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Sharp, Richard

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Critical exponents for groups of isometries

Richard Sharp

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Abstract Let Γ be a convex co-compact group of isometries of a CAT(-1) space X and let Γ_0 be a normal subgroup of Γ. We show that, provided Γ is a free group, a sufficient condition for Γ and Γ_0 to have the same critical exponent is that Γ/Γ_0 is amenable.

Keywords CAT(-1)space · Riemannian manifold · Negative curvature · Group of isometrics · Critical exponent · Amenable group

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1 Introduction and results

Let Γ be a group of isometries acting freely and properly discontinuously on a CAT(-1) space X. Roughly speaking, a CAT(-1) space is a path metric space for which every geodesic triangle is more pinched than a congruent triangle in the hyperbolic plane; see [5] for a formal definition. Prototypical examples of CAT(-1) spaces are simply connected Riemannian manifold with sectional curvatures bounded above by -1 and (simplicial or non-simplicial) $\mathbb{R}$-trees.

A fundamental quantity associated to Γ is its critical exponent $\delta(\Gamma)$. This is defined to be the abscissa of convergence of the Poincaré series

$$\wp_\Gamma(s) = \sum_{\gamma \in \Gamma} e^{-s d_X(o,\gamma o)}, \tag{1.1}$$

R. Sharp (✉)
University of Manchester, School of Mathematics, Oxford Road,
Manchester M13 9PL, U.K.
e-mail: sharp@maths.man.ac.uk

where $o \in X$ and $d_X(\cdot,\cdot)$ denotes the distance in X. In other words, the series converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$. An equivalent definition is that

$$\delta(\Gamma) = \limsup_{T\to+\infty} \frac{1}{T} \log \#\{\gamma \in \Gamma : d_X(o,\gamma o) \leq T\}. \tag{1.2}$$

A simple calculation shows that $\delta(\Gamma)$ is independent of the choice of $x \in X$.

Let ∂X denote the ideal boundary of X. The set $\{\gamma o : \gamma \in \Gamma\}$ accumulates on a subset $\Lambda_\Gamma \subset \partial X$ (independent of o) called the limit set of Γ. Let $\mathcal{C}_\Gamma = \text{c.h.}(\Lambda_\Gamma) \cap \mathcal{X}$, where c.h.$(\Lambda_\Gamma)$ is the geodesic convex hull of Λ_Γ. We say that Γ is convex co-compact if $\mathcal{C}_\Gamma/\Gamma$ is compact. (If Γ is a Kleinian group, this agrees with the classical notion of convex co-compactness.) In addition, we say that Γ is non-elementary if it is not a finite extension of a cyclic group. These two conditions ensure that $\delta(\Gamma) > 0$ and the limit in (1.2) exists.

Now suppose that Γ_0 is a normal subgroup of a convex co-compact group Γ. Then Γ_0 itself has a critical exponent $\delta(\Gamma_0)$ and, clearly, $\delta(\Gamma_0) \leq \delta(\Gamma)$. Our main result addresses the question of when we have equality.

Theorem 1 *If* Γ/Γ_0 *is amenable then* $\delta(\Gamma_0) = \delta(\Gamma)$.

The definition of amenable group is given in the next section.

Remark Equality of $\delta(\Gamma_0)$ and $\delta(\Gamma)$ was previously known to hold when Γ/Γ_0 is finite or abelian [15]. (In fact, the results in ref. [15] are stated in the case where X is real hyperbolic space but the proofs given there apply more generally.)

Since obtaining the results in this paper, we have learned that Theorem 1 has been proved by Roblin [16], without the restriction that Γ is a free group, using completely different methods. However, we feel that our alternative approach, based on approximating $\delta(\Gamma)$ and $\delta(\Gamma_0)$ by quantities related to random walks on graphs, has independent interest. It is worth remarking that the equality of the two critical exponents has been used recently in ref. [10].

We shall now outline the contents of the paper. In Sect. 1, we give definition of amenable groups and introduce Grigorchuk's co-growth criterion, interpreting it in terms of a graph. In Sect. 2, we describe how to write the Poincaré series $\wp_\Gamma(s)$ and $\wp_{\Gamma_0}(s)$ in terms of a subshift of finite type. We also introduce sequences of matrices which are used to approximate $\delta(\Gamma)$ and $\delta(\Gamma_0)$. In Sect. 3, we use ideas from the theory of random walks on graphs, in particular [12], to show that, if Γ/Γ_0 is amenable then the respective approximations to $\delta(\Gamma)$ and $\delta(\Gamma_0)$ agree at each stage, from which Theorem 1 follows. In the final section, we consider that special case of $X = \mathbb{H}^{n+1}$.

I am very grateful to the referee for suggesting numerous improvements to the exposition.

2 Amenable groups and co-growth

Amenable groups were defined by von Neumann. A group G is said to be amenable if there is an invariant mean on $L^\infty(G,\mathbb{R})$, i.e. a bounded linear functional $\mu: L^\infty(G,\mathbb{R}) \to \mathbb{R}$ such that, for any $f \in L^\infty(G,\mathbb{R})$,

(1) $\inf_{g\in G} f(g) \leq \mu(f) \leq \sup_{g\in G} f(g)$; and
(2) for all $g \in G$, $\mu(g\cdot f) = \mu(f)$, where $g \cdot f(x) = f(g^{-1}x)$.

