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ON THE PROBLEM OF STOCHASTIC INTEGRAL REPRESENTATIONS OF FUNCTIONALS OF THE BROWNIAN MOTION. I*

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(*Translated by M. V. Khatuntseva*)

Abstract. For functionals $S = S(\omega)$ of the Brownian motion B, we propose a method for finding stochastic integral representations based on the Itô formula for the stochastic integral associated with B. As an illustration of the method, we consider functionals of the "maximal" type: S_T, $S_{T_{-a}}$, S_{g_T}, and S_{θ_T}, where $S_T = \max_{t \leqq T} B_t$, $S_{T_{-a}} = \max_{t \leqq T_{-a}} B_t$ with $T_{-a} = \inf\{t > 0 : B_t = -a\}$, $a > 0$, and $S_{g_T} = \max_{t \leqq g_T} B_t$, $S_{\theta_T} = \max_{t \leqq \theta_T} B_t$, g_T and θ_T are *non*-Markov times: g_T is the time of the last zero of Brownian motion on $[0, T]$ and θ_T is a time when the Brownian motion achieves its maximal value on $[0, T]$.

Key words. Brownian motion, Markov time, non-Markov time, stochastic integral, stochastic integral representation, Itô formula

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1. Introduction.

1.1. Let $B = (B_t)_{t \geqq 0}$ be a standard Brownian motion given on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geqq 0}$ a filtration generated by the Brownian motion ($\mathcal{F}_t = \sigma(B_s, s \leqq t)$, $t \geqq 0$, where all σ-algebras under consideration are supposed to be completed with sets from $\mathcal{F}$ of $\mathbf{P}$-probability zero; we also put $\mathcal{F}_\infty = \sigma(\bigcup_{t \geqq 0} \mathcal{F}_t)$).

It is well known (see [1], [2], and [3]) that if $S = S(\omega)$ is an $\mathcal{F}_\infty$-measurable random variable (i.e., an $\mathcal{F}_\infty$-measurable functional of a Brownian motion) with $\mathbf{E}S^2 < \infty$ ($S \in L^2(\Omega, \mathcal{F}_\infty, \mathbf{P})$), then there exists a process $Y = (Y_t)_{t \geqq 0}$ adapted with the filtration $\mathbb{F}$ (i.e., Y_t are $\mathcal{F}_t$-measurable, $t \geqq 0$) such that the following *stochastic integral representation* holds:

$$S = \mathbf{E}S + \int_0^\infty Y_t \, dB_t, \tag{1}$$

where

$$\mathbf{E} \int_0^\infty Y_t^2 \, dt < \infty. \tag{2}$$

It is necessary to emphasize that although relation (1) essentially solves the problem of representing S in the form of a stochastic integral, finding *explicit* expressions for Y_t, $t \geqq 0$, is a rather uneasy business.

One general enough result is known. It is called the "Clark–Ocone formula." It gives the following expression for Y_t:

$$Y_t = \mathbf{E}\big(DS(t, \infty] \,\big|\, \mathcal{F}_t\big), \tag{3}$$

where DS is the Malliavin derivative (see, for example, [4]).

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However, already in rather simple cases calculating Y_t with the help of (3) requires significant efforts.

1.2. In this paper we use a different approach in order to find *explicit* formulas for Y_t, $t \geqq 0$, and the idea is the following.

We connect with functional S the *associated Lévy martingale*

$$\mathbf{M}_t = \mathbf{E}(S \mid \mathcal{F}_t), \qquad t \geqq 0,$$

and try to represent $\mathbf{E}(S \,|\, \mathcal{F}_t)(\omega)$ in the form $F(t, \omega; B_t(\omega))$, where the last functional is such that one can apply the (perhaps generalized) *Itô formula.* (See, for example, [8], [9]; compare also with the corresponding methods in the theory of optimal nonlinear filtration [5].)

On its structure, the Itô formula contains a *stochastic integral* with respect to Brownian motion which should give (up to a constant) a stochastic integral representation for the martingale $(\mathbf{M}_t)_{t \geqq 0}$. Since by the Lévy theorem

$$\mathbf{M}_t = \mathbf{E}(S \mid \mathcal{F}_t) \longrightarrow \mathbf{E}(S \mid \mathcal{F}_\infty) = S \qquad (\mathbf{P}\text{-a.s.}),$$

we receive as a consequence the required stochastic integral representation for the functionals S.

