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The Bender method in groups of finite Morley rank

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0. Introduction

The algebraicity conjecture for simple groups of finite Morley rank, also known as the Cherlin–Zilber conjecture, states that simple groups of finite Morley rank are simple algebraic groups over algebraically closed fields. In the last 15 years, the main line of attack on this problem has been Borovik’s program of transferring methods from finite group theory. Borovik’s program has led to considerable progress; however, the conjecture itself remains decidedly open. In Borovik’s program, groups of finite Morley rank are divided into four types, odd, even, mixed, and degenerate, according to the structure of their Sylow 2-subgroup. For *even* and *mixed type* the algebraicity conjecture has been proven. The present paper is part of the program to bound the *Prüfer rank* of minimal simple groups of finite Morley rank and *odd type*.

In [1], Cherlin and Jaligot achieved a bound of Prüfer rank two for *tame* minimal simple groups. Here a group of finite Morley rank is said to be tame if it does not involve a field of finite Morley rank with a proper infinite definable subgroup of its multiplicative group. Cherlin, Jaligot, and the present author will bound the Prüfer rank at two in [2].

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Tameness is used in two important ways in [1]. The final number theoretic contradiction of [1] uses tameness in an essential way, and [2] will completely replace this argument. However, the very first use of tameness in [1] produces the following fact, which shows that intersections of Borel subgroups are Abelian.

Jaligot’s lemma. (See [1, Lemma 3.11].) *Let G be a tame minimal connected simple group of finite Morley rank. Let B_1 and B_2 be two distinct Borel subgroups of G with $O(B_1) \neq 1$ and $O(B_2) \neq 1$. Then $F(B_1) \cap F(B_2) = 1$.*

The present paper examines the worst violations of Jaligot’s lemma in the *nontame* context, i.e., those involving non-Abelian intersections of Borel subgroups. We fail to exclude such non-Abelian intersections outright, but we gain much information about the specific local configuration responsible for non-Abelian intersections.

In the context of minimal simple groups, the present paper provides an analog of Bender’s uniqueness theorem [3, Theorem 28.2] (see also [4, Chapter 5] and [5, Section 9]), a result underlying the Bender method [3, Section 28] of analyzing the maximal subgroups containing the centralizer of an involution. Both the Bender uniqueness theorem and the present paper provide information about the normalizers of various subgroups of the intersection of two distinct maximal subgroups. However, our situation will be simplified by two facts: torsion behaves extremely well (see Section 2), and our “torsion-free primes,” so-called reduced ranks, are naturally ordered by their degree of unipotence (see Fact 1.16).

In [2], much of the information about this non-Abelian configuration, plus analysis of the relevant Abelian intersections, is used to prove the following.

Theorem. *Let G be a minimal connected simple group of finite Morley rank and odd type with a strongly embedded subgroup. Then G has Prüfer rank one.*

One proves the bound on Prüfer rank by showing that simple groups of finite Morley rank and Prüfer rank at least three have strongly embedded subgroups [6].

The bulk of this paper consists of the analysis of non-Abelian maximal intersections of Borel subgroups in a minimal connected simple group of finite Morley rank (see Section 3). A priori, the analysis of these maximal intersections yields only a limited description of nonmaximal intersections.

Proposition 4.1. *Let G be a minimal connected simple group of finite Morley rank. Let B_1, B_2 be two distinct Borel subgroups of G , and H a connected subgroup of the intersection $B_1 \cap B_2$. Then the following hold.*

- (1) H' is rank homogeneous for $r' := \bar{r}_0(H')$.
- (2) Every connected nilpotent subgroup of H is Abelian.
- (3) $F_{r'}(H) = U_{0,r'}(H)$ is a Sylow $U_{0,r'}$ -subgroup of H .
(etc.)

In high Prüfer rank, the experience of [2] suggests that Proposition 4.1 itself is insufficient, but that the results of Section 3 which describe the configuration arising from

maximal non-Abelian intersections *are* sufficient. In practice, these results can be used because of the following equivalence between different characterizations of non-Abelian maximal intersections.

Theorem 4.3. *Let G be a minimal connected simple group of finite Morley rank, and let B_1, B_2 be two distinct Borel subgroups of G . Suppose that $H := (B_1 \cap B_2)^\circ$ is non-Abelian. Then the following are equivalent.*

- (1) B_1 and B_2 are the only Borel subgroups of G containing H .
- (2) If B_3 and B_4 are distinct Borel subgroups of G containing H , then $(B_3 \cap B_4)^\circ = H$.
- (3) If $B_3 \neq B_1$ is a Borel subgroup containing H , then $(B_3 \cap B_4)^\circ = H$.
- (4) $C_G^\circ(H')$ is contained in B_1 or B_2 .
- (5) B_1 and B_2 are not conjugate under $C_G^\circ(H')$.
- (6) $\bar{r}_0(B_1) \neq \bar{r}_0(B_2)$.

The results presented here do not assume the presence of 2-torsion. As such, we expect these results to play a significant role in the study of both odd and degenerate type groups.

The paper begins by recalling the necessary background in Section 1, including the definition of 0-unipotence. Section 2 proves Jaligot’s lemma for Borel subgroups with p -unipotent radicals, and thus eliminates most concerns with connected torsion. Section 3 carries out the core of our analysis of a maximal non-Abelian intersection of Borel subgroups. Section 4 proves the equivalence of the various notions of non-Abelian maximal intersection (see Theorem 4.3), and summarizes the results of Section 3 in that context (see Theorem 4.5). Section 5 discusses possible some future directions related to Carter subgroups.

1. Background

1.1. Unipotent groups

While there is no intrinsic definition of unipotence in a group of finite Morley rank, there are various analogs of the “unipotent radical”: the Fitting subgroup, the p -unipotent operators U_p , for p prime, and their “characteristic zero” analogs $U_{0,r}$ from [6,7]. We recall their definitions.

Definition 1.1. The *Fitting subgroup* $F(G)$ of a group G of finite Morley rank is the subgroup generated by all its nilpotent normal subgroups.

The Fitting subgroup is itself nilpotent and definable [8–10, Theorem 7.3], and serves as a notion of unipotence in some contexts. However, since the Fitting subgroup of a solvable group may not be contained in the Fitting subgroups of a solvable group containing it, it is not a robust notion.

Fact 1.2. (See [10, Corollary 9.9, Theorem 9.21].) *Let G be a connected solvable group of finite Morley rank. Then $G' \leq F^\circ(G)$ and $G/F^\circ(G)$ is a divisible Abelian group.*

Fact 1.3. (See [11, Lemma 4.20]; compare [10, Example 3, p. 148].) Let G be a connected solvable group of finite Morley rank, and let T be a divisible torsion subgroup of G . Then $T \cap F(G)$ is central in G .

Definition 1.4. A subgroup of a connected solvable group H of finite Morley rank is said to be p -unipotent if it is a definable connected p -group of bounded exponent.

Fact 1.5. (See [1, Corollary 2.16]; [12, Fact 2.36].) Let H be a connected solvable group of finite Morley rank. Then there is a unique maximal p -unipotent subgroup of H , denoted $U_p(H)$, and $U_p(H) \leq F^\circ(H)$.

The p -unipotent radical U_p will automatically behave well under intersections with other solvable groups.

Fact 1.6. (See [10, Theorem 9.29 and Section 6.4].) Let G be a connected solvable group of finite Morley rank. Then a Sylow p -subgroup P of G is connected, and $P = U_p(G) * T$ for a divisible Abelian p -group T .

The present paper relies on the theory of “characteristic zero” unipotence introduced in [7]. We now turn our attention to this (long) definition, as well as some facts from [6,7,13].

We say that a connected Abelian group of finite Morley rank is *indecomposable* if it has a unique maximal proper definable connected subgroup, denoted $J(A)$.

Fact 1.7. (See [7, Lemma 2.4].) Every connected Abelian group of finite Morley rank can be written as a finite sum of definable indecomposable Abelian subgroups.

Definition 1.8. We define the *reduced rank* $\bar{r}(A)$ of a definable Abelian group A to be the Morley rank of the quotient $A/J(A)$, i.e., $\bar{r}(A) = \text{rk}(A/J(A))$. For a group G of finite Morley rank, and any integer r , we define

$$U_{0,r}(G) = \{A \leq G \mid A \text{ is a definable indecomposable group, } \bar{r}(A) = r, \text{ and } A/J(A) \text{ is torsion-free}\}.$$

We say G is a $U_{0,r}$ -group (alternatively $(0, r)$ -unipotent) if $U_{0,r}(G) = G$. The 0 -unipotent radical $U_0(G)$ is the nontrivial $U_{0,r}(G)$ with r maximal. We also set $\bar{r}_0(G) = \max\{r \mid U_{0,r}(G) \neq 1\}$. So $U_0(G) = U_{0,\bar{r}_0(G)}(G)$.

As a notational convention, we define $F_r(G) = U_{0,r}(F(G))$, and use $F_{\text{tor}}(G)$ to denote the definable closure of the torsion subgroup of $F(G)$.

We view the reduced rank parameter r as a *scale of unipotence*, with larger values being more unipotent. By the following fact, the “most unipotent” groups, in this scale, are nilpotent.

Fact 1.9. (See [6, Theorem 2.21]; [7, Theorem 2.16].) *Let H be a connected solvable group of finite Morley rank. Then $U_0(H) \leq F(H)$.*

Fact 1.10. (See [13, Lemma 2.5].) *Let H be a connected solvable group of finite Morley rank. Let r be the maximal reduced rank such that $U_{0,r}(H) \not\leq Z_n(H)$ for all n . Then $U_{0,r}(H) \leq F(H)$.*

The two preceding facts prove nilpotence of $U_{0,r}(H)$ for the largest values of r . However, H may intersect a solvable group K with $\bar{r}_0(K) > \bar{r}_0(H)$. Many other facts about $U_{0,r}$ -groups hold for arbitrary values of r .