It is immediate from the definition that any finite group is amenable by setting

$$\mu(f) = \frac{1}{|G|} \sum_{g \in G} f(g).$$

The situation for infinite groups is more subtle and we shall restrict our discussion to finitely generated groups.

A group with subexponential growth is amenable [2,7]. In particular, any abelian or nilpotent group is amenable. However, there are examples of amenable groups with exponential growth (e.g. the lamplighter groups [8]). In contrast, non-abelian free groups and, more generally, non-elementary Gromov hyperbolic groups are not amenable. It was conjectured by von Neumann that a group fails to be amenable only if it contains the free group on two generators; however, a counterexample to this was constructed by Ol'shanskii [11].

Grigorchuk related amenability to the property of co-growth of subgroups of free groups. Let Γ (considered as an abstract group) be the free group on k generators $\{a_1, \ldots, a_k\}$ and let $|\gamma|$ denote the word length of γ, i.e. the length of the shortest representation of γ as a word in $a_1^{\pm 1}, \ldots, a_k^{\pm 1}$. Clearly, we have that

$$\lim_{n \to +\infty} (\#\{\gamma \in \Gamma : |g| = n\})^{1/n} = 2k - 1.$$

Now suppose that Γ_0 is a normal subgroup of Γ. Grigorchuk showed that the co-growth $c(\Gamma_0)$, defined by

$$c(\Gamma_0) := \limsup_{n \to +\infty} (\#\{g \in \Gamma_0 : |g| = n\})^{1/n}$$

is equal to $2k - 1$ if and only if $G = \Gamma/\Gamma_0$ is amenable [6] (see also [4]).

Grigorchuk's result may be reinterpreted in terms of graphs. Let $\mathcal{G}$ denote the graph consisting of one vertex and k oriented edges, labelled by $a_1, \ldots, a_k$. The same edges with the reverse orientation will be labelled $a_1^{-1}, \ldots, a_k^{-1}$, respectively. Write $\mathcal{T}$ for the universal cover of $\mathcal{G}$; then $\mathcal{T}$ is a $2k$-regular tree. It is an easy observation that Γ acts freely on $\mathcal{T}$ with quotient $\mathcal{G}$. Furthermore, we may identify elements of word length n in Γ with non-backtracking paths of length n in $\mathcal{G}$. (A path $(e_1, \ldots, e_n)$ is said to be non-backtracking if, for each $i = 2, \ldots, n$, the edge e_i is not equal to e_{i-1} with the reversed orientation.)

Now consider the action of the subgroup Γ_0 on $\mathcal{T}$ and write $\widetilde{\mathcal{G}} = \mathcal{T}/\Gamma_0$, for the quotient graph; this is a G-cover of $\mathcal{G}$. (In fact, $\widetilde{\mathcal{G}}$ is the Cayley graph of G with respect to the generators obtained from $a_1, \ldots, a_k$.) Then we may identify elements of word length n in Γ_0 with non-backtracking paths of length n in $\widetilde{\mathcal{G}}$ starting from and ending at some fixed vertex. Grigorchuk's result may then be reformulated as saying that the growth rate of the number of paths of length n in $\widetilde{\mathcal{G}}$, starting from and ending at a fixed vertex, is equal to the corresponding growth rate for paths in $\mathcal{G}$ if and only if Γ/Γ_0 is amenable.

The parallels between equality of these growth rates and equality of the critical exponents is apparent. However, the "lengths" are different: word length $|\gamma|$ in one setting and the displacement $d(o, \gamma o)$ for the action on X in the other. Nevertheless, this will provide the basis for our approach. In this context, we note that there exists $A > 1$ such that

$$A^{-1}|\gamma| \leq d(o, \gamma o) \leq A|\gamma|. \tag{2.1}$$

We shall use several properties of the graph $\widetilde{\mathcal{G}}$. Firstly, provided it is not itself a tree (which only occurs if Γ_0 is trivial) $\widetilde{\mathcal{G}}$ has the property that "small cycles are dense" [12]: there exists $R > 0$ such that, for each vertex u in $\widetilde{\mathcal{G}}$, the set $B(u,R) = \{v : d_{\mathcal{G}}(u,v) \le R\}$ contains a cycle. We also note that there is a number $L(R) > 0$ such that, for every vertex u in $\widetilde{\mathcal{G}}$, $\#B(u,R) \le L(R)$.

Later we shall need to find paths joining vertices in $\widetilde{\mathcal{G}}$. Let $c_n(u,v)$ denote the number of non-backtracking paths of length n in $\widetilde{\mathcal{G}}$ from u to v.

Lemma 2.1 [17] *Let u, v be vertices of $\widetilde{\mathcal{G}}$. Then either*

$$\lim_{n\to+\infty} c_n(u,v)^{1/n} = c(\Gamma_0)$$

or

$$\lim_{n\to+\infty} c_{2n+\delta(u,v)}(u,v)^{1/2n} = c(\Gamma_0) \ \ \textit{and} \ \ c_{2n+\delta(u,v)-1}(u,v) = 0,$$

where $\delta(u,v) = 0$ if $d_{\mathcal{G}}(u,v)$ is even and $\delta(u,v) = 1$ if $d_{\mathcal{G}}(u,v)$ is odd.

