the matrix of multiplicities for the opposite weak M-structure.

Theorem 4.1. *Let $\mathcal{L}$ be a weak M-structure with semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$ given by formula* (4.21) *and with $\mathcal{L}$ given by* (4.22). *Then there exists a simple laced affine Dynkin diagram [18] with vector spaces from the set $\{V_1, \dots, V_r, W_1, \dots, W_s\}$ assigned to each vertex in such a way that:*

1. *there is one-to-one correspondence between this set and the set of vertices,*
2. *for any i, j the spaces V_i, V_j are not connected by edges as well as the spaces W_i, W_j,*
3. *$a_{i,j}$ is equal to the number of edges between V_i and W_j,*
4. *the vector* $(\dim V_1, \dots, \dim V_r, \dim W_1, \dots, \dim W_s)$ *is a positive imaginary root of the diagram.*

Proof. Consider a linear space with a basis $\{v_1, \dots, v_r, w_1, \dots, w_s\}$ and the symmetric bilinear form $(v_i, v_j) = (w_i, w_j) = 2\delta_{i,j}, (v_i, w_j) = -a_{i,j}$. Let $J = m_1 v_1 + \dots + m_r v_r + n_1 w_1 + \dots + n_s w_s$. It is clear that Eqs. (4.23) can be written as $(v_i, J) = (w_j, J) = 0$, which means that J belongs to the kernel of the form $(\cdot, \cdot)$. Therefore (see [19]) the matrix of the form is the Cartan matrix of a simple laced affine Dynkin diagram. It is also clear that J is a positive imaginary root.

On the other hand, consider a simple laced affine Dynkin diagram with a partition of the set of vertices into two subsets such that vertices of the same subset are not connected. It is clear that if such a partition exists, then it is unique up to transposition of subsets. Let $v_1, \dots, v_r$ be roots corresponding to vertices of the first subset and $w_1, \dots, w_s$ be roots corresponding to the second subset. We have $(v_i, v_j) = (w_i, w_j) = 2\delta_{i,j}$. Let $J = m_1 v_1 + \dots + m_r v_r + n_1 w_1 + \dots + n_s w_s$ be an imaginary root and $a_{i,j} = -(v_i, w_j)$. Then it is easy to see that (4.23) holds.

Remark. The interchanging of the subsets corresponds to the transposition of the matrix $(a_{i,j})$.

It is easily seen that among simple laced affine Dynkin diagrams only diagrams of the $\tilde{A}_{2k-1}$, $\tilde{D}_k$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$-type admit a partition of the set of vertices into two subsets such that vertices of the same subset are not connected. The natural question arises: to describe all M-structures with the algebras $\mathcal{A}$ and $\mathcal{B}$ given by (4.21) and $\mathcal{L}$ given by (4.22), where the matrix $(a_{i,j})$ is constructed by an affine Dynkin diagram of the $\tilde{A}_{2k-1}$, $\tilde{D}_k$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$-type. It turns out that these M-structures exist iff J is the minimal positive imaginary root.

4.2. M-structures related to affine Dynkin diagrams and quiver representations. We recall that the quiver is just a directed graph $Q = (Ver, E)$, where Ver is a finite set of vertices and E is a finite set of arrows between them. If $a \in E$ is an arrow, then t_a and h_a denote its tail and its head, respectively. Note that loops and several arrows with the same tail and head are allowed. A representation of the quiver Q is a set of vector spaces L_x attached to each vertex $x \in Ver$ and linear maps $f_a : L_{t_a} \to L_{h_a}$ attached to each arrow $a \in E$. The set of natural numbers $\dim L_x$ attached to each vertex $x \in Ver$ is called the dimension of the representation. By affine quiver we mean such a quiver that the corresponding graph is an affine Dynkin diagram of ADE-type.