Fact 1.11. (See [6, Lemma 2.12]; [7, Lemma 2.11].) *Let $f : G \rightarrow H$ be a definable homomorphism between two groups of finite Morley rank. Then the following hold.*

- (1) (Push-forward) $f(U_{0,r}(G)) \leq U_{0,r}(H)$ is a $U_{0,r}$ -group.
- (2) (Pull-back) If $U_{0,r}(H) \leq f(G)$ then $f(U_{0,r}(G)) = U_{0,r}(H)$.

In particular, an extension of a $U_{0,r}$ -group by a $U_{0,r}$ -group is a $U_{0,r}$ -group.

Fact 1.12. (See [6, Lemma 2.26]; compare [13, Lemma 2.3].) *Let G be a nilpotent group of finite Morley rank satisfying $U_{0,r}(G) \neq 1$ or $U_p(G) \neq 1$. Then $U_{0,r}(Z(G)) \neq 1$ or $U_p(Z(G)) \neq 1$, respectively.*

We have a 0-unipotent analog of the connected normalizer condition of [10, Lemma 6.3].

Fact 1.13. (See [6, Lemma 2.28]; [13, Lemma 2.4].) *Let G be a nilpotent $U_{0,r}$ -group of finite Morley rank. If $H < G$ is a definable subgroup then $U_{0,r}(N_G(H)/H) > 1$.*

Our next result generalizes the fact that a finite nilpotent group is the product of its Sylow p -subgroups.

Fact 1.14. (See [6, Theorem 2.31]; [13, Corollary 3.6].) *Let G be a nilpotent group of finite Morley rank. Then $G = D * B$ is a central product of definable characteristic subgroups $D, B \leq G$ where D is divisible and B has bounded exponent (which is connected iff G is connected). Let T be the torsion part of D . Then we have decompositions of D and B as follows:*

$$D = d(T) * U_{0,1}(G) * U_{0,2}(G) * \dots,$$

$$B = U_2(G) \oplus U_3(G) \oplus U_5(G) \oplus \dots$$

Here $d(T)$ denotes the definable closure of T , which is defined to be the intersection of all definable subgroups containing T .

Fact 1.15. (See [6, Lemma 2.32]; [13, Corollary 3.7].) *Let G be a solvable group of finite Morley rank, let $S \subseteq G$ be any subset, and let H be a nilpotent $U_{0,r}$ -group which is normal in G . Then $[H, S] \leq H$ is a $U_{0,r}$ -group.*

Olivier Frécon has improved this result in [14].

Our next fact says that “more unipotent” groups do not act on “less unipotent” groups.

Fact 1.16. (See [13, Lemma 4.4]; see also [15, Corollary 3.8].) *Let $G = HT$ be a group of finite Morley rank with $H \triangleleft G$ a nilpotent $U_{0,r}$ -group and T a nilpotent $U_{0,s}$ -group for some $s \geq r$. Then G is nilpotent.*

In [16], Wagner showed that fields of finite Morley rank and characteristic $p \neq 0$ have no torsion free sections of their multiplicative groups [16]. The Zilber field theorem [10, Theorem 9.1] allows us to rephrase Wagner’s result as follows.

Fact 1.17. (See [13, Lemma 4.3].) *Let G be a connected solvable group of finite Morley rank. Suppose that S is a nilpotent $U_{0,r}$ -subgroup of G , and that $G = U_p(G)S$ for some p prime. Then G is nilpotent, and $[U_p(G), S] = 1$.*

1.2. *Toral groups*

Definition 1.18. A definable subgroup C of a group G of finite Morley rank which is nilpotent and almost self-normalizing in G is called a *Carter subgroup* of G .

The following result is a summary, in order, of [17, Proposition 3.2, Corollary 4.8], [18, 19, Theorem 5.5.12], and [17, Corollary 7.15].

Fact 1.19. *Let H be a connected solvable group of finite Morley rank. Then the following hold.*

- (1) H has a Carter subgroup.
- (2) The Carter subgroups of H are the definable nilpotent subgroup of H with $N_H^\circ(Q) = Q$. In particular, Carter subgroups of H are connected.
- (3) The Carter subgroups of H are conjugate in H .
- (4) The Carter subgroups of H cover H/H' .
- (5) Let R be a Sylow p -subgroup of H . Then $N_H(R)$ contains a Carter subgroup of H .

Definition 1.20. A Sylow $U_{0,r}$ -subgroup of a group G of finite Morley rank is a maximal definable nilpotent $U_{0,r}$ -subgroup of G .

Sylow $U_{0,r}$ -subgroups are an analog of Carter subgroups in the following sense.

Lemma 1.21. (See [6, Lemma 4.18]; [13, Lemma 5.2].) *Let H be a group of finite Morley rank. Then the Sylow $U_{0,r}$ -subgroups of H are exactly those nilpotent $U_{0,r}$ -subgroups S such that $U_{0,r}(N_G(S)) = S$.*

Fact 1.22. (See [6, Lemma 4.19]; [13, Theorem 5.7].) *Let H be a connected solvable group of finite Morley rank and let Q be a Carter subgroup of H . Then $U_{0,r}(H')U_{0,r}(Q)$ is a Sylow $U_{0,r}$ -subgroup of H , and every Sylow $U_{0,r}$ -subgroup has this form for some Carter subgroup Q .*

Fact 1.23. (See [6, Theorem 4.16]; [13, Theorem 5.5].) *Let H be a connected solvable group of finite Morley rank. Then the Sylow $U_{0,r}$ -subgroups of H are H -conjugate.*

2. p -Unipotence

In this section, we show that intersections of the Fitting subgroups of distinct Borel subgroups are torsion free, and thus eliminate many concerns about torsion from the main analysis to follow. The arguments of this section are based directly on the original proof of Jaligot’s lemma [1, Lemma 3.11] (see Introduction).

Lemma 2.1. *Let G be a minimal connected simple group of finite Morley rank. Let B_1, B_2 be two distinct Borel subgroups satisfying $U_{p_i}(B_i) \neq 1$ for some prime p_i ($i = 1, 2$). Then $F(B_1) \cap F(B_2) = 1$.*

Proof. We first show that $U_p(B_1 \cap B_2) = 1$ for all p , prime. Suppose toward a contradiction that $X := U_p(B_1 \cap B_2) \neq 1$. We may assume that $\text{rk}(X)$ is maximal among all choices of B_1 and B_2 . Let B be a Borel subgroup of G containing $N_G^\circ(X)$.

We now show that $B_i = B$ for $i = 1, 2$. If $X = U_p(B_i)$, then $B \geq N_G^\circ(U_p(B_i)) = B_i$, and $B = B_i$. So we consider the case where $X < U_p(B_i)$. By Fact 1.5, the group $U_p(B_1)$ is nilpotent. By the normalizer condition [10, Lemma 6.3],

$$X < U_p(N_{U_p(B_i)}^\circ(X)) \leq U_p(B).$$

Since B is a Borel subgroup containing $N_{U_p(B_i)}^\circ(X)$, $B_i = B$ by the maximality of $\text{rk}(X)$. Thus $B_1 = B = B_2$, a contradiction. So $U_p(B_1 \cap B_2) = 1$.

We now prove the lemma. Suppose toward a contradiction that there is an $f \in F(B_1) \cap F(B_2)$ with $f \neq 1$. By Fact 1.12, $Z_i := U_{p_i}(Z(F(B_i)))$ is a nontrivial U_{p_i} -subgroup of $Z(F(B_i))$ for $i = 1, 2$. Let B be a Borel subgroup containing $C_G^\circ(f)$. As $Z_i \leq Z(F(B_i)) \leq C_G^\circ(f)$ for $i = 1, 2$, we find $Z_i \leq U_{p_i}(B_i \cap B)$ for $i = 1, 2$. Thus $B_1 = B = B_2$ by the first part. \square

The above conclusion holds, with a similar proof, if we replace U_p by $U_{0,\bar{r}_0(G)}$; however, one does not know that all Borel subgroups satisfy $\bar{r}_0(B) = \bar{r}_0(G)$, just as one does not know that $U_p(B) \neq 1$. These techniques can be extended to eliminate all torsion from the intersection of the Fitting subgroups.

Corollary 2.2. *Let G be a minimal connected simple group of finite Morley rank. Let B_1, B_2 be two distinct Borel subgroups of G . Then $F(B_1) \cap F(B_2)$ is torsion free.*

Proof. Suppose towards a contradiction that, for some prime p , there is a nontrivial Sylow p -subgroup P of $X := F(B_1) \cap F(B_2)$. We may assume $U_p(B_2) = 1$ by Lemma 2.1. So P is central in B_2 by Fact 1.3, and hence $C_G^\circ(P) = B_2$. If $U_p(B_1) = 1$ too, then P is central in B_1 by Fact 1.3. Thus $U_p(B_1) \neq 1$. Let R be a Sylow p -subgroup of B_1 containing P . By Fact 1.6, $R = U_p(B_1) * T$ for some divisible Abelian p -group T . By Fact 1.12, $Y := U_p(C_{U_p(B_1)}(P)) \neq 1$. But $Y \leq B_2$, a contradiction. \square

3. Maximal intersections

In this section, we analyze intersections of Borel subgroups which are maximal in the sense of the following definition.