Corollary 2.1.1 *Suppose that G is amenable (or even that $c(\Gamma_0) > 0$) and let u, v be vertices of $\widetilde{\mathcal{G}}$. Then there exists $l(u,v) > 0$ such that either $c_{l(u,v)}(u,v) > 0$ or $c_{l(u,v)-1}(u,v) > 0$.*

3 Shifts of Finite Type and Approximation

Recall that the free group Γ is given in terms of generators $\mathcal{A} = \{a_1^{\pm 1}, \dots, a_k^{\pm 1}\}$. We shall form a subshift of finite type $\sigma : \Sigma \to \Sigma$, where

$$\Sigma = \{x = (x_i)_{i=0}^{\infty} \in \mathcal{A}^{\mathbb{Z}^+} : x_{i+1} \ne x_i^{-1}, \forall i \in \mathbb{Z}^+\}$$

and σ is the shift map: $(\sigma x)_i = x_{i+1}$. We call $(x_0, \dots, x_{n-1}) \in \mathcal{A}^n$ an allowed string of length n if $x_{i+1} \ne x_i^{-1}$, $i = 0, \dots, n-2$. We write Σ_n for the set of all allowed strings of length n, $\Sigma_{\le n} = \bigcup_{m=0}^{n} \Sigma_m$ and $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma_n$, where Σ_0 is defined to be a singleton consisting of an "empty string" ω. There is an obvious bijection between Σ_n and elements of Γ with word length n (and hence between Γ and Σ^*).

We make $\Sigma \cup \Sigma^*$ into a metric space by setting $d(x,y) = 2^{-n(x,y)}$, where

$$n(x,y) = \begin{cases} 0, & \text{if } x_0 \ne y_0, \\ \sup\{n \ge 0 : x_m = y_m, 0 \le m \le n\}, & \text{otherwise} \end{cases}$$

If $f : \Sigma \cup \Sigma^* \to \mathbb{R}$ is Hölder continuous with Hölder exponent $\alpha > 0$ then we write

$$|f|_\alpha = \sup\left\{\frac{f(x) - f(y)}{d(x,y)^\alpha} : x \ne y\right\}.$$

If we define $\sigma(\omega) = \omega$, the shift map extends to $\sigma : \Sigma \cup \Sigma^* \to \Sigma \cup \Sigma^*$ and $\sigma(\Sigma_n) = \Sigma_{n-1}$, $n \ge 1$. For a function $f : \Sigma \cup \Sigma^* \to \mathbb{R}$, we write $f^n(x) = f(x) + f(\sigma x) + \cdots + f(\sigma^{n-1} x)$.

Proposition 3.1 [9,13,14] *There is a strictly positive Hölder continuous function $r : \Sigma \cup \Sigma^* \to \mathbb{R}$ such that, if $\gamma = x_0 \cdots x_{n-1}$ then*

$$r^n(x_0, \dots, x_{n-1}) = d_X(o, \gamma o).$$

Remark An examination of the proof in ref. [14] shows that what is essential for the proof is that X satisfies the Aleksandrov–Toponogov Comparison property. Thus, the result holds if X is a CAT(-1) space.

An easy calculation then shows that

$$\wp_\Gamma(s) = 1 + \sum_{n=1}^{\infty} \sum_{x \in \sigma^{-n}(\omega)\setminus\{\omega\}} \mathrm{e}^{-sr^n(x)}.$$

Let $\psi : \Gamma \to G = \Gamma/\Gamma_0$ be the natural homomorphism and, for $x = (x_0, \ldots, x_{n-1}) \in \Sigma_n$, write $\psi_n(x) = \psi(x_0)\cdots\psi(x_{n-1})$. We have

$$\wp_{\Gamma_0}(s) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{x \in \sigma^{-n}(\omega)\setminus\{\omega\} \\ \psi_n(x)=e}} \mathrm{e}^{-sr^n(x)}.$$

We shall study the abscissas of convergence of the above two series via a sequence of approximations to r. We define

$$r_N(x) = \begin{cases} r(x), & \text{if } x \in \Sigma_n, n \le N, \\ r(x_0, \ldots, x_{N-1}), & \text{otherwise.} \end{cases}$$

Then $\|r - r_N\|_\infty \le |r|_\alpha 2^{-\alpha(N+1)}$, where $\alpha > 0$ is the Hölder exponent of r. Hence, given $\epsilon > 0$, we can choose N sufficiently large so that, for each $x \in \Sigma \cup \Sigma^*$ and $n \ge 1$, $|r^n(x) - r_N^n(x)| < n\epsilon$.

We define δ_N and δ_N^0 to be the abscissas of convergence of $\wp_N(s)$ and $\wp_N^0(s)$, respectively, where

$$\wp_N(s) = 1 + \sum_{n=1}^{\infty} \sum_{x \in \sigma^{-n}(\omega)\setminus\{\omega\}} \mathrm{e}^{-sr_N^n(x)}, \quad \wp_N^0(s) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{x \in \sigma^{-n}(\omega)\setminus\{\omega\} \\ \psi_n(x)=e}} \mathrm{e}^{-sr_N^n(x)}.$$

Lemma 3.1 *We have* $\lim_{N\to+\infty} \delta_N = \delta(\Gamma)$ *and* $\lim_{N\to+\infty} \delta_N^0 = \delta(\Gamma_0)$.