Theorem 4.2. *Let $\mathcal{L}$ be an M-structure with semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$ given by* (4.21)*. Then there exists a representation of an affine Dynkin quiver such that:*

1. *There is an one-to-one correspondence between the set of vector spaces attached to vertices of the quiver and the set of vector spaces $\{V_1, \ldots, V_r, W_1, \ldots, W_s\}$. Each vector space from this set is attached to only one vertex.*
2. *For any $a \in E$ the space attached to its tail t_a is some of V_i and the space attached to its head h_a is some of W_j.*
3. *$\mathcal{L}$ as $\mathcal{A}^{op} \otimes \mathcal{B}$-module is isomorphic to $\oplus_{a\in E} V^{\star}_{t_a} \otimes W_{h_a}$.*
4. *The vector* $(\dim V_1, \ldots, \dim V_r, \dim W_1, \ldots, \dim W_s)$ *is the minimal imaginary positive root of the Dynkin diagram.*
5. *The element $1 \in \mathcal{L} = \oplus_{a\in E} Hom(V_{t_a}, W_{h_a})$ is just $\sum_{a\in E} f_a$, where f_a is the linear map attached to the arrow a.*

Proof. In Theorem 4.1 we have already constructed the affine Dynkin diagram corresponding to $\mathcal{L}$ with vector spaces $\{V_1, \ldots, V_r, W_1, \ldots, W_s\}$ attached to the vertices. Note that each edge of this affine Dynkin diagram links some linear spaces V_i and W_j. By definition, the direction of this edge is from V_i to W_j. The decomposition of the element $1 \in \mathcal{L} = \oplus_{1\le i\le r, 1\le j\le s}(V_i^{\star} \otimes W_j)^{a_{i,j}}$ defines the element from $V_i^{\star} \otimes W_j$. Since $V_i^{\star} \otimes W_j = Hom(V_i, W_j)$, we obtain a representation of the quiver. We know already that $J = (\dim V_1, \ldots, \dim V_r, \dim W_1, \ldots, \dim W_s)$ is an imaginary positive root. It is easy to see that if it is not minimal, then $\dim \mathcal{A} \cap \mathcal{B} > 1$.

Now we can use known classification of representations of affine quivers [10–12] to describe the corresponding M-structures. Note that each vertex of our quiver can not be a tail of one arrow and a head of another arrow at the same time. Given a representation of such a quiver, it remains to construct an embedding $\mathcal{A} \to \mathcal{L}$, $\mathcal{B} \to \mathcal{L}$ and a scalar product $(\cdot, \cdot)$ on the space $\mathcal{L}$. We can construct the embedding $\mathcal{A} \to \mathcal{L}$, $\mathcal{B} \to \mathcal{L}$ by the formula $a \to 1a$, $b \to b1$ for $a \in \mathcal{A}$, $b \in \mathcal{B}$ whenever we know the element $1 \in \mathcal{L}$. After that it is not difficult to construct the scalar product.

Example. Consider the case $\tilde{A}_{2k-1}$. We have $\dim V_i = \dim W_i = 1$ for $1 \le i \le k$. Let $\{v_i\}$ be a basis of $V_i^{\star}$ and $\{w_i\}$ be a basis of W_i. Let $\{e_i\}$ be a basis of $End(V_i)$ such that $v_i e_i = v_i$ and $\{f_i\}$ be a basis of $End(W_i)$ such that $f_i w_i = w_i$. A generic element $1 \in \mathcal{L}$ in a suitable basis in V_i, W_i can be written in the form $1 = \sum_{1\le i\le k}(v_i \otimes w_i + \lambda v_{i+1} \otimes w_i)$, where index i is taken modulo k and $\lambda \in \mathbb{C}$ is a generic complex number. The embedding $\mathcal{A} \to \mathcal{L}$, $\mathcal{B} \to \mathcal{L}$ is the following: $e_i \to 1e_i = v_i \otimes w_i + \lambda v_i \otimes w_{i-1}$, $\quad f_i \to f_i 1 = v_i \otimes w_i + \lambda v_{i+1} \otimes w_i$. It is clear that the vector space $\mathcal{A} \cap \mathcal{B}$ is spanned by the vector $\sum_i (v_i \otimes w_i + \lambda v_i \otimes w_{i-1})$ and that the algebra $\mathcal{A} \cap \mathcal{B}$ is isomorphic to $\mathbb{C}$.

Let $Q = (Ver, E)$ be an affine quiver and ρ be its representation constructed by a given M-structure $\mathcal{L}$ with semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$. Let $Ver = Ver_t \sqcup Ver_h$, where Ver_t is the set of tails and Ver_h is the set of heads of arrows. We have $\rho : x \to V_x$, $y \to W_y$, $a \to f_a$ for $x \in Ver_t$, $y \in Ver_h$ and $a \in E$. It turns out that representations of the algebra $U(\mathcal{L})$ can also be described in terms of representations of the quiver Q.