Definition 3.1. Let G be a minimal connected simple group of finite Morley rank, and let B_1, B_2 be two distinct Borel subgroups of G . We say that $H := (B_1 \cap B_2)^\circ$ is a *maximal intersection* if H is maximal among all choices of distinct Borel subgroups B_1 and B_2 . In this situation, we refer to B_1, B_2 as a *maximal pair* of Borel subgroups of G .

The analysis of maximal intersections will not always directly produce information about general intersections of Borel subgroups. One hopes to produce results which translate down, in some form, to nonmaximal intersections. For example, our lives would be simple if the intersection H turned out to be Abelian. However, such a simple analog of Jaligot’s lemma eludes us. Instead, we discover a plausible non-Abelian maximal configuration, which this section explores in detail.

In the specific case of [2], the configuration below survives until the end of the analysis, and eventually dies for the same reasons as the case of an Abelian intersection does. Since roughly half of the facts below are important in [2], we view them collectively as a machine for handling non-Abelian intersections.

Throughout this section, we consider a minimal connected simple group G of finite Morley rank, and a maximal pair B_1, B_2 of Borel subgroups of G which violate Jaligot’s lemma.

Hypothesis 3.2. We assume the following.

- (a) G is a minimal connected simple group of finite Morley rank.
- (b) B_1 and B_2 are a maximal pair of Borel subgroups of G .
- (c) The intersection $F(B_1) \cap F(B_2)$ of their Fitting subgroups is nontrivial.

Notation 3.3. We also adopt the following notations.

- (a) $H := (B_1 \cap B_2)^\circ$ denotes the maximal intersection.
- (b) $X := F(B_1) \cap F(B_2)$ denotes the intersection of the Fitting subgroups.
- (c) $r' := \bar{r}_0(X)$ denotes the reduced rank of X .

In addition, we frequently discuss the reduced rank r' piece $Y := F_{r'}(H)$ of the Fitting subgroup of H . A Carter subgroup Q of H will also play a central role.

We observe that $\bar{r}_0(H') = r'$ if $H' \neq 1$ by Theorem 3.10 below, which explains our choice of notation.

3.1. Examples

Before beginning our analysis in Section 3.2, we describe a few “nearly algebraic” configurations which survive. This material will not be used below, but may shed some light on our goals.

We consider two Borel subgroups B_1 and B_2 such that $H := (B_1 \cap B_2)^\circ$ is a maximal intersection. We suppose that $\bar{r}_0(B_1) \geq \bar{r}_0(B_2)$, and that H is non-Abelian. The “light” Borel subgroup B_2 could be the group of upper triangular 3×3 matrices over an algebraically closed field k of characteristic zero, and say of determinant one. The important requirement here is that $F(B_2)$ is non-Abelian.

It will be shown that in fact $\bar{r}_0(B_1) > \bar{r}_0(B_2)$. After breaking the symmetry in this way, it turns out that there are striking differences between the “heavy” B_1 and the “light” B_2 . In particular, the Borel subgroup B_2 can be algebraic over an algebraically closed field k of characteristic zero, while B_1 necessarily interprets a bad field.

One way to proceed is as follows. Let B_2 be the subgroup of upper triangular 3×3 matrices with $a_{11} = a_{33} = 1$,

$$B_2 = \begin{pmatrix} 1 & k_+ & k_+ \\ & T & k_+ \\ & & 1 \end{pmatrix}.$$

So $B_2 = U \rtimes T$ where U is the (unipotent) group of strictly upper triangular matrices, and T is a one-dimensional torus.

Before specifying B_1 , we need to choose $H = (B_1 \cap B_2)^\circ$ so that $H = N_G(H')$ and $F(H)$ is Abelian. A suitable choice of H is the subgroup of B_2 given by $a_{23} = 0$. Then $H = F(H) \rtimes T$ with $F(H) = H' \times Z(U)$,

$$H = \begin{pmatrix} 1 & H' & Z(U) \\ & T & 0 \\ & & 1 \end{pmatrix}.$$

One possibility for B_1 would be the direct product

$$B_1 = [H' \rtimes T] \times [U_0(B_1) \rtimes Z(U)],$$

where $H' \rtimes T$ behaves as it does in B_2 , while $Z(U)$ has become part of a bad field, and $U_0(B_1)$ is an additive group of larger reduced rank.

In the above situation, a Carter subgroup $Q = T \times Z(U)$ of H is a Carter subgroup of both B_1 and B_2 . We will see below that Q will always be a Carter subgroup of B_1 . However, a Carter subgroup of H need not be a Carter subgroup of B_2 , in general.

For example, we may take B_2 to be the Borel subgroup of $SL_3(k)$. Here we must take H' to be a one-dimensional unipotent subgroup of B_2 which is normalized by some one-dimensional torus T , but not by the full torus. As above, $H = N_{B_2}(H')$ and $T \cong (H' \rtimes T) \times Z(U)$. Indeed, $Q = T \times Z(U)$ is a Carter subgroup of H , but $Z(U)$ is no longer part of a Carter subgroup of B_2 , and T is a proper subgroup of a Carter subgroup of B_2 . As a consequence T may not centralize $U_0(B_1)$. Indeed, the Carter subgroup $Q = T \times Z(U)$ of H (and B_1) may be the full multiplicative group of a bad field, while only T acts on H' !

3.2. Rank homogeneity of X

We have two goals in the first stage of the analysis. First, we will show that $\bar{r}_0(B_1) \neq \bar{r}_0(B_2)$, a key fact in the remainder of the analysis. Second, we will show that the subgroup $X := F(B_1) \cap F(B_2)$ is rank homogeneous in the sense of the following definition.

Definition 3.4. A group K of finite Morley rank is said to be *rank homogeneous* if K is torsion free and $U_{0,r}(K) = 1$ for $r \neq \bar{r}_0(K)$.

We observe that $X \triangleleft H$, and that $H' \leq X$. So rank homogeneity of X will imply $r' = \bar{r}_0(H')$ too, if $H' \neq 1$. Clearly $r' > 0$ by Corollary 2.2. In particular, $\bar{r}_0(H) > 0$.

To begin our analysis, we may assume that

$$\bar{r}_0(B_1) \geq \bar{r}_0(B_2). \tag{*}$$

As a first step, the normalizer condition shows that the reduced rank must grow on one side of our intersection.

Lemma 3.5. $\bar{r}_0(H) < \bar{r}_0(B_1)$.

Proof. Suppose toward a contradiction that $\bar{r}_0(H) = \bar{r}_0(B_1)$. Since $\bar{r}_0(H) \leq \bar{r}_0(B_2) \leq \bar{r}_0(B_1)$ by our assumption, all are equal and $U_0(H) \leq U_0(B_1) \cap U_0(B_2)$. Let B_3 be a Borel subgroup of G containing $N_G^\circ(U_0(H))$.

We now show that $B_i = B_3$ for $i = 1, 2$. If $U_0(H) = U_0(B_i)$, then $B_3 \geq N_G^\circ(U_0(H)) = B_i$, and $B_3 = B_i$. So we suppose that $U_0(H) < U_0(B_i)$. By Fact 1.9, $U_0(B_i)$ is nilpotent. By Fact 1.13,

$$U_0(H) < U_{0,\bar{r}_0(H)}(N_{U_0(B_i)}(U_0(H))) \leq B_3.$$

Since $U_0(H) \triangleleft H$, $(B_i \cap B_3)^\circ > H$. By maximality of H , $B_i = B_3$ here too. Thus $B_1 = B_3 = B_2$, a contradiction. \square

Starting with our next lemma, we use the decomposition of nilpotent groups, given in Fact 1.14, to “blow up” the centralizers of various subgroups of H . This is a variation on the normalizer condition based argument used above.

Some of our next lemmas use the decomposition of nilpotent groups (Fact 1.14) instead of the connected normalizer condition of [10, Lemma 6.3]. Such arguments resemble the use of Fact 1.3 in Corollary 2.2.

Lemma 3.6. $\bar{r}_0(H) = \bar{r}_0(B_2)$.

Proof. Suppose toward a contradiction that $\bar{r}_0(H) < \bar{r}_0(B_2)$. Since $U_0(X) \neq 1$ by Corollary 2.2, there is a Borel subgroup B_3 of G containing $N_G^\circ(U_0(X))$. Since $U_0(X) \triangleleft H$, we have $H \leq B_3$. Since $\bar{r}_0(H) < \bar{r}_0(B_i)$ for $i = 1, 2$ by Lemma 3.5 and our hypothesis, we have $U_0(B_i) \leq C_G^\circ(U_0(X)) \leq B_3$ by Fact 1.14. So $(B_i \cap B_3)^\circ > H$ for $i = 1, 2$, and hence $B_1 = B_3 = B_2$, a contradiction. \square

So $\bar{r}_0(B_1) > \bar{r}_0(B_2)$ by Lemmas 3.5 and 3.6. In the event that H is non-Abelian, this rank inequality prevents other Borel subgroups from containing H .

Proposition 3.7. *If H is non-Abelian, then B_1 and B_2 are the only Borel subgroups containing H .*

Proof. Suppose toward a contradiction that there is a Borel subgroup B_3 , distinct from B_1 and B_2 , which contains H . By the maximality of H , $(B_1 \cap B_3)^\circ = H$ and $(B_2 \cap B_3)^\circ = H$. Since $H' \leq F(B_3)$, the maximal pairs B_1, B_3 and B_3, B_2 satisfy Hypothesis 3.2. Since $\bar{r}_0(B_1) > \bar{r}_0(H)$ by Lemma 3.5, $\bar{r}_0(B_3) = \bar{r}_0(H)$ by Lemma 3.6. But, since $\bar{r}_0(B_2) = \bar{r}_0(H)$ by Lemma 3.6, $\bar{r}_0(B_3) > \bar{r}_0(H)$ by Lemma 3.5, a contradiction. \square

The groups B_1 and B_2 are not conjugate, since $\bar{r}_0(B_1) > \bar{r}_0(B_2)$, a point exploited by the following lemma.