Proof For $\gamma = x_0 \cdots x_{|\gamma|-1} \in \Gamma$, let $x_\gamma = (x_0, \ldots, x_{|\gamma|-1}) \in \Sigma^*$. Then, $r^{|\gamma|}(x_\gamma) = d(o, \gamma o)$, so, using this notation,

$$\delta(\Gamma) = \limsup_{T\to+\infty} \frac{1}{T} \log \#\{\gamma : r^{|\gamma|}(x_\gamma) \le T\}, \ \delta_N = \limsup_{T\to+\infty} \frac{1}{T} \log \#\{\gamma : r_N^{|\gamma|}(x_\gamma) \le T\}.$$

Fix $\epsilon > 0$ sufficiently small that $A\epsilon < 1$, where A is given by (2.1). Then, provided N is sufficiently large, $r^{|\gamma|}(x_\gamma) \le r_N^{|\gamma|}(x_\gamma) + |\gamma|\epsilon \le r_N^{|\gamma|}(x_\gamma) + Ar^{|\gamma|}(x_\gamma)\epsilon$ and so

$$r^{|\gamma|}(x_\gamma) \le \frac{r_N^{|\gamma|}(x_\gamma)}{1 - A\epsilon}.$$

Hence

$$\#\{\gamma : r_N^{|\gamma|}(x_\gamma) \le T\} \le \#\{\gamma : r^{|\gamma|}(x_\gamma) \le (1 - A\epsilon)^{-1} T\}$$

and so $\delta_N \le (1 - A\epsilon)^{-1}\delta(\Gamma)$. Since we may take ϵ arbitrarily small, we conclude that $\limsup_{N\to+\infty} \delta_N \le \delta(\Gamma)$. A similar argument gives the corresponding lower bound, so we have $\lim_{N\to+\infty} \delta_N = \delta(\Gamma)$. The same proof gives the result for δ_N^0.

Hence, to prove Theorem 1, it suffices to show that if G is amenable then $\delta_N = \delta_N^0$, for each $N \geq 1$. We shall do this in the next section. First we need to rewrite $\wp_N(s)$ and $\wp_N^0(s)$ in matrix form.

For $N \geq 1$, define matrices P_N, indexed by $\Sigma_N \times \Sigma_N$, by

$$P_N(x,y) = \begin{cases} e^{-\delta_N r_N(x_0,x_1,\ldots,x_{N-1},y_{N-1})}, & \text{if } x_n = y_{n-1}, \quad n = 1,\ldots,N-1, \\ 0, & \text{otherwise,} \end{cases}$$

where $x = (x_0, x_1, \ldots, x_{N-1})$, $y = (y_0, y_1, \ldots, y_{N-1})$. (For $N = 1$, we set $P_1(x_0, y_0) = 0$ whenever $y_0 = x_0^{-1}$. For $N \geq 2$ this is automatically avoided.) Each P_N is irreducible (and aperiodic). Also define another sequence of matrices Q_N, indexed by $\Sigma_{\leq N} \times \Sigma_{\leq N}$, by

$$Q_N(x,y) = \begin{cases} e^{-\delta_N r_N(x_0,x_1,\ldots,x_{N-1},y_{N-1})}, & \text{if } x_n = y_{n-1}, \quad n = 1,\ldots,N-1, \\ 0, & \text{otherwise,} \end{cases}$$

where, for $x \in \Sigma_m$, we write $x = (x_0, \ldots, x_{m-1}, \underbrace{\omega, \ldots, \omega}_{N-m})$. The matrices Q_N are not irreducible. Note that P_N is the restriction of Q_N to $\Sigma_N \times \Sigma_N$.

From the definition of Q_N, we have that, for $n > N$,

$$\sum_{x \in \sigma^{-n}(\omega) \setminus \{\omega\}} e^{-\delta_N r_N^n(x)} = \sum_{x \in \Sigma_N} \sum_{a \in \Sigma_1} Q_N^n(x, (a, \omega, \ldots, \omega)).$$

Now, since P_N is irreducible, the value of $\limsup_{n \to +\infty} (P_N^n(x,y))^{1/n}$ is independent of $x, y \in \Sigma_N$ (in fact it is the spectral radius of P_N). □

Lemma 3.2 *For any $x, y \in \Sigma_N$ and $a \in \Sigma_1$,*

$$\limsup_{n \to +\infty} (P_N^n(x,y))^{1/n} = \limsup_{n \to +\infty} \left(Q_N^n\,(x, (z, \omega, \ldots, \omega))\right)^{1/n}.$$

Proof We have

$$\begin{aligned} Q_N^n(x, (a, \omega, \ldots, \omega)) &= \sum_{y \in \Sigma_N} Q_N^{n-N}(x,y)\, Q_N^N\,(y, (a, \omega, \ldots, \omega)) \\ &= \sum_{y \in \Sigma_N} P_N^{n-N}(x,y)\, Q_N^N(y, (a, \omega, \ldots, \omega)). \end{aligned}$$

Since δ_N is the abscissa of convergence of $\wp_N(s)$, we deduce that, for each $x, y \in \Sigma_N$, $\limsup_{n \to +\infty} (P_N^n(x,y))^{1/n} = 1$.

By the Perron–Frobenius Theorem, P_N has 1 as an eigenvalue and an associated strictly positive (row) eigenvector v_N: $v_N P_N = v_N$. In addition, we may suppose that P_N is normalized so that

$$\sum_{y \in \Sigma_N} P_N(x,y) = 1.$$

In other words, P_N may be regarded as a matrix of transition probabilities between elements of Σ_N.