Theorem 4.3. *Suppose we have a representation of the algebra $U(\mathcal{L})$ in a linear space N; then there exists a representation $\tau : x \to N_x, a \to \phi_a;\ \ x \in Ver, a \in E$ of the quiver Q such that*

1. *The restriction of the representation of the algebra $U(\mathcal{L})$ on the subalgebra $\mathcal{A} \subset U(\mathcal{L})$ is isomorphic to $\oplus_{x\in Ver_t} V_x \otimes N_x$.*

2. *The restriction of the representation of the algebra* $U(\mathcal{L})$ *on the subalgebra* $\mathcal{B} \subset U(\mathcal{L})$ *is isomorphic to* $\oplus_{x \in Ver_h} W_x \otimes N_x$.
3. *The formula* $f = \sum_{a \in E} f_a \otimes \phi_a$ *defines an isomorphism* $f : \oplus_{x \in Ver_t} V_x \otimes N_x \to \oplus_{x \in Ver_h} W_x \otimes N_x$.

Proof. It is known that any representation of the algebra $End(V)$ has the form $V \otimes S$, where S is a linear space. The action is given by $f(v \otimes s) = (fv) \otimes s$. Therefore N has the form $N^a = \oplus_{x \in Ver_t} V_x \otimes N_x$ with respect to the action of $\mathcal{A} = \oplus_{1 \le i \le r} End(V_i)$ and has the form $N^b = \oplus_{x \in Ver_h} W_x \otimes N_x$ with respect to the action of $\mathcal{B} = \oplus_{1 \le j \le s} End(W_j)$ for some linear spaces N_x. Both linear spaces N^a and N^b are isomorphic to N. Thus we have linear spaces N_x attached to each $x \in Ver$ and isomorphism $f : \oplus_{x \in Ver_t} V_x \otimes N_x \to \oplus_{x \in Ver_h} W_x \otimes N_x$. Let $f = \sum_{x,y \in Ver} f_{x,y}$. It is easy to see that $f_{x,y} = 0$ if x and y are not linked by arrow and $f_{x,y} = f_a \otimes \phi_a$ for some ϕ_a if $x = t_a$, $y = h_a$. Here f_a is defined by Theorem 4.2 (see property 5). This gives us a linear map ϕ_a attached to each arrow $a \in E$.

Remark 1. It is clear that all statements of this section are valid for weak M-structures with semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$. However, it is possible to check that any such weak M-structure has a quadratic central element K and therefore is an M-structure.

Remark 2. It follows from Theorem 4.3 (see property 3) that

$$\dim N = \sum_{x \in Ver_t} m_x \dim N_x = \sum_{x \in Ver_h} n_x \dim N_x. \tag{4.24}$$

Moreover, if the representation τ is decomposable, then the representation of $U(\mathcal{L})$ is also decomposable. Therefore, if the representation of $U(\mathcal{L})$ is indecomposable, then $\dim \tau$ must be a positive root with the property (4.24). If this root is real, then the representation does not depend on parameters and corresponds to some special value of K. If this root is imaginary, then the representation depends on one parameter and the action of K depends on this parameter also. In the Appendix we describe these representations for imaginary roots explicitly.

5. Appendix

In this Appendix we present explicit formulas for M-algebras with semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$ based on known classification results on affine quiver representations. We give also formulas for the operator R with values in $End(U(\mathcal{L}))$. Note that $K = R(1)$. It turns out that in all cases

$$S_\lambda = 1 + \lambda R. \tag{5.25}$$

Moreover, the operator R satisfies a polynomial equation of degree 3 in the case $\tilde{A}_{2k-1}$ and degree 4 in other cases. Using these equations, one can define $(v + R)^{-1}$ with values in the localization $\mathbb{C}(K) \otimes U(\mathcal{L})$, where $\mathbb{C}(K)$ is the field of rational functions in K. Formula (2.18) gives us the corresponding universal r-matrix with values in $\mathbb{C}(K) \otimes U(\mathcal{L})$. For any representation of $U(\mathcal{L})$ in a vector space N the image of this r-matrix is an r-matrix with values in $End(N)$.