Lemma 3.8. $F^\circ(B_i) \not\leq H$ for $i = 1, 2$.

Proof. Since $\bar{r}_0(H) < \bar{r}_0(B_1)$ by Lemma 3.5, we have $F^\circ(B_1) \not\leq H$. Suppose toward a contradiction that $F^\circ(B_2) \leq H$. Then $H \triangleleft B_2$ by Fact 1.2. So $H \leq B_1 \cap B_1^g$ for $g \in B_2 \setminus B_1$, contradicting Lemmas 3.5 and 3.6. \square

Our analysis hinges upon understanding the behavior of the normalizers of various $U_{0,r'}$ -subgroups of H .

Lemma 3.9. *For any nontrivial definable $X_1 \leq X$ with $X_1 \triangleleft H$, we have $N_G^\circ(X_1) \leq B_1$.*

Proof. Let B_3 be a Borel subgroup containing $N_G^\circ(X_1)$. Then $H \leq B_3$. Since $\bar{r}_0(H) < \bar{r}_0(B_1)$ by Lemma 3.5, Fact 1.14 yields $U_0(B_1) \leq C_G^\circ(X_1) \leq B_3$; and thus $(B_1 \cap B_3)^\circ > H$. By the maximality of H , $B_1 = B_3$. So $N^\circ(X_1) \leq B_1$. \square

In particular, the previous two lemmas show that $X \cap Z(F(B_2)) = 1$.

Our results now diverge from the conclusions of Bender’s uniqueness theorem [3, Theorem 28.2] in that only one “prime” may occur in X .

Theorem 3.10. *X is rank-homogeneous. In particular, $X = U_0(X)$.*

Proof. By Corollary 2.2, X is torsion free. Suppose toward a contradiction that $U_{0,r}(X) \neq 1$ for some $r < r'$. By Fact 1.14 and Lemma 3.9,

$$F(B_2) \leq C^\circ(U_{0,r'}(X))C^\circ(U_{0,r}(X)) \leq B_1$$

and $F(B_2) \leq H$, contradicting Lemma 3.8. Hence $X = U_0(X)$. \square

3.3. Fitting subgroup of B_2

Our next goal is to understand the Fitting subgroup of B_2 . In particular, we will determine which factors of $F(B_2)$ are contained in H .

Lemma 3.11. $F^\circ(B_2)$ is divisible, and $F_{\text{tor}}^\circ(B_2) \leq Z(H)$.

Proof. Of course, $U_p(B_2) \leq F_{\text{tor}}^\circ(B_2)$ by Fact 1.5. By Theorem 3.10 and Fact 1.14, $F_{\text{tor}}^\circ(B_2) \leq N^\circ(X)$. Since $N^\circ(X) \leq B_1$ by Lemma 3.9, $F_{\text{tor}}^\circ(B_2) \leq H$. So $U_p(B_2) \leq H$ and $F_{\text{tor}}^\circ(B_2) \leq F^\circ(H)$. But $U_p(H) = 1$ for all primes p , by Lemma 2.1. By Fact 1.6, $F^\circ(B_2)$ is divisible. So $F_{\text{tor}}^\circ(B_2)$ is central in H by Fact 1.3. \square

Lemma 3.12. $F_r(B_2) \leq Z(H)$ for $r \neq r'$.

Proof. By Fact 1.14 and Theorem 3.10, $F_r(B_2) \leq C_G^\circ(X)$. Since $C_G^\circ(X) \leq B_1$ by Lemma 3.9, $F_r(B_2) \leq H$. Since $F_r(B_2)$ is normalized by $H \leq B_2$, we have $F_r(B_2) \leq F(H)$. By Fact 1.15 and Theorem 3.10,

$$[H, F_r(B_2)] \leq U_{0,r}(H') \leq U_{0,r}(X) = 1. \quad \square$$

We know that one part of the Fitting subgroup of B_2 is not contained in H .

Lemma 3.13. $F_{r'}(B_2) \not\leq H$ is not Abelian.

Proof. By Lemma 3.8, $F^\circ(B_2) \not\leq H$. Since $F_{\text{tor}}^\circ(B_2) \leq H$ by Lemma 3.11, $F_s(B_2) \not\leq H$ for some s by Fact 1.14. So $F_{r'}(B_2) \not\leq H$ by Lemma 3.12. Since $N_G^\circ(X) \leq B_1$ by Lemma 3.9, $F_{r'}(B_2)$ is not Abelian. \square

Lemmas 3.12 and 3.13 tell us that r' is uniquely determined by B_2 .

Corollary 3.14. $F_r(B_2)$ is non-Abelian iff $r = r'$.

Proof. Immediate from Lemmas 3.12 and 3.13. \square

The first two lemmas of this section give us a measure of control over the large reduced ranks in B_2 .

Lemma 3.15. $U_{0,r}(B_2) \leq F(B_2)$ for $r > r'$.

Proof. Let A be an indecomposable $U_{0,r}$ -subgroup of B_2 . Since $r > r'$, the group $A \cdot F_{r'}(B_2)$ is nilpotent by Fact 1.16. By Fact 1.14, A centralizes $F_{r'}(B_2)$, including X . So $A \leq H$ by Lemma 3.9. By Lemma 3.11, A centralizes $F_{\text{tor}}^\circ(B_2)$. By Lemma 3.12, A centralizes $F_s(B_2)$ for $s \neq r'$. So A centralizes $F^\circ(B_2)$ by Fact 1.14, and hence $A \cdot F^\circ(B_2)$ is nilpotent. By Fact 1.2, $A \cdot F^\circ(B_2) \triangleleft B_2$, and hence $A \leq F(B_2)$. \square

In particular, $F_r(B_2) = U_{0,r}(B_2)$ is the unique Sylow $U_{0,r}$ -subgroup of H . We also observe that, if H contains a Carter subgroup of B_2 , then $r = r'$ is the maximal reduced rank such that $U_{0,r}(H) \not\leq Z(H)$, and hence $U_{0,r'}(B_2) \leq F(B_2)$ too, by Fact 1.10. One may hope to show that $U_{0,r'}(B_2) \leq F(B_2)$ without assuming that H contains a Carter subgroup of B_2 .

3.4. Structure of H

We now turn our attention towards various subgroups of H . First, if we restrict ourselves to $U_{0,r'}(H)$, the argument of Lemma 3.15 applies to the reduced rank r' itself.

Lemma 3.16. $U_{0,r'}(H) \leq F(B_2)$. In particular, $F_{r'}(H) = U_{0,r'}(H)$ is the unique Sylow $U_{0,r'}$ -subgroup of H .

Proof. Let A be an indecomposable $U_{0,r'}$ -subgroup of H . By Fact 1.16, $A \cdot F_{r'}(B_2)$ is nilpotent. For $r \neq r'$, A centralizes $F_r(B_2)$, by Lemma 3.12. By Lemma 3.11, A centralizes $F_{\text{tor}}^\circ(B_2)$ too. So $A \cdot F^\circ(B_2)$ is nilpotent by Fact 1.14. By Fact 1.2, $A \cdot F(B_2) \triangleleft B_2$, and hence $A \leq F(B_2) \cap H \leq F(H)$. So $U_{0,r'}(H) \leq F_{r'}(H)$, and these two subgroups are equal. \square

We adopt the notation $Y := U_{0,r'}(H) (= F_{r'}(H))$. We find that Y opposes the pull of X .

Lemma 3.17. $N_G^\circ(Y) \leq B_2$ and $Y > X$. In addition, $U_{0,r'}(N_{F(B_2)}(Y)) \not\leq H$.

Proof. Let $P := F_{r'}(B_2)$. Then $Y \leq P$ by Lemma 3.16. By Lemma 3.13, $F_{r'}(B_2) \not\leq H$, so $Y < P$. By Fact 1.13, $Y < U_{0,r'}(N_P^\circ(Y))$. Now $Y > X$ by Lemma 3.9, and $N_G^\circ(Y) \leq B_2$ by maximality of H . \square

We can now prove one of our main results.

Theorem 3.18. Every connected definable nilpotent subgroup of H is Abelian.

Proof. Since $U_p(H) = 1$ for any prime p by Lemma 2.1, the Sylow p -subgroups of H are Abelian, by Fact 1.6. For $r \neq r'$, $U_{0,r}(H) = 1$ by Theorem 3.10, so a Sylow $U_{0,r}$ -subgroup of H is Abelian by Fact 1.15 (or via Fact 1.11). If $Y' \neq 1$, then $N_{B_2}^\circ(Y') \leq B_1$ by Lemma 3.9, contradicting Lemma 3.17. So Y is Abelian. By Lemma 3.16, Y is the unique Sylow $U_{0,r'}$ -subgroup of H . Thus all Sylow subgroups of H are Abelian. For any connected definable nilpotent subgroup K of H , K is the central product of its (generalized) Sylow subgroups, by Fact 1.14, so K is Abelian. \square

We observe that H tends to be almost self-normalizing.