Now we define another sequence of (infinite) matrices $\widetilde{P}_N$, $N \geq 1$, indexed by $(\Sigma_N \times G) \times (\Sigma_N \times G)$, by

$$\widetilde{P}_N((x,g),(y,h)) = \begin{cases} P_N(x,y), & \text{if } \psi(x_0) = g^{-1}h, \\ 0, & \text{otherwise.} \end{cases}$$

(Note that the exponent in the entries of $\widetilde{P}_N$ is δ_N not δ_N^0.) Each $\widetilde{P}_N$ is locally finite in the sense that, for each (x,g), there are only finitely many (y,h) such that $\widetilde{P}_N((x,g),(y,h)) > 0$.

We also define a corresponding sequence of infinite matrices $\widetilde{Q}_N$, $N \geq 1$, indexed by $\left(\Sigma_{\leq N} \times G\right) \times \left(\Sigma_{\leq N} \times G\right)$, by

$$\widetilde{Q}_N((x,g),(y,h)) = \begin{cases} Q_N(x,y), & \text{if } \psi(x_0) = g^{-1}h, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\sum_{\substack{x \in \sigma^{-n}(\omega)\setminus\{\omega\} \\ \psi_n(x) = e}} \mathrm{e}^{-s r_N^n(x)} = \sum_{x \in \Sigma_N} \sum_{y \in \Sigma_1} \widetilde{Q}_N^n((x,e),\big((y,\omega,\ldots,\omega),e\big)).$$

In Sect. 4, we shall prove the following lemma. □

Lemma 3.3 *G is amenable if and only if* $\limsup_{n\to+\infty} (\widetilde{P}_N^n((x,e),(y,e)))^{1/n} = 1$.

This lemma implies that, provided G is amenable, $\delta_N = \delta_N^0$, $N \geq 1$. Combining this with Lemma 2.1 gives Theorem 1.

4 An Auxiliary Estimate

In this section, we establish an estimate needed to complete the proof of Lemma 2.3 in Sect. 4.

Write $\mathrm{Fix}_n = \{x \in \Sigma : \sigma^n x = x\}$. If $x = (x_0, x_1, \ldots, x_{n-1}, x_0, \ldots) \in \mathrm{Fix}_n$, write $x^{-1} = (x_{n-1}^{-1}, \ldots, x_1^{-1}, x_0^{-1}, x_{n-1}^{-1}, \ldots) \in \mathrm{Fix}_n$.

Lemma 4.1 *For each* $N \geq 1$, $r_N^n(x) = r_N^n(x^{-1})$ *whenever* $x \in \mathrm{Fix}_n$, $n \geq 1$.

Proof For $n \geq N$,

$$\begin{aligned} r_N^n(x) &= r(x_0, x_1, \ldots, x_{N-1}) + r(x_1, x_2, \ldots, x_N) + \cdots + r(x_{n-1}, x_0, \ldots, x_{N-2}) \\ &= d(o, x_0 x_1 \cdots x_{N-1} o) - d(o, x_1 \cdots x_{N-1} o) \\ &\quad + d(o, x_1 x_2 \cdots x_N o) - d(o, x_2 \cdots x_N o) \\ &\quad + \cdots + d(o, x_{n-1} x_0 \cdots x_{N-2} o) - d(o, x_0 \cdots x_{N-2} o). \end{aligned}$$

On the other hand,

$$\begin{aligned}
r_N^n(x^{-1}) &= r(x_{n-1}^{-1}, x_{n-2}^{-1}, \dots, x_{n-N}^{-1}) + r(x_{n-2}^{-1}, x_{n-3}^{-1}, \dots, x_{n-N-1}^{-1}) \\
&\quad + \cdots + r(x_0^{-1}, x_{n-1}^{-1}, \dots, x_{n-N+1}^{-1}) \\
&= d(o, x_{n-1}^{-1} x_{n-2}^{-1} \cdots x_{n-N}^{-1} o) - d(o, x_{n-2}^{-1} \cdots x_{n-N}^{-1} o) \\
&\quad + d(o, x_{n-2}^{-1} x_{n-3}^{-1} \cdots x_{n-N-1}^{-1} o) - d(o, x_{n-3}^{-1} \cdots x_{n-N-1}^{-1} o) \\
&\quad + \cdots + d(o, x_0^{-1} x_{n-1}^{-1} \cdots x_{n-N+1}^{-1} o) - d(o, x_{n-1}^{-1} \cdots x_{n-N+1}^{-1} o) \\
&= d(o, x_{n-N} \cdots x_{n-2} x_{n-1} o) - d(o, x_{n-N} \cdots x_{n-2} o) \\
&\quad + d(o, x_{n-N-1} \cdots x_{n-3} x_{n-2} o) - d(o, x_{n-N-1} \cdots x_{n-3} o) \\
&\quad + \cdots + d(o, x_{n-N+1} \cdots x_{n-1} x_0 o) - d(o, x_{n-N+1} \cdots x_{n-1} o) = r_N^n(x).
\end{aligned}$$

If $n < N$, the calculations become easier.