The case $\tilde{A}_{2k-1}$. The algebras $\mathcal{A}$ and $\mathcal{B}$ have bases $\{e_i; i \in \mathbb{Z}/k\mathbb{Z}\}$ and $\{f_i; i \in \mathbb{Z}/k\mathbb{Z}\}$ correspondingly such that the multiplications are given by

$$e_i e_j = \delta_{i,j} e_i, \qquad f_i f_j = \delta_{i,j} f_i. \tag{5.26}$$

The M-algebra $U(\mathcal{L})$ is generated by $e_1, \ldots, e_k, f_1, \ldots, f_k$ with defining relations (5.26) and

$$\begin{aligned} e_1 + \cdots + e_k = f_1 + \cdots + f_k &= 1, \\ f_i e_j &= 0, \quad j - i \neq 0, 1. \end{aligned}$$

The operator R can be written in the form:

$$R(x) = \sum_{1 \le i \le j \le k-1} e_i x f_j + f_k e_k x.$$

This operator satisfies the following equation:

$$K R(x) - (K+1) R^2(x) + R^3(x) = 0.$$

From this equation we obtain

$$(v + R)^{-1}(x) = \frac{1}{v} x + \frac{1}{v(v+1)} (v + K)^{-1} (R^2(x) - (1 + v + K) R(x)).$$

The corresponding r-matrix is given by (2.18).

For any generic value of K the algebra $U(\mathcal{L})$ has the following irreducible representation V. There exist two bases $\{v_i; i \in \mathbb{Z}/k\mathbb{Z}\}$ and $\{w_i; i \in \mathbb{Z}/k\mathbb{Z}\}$ of the space V such that

$$e_i v_j = \delta_{i,j} v_i, \quad f_i w_j = \delta_{i,j} w_i, \quad v_i = w_i - t w_{i-1}, \quad i, j \in \mathbb{Z}/k\mathbb{Z}.$$

Here $t \in \mathbb{C}$ is a parameter of representation. In this representation K acts as multiplication by $1/(1 - t^k)$.

The case $\tilde{D}_{2k}$. The algebra $\mathcal{A} \cong \mathbb{C} \oplus \mathbb{C} \oplus (Mat_2)^{k-2} \oplus \mathbb{C} \oplus \mathbb{C}$ has a basis $\{e_1, e_2, e_{2k}, e_{2k+1}, e_{2\alpha,i,j}; 2 \le \alpha \le k-1, 1 \le i, j \le 2\}$ with multiplication

$$e_\alpha e_\beta = \delta_{\alpha,\beta} e_\beta, \quad e_\alpha e_{\beta,i,j} = e_{\beta,i,j} e_\alpha = 0, \quad e_{\alpha,i,j} e_{\beta,i',j'} = \delta_{\alpha,\beta} \delta_{j,i'} e_{\alpha,i,j'}. \tag{5.27}$$

The algebra $\mathcal{B} \cong (Mat_2)^{k-1}$ has a basis $\{e_{2\alpha-1,i,j}; 2 \le \alpha \le k, 1 \le i, j \le 2\}$ with multiplication

$$e_{\alpha,i,j} e_{\beta,i',j'} = \delta_{\alpha,\beta} \delta_{j,i'} e_{\alpha,i,j'}. \tag{5.28}$$

The M-algebra $U(\mathcal{L})$ is generated by $e_1, e_2, e_{2k}, e_{2k+1}, e_{\alpha,i,j}; 3 \le \alpha \le 2k-1$, $1 \le i, j \le 2$ with defining relations (5.27), (5.28) and

$$\begin{aligned} e_1 + e_2 + e_{2k} + e_{2k+1} + \sum_{2 \le \alpha \le k-1, 1 \le i \le 2} e_{2\alpha,i,i} &= \sum_{2 \le \alpha \le k, 1 \le i \le 2} e_{2\alpha-1,i,i} = 1, \\ e_{2\alpha-1,i,j} e_\beta &= 0, \quad 2 < \alpha < k, \quad \beta = 1, 2, 2k, 2k+1, \\ e_{2\alpha-1,i,j} e_{2\beta,i',j'} &= 0, \quad \alpha \neq \beta, \beta + 1, \\ e_{3,1,2} e_1 &= e_{3,2,2} e_1 = e_{3,1,1} e_2 = e_{3,2,1} e_2 = 0, \\ e_{2\alpha-1,i,j} e_{2\alpha,i',j'} &= e_{2\alpha+1,i,j} e_{2\alpha,i',j'} = 0, \quad j \neq i', \\ e_{2\alpha-1,i,1} e_{2\alpha,1,j} &= e_{2\alpha-1,i,2} e_{2\alpha,2,j}, \quad e_{2\alpha+1,i,1} e_{2\alpha,1,j} = e_{2\alpha+1,i,2} e_{2\alpha,2,j}, \\ e_{2k-1,1,1} e_{2k} &= e_{2k-1,1,2} e_{2k}, \quad e_{2k-1,2,1} e_{2k} = e_{2k-1,2,2} e_{2k}, \\ e_{2k-1,1,2} e_{2k+1} &= \lambda e_{2k-1,1,1} e_{2k+1}, \quad e_{2k-1,2,2} e_{2k+1} = \lambda e_{2k-1,2,1} e_{2k+1}. \end{aligned}$$