Lemma 3.19. *If H is non-Abelian, then $N_G^\circ(H) = H$.*

Proof. By Lemma 3.17, $N_G^\circ(H) \leq N_G^\circ(Y) \leq B_2$. Since H is non-Abelian, $N_G^\circ(H) \leq N_G^\circ(H') \leq B_1$ too, by Lemma 3.9. \square

3.5. Structure of B_1

We now turn our attention toward the Fitting subgroup of B_1 , and a Carter subgroup Q of H .

Lemma 3.20. *$F^\circ(B_1)$ is divisible, and $F_{\text{tor}}^\circ(B_1) \leq Z(H)$.*

Proof. By Fact 1.17, $U_p(B_1)$ is centralized by Y . By Lemma 3.17, $U_p(B_1) \leq H$. But $U_p(H) = 1$ for any prime p , by Lemma 2.1. By Fact 1.6, $F_{\text{tor}}^\circ(B_1)$, and $F^\circ(B_1)$, are divisible. So $F_{\text{tor}}^\circ(B_1)$ is central in B_1 by Fact 1.3. By Lemma 3.17, $F_{\text{tor}}^\circ(B_1) \leq N_G^\circ(Y) \leq B_2$, and $F_{\text{tor}}^\circ(B_1) \leq Z(H)$. \square

We observe that $U_p(B_i) = 1$ for $i = 1, 2$ and p prime, by Lemmas 3.11 and 3.20.

Lemma 3.21. *$F_{r'}(B_1) = X$. So $B_1 = N_G^\circ(X)$.*

Proof. By Fact 1.16, $Y \cdot F_{r'}(B_1)$ is nilpotent. By Lemma 3.17, $N_{Y \cdot F_{r'}(B_1)}^\circ(Y) \leq H$. So we obtain $F_{r'}(B_1) \leq Y$ by Fact 1.13. By Lemma 3.16, $Y \leq F(B_2)$, so $F_{r'}(B_1) \leq X$. \square

Corollary 3.22. *$F_{r'}(B_1)$ is Abelian, and $F^\circ(B_1) \leq C_G^\circ(X)$.*

Proof. By Theorem 3.18 and Lemma 3.21, $F_{r'}(B_1)$ is Abelian, and $F_{r'}(B_1) \leq C_G^\circ(X)$. For $r \neq r'$, $F_r(B_1) \leq C_G^\circ(X)$ by Fact 1.14. Also $F_{\text{tor}}^\circ(B_1) \leq C_G^\circ(X)$ by Fact 1.14. So $F(B_1) \leq C_G^\circ(X)$ by Fact 1.14. \square

We now examine a Carter subgroup Q of H .

Lemma 3.23. *$U_{0,r'}(Q) = U_{0,r'}(Z(H))$, and this group is nontrivial.*

Proof. By Lemma 3.17, $U_{0,r'}(H/H') \neq 1$. So $U_{0,r'}(Q) \neq 1$ by Facts 1.19(4) and 1.11. By Theorem 3.18, Q and Y are Abelian. By Lemma 3.16, $U_{0,r'}(Q) \leq Y$. So $U_{0,r'}(Q)$ centralizes both Q and $H' \leq Y$. Thus $U_{0,r'}(Q) \leq Z(H)$ by Fact 1.19(4). Conversely, $Z^\circ(H) \leq Q$. \square

Theorem 3.24. *$N_G^\circ(U_{0,r'}(Q)) \leq B_2$. So $N_G^\circ(Q) \leq B_2$, and Q is a Carter subgroup of B_1 .*

Proof. We first show that $N_G^\circ(U_{0,r'}(Q)) \leq B_2$. By Lemma 3.17, $N_G^\circ(Y) \leq B_2$. So we may assume that $U_{0,r'}(Q) < Y$, and hence H is non-Abelian by Lemma 3.23. So B_1 and B_2

are the only Borel subgroups of G which contain H , by Proposition 3.7. By Lemma 3.23, $H \leq N_G^{\circ}(U_{0,r'}(Q)) < G$, and hence $N_G^{\circ}(U_{0,r'}(Q)) \leq B_1 \cup B_2$. By Lemma 3.21, $N_G^{\circ}(X) = B_1$. Since $Y = XU_{0,r'}(Q)$ by Fact 1.22, $N_{B_1}^{\circ}(U_{0,r'}(Q)) \leq N_G^{\circ}(Y) \leq B_2$ (by Lemma 3.17). So $N_G^{\circ}(U_{0,r'}(Q)) \leq B_2$.

Thus $N_G^{\circ}(Q) \leq N_G^{\circ}(U_{0,r'}(Q)) \leq B_2$. By Fact 1.19(2), Q is a Carter subgroup of B_1 . \square

We can now show that r' is the only reduced rank appearing in both $F(B_1)$ and $F(B_2)$.

Lemma 3.25. $F_r(B_1) = 1$ for $r \leq \bar{r}_0(B_2)$ with $r \neq r'$.

Proof. Let $T := F_r(B_1)$. We claim that $T \leq H$. First suppose that $\bar{r}_0(B_2) = r'$. Then $Y \cdot T$ is nilpotent by Fact 1.16. By Fact 1.14, Y centralizes T . So $T \leq N_G^{\circ}(Y) \leq B_2$ by Lemma 3.17, and $T \leq H$. Next, suppose that $\bar{r}_0(B_2) > r'$. Then $U_0(B_2) \leq Z(H)$ by Lemma 3.12. By Fact 1.16, $U := U_0(B_2) \cdot T$ is nilpotent. If $r \neq \bar{r}_0(B_2)$, then $T \leq C_G^{\circ}(U_0(B_2)) \leq B_2$ by Fact 1.14, and $T \leq H$. So we may assume that $r = \bar{r}_0(B_2)$. If $T \not\leq B_2$, then $U_{0,r}(N_U^{\circ}(U_0(B_2))) > U_0(B_2)$ by Fact 1.13, but $N_G^{\circ}(U_0(B_2)) = B_2$, a contradiction. Thus $T \leq H$.

Since $T \leq H$, and $U_{0,r}(H') = 1$ by Theorem 3.10, T is contained in a Carter subgroup of H , by Fact 1.22. Now $T \leq Q$ because $T \triangleleft H$. Clearly $T \leq F^{\circ}(H)$ too. By Theorem 3.18 and Fact 1.19(4), $H = QF^{\circ}(H) \leq C_G^{\circ}(T)$, and hence $T \leq Z(H)$.

Now consider the case where $r > r'$. Then $F_{r'}(B_2) \cdot T$ is nilpotent by Fact 1.16. So $F_{r'}(B_2) \leq C_G^{\circ}(T)$ by Fact 1.14, a contradiction to $N_G^{\circ}(T) = B_2$ if $T \neq 1$.

Finally consider the case where $r < r'$. Since $T \leq Z(H)$, TY is Abelian by Theorem 3.18. Recall that $Y \leq F(B_2)$ by Lemma 3.16. Let $P := U_{0,r'}(N_{F(B_2)}(Y))$. Then $[x, h] \in Y$ for any $x \in X$ and any $h \in P$, and hence $[x, h] = [x, h]^t = [x, h^t]$ for any $t \in T$. So $[h, t] = h^{-1}h^t \in C_G(X)$. Now $[P, T] \leq Y$ by Lemma 3.9 and Fact 1.15. Since P is nilpotent, and T commutes with Y , the product TP is nilpotent. By Fact 1.14, $P \leq N_G^{\circ}(T) = B_1$ if $T \neq 1$, in contradiction with Lemma 3.17. \square

As a result, r' is also uniquely determined by B_1 .

Corollary 3.26. r' is the minimal reduced rank in $F(B_1)$.

Corollary 3.27. For $r \leq \bar{r}_0(B_2)$, a Sylow $U_{0,r}$ -subgroup of H is a Sylow $U_{0,r}$ -subgroup of B_1 .

Proof. By Lemmas 3.25 and 3.21, $F_r(B_1) \leq H$. Since Q is a Carter subgroup of B_1 by Theorem 3.24, $U_{0,r}(Q)F_r(B_1) \leq H$ is a Sylow $U_{0,r}$ -subgroup of B_1 by Fact 1.22. \square

3.6. Non-Abelian intersections

In closing, we can build upon Corollary 3.22 to produce a characterization of B_1 . In Section 4, this fact will be used, together with Proposition 3.7, to show that all reasonable notions of maximal intersection are equivalent.

Lemma 3.28. *The subgroup $C_G^\circ(X)$ is nonnilpotent. If H is non-Abelian, then B_1 is the only Borel subgroup containing $C_G^\circ(X)$.*

Proof. By Lemma 3.9, $C_G^\circ(X) \leq B_1$. By Lemma 3.16 and Theorem 3.18, $U_{0,r'}(Q) \leq C_G^\circ(X)$. By Fact 1.14, $U_0(B_1) \leq C_G^\circ(X)$ too. By Theorem 3.24 and Fact 1.14, $U_{0,r'}(Q) \cdot U_0(B_1)$ is nonnilpotent. So $C_G^\circ(X)$ is nonnilpotent.