Consider the restriction $r_N \colon \Sigma_N \to \mathbb{R}$. We can define another function $\check{r}_N : \Sigma_N \to \mathbb{R}$ by $\check{r}_N(x_0, \dots, x_{N-1}) = r_N(x_{N-1}^{-1}, \dots, x_0^{-1})$. Applying Livsic's theorem for finite directed graphs to the above result, we may deduce: □

Corollary 4.1.1 *There exists $u : \Sigma_{N-1} \to \mathbb{R}$ such that*

$$r_N(x_0, x_1, \dots, x_{N-1}) = r_N(x_{N-1}^{-1}, \dots, x_1^{-1}, x_0^{-1}) + u(x_1, \dots, x_{N-1}) - u(x_0, \dots, x_{N-2}).$$

Lemma 4.2 *There exists a constant $C_0 > 0$ such that, for all $(x, g), (y, h) \in \Sigma_N \times G$ and $n \geq 1$,*

$$P_N^n((x, g), (y, h)) \leq C_0 P_N^n((\check{y}, h^{-1}), (\check{x}, g^{-1})),$$

where, if $x = (x_0, x_1, \dots, x_{N-1})$ and $y = (y_0, y_1, \dots, y_{N-1})$, we use the notation $\check{x} = (x_{N-1}^{-1}, \dots, x_1^{-1}, x_0^{-1})$ and $\check{y} = (y_{N-1}^{-1}, \dots, y_1^{-1}, y_0^{-1})$.

We may take

$$C_0 = \exp\left(2\delta_N \sup\{|u(x)| : x \in \Sigma_{N-1}\}\right).$$

5 Random Walks on Graphs

In order to prove Lemma 3.3, we shall adapt work of Ortner and Woess on non-backtracking random walks on graphs contained in ref. [12].

For each $N \geq 1$, we define an (undirected) graph $\mathcal{S}_N$ with vertex set $\Sigma_N \times G$. Two vertices (x, g) and (y, h) will be joined by an edge if and only if either $\widetilde{P}_N((x, g), (y, h)) > 0$ or $\widetilde{P}_N((y, h), (x, g)) > 0$. We note that $\mathcal{S}_N$ is connected and that each vertex has degree $2k$.

We may think of $\widetilde{P}_N$ as defining a Markov process on $\mathcal{S}_N$. As part of the proof of Lemma 3.3, we will show that $\widetilde{P}_N$ has the following three properties [12]:

(1) $\widetilde{P}_N$ has bounded range, i.e. there exists $R > 0$ such that if $\widetilde{P}_N((x, g), (y, h)) > 0$ then (x, g) and (y, h) are at distance $\leq R$ in $\mathcal{S}_N$.
(2) $\widetilde{P}_N$ has a bounded invariant measure; i.e. there exists a function $\nu : \Sigma_N \times G \to \mathbb{R}^+$, bounded above and below away from zero, such that, for all $(y, h) \in \Sigma_N \times G$,

$$\sum_{(x,g) \in \Sigma_N \times G} \widetilde{P}_N((x, g), (y, h))\, \nu((x, g)) = \nu((y, h)).$$

(3) $\widetilde{P}_N$ is *uniformly irreducible*, i.e. there exist constants $K > 0$, $\epsilon > 0$ such that, for any pair of neighbouring vertices $(x, g), (y, h)$ in $\mathcal{S}_N$, one can find $k \leq K$ such that $\widetilde{P}_N^k((x, g), (y, h)) \geq \epsilon$.

We note that (1) holds immediately with $R = 1$.

To show (2), let recall that there is a strictly positive row vector $v_N = (v_N(x))_{x \in \Sigma_N}$ such that $v_N P_N = v_N$. Define v by $v((x, g)) = v_N(x)$. Clearly this is bounded above and below away from zero. A simple calculation shows it has the desired $\widetilde{P}_N$-invariance.

Finally, we show that $\widetilde{P}_N$ is uniformly irreducible.

Lemma 5.1 *$\widetilde{P}_N$ is uniformly irreducible.*

Proof Fix a number K (to be determined later). Let $\epsilon_0 < 1$ denote the smallest positive entry of $\widetilde{P}_N$ and let $\epsilon = \epsilon_0^K$; then, for every $k \leq K$, each positive entry of $\widetilde{P}_N^k$ is greater than or equal to ϵ. Let (x, g) and (y, h) be neighbouring vertices in $\mathcal{S}_N$. Without lose of generality, $\widetilde{P}_N((x, g), (y, h)) > \epsilon$ and $\widetilde{P}_N((y, h), (x, g)) = 0$. To complete the proof we need to find a positive probability path of length at most K from (y, h) to (x, g).

Observe that we can identify $\Sigma_N \times G$ with the set of non-backtracking paths of length N in $\widetilde{\mathcal{G}}$ and a positive probability path of length k in $\mathcal{S}_N$ corresponds to a non-backtracking path of length $N + k$ in $\widetilde{\mathcal{G}}$. We therefore need to show that, for any two non-backtracking paths (given by sequences of vertices) $(u_0, u_1, \ldots, u_N)$ and $(v_0, v_1, \ldots, v_N)$ in $\widetilde{\mathcal{G}}$, there exists $k \leq K$ such that there is a non-backtracking path of length k joining them to give a non-backtracking path from u_0 to v_N. It follows from Corollary 2.1.1 that there is a non-backtracking path $(u_N, w_1, \ldots, w_{\kappa-1}, v_0)$, with $\kappa \leq l(u_N, v_0)$, joining u_N to v_0. However, it is possible then when this is inserted between the other two paths, backtracking occurs. To avoid this we shall use the "small cycles are dense" property of $\widetilde{\mathcal{G}}$. (The following part of the proof is adapted from the proof of Lemma 4.7 in ref. [12].)