The operator R can be written in the form:

$$
\begin{aligned}
R(x) = &\sum_{1\le\alpha\le k-1} (\lambda e_1 x e_{2\alpha+1,2,2} - \lambda e_1 x e_{2\alpha+1,2,1} + e_2 x e_{2\alpha+1,1,1} - e_2 x e_{2\alpha+1,1,2} \\
&+ e_{2k} x e_{2\alpha+1,1,1} + \lambda e_{2k} x e_{2\alpha+1,2,2} + \lambda e_{2k+1} x e_{2\alpha+1,1,1} + \lambda e_{2k+1} x e_{2\alpha+1,2,2}) \\
&+ \sum_{2\le\alpha\le k-1,\ 2\le\beta\le k} (\lambda e_{2\alpha,1,1} x e_{2\beta-1,2,2} + e_{2\alpha,2,2} x e_{2\beta-1,1,1}) \\
&- \sum_{2\le\alpha<\beta\le k} (\lambda e_{2\alpha,1,1} x e_{2\beta-1,2,1} + e_{2\alpha,2,2} x e_{2\beta-1,1,2}) \\
&+ \sum_{2\le\beta\le\alpha\le k-1} (\lambda e_{2\alpha,2,1} x e_{2\beta-1,2,2} + e_{2\alpha,1,2} x e_{2\beta-1,1,1}) + (1-\lambda) e_{2k-1,2,2} e_{2k+1} x.
\end{aligned}
$$

This operator satisfies the following equation:

$$R^4(x) - (1+\lambda+K)R^3(x) + (\lambda+K+\lambda K)R^2(x) - \lambda K R(x) = 0.$$

From this equation we obtain

$$
\begin{aligned}
(v+R)^{-1}(x) = -\frac{1}{v}x + \frac{1}{v(v+1)(v+\lambda)}(v+K)^{-1}\Big(&R^3(x) - (1+v+\lambda+K)R^2(x) \\
&+ (v^2+\lambda v+v+\lambda+(1+v+\lambda)K)R(x)\Big),
\end{aligned}
$$

and the r-matrix is given by (2.18).

For any generic value of K the algebra $U(\mathcal{L})$ has the following irreducible representation V of dimension $4k-4$. There exist two bases $\{v_1, v_2, v_{2k}, v_{2k+1}, v_{2\alpha,i,j};\ 2\le\alpha\le k-1,\ 1\le i,j\le 2\}$ and $\{v_{2\alpha-1,i,j};\ 2\le\alpha\le k,\ 1\le i,j\le 2\}$ of the space V such that

$$
\begin{aligned}
e_\alpha v_\beta &= \delta_{\alpha,\beta} v_\beta, \quad \alpha,\beta = 1,2,2k,2k+1, \\
e_\alpha v_{2\beta,i,j} &= e_{2\beta,i,j} v_\alpha = 0, \quad \alpha = 1,2,2k,2k+1 \quad 2\le\beta\le k-1, \\
e_{2\alpha,i,j} v_{2\beta,i',j'} &= \delta_{\alpha,\beta}\delta_{j,i'} v_{2\alpha,i,j'}, \quad 2\le\alpha,\beta\le k-1, \\
e_{2\alpha-1,i,j} v_{2\beta-1,i',j'} &= \delta_{\alpha,\beta}\delta_{j,i'} v_{2\alpha-1,i,j'}, \quad 2\le\alpha,\beta\le k,
\end{aligned}
$$

and

$$
\begin{aligned}
v_1 &= v_{3,1,1}, \quad v_2 = v_{3,2,2}, \\
v_{2\alpha,i,j} &= v_{2\alpha+1,i,j} - v_{2\alpha-1,i,j}, \quad 2\le\alpha\le k-1, \quad i,j = 1,2, \\
v_{2k} &= v_{2k-1,1,1} + v_{2k-1,2,1} + v_{2k-1,1,2} + v_{2k-1,2,2}, \\
v_{2k+1} &= v_{2k-1,1,1} + \lambda v_{2k-1,2,1} + t v_{2k-1,1,2} + \lambda t v_{2k-1,2,2}.
\end{aligned}
$$

Here $\lambda\in\mathbb{C}$ is a parameter of the algebra $U(\mathcal{L})$ and $t\in\mathbb{C}$ is a parameter of representation. In this representation K acts as multiplication by $\mu = \lambda(t-1)/(t-\lambda)$.