We now suppose that H is non-Abelian. Suppose also that another Borel subgroup of G , distinct from B_1 , contains $C_G^\circ(X)$. So there is a maximal pair B_3, B_4 whose intersection contains $C_G^\circ(X)$. We may assume that $\bar{r}_0(B_3) \geq \bar{r}_0(B_4)$. Let $K := C_G^\circ(X)'$. By Corollary 3.22, $F^\circ(B_1) \leq C_G^\circ(X)$. So $C_G^\circ(X) \triangleleft B_1$ by Fact 1.2. Now $N_G^\circ(K) = B_1$. Since $K \leq F(B_3) \cap F(B_4)$, we have $F^\circ(B_3) \leq C_G^\circ(K)$, by Corollary 3.22 (applied to the pair B_3, B_4). Thus $\bar{r}_0(B_1) \geq \bar{r}_0(B_3) > \bar{r}_0(B_4)$ by Lemma 3.5. But, as $F^\circ(B_1) \leq B_4$, $\bar{r}_0(B_4) \geq \bar{r}_0(B_1)$ by Fact 1.9, a contradiction. \square

Corollary 3.29. *Suppose that H is not Abelian. Then, for nontrivial definable $X_1 \leq X$, B_1 is the only Borel subgroup containing $C_G^\circ(X_1)$.*

Lemma 3.30. *Let B be a Borel subgroup of G , distinct from B_1 . Suppose that $H_1 := (B \cap B_1)^\circ$ is non-Abelian, B_1, B is a maximal pair, and $\bar{r}_0(B_1) \geq \bar{r}_0(B)$. Then B is $F^\circ(B_1)$ -conjugate to B_2 .*

Proof. We will apply the preceding results with the maximal pair B_1, B . We observe that $H_1' \leq F(B_1) \cap F(B)$. By Corollary 3.26 and Theorem 3.10, $r' = \bar{r}_0(H_1')$, and $H', H_1' \leq F_{r'}(B_1)$. By Lemma 3.21, $F_{r'}(B_1) \leq H, H_1$. Let Q and Q_1 be Carter subgroups of H and H_1 , respectively. By Theorem 3.24, Q and Q_1 are Carter subgroups of B_1 . By Fact 1.19(3), $Q^h = Q_1$ for some $h \in B_1$. We may assume $h \in F^\circ(B_1)$ by Fact 1.19(4). By Fact 1.19(4) and Lemma 3.21, $H^h = F_{r'}(B_1)Q^h = F_{r'}(B_1)Q_1 = H_1$. Since H_1 is non-Abelian, $B_2^h = B$ by Proposition 3.7. \square

4. Conclusions

Our main task is to understand how the results of Section 3 translate down to nonmaximal intersections. This translation is immediate for results which describe the internal structure of H , such as Theorems 3.10 (applied to H' , not X) and 3.18, but such direct translations are not possible for a number of results, such as Lemmas 3.5 and 3.12. The following summarizes the most useful consequences for arbitrary intersections for Borel subgroups.

Proposition 4.1. *Let G be a minimal connected simple group of finite Morley rank, let B_1, B_2 be two distinct Borel subgroups of G , and let H be a connected subgroup of the intersection $B_1 \cap B_2$. Then the following hold.*

- (1) H' is rank homogeneous for $r' := \bar{r}_0(H')$, or trivial.
- (2) Every connected nilpotent subgroup of H is Abelian.

- (3) $F_{r'}(H) = U_{0,r'}(H)$ is the unique Sylow $U_{0,r'}$ -subgroup of H .
- (4) $F_r(H) \leq Z(H)$ for $r \neq r'$.
- (5) $\bar{r}_0(C_G(H')) > \bar{r}_0(H)$ if H is non-Abelian.

Proof. We may assume H is non-Abelian because all five statements are trivial if H is Abelian. Let B_3, B_4 be a maximal pair, containing H , with $\bar{r}_0(B_3) \geq \bar{r}_0(B_4)$. The first two conclusions follow immediately from Theorems 3.10 and 3.18. The third conclusion follows from Lemma 3.16. For the fourth conclusion, $F_r(H)$ lies in a Carter subgroup Q by Fact 1.22, and $H \leq QH' \leq C_H^\circ(F_r(H))$ by Fact 1.19(4). By Lemma 3.5, $\bar{r}_0(B_3) > \bar{r}_0(B_4) \geq \bar{r}_0(H)$. By Fact 1.14, $U_0(B_3) \leq C_G(H')$, and conclusion five follows. \square

Corollary 4.2. *Let G be a minimal connected simple group of finite Morley rank. Then a definable connected non-Abelian nilpotent subgroup H of G is contained in exactly one Borel subgroup of G . In particular, a non-Abelian Carter subgroup of any Borel subgroup of G is a Carter subgroup of G itself.*

In Section 3, Lemmas 3.5 and 3.12 told us much about the Borel subgroups involved, and this information is lost in Proposition 4.1(4), (5). Instead, we prove that, in the non-Abelian case, all reasonable notions of maximal intersections are equivalent. In practice, this allows one to make direct use of the analysis of Section 3.

Theorem 4.3. *Let G be a minimal connected simple group of finite Morley rank, and let B_1, B_2 be two distinct Borel subgroups of G . Suppose that $H := (B_1 \cap B_2)^\circ$ is non-Abelian. Then the following are equivalent.*

- (1) B_1 and B_2 are the only Borel subgroups of G containing H .
- (2) If B_3 and B_4 are distinct Borel subgroups of G containing H , then $(B_3 \cap B_4)^\circ = H$.
- (3) If $B_3 \neq B_1$ is a Borel subgroup containing H , then $(B_3 \cap B_1)^\circ = H$.
- (4) $C_G^\circ(H')$ is contained in B_1 or B_2 .
- (5) B_1 and B_2 are not conjugate under $C_G^\circ(H')$.
- (6) $\bar{r}_0(B_1) \neq \bar{r}_0(B_2)$.

The second and third clauses express the maximality of H in two different senses. The second clause corresponds to the sense of maximality used in Section 3, while the third clause has an a priori weaker sense. The first clause goes far beyond maximality, to assure uniqueness of the Borel subgroups, and the fourth and fifth clauses provide means to recognize maximal intersections “in the wild.” We observe that the first and second clauses are equivalent, thanks to Proposition 3.7. To prove this theorem, we first treat the most subtle implication (4) \Rightarrow (1) with a lemma.

Lemma 4.4. *Let G be a minimal connected simple group of finite Morley rank, and let B_1, B_2 be two distinct Borel subgroups of G . Suppose that the intersection $H := (B_1 \cap B_2)^\circ$ is non-Abelian, and that $C_G^\circ(H')$ is contained in B_1 . Then B_1 and B_2 are the only Borel subgroups containing H .*

Proof. Suppose toward a contradiction that B is a Borel subgroup of G containing H which is distinct from B_1 and B_2 . We may choose B such that $H_2 := (B_2 \cap B)^\circ$ is maximal, subject to $B \neq B_1, B_2$ and $B \geq H$. Consider a maximal pair B_3, B_4 containing H_2 , with $\bar{r}_0(B_3) \geq \bar{r}_0(B_4)$. By Corollary 3.29, B_3 is the only Borel subgroup containing $C_G^\circ(H')$, so $B_1 = B_3$. Thus $H = H_2$. By Proposition 3.7, $B_1 = B_3$ and B_4 are the only Borel subgroups containing their intersection. So we may assume that $B_4 \neq B_2$, as otherwise we are done. Therefore we may also assume that $B = B_4$. So $H_1 := (B_1 \cap B)^\circ = (B_3 \cap B_4)^\circ$ is a maximal intersection containing H_2 , and we are free to apply Section 3 here. We observe that $r' := \bar{r}_0(H') = \bar{r}_0(F(B_1) \cap F(B))$, by Theorem 3.10.

We first consider the case where $F_{r'}(B_2) \leq B_1$. Since H' is rank homogeneous, Fact 1.14 says $F_r(B_2) \leq C_G^\circ(H') \leq B_1$ for $r \neq r'$, and $F_{\text{tor}}^\circ(B_2) \leq C_G^\circ(H') \leq B_1$. So $F^\circ(B_2) \leq B_1$. Hence $F^\circ(B_2) \leq H$, and $H \triangleleft B_2$. By Corollary 3.22 (for B_1, B), $F^\circ(B_1) \leq N_G^\circ(H') = B_2$. So $U_0(B_1) \leq H$ by Fact 1.9, and $\bar{r}_0(B_1) \leq \bar{r}_0(H_1)$. But this contradicts Lemma 3.5.

We next consider the case where $F_{r'}(B_2) \not\leq B_1$. Let $P := F_{r'}(H)$, $Y := F_{r'}(H_1)$, and $M := N_G^\circ(P)$. By Proposition 4.1(3), $P = U_{0,r'}(H)$ is a Sylow $U_{0,r'}$ -subgroup of H . Since P normalizes $F_{r'}(B_2)$, $P \cdot F_{r'}(B_2)$ is nilpotent by Fact 1.16. By Fact 1.13, $U_{0,r'}(N_{P \cdot F_{r'}(B_2)}^\circ(P)) > P$. Since $P = U_{0,r'}(H)$, $(M \cap B_2)^\circ > H$. Since $H = (B_1 \cap B_2)^\circ$, $(M \cap B_2)^\circ \not\leq B_1$. So M is contained in B_2 , by maximality of $H_2 (= H)$.

By Lemma 3.13, $F_{r'}(B) \not\leq H_1$. Since P normalizes $F_{r'}(B)$, $P \cdot F_{r'}(B)$ is nilpotent by Fact 1.16. By Fact 1.13, $U_{0,r'}(N_{P \cdot F_{r'}(B)}^\circ(P)) > P$ too. As $M \leq B_2$, we have $U_{0,r'}(N_{P \cdot F_{r'}(B)}^\circ(P)) \leq H_2 = H$, contradicting the fact that P is a Sylow $U_{0,r'}$ -subgroup of H . \square

Proof of Theorem 4.3. The first three clauses are successively weaker, and clause (4) implies clause (5). By Lemmas 3.5 and 3.6, clause (2) implies clause (6). Clearly, clause (6) implies clause (5). So we shall examine the implications (3) to (4), and (5) to (1). Let B_c be a Borel subgroup containing $N_G^\circ(H')$.