First, we consider the beginning of the inserted path. If $w_1 \neq u_{N-1}$ there is nothing to do, so suppose that $w_1 = u_{N-1}$. Choose a neighbour z_1 of u_N which is not equal to u_{N-1}. By Lemma 4.3 of [12], (u_N, z_1) may be extended into non-backtracking paths which reach infinitely many vertices. Since $B(u_{N-1}, R)$ is finite, we may choose one of these paths, $(u_N, z_1, \ldots, z_r)$, so that $z_r \notin B(u_{N-1}, R)$ but $z_i \in B(u_{N-1}, R)$, $i = 1, \ldots, r - 1$ (with $r \leq L(R) + 1$. By the "small cycles are dense" property, there is a cycle $(c_0, c_1, \ldots, c_{p-1}, c_0)$ in $B(z_r, R)$ (with $p \leq L(R)$). Either

(a) $z_r = c_i$ for some $i = 0, 1, \ldots, p - 1$, or,
(b) by the definition of $B(z_r, R)$, there is a non-backtracking path $(z_r, a_1, \ldots, a_{q-1}, c_0)$ $(a_1 \neq z_{r-1})$ joining z_r to c_0 (with $q \leq R$).

In case (a), we insert

$$(u_N, z_1, \ldots, z_r, c_{i+1}, \ldots, c_{p-1}, c_0, \ldots, c_{i-1}, z_r, z_{r-1}, \ldots, z_1, u_N)$$

and in case (b), we insert

$$(u_N, z_1, \ldots, z_r, a_1, \ldots, a_{q-1}, c_0, c_1, \ldots, c_{p-1}, c_0, a_{q-1}, \ldots, a_1, z_r, z_{r-1}, \ldots, z_1, u_N)$$

between $(u_0, u_1, \ldots, u_N)$ and $(u_N, w_1, \ldots, w_{\kappa-1}, v_0)$.

Now consider the end of the path $(u_N, w_1, \ldots, w_{\kappa-1}, v_0)$. If $w_{k-1} \neq v_1$ there is nothing to do. On the other hand, if $w_{k-1} = v_1$ then we carry out a similar construction to that in the paragraph above.

In this way, we have obtained a non-backtracking path starting with $(u_0, u_1, \ldots, u_N)$ and ending with $(v_0, v_1, \ldots, v_N)$ with u_N and v_0 being joined in at most $l(u_N, v_0) + 4(L(R)+1) + 4R + 4L(R)$ steps.

To complete the proof, we need to show that this number may be bounded independently of our initial choice of (x, g) and (y, h) (which determine u_N and v_0). First, we note that there are only finitely many x and y in Σ_N. Second, we observe that, for any $a \in G$, $\widetilde{P}_N((x, ag), (y, ah)) = \widetilde{P}_N((x, g), (y, h))$, so, without loss of generality, we may suppose that $g = e$. Since (y, h) is a neighbour of (x, g) in $\mathcal{S}_N$, this forces h to be one of the finitely many elements $\psi(a_1^{\pm 1}), \ldots, \psi(a_k^{\pm 1})$. Therefore, we may choose K to be the maximum of $l(u_N, v_0) + 8L(R) + 4R + 4$, taken over this finite number of choices.

Since $\widetilde{P}_N$ has an invariant measure ν, it acts on the Hilbert space $l^2(\mathcal{S}_N, \nu)$. Let $\rho_2(\widetilde{P}_N)$ denote the spectral radius. Also, since $\widetilde{P}_N$ is irreducible,

$$\rho(\widetilde{P}_N) = \limsup_{n \to +\infty} (\widetilde{P}_N^n((x, g), (y, h)))^{1/n}$$

is independent of (x, g) and (y, h) and $\rho(\widetilde{P}_N) \leq \rho_2(\widetilde{P}_N)$.

To complete the proof of Lemma 3.3 (and hence of Theorem 1) we use the following results from [12]. (See page 112 of [18] for the definition of an amenable graph.) □

Proposition 5.1 [12,Theorem 4.6] *If $\mathcal{S}_N$ is connected with bounded vertex degrees and $\widetilde{P}_N$ satisfies* (1)–(3) *then $\rho_2(\widetilde{P}_N) = 1$ if and only if $\mathcal{S}_N$ is amenable.*

We have already seen that the hypotheses used in Proposition 5.1 are satisfied. The next result relates $\rho_2(\widetilde{P}_N)$ and $\rho(\widetilde{P}_N)$.

Proposition 5.2 $\rho(\widetilde{P}_N) = \rho_2(\widetilde{P}_N)$.

Proof The proof is a simple modification of the proof of Proposition 2.6 in ref. [12]. The hypothesis there is that one has a graph for which "small cycles are dense"; since this holds for $\widetilde{\mathcal{G}}$, it also holds for $\mathcal{S}_N$. There are two differences from the proof in ref. [12]:

(1) we consider a matrix $\overline{P}_N = \frac{1}{2}(I + \widetilde{P}_N)$, where I is the identity matrix, and observe that $\overline{P}_N$ preserves ν (rather than the counting measure as in ref. [12]);
(2) we use Lemma 4.2: there exists a constant $C_0 > 0$ such that, for all $(x, g), (y, h) \in \Sigma_N \times G$ and $n \geq 1$,

$$P_N^n((x, g), (y, h)) \leq C_0 P_N^n((\check{y}, h^{-1}), (\check{x}, g^{-1})),$$

where, if $x = (x_0, x_1, \ldots, x_{N-1})$ and $y = (y_0, y_1, \ldots, y_{N-1})$, we use the notation $\check{x} = (x_{N-1}^{-1}, \ldots, x_1^{-1}, x_0^{-1})$ and $\check{y} = (y_{N-1}^{-1}, \ldots, y_1^{-1}, y_0^{-1})$. (In ref. [12], the inequality is an equality with $C_0 = 1$.)