The case $\tilde{D}_{2k-1}$. The algebra $\mathcal{A} \cong \mathbb{C} \oplus \mathbb{C} \oplus (Mat_2)^{k-2}$ has a basis $\{e_1, e_2, e_{2\alpha,i,j};\ 2 \le \alpha \le k-1, 1 \le i, j \le 2\}$ with multiplication

$$e_\alpha e_\beta = \delta_{\alpha,\beta} e_\beta, \quad e_\alpha e_{\beta,i,j} = e_{\beta,i,j} e_\alpha = 0, \quad e_{\alpha,i,j} e_{\beta,i',j'} = \delta_{\alpha,\beta}\delta_{j,i'} e_{\alpha,i,j'}. \tag{5.29}$$

The algebra $\mathcal{B} \cong \mathbb{C} \oplus \mathbb{C} \oplus (Mat_2)^{k-2}$ has a basis $\{e_{2k-1}, e_{2k}, e_{2\alpha-1,i,j};\ 2 \le \alpha \le k-1, 1 \le i, j \le 2\}$ with multiplication

$$e_\alpha e_\beta = \delta_{\alpha,\beta} e_\beta, \quad e_\alpha e_{\beta,i,j} = e_{\beta,i,j} e_\alpha = 0, \quad e_{\alpha,i,j} e_{\beta,i',j'} = \delta_{\alpha,\beta}\delta_{j,i'} e_{\alpha,i,j'}. \tag{5.30}$$

The M-algebra $U(\mathcal{L})$ is generated by $e_1, e_2, e_{2k-1}, e_{2k}, e_{\alpha,i,j};\ 3 \le \alpha \le 2k-2$, $1 \le i, j \le 2$ with defining relations (5.29), (5.30) and

$$\begin{aligned}
e_\alpha e_\beta &= 0, \quad \alpha = 2k-1, 2k, \quad \beta = 1, 2,\\
e_1 + e_2 &= \sum_{2\le\alpha\le k-1, 1\le i\le 2} e_{2\alpha,i,i} = e_{2k-1} + e_{2k} + \sum_{2\le\alpha\le k-1, 1\le i\le 2} e_{2\alpha-1,i,i} = 1,\\
e_{2\alpha-1,i,j} e_\beta &= 0, \quad \alpha > 2, \quad \beta = 1, 2,\\
e_{2\alpha-1,i,j} e_{2\beta,i',j'} &= 0, \quad \alpha \ne \beta, \beta+1,\\
e_\alpha e_{2\beta,i,j} &= 0, \quad \beta < k-1, \quad \alpha = 2k-1, 2k,\\
e_{3,1,2} e_1 &= e_{3,2,2} e_1 = e_{3,1,1} e_2 = e_{3,2,1} e_2 = 0,\\
e_{2\alpha-1,i,j} e_{2\alpha,i',j'} &= e_{2\alpha+1,i,j} e_{2\alpha,i',j'} = 0, \quad j \ne i',\\
e_{2\alpha-1,i,1} e_{2\alpha,1,j} &= e_{2\alpha-1,i,2} e_{2\alpha,2,j}, \quad e_{2\alpha+1,i,1} e_{2\alpha,1,j} = e_{2\alpha+1,i,2} e_{2\alpha,2,j},\\
e_{2k-1} e_{2k-2,1,1} &= e_{2k-1} e_{2k-2,2,1}, \quad e_{2k-1} e_{2k-2,1,2} = e_{2k-1} e_{2k-2,2,2}\\
e_{2k} e_{2k-2,2,1} &= \lambda e_{2k} e_{2k-2,1,1}, \quad e_{2k} e_{2k-2,2,2} = \lambda e_{2k} e_{2k-2,1,2}.
\end{aligned}$$