We first assume clause (3), and show clause (4). Let B_x denote B_1 , unless $B_c = B_1$, in which case we let B_x denote B_2 . By clause (3), $H := (B_c \cap B_x)^\circ$. By Lemma 4.4 (for B_c, B_x), $B_c \geq C_G^\circ(H')$ must be one of B_1 or B_2 , so clause (4) holds.

We now assume clause (1) fails, and show that clause (5) fails. Then, for $i = 1, 2$, $C_G^\circ(H') \not\leq B_i$, by Lemma 4.4. But B_c, B_1 and B_c, B_2 are maximal pairs, by Lemma 4.4 again. So $\bar{r}_0(B_c) \geq \bar{r}_0(B_1), \bar{r}_0(B_2)$, by Lemma 3.28. By Lemma 3.30, B_1 is $F^\circ(B_c)$ -conjugate to B_2 . By Corollary 3.22, $F^\circ(B_c) \leq C_G^\circ(H')$, as desired. \square

We can summarize Section 3, in the non-Abelian case, as follows.

Theorem 4.5. *Let G, B_1, B_2 , and H satisfy the hypotheses and conditions of Theorem 4.3, and let $r' = \bar{r}_0(H')$. Suppose that $\bar{r}_0(B_1) \geq \bar{r}_0(B_2)$. Then the following hold.*

- (1) $\bar{r}_0(B_1) > \bar{r}_0(H) = \bar{r}_0(B_2)$, and $N_G^\circ(H) = H$.
- (2) Every connected nilpotent subgroup of H is Abelian.
- (3) $F_{r'}(H) = U_{0,r'}(H)$ is the unique Sylow $U_{0,r'}$ -subgroup of H . It is contained in $F(B_2)$, and its normalizer $^\circ$ is contained in B_2 .

- (4) $F_r(B_2) \leq Z(H)$ for $r \neq r'$, and $F_{r'}(B_2)$ is non-Abelian. So $F_r(B_2)$ is non-Abelian iff $r = r'$.
- (5) Carter subgroups of H are Carter subgroups of B_1 , and their normalizers $^\circ$ are contained in B_2 .
- (6) $F_{r'}(B_1) = F(B_1) \cap F(B_2)$ is rank homogeneous for r' , and B_1 is the only Borel subgroup containing $C_G^\circ(F_{r'}(B_1))$.
- (7) $F_r(B_1) = 1$ for $r \leq \bar{r}_0(B_2)$ with $r \neq r'$. So r' is the minimal reduced rank in $F(B_1)$.
- (8) $F^\circ(B_1)$ and $F^\circ(B_2)$ are divisible.

Proof. Part (1) follows from Lemmas 3.5, 3.6, and 3.19. Part (2) is Theorem 3.18. Part (3) consists of Lemmas 3.16 and 3.17. Part (4) includes Lemmas 3.12 and 3.13, along with Corollary 3.14. Part (5) restates Theorem 3.24. Part (6) summarizes Lemma 3.21, Theorem 3.10, and Lemma 3.28. Part (7) is Lemma 3.25, along with Corollary 3.26. Part (8) is Lemmas 3.11 and 3.20. \square

5. Genericity of Carter subgroups

We conclude this article by discussing an important open question: whether a group G of finite Morley rank possesses a Carter subgroup whose conjugates are generic in G . It is natural to ask whether the structure imposed by a non-Abelian intersection can be used to prove the genericity of any related Carter subgroup. We show that genericity of conjugates holds for a Carter subgroups of a non-Abelian intersection iff it is a Carter subgroup of both sides.

Theorem 5.1. *Let G be a minimal simple group of finite Morley rank, and let B_1, B_2 be a maximal pair of Borel subgroups of G . Suppose that $H := (B_1 \cap B_2)^\circ$ is non-Abelian, and that $\bar{r}_0(B_1) > \bar{r}_0(B_2)$. If Q is a Carter subgroup of H , then the following are equivalent.*

- (1) Q is a Carter subgroup of B_2 .
- (2) Q is a Carter subgroup of G .
- (3) $\bigcup Q^G$ is generic in G .
- (4) $\bigcup H^G$ is generic in G .

To prove this, we recall the following fact.

Fact 5.2. *(Cf. [1, Lemma 3.3].) Let G be a connected group of finite Morley rank, and let C be a definable almost self-normalizing subgroup. Suppose there is a definable subset J of C , not generic in C , such that $C \cap C^g \subseteq J$ whenever $g \notin N_G(C)$. Then $\bigcup (C \setminus J)^G$ is generic in G .*

We also need a lemma due to Frécon.

Fact 5.3. (See Frécon, cf. [1, Lemma 3.5].) *Let B be a connected solvable group of finite Morley rank, and let C be a Carter subgroup of B . Then there is a definable subset J of C , not generic in C , such that $C \cap C^b \subseteq J$ whenever $b \notin N_B(C)$.*

These two facts can be used to prove the following.

Lemma 5.4. *Let G be a minimal connected simple group of finite Morley rank. Let H be a connected solvable subgroup of G , and let Q be a Carter subgroup of H . Then $\bigcup H^G$ is generic in G iff $\bigcup Q^G$ is generic in G .*

Proof. The if direction is immediate because $\text{rk}(\bigcup H^G) \geq \text{rk}(\bigcup Q^G)$. So we assume $\text{rk}(\bigcup Q^G) < \text{rk}(G)$, and show that $\text{rk}(\bigcup H^G) < \text{rk}(G)$. Let $H_* := H \setminus \bigcup Q^H$. By Fact 5.3 there is a set $J \subset Q$, not generic in Q , such that $Q \cap Q^h \subset J$ for $h \in H$. By Fact 5.2, $\bigcup Q^H$ is generic in H , and $\text{rk}(H_*) < \text{rk}(H)$. So $\text{rk}(\bigcup H_*^G) \leq \text{rk}(G/N_G(H) \times H_*) = \text{rk}(G) - \text{rk}(N_G(H)) + \text{rk}(H_*) < \text{rk}(G)$. Since $H = H_* \sqcup \bigcup Q^G$, we have $\text{rk}(\bigcup H^G) \leq \max(\text{rk}(\bigcup H_*^G), \text{rk}(\bigcup Q^G)) < \text{rk}(G)$, as desired. \square

We will also need the following two tools.

Fact 5.5. *Let G be a minimal simple group of finite Morley rank, and let Q be a Carter subgroup of G with divisible p -torsion. Then $\bigcup Q^G$ is generic in G .*

Proof. Let P be a divisible Abelian p -subgroup of Q . Then P is contained in a maximal decent torus T (see [20]). By Fact 1.19(5), $N_G^\circ(T)$ contains a Carter subgroup Q_1 of $C_G^\circ(P)$. Since Q is a Carter subgroup of $C_G^\circ(P)$ too, we may assume $Q = Q_1$ by Fact 1.19(3). So $T \leq Q$ by Fact 1.3, and Q is a Carter subgroup of $C_G^\circ(T)$. By [20], $\bigcup C_G^\circ(T)^G$ is generic in G . So $\bigcup Q^G$ is generic in G , by Lemma 5.4. \square

Fact 5.6. *Let Q be a nilpotent group of finite Morley rank. Then*

$$Q' = U_{0,1}(Q)' \cdot U_{0,2}(Q)' \cdots U_{0,\text{rk}(Q)}(Q)' \cdot U_2(Q)' \cdot U_3(Q)' \cdots$$

Proof. Let P denote the product on the right-hand side. Clearly $P \leq Q'$. By Fact 1.14, Q/P is Abelian, so $Q' \leq P$. \square

Lemma 5.7. *Let Q be a nilpotent group of finite Morley rank without divisible torsion, and suppose that $U_{0,r}(Q) \neq 1$. Then there is a generic subset Q_r^* of Q such that $U_{0,r}(d(k)) \neq 1$ for all $k \in Q_r^*$.*

Proof. There is a (not necessarily unique) maximal connected normal subgroup P of Q such that $U_{0,r}(Q/P) \neq 1$. Then P contains $U_{0,r}(Q)'$ for any r , and $U_p(Q)'$ for any prime p . So P contains Q' by Fact 5.6. Hence the normality assumption on P is superfluous. By Fact 1.5, $U_p(Q) \leq P$ for any prime p . By Fact 1.6, Q/P is torsion-free. If $U_{0,r}(d(xP/P)) = 1$ for some $x \in Q_r^*$, then $U_{0,r}((Q/P)/d(xP/P)) \neq 1$ by Fact 1.11. So $U_{0,r}(d(xP/P)) \neq 1$ by the maximality of P . By Fact 1.11, $U_{0,r}(d(x)) \neq 1$, as desired. \square

Proof of Theorem 5.1. We first observe that B_1 and B_2 satisfy Hypothesis 3.2. So clause (1) and clause (2) are equivalent, by Lemma 3.17. Lemma 5.4 proves that clauses (3) and (4) are equivalent.