Neither of these affect the proof. □

Together, these two results show that $\rho(\widetilde{P}_N) = 1$ if and only if $\mathcal{S}_N$ is amenable. To finish things off, we show that the latter condition is equivalent to the amenability of G.

Recall that a map $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called a *quasi-isometry* if there exist $A \geq 1, B, C \geq 0$ such that,

(1) for all $x, x' \in X$, $A^{-1} d_X(x, x') - B \leq d_Y(f(x), f(x')) \leq A d_X(x, x') + B$; and
(2) for every $y \in Y$, there exists $x \in X$ such that $d_Y(y, f(x)) \leq C$.

Proposition 5.3 *$\mathcal{S}_N$ is amenable if and only if G is amenable.*

Proof We identify G with its Cayley graph $\mathcal{C}(\mathcal{G})$; G is an amenable group if and only if $\mathcal{C}(\mathcal{G})$ is an amenable graph. Define a map $f_N : \mathcal{S}_N \to \mathcal{C}(\mathcal{G})$ on the vertices by $f_N(x,g) = g$ and extend it to the edges by $f_N((x,g),(y,h)) = (g,h)$. This map is clearly a quasi-isometry. Since, for graphs with bounded vertex degree, amenability is an invariant of quasi-isometry [18,Theorem 4.7], the result is proved. □

6 Kleinian Groups

In this section, we shall discuss the relevance of our results for Kleinian groups acting on the hyperbolic space $\mathbb{H}^{n+1}$ and, in particular, for finitely generated Fuchsian results. (These results are subsumed by those in ref. [16].)

We begin be describing the results of Brooks on amenability and the spectrum of the Laplacian. Let N be a complete Riemannian manifold and let Δ_N denote the Laplace–Beltrami operator acting on $L^2(N)$. Then $-\Delta_N$ is a positive self-adjoint operator on $L^2(N)$. If $\sigma(-\Delta_N)$ denotes the spectrum of $-\Delta_N$ then $\sigma(-\Delta_N) \subset [0,+\infty)$. Let $\lambda_0(N)$ denote the bottom of the spectrum, i.e.

$$\lambda_0(N) = \inf \sigma(-\Delta_N).$$

If $\widetilde{N}$ is a Riemannian cover of N then $\lambda_0(\widetilde{N}) \geq \lambda_0(N)$.

Theorem (Brooks [3]) *Suppose that $\widetilde{N}$ is a Riemannian cover of N. If $\pi_1(N)/\pi_1(\widetilde{N})$ is amenable then $\lambda_0(\widetilde{N}) = \lambda_0(N)$.*

Remark Subject to certain conditions, in particular, if N is compact, Brooks also showed the converse.

Let Γ be a Kleinian group, i.e. a discrete group of isometries of the real $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$. We say that Γ is geometrically finite if it is possible to choose a fundamental domain which is a finite sided polyhedron. We shall suppose that Γ acts freely so that $\mathbb{H}^{n+1}/\Gamma$ is a smooth manifold and that Γ is non-elementary. Then $0 < \delta(\Gamma) \leq n$, with equality if and only if $\mathbb{H}^{n+1}/\Gamma$ has finite volume. As before, Γ_0 will be a normal subgroup of Γ.

In this setting, $\delta(\Gamma)$ is related to $\lambda_0(\mathbb{H}^{n+1}/\Gamma)$ by the formula

$$\lambda_0(\mathbb{H}^{n+1}/\Gamma) = \begin{cases} \delta(\Gamma)(n-\delta(\Gamma)) & \text{if } \delta(\Gamma) > n/2, \\ n^2/4 & \text{if } \delta(\Gamma) \leq n/2 \end{cases}$$

with an identical formula holding for Γ_0. Thus, in the range $\delta(\Gamma) > n/2$, the critical exponent may be read off from the λ_0 and vice versa, while for $\delta(\Gamma) \leq n/2$ the critical exponent is a more subtle quantity.

Using the above relation, Brooks was able to deduce that, if Γ is geometrically finite and $\delta(\Gamma) > n/2$ then amenability of Γ/Γ_0 implies that $\delta(\Gamma_0) = \delta(\Gamma)$ [3]. In the case where Γ is a free group, we can remove the restriction that $\delta(\Gamma) > n/2$. In particular, this gives a complete result for finitely generated Fuchsian groups.

Theorem 2 *Let Γ be a finitely generated Fuchsian group and let Γ_0 be a normal subgroup. If Γ/Γ_0 is amenable then $\delta(\Gamma_0) = \delta(\Gamma)$.*

Proof First, we note that, for Fuchsian groups, if Γ is finitely generated then it is geometrically finite. If $\mathbb{H}^2/\Gamma$ is compact then $\delta(\Gamma) = 1$, so Brooks's result applies. If $\mathbb{H}^2/\Gamma$ is not compact then Γ is a free group. If $\mathbb{H}^2/\Gamma$ has a cusp then $\delta(\Gamma) > 1/2$ [1], so again Brooks's result applies. In the remaining case, the result follows from Theorem 1. □

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