The operator R can be written in the form:

$$\begin{aligned}
R(x) = (\lambda-1) e_1 x e_{2k-1} &+ \sum_{2\le\alpha\le k-1} ((\lambda-1) e_1 x e_{2\alpha-1,2,2} + (\lambda-1) e_{2\alpha,1,1} x e_{2k-1}\\
&-\lambda e_2 x e_{2\alpha-1,1,2} - e_1 x e_{2\alpha-1,2,1} + \lambda e_2 x e_{2\alpha-1,2,2} + \lambda e_1 x e_{2\alpha-1,1,1})\\
&+ \sum_{2\le\alpha,\beta\le k-1} ((\lambda-1) e_{2\alpha,1,1} x e_{2\beta-1,2,2}\\
&+\lambda e_{2\alpha,1,1} x e_{2\beta-1,1,1} + \lambda e_{2\alpha,2,2} x e_{2\beta-1,2,2})\\
&+ \sum_{2\le\beta\le\alpha\le k-1} (\lambda e_{2\alpha,1,2} x e_{2\beta-1,1,1} + e_{2\alpha,2,1} x e_{2\beta-1,2,2})\\
&- \sum_{2\le\alpha<\beta\le k-1} (\lambda e_{2\alpha,2,2} x e_{2\beta-1,1,2} + e_{2\alpha,1,1} x e_{2\beta-1,2,1}) + (\lambda-1) x e_{2k} e_{2k-2,2,2}.
\end{aligned}$$

This operator satisfies the following equation:

$$R^4(x) - R^3(x)(2\lambda - 1 + K) + R^2(x)(\lambda^2 - \lambda - K + 2\lambda K) - \lambda(\lambda-1) R(x) K = 0.$$

From this equation we obtain

$$\begin{aligned}
(v+R)^{-1}(x) = -\frac{1}{v} x &+ \frac{1}{v(v+\lambda)(v+\lambda-1)} \Big(R^3(x) - R^2(x)(v + 2\lambda - 1 + K)\\
&+R(x)(v^2 + 2\lambda v + \lambda^2 - v - \lambda + (v - 1 + 2\lambda)K) \Big) (v+K)^{-1},
\end{aligned}$$

and the r-matrix is given by (2.18).

For any generic value of K the algebra $U(\mathcal{L})$ has the following irreducible representation V of dimension $4k-6$. There exist two bases $\{v_1, v_2, v_{2\alpha,i,j}; 2 \le \alpha \le k-1, 1 \le i, j \le 2\}$ and $\{v_{2k-1}, v_{2k}, v_{2\alpha-1,i,j}; 2 \le \alpha \le k-1, 1 \le i, j \le 2\}$ of the space V such that

$$
\begin{aligned}
e_\alpha v_\beta &= \delta_{\alpha,\beta} v_\beta, \quad \alpha, \beta = 1, 2,\\
e_\alpha v_{2\beta,i,j} &= e_{2\beta,i,j} v_\alpha = 0, \quad \alpha = 1, 2, \quad 2 \le \beta \le k-1,\\
e_{2\alpha,i,j} v_{2\beta,i',j'} &= \delta_{\alpha,\beta}\delta_{j,i'} v_{2\alpha,i,j'}, \quad 2 \le \alpha, \beta \le k-1,\\
e_\alpha v_\beta &= \delta_{\alpha,\beta} v_\beta, \quad \alpha, \beta = 2k-1, 2k,\\
e_\alpha v_{2\beta-1,i,j} &= e_{2\beta-1,i,j} v_\alpha = 0, \quad \alpha = 2k-1, 2k, \quad 2 \le \beta \le k-1,\\
e_{2\alpha-1,i,j} v_{2\beta-1,i',j'} &= \delta_{\alpha,\beta}\delta_{j,i'} v_{2\alpha-1,i,j'}, \quad 2 \le \alpha, \beta \le k-1,
\end{aligned}
$$

and

$$
\begin{aligned}
v_1 &= v_{3,1,1}, \quad v_2 = v_{3,2,2},\\
v_{2\alpha,i,j} &= v_{2\alpha+1,i,j} - v_{2\alpha-1,i,j}, \quad 2 \le \alpha < k-1, \quad i, j = 1, 2,\\
v_{2k-2,1,1} &= v_{2k-1} + v_{2k} - v_{2k-3,1,1}, \quad v_{2k-2,2,2} = v_{2k-1} + \lambda t v_{2k} - v_{2k-3,2,2},\\
v_{2k-2,1,2} &= v_{2k-1} + t v_{2k} - v_{2k-3,1,2}, \quad v_{2k-2,2,1} = v_{2k-1} + \lambda v_{2k} - v_{2k-3,2,1}.
\end{aligned}
$$

Here $\lambda \in \mathbb{C}$ is a parameter of the algebra $U(\mathcal{L})$ and $t \in \mathbb{C}$ is a parameter of representation. In this representation K acts as multiplication by $\mu = t\lambda(1-\lambda)/(1-t\lambda)$.