We next assume clause (2) and prove clause (3). Suppose toward a contradiction that $\bigcup Q^G$ is not generic in G . Then Q contains no divisible torsion, by Fact 5.5. By Theorem 3.24, $U_{0,r'}(Q) \neq 1$. So there is a generic subset $Q_{r'}^*$ of Q such that $U_{0,r'}(d(h)) \neq 1$ for any $h \in Q_{r'}^*$, by Lemma 5.7. For $i = 1, 2$, there is a set $J_i \subset Q$, not generic in Q , such that $Q \cap Q^b \subset J_i$ for $b \in B_i$, by Fact 5.3. Then $Q_* := Q_{r'}^* \setminus (J_1 \cup J_2)$ is a generic subset of Q . By Fact 5.2, there is an $h \in Q_* \cap Q^g$, for some $g \notin N_G(Q)$. Since $h \in Q_{r'}^*$, $K := U_{0,r'}(d(h)) \neq 1$. By Theorem 3.18 and Lemma 3.21, $H', (H^g)' \leq C_G^\circ(K)$. Since $h \in Q \cap Q^g$, and Q is Abelian, we have $Q, Q^g \leq C_G^\circ(K)$ too, so $H, H^g \leq C_G^\circ(K)$. So $Q^g \leq H^g$ lies in either B_1 or B_2 by Proposition 3.7. Since Q^g is a Carter subgroup of either B_1 or B_2 , $h \in Q \cap Q^g$ lies in either J_1 or J_2 by Fact 1.19(3), a contradiction.

We next assume clause (2) fails, and show that clause (3) fails. For any group Q , $\text{rk}(\bigcup_{g \in G} Q) \leq \text{rk}(G/N_G(Q) \times Q) = \text{rk}(G) - \text{rk}(N_G(Q)) + \text{rk}(Q)$. So $\text{rk}(G) - \text{rk}(\bigcup_{g \in G} Q) \geq \text{rk}(N_G(Q)/Q) > 0$, as desired. \square

Corollary 5.8. *Let G be a minimal connected simple group of finite Morley rank, and let B_1, B_2 be two distinct Borel subgroups of G . Suppose that $H := (B_1 \cap B_2)^\circ$ is non-Abelian, and contains a Carter subgroup Q of both B_1 and B_2 . Then $\bar{r}_0(B_1) \neq \bar{r}_0(B_2)$, and $\bigcup Q^G$ is generic in G .*

Proof. We may assume that $\bar{r}_0(B_1) \geq \bar{r}_0(B_2)$. If $\bar{r}_0(B_1) \neq \bar{r}_0(B_2)$, then B_1 and B_2 are the only Borel subgroups containing H , by Theorem 4.3, and hence Theorem 5.1 applies.

Suppose toward a contradiction that $\bar{r}_0(B_1) = \bar{r}_0(B_2)$. Let B_3 be a Borel subgroup containing $N_G^\circ(H')$. For $i = 1, 2$, B_i and B_3 are the only Borel subgroups containing their intersection $H_i := (B_i \cap B_3)^\circ$, by Theorem 4.3. Since Q is a Carter subgroup of B_i , and $Q \leq H \leq B_3$, Q is a Carter subgroup of H_i too. By Theorem 3.24, Q is a Carter subgroup of B_3 . By Lemma 3.21, $H'_i \leq F_{r'}(B_3) \leq H_i$. By Fact 1.19(4), $H_1 = F_{r'}(B_3)Q = H_2$, contradicting $B_1 \neq B_2$. \square

Even if Q fails to be a Carter subgroup of B_2 , one might hope to show that the conjugates of a Carter subgroup C of B_2 are generic. However, a non-Abelian intersection appears to place few constraints on C .

On the positive side, Corollary 4.2 and Fact 1.19(2) show that a non-Abelian Carter subgroup of any Borel subgroup of a minimal connected simple group G of finite Morley rank is actually a Carter subgroup of G itself. So it is natural to ask when these Carter subgroups are generic in G .

Proposition 5.9. *Let G be a minimal simple group of finite Morley rank, and let Q be a non-Abelian Carter subgroup of G . If Q is not rank homogeneous, then $\bigcup Q^G$ is generic in G .*

We observe that nilpotent Borel subgroups with unipotent torsion are generic.

Fact 5.10. *Let G be a minimal simple group of finite Morley rank with a nilpotent Borel subgroup B which is not divisible. Then $\bigcup B^G$ is generic in G .*

Proof. By Lemma 2.1, $B \cap B^g = 1$ for $g \notin N_G(B)$. So genericity follows by Fact 5.2. \square

Proof of Proposition 5.9. Suppose toward a contradiction that $\bigcup Q^G$ is not generic in G , but that Q is not rank homogeneous. Since Q is non-Abelian, there is a unique Borel subgroup B containing Q , by Corollary 4.2. By Fact 5.3, there is a set $J_0 \subset Q$, not generic in Q , such that $Q \cap Q^b \subset J_0$ for any $b \in B \setminus N_G(Q)$.

By Fact 5.5, Q contains no divisible torsion. If Q had bounded exponent, Q would be a Borel subgroup, by Fact 1.5, in contradiction with Fact 5.10. So Q is not of bounded exponent.

Since Q is non-Abelian, either $U_{0,r}(Q)' \neq 1$ for some r , or $U_p(Q)' \neq 1$ for some p prime, by Fact 1.14. We take $p = 0$ in the former case and $r = 0$ in the latter. First consider the case where $U_{0,s}(Q) \neq 1$ for some $s \neq r$. Then there is a generic subset Q_s^* of Q such that $U_{0,s}(d(h)) \neq 1$ for all $h \in Q_s^*$, by Lemma 5.7. Next consider the case where $U_{0,s}(Q) = 1$ for all $s \neq r$. We may assume that $p = 0$, since Q is not of bounded exponent. Since Q is not rank homogeneous, there is a prime q such that $U_q(Q) \neq 1$. So there is a maximal connected subgroup P of Q such that $U_q(Q/P) \neq 1$. Clearly, $U_q(Q/P) = Q/P$. So, for any $h \notin P$, $d(h)$ contains a q -element. In either case, there is a generic subset Q_* of Q given by either $Q_* := Q_s^* \setminus J_0$ or $Q_* := Q \setminus (P \cup J_0)$.

By Fact 5.2, there is an $h \in Q_* \cap Q^g$, for some $g \notin N_G(Q)$. Since $h \notin P$, either $K := U_{0,s}(d(h)) \neq 1$, or $d(h)$ contains a q -torsion subgroup K . By Fact 1.14, $C_B^\circ(K)$ contains the non-Abelian nilpotent subgroup $U_{0,r}(Q)$, or $U_p(Q)$. So B is the only Borel subgroups containing $C_G^\circ(K)$, by Proposition 4.1. But $Q, Q^g \leq C_G^\circ(K)$. So they are B -conjugate by Fact 1.19(3), a contradiction to $h \notin J_0$. \square

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References

- [1] G. Cherlin, E. Jaligot, Tame minimal simple groups of finite Morley rank, *J. Algebra* 276 (1) (2004) 13–79.
- [2] J. Burdges, G. Cherlin, E. Jaligot, Minimal connected simple groups of finite Morley rank with strongly embedded subgroups, *J. Algebra* (2007), in press.
- [3] D. Gorenstein, R. Lyons, R. Solomon, *The Classification of the Finite Simple Groups*, Math. Surveys Monogr., vol. 40, Amer. Math. Soc., Providence, RI, 1994.
- [4] T.M. Gagen, *Topics in Finite Groups*, London Math. Soc. Lecture Note Ser., vol. 16, Cambridge Univ. Press, Cambridge, 1976.
- [5] H. Bender, G. Glauberman, *Local Analysis for the Odd Order Theorem*, London Math. Soc. Lecture Note Ser., vol. 188, Cambridge Univ. Press, Cambridge, 1994, with the assistance of Walter Carlip.
- [6] J. Burdges, Simple groups of finite Morley rank of odd and degenerate type, PhD thesis, Rutgers University, New Brunswick, NJ, 2004.
- [7] J. Burdges, A signalizer functor theorem for groups of finite Morley rank, *J. Algebra* 274 (1) (2004) 215–229.
- [8] O.V. Belegradek, On groups of finite Morley rank, in: *Abstracts of the Eight International Congress of Logic, Methodology and Philosophy of Science LMPS'87*, Moscow, 17–22 August 1987, pp. 100–102.
- [9] A. Nesin, Generalized Fitting subgroup of a group of finite Morley rank, *J. Symbolic Logic* 56 (4) (1991) 1391–1399.
- [10] A. Borovik, A. Nesin, *Groups of Finite Morley Rank*, Oxford Sci. Publ., Oxford Univ. Press, New York, 1994.
- [11] O. Frécon, Sous-groupes de Hall généralisés dans les groupes résolubles de rang de Morley fini, *J. Algebra* 233 (1) (2000) 253–286.
- [12] T. Altinel, A. Borovik, G. Cherlin, Groups of mixed type, *J. Algebra* 192 (2) (1997) 524–571.
- [13] J. Burdges, Sylow 0-unipotent subgroups in groups of finite Morley rank, *J. Group Theory*, in press.
- [14] O. Frécon, Around unipotence in groups of finite Morley rank, *J. Group Theory*, submitted for publication.
- [15] O. Frécon, E. Jaligot, The existence of Carter subgroups in groups of finite Morley rank, *J. Group Theory* 8 (2005) 623–633.
- [16] F. Wagner, Fields of finite Morley rank, *J. Symbolic Logic* 66 (2) (2001) 703–706.
- [17] O. Frécon, Sous-groupes anormaux dans les groupes de rang de Morley fini résolubles, *J. Algebra* 229 (1) (2000) 118–152.
- [18] F.O. Wagner, *Stable Groups*, Cambridge Univ. Press, Cambridge, 1997.
- [19] F.O. Wagner, Nilpotent complements and Carter subgroups in stable \mathcal{R} -groups, *Arch. Math. Logic* 33 (1) (1994) 23–34.
- [20] G. Cherlin, Good tori in groups of finite Morley rank, *J. Group Theory* 8 (2005) 613–621.