Acknowledgements. The authors are grateful to I.Z. Golubchik, A. Vaintrob and M.A. Semenov-Tian-Shansky for useful discussions. The research was supported by the Manchester Institute for Mathematical Sciences (MIMS). The research was partially supported by: RFBR grant 05-01-00189, NSh grants 1716.2003.1 and 2044.2003.2.

References

1. Odesskii, A.V., Sokolov, V.V.: Algebraic structures connected with pairs of compatible associative algebras. *International Mathematics Research Notices*, **2006**, Article ID 43734, 35 pages (2006)
2. Magri, F.: A simple model of the integrable Hamiltonian equation. J. Math. Phys. **19**, 1156–1162 (1978)
3. Golubchik, I.Z., Sokolov, V.V.: Factorization of the loop algebra and integrable top-like systems. Theoret. Math. Phys. **141**(1), 1329–1347 (2004)
4. Odesskii, A.V., Sokolov, V.V.: Integrable matrix equations related to pairs of compatible associative algebras. J. Phys. A: Math. Gen. **39**, 12447–12456 (2006)
5. Mikhailov, A.V., Sokolov, V.V.: Integrable ODEs on associative algebras. Commun. Math. Phys. **211**(1), 231–251 (2000)
6. Drinfeld, V.G.: Quantum groups. In: *Proceedings of the International Congress of Mathematicians*, Vols. **1**, **2** (Berkeley, Calif., 1986), Providence, RI: Amer. Math. Soc., 1987, pp. 798–820
7. Belavin, A.A., Drinfeld, V.G.: Solutions of the classical Yang-Baxter equation for simple Lie algebras (Russian). Funks. Anal. i Pril. **16**(3), 1–29 (1982)
8. Semenov-Tian-Shansky, M.A.: What a classical r-matrix is (Russian). Funkts. Anal. i Pril. **17**(4), 17–33 (1983)
9. Sklyanin, E.K.: Separation of variables—new trends. In: *Quantum field theory, integrable models and beyond (Kyoto, 1994)*. Progr. Theoret. Phys. Suppl. No. **118**, 35–60 (1995)
10. Schofield, A.: General representations of quivers. Proc. London Math. Soc. **65**, 46–64 (1992)
11. Dlab, V., Ringel, C.M.: *Indecomposable representations of graphs and algebras*. Mem. Amer. Math. Soc., **173** providence, RI: Amer. Math. Soc., 1976
12. Ringel, C.M.: *Tame algebras and integral quadratic forms*. Springer Lect. Notes **1099**, Berlin-Heidelberg-New York: Springer, 1984

13. Golubchik, I.Z., Sokolov, V.V.: Compatible Lie brackets and integrable equations of the principle chiral model type. Func. Anal. and Appl. **36**(3), 172–181 (2002)
14. Golubchik, I.Z., Sokolov, V.V.: Factorization of the loop algebras and compatible Lie brackets. J. Nonlin. Math. Phys. **12**(1), 343–350 (2005)
15. Golubchik, I.Z., Sokolov, V.V.: Compatible Lie brackets and Yang-Baxter equation. Theoret. Math. Phys. **146**(2), 159–169 (2006)
16. Odesskii, A.V., Sokolov, V.V.: Compatible Lie brackets related to elliptic curve. J. Math. Phys. **47**, 013506 (2006)
17. Van Der Waerden, B.L.: *Algebra*. Berlin-Heidelberg-New York: Springer-Verlag, 1967
18. Bourbaki, N.: *Groupes et algébres de Lie*. Paris: Hermann, 1968
19. Vinberg, E.B.: Disete linear groups that are generated by reflections. Izv. Akad. Nauk SSSR, Ser. Mat. **35**, 1072–1112 (1971)
20. Aquiar, M.: On the associative analog of Lie bialgebras. J. Algebra. **244**(2), 492–532 (2001)

Communicated by A. Connes