1.3. The realization of that program will be shown in what follows with examples of several functionals S of "maximal" type. Namely, let

$$S_t = \max_{u \leqq t} B_u.$$

In section 2 of this first part of the paper, the stochastic integral representation for the functional $S = S_T$ is considered, where T is some *constant*, i.e., for the case $S = \max_{u \leqq T} B_u$.

In section 3 the case $S = S_{T_{-a}}$, i.e., $S = \max_{u \leqq T_{-a}} B_u$, is investigated, where

$$T_{-a} = \inf\{t > 0\colon\ B_t = -a\}, \qquad a > 0.$$

The times T and T_{-a} are, obviously, *Markov times.* In the second part of the paper the problems of the integral representations for the functionals

$$S = S_{g_T} \quad \text{and} \quad S = S_{\theta_T}$$

are considered, where g_T and θ_T are the following *non-Markov* times:

g_T is the time *of the last zero* of the Brownian motion B on $[0, T]$ and $g_T = T$ if there is no such time; and

θ_T is the time when Brownian motion B achieves its *maximal value* on $[0, T]$.

These two cases are considered, respectively, in sections 4 and 5 of the second part of the paper.

In the conclusion of this introduction we note that, besides the results concerning stochastic integral representations for "partial" maxima S_T, $S_{T_{-a}}$, S_{g_T}, and S_{θ_T}, this paper also contains a number of nontraditional methods and tools of stochastic analysis (see, for example, Lemma 1 on a stochastic integral representation for the indicator $I(T_a < T)$ of the Markov time T_a, $a > 0$), which, we think, can also be useful in other problems in the theory of stochastic processes.

2. Case $S_T = \max_{t \leqq T} B_t$.

2.1. The problem of stochastic integral representation of the functional S_T was considered in a number of papers (see, for example, [1], [2], [4], [6]) according to which the following statement is known.

THEOREM 1. *For S_T the stochastic integral representation*

$$S_T = \mathbf{E}S_T + 2\int_0^T \left[1 - \Phi\left(\frac{S_t - B_t}{\sqrt{T-t}}\right)\right] dB_t \tag{4}$$

is valid, where $S_t = \max_{u \leqq t} B_u$,

$$\mathbf{E}S_T = \mathbf{E}\max_{t \leqq T} B_t = \mathbf{E}|B_T| = \sqrt{\frac{2T}{\pi}}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-y^2/2}\,dy.$$

In what follows we give two ways of proving representation (4) based on the strategy stated above using the application of the Itô formula to the associated martingale $(\mathbf{E}[S \mid \mathcal{F}_t])_{t \geqq 0}$.

Proof. The first way. For any nonnegative random variable ξ, the representation

$$\xi = \int_0^\infty I(a < \xi)\,da$$

is valid.

Hence,

$$S_T = \max_{t \leqq T} B_t = \int_0^\infty I\Big(a < \max_{t \leqq T} B_t\Big)\,da = \int_0^\infty I(T_a < T)\,da, \tag{5}$$

where

$$T_a = \inf\{t \geqq 0\colon\ B_t = a\}, \qquad a \geqq 0.$$

The following result, interesting in itself, gives the stochastic integral *representation* for the stopping time T_a. (Note that $\mathbf{P}\{T_a < \infty\} = 1$, but $\mathbf{E}T_a = \infty$, $a > 0$.)

LEMMA 1. *For any $a > 0$ ($\mathbf{P}$-a.s.)*

$$I(T_a < T) = \mathbf{P}\{T_a < T\} + 2\int_0^{T_a \wedge T} \varphi_{T-t}(B_t - a)\,dB_t, \tag{6}$$

where

$$\varphi_t(a) = \frac{1}{\sqrt{2\pi t}}\,e^{-a^2/(2t)}. \tag{7}$$

Proof. We consider the stochastic exponent

$$\mathcal{E}_t(\lambda) = e^{\lambda B_t - \lambda^2 t/2}, \qquad \lambda \in \mathbf{R}. \tag{8}$$

Under the Doob stopping theorem for martingales we find that

$$\mathbf{E}\big(\mathcal{E}_{T_a}(\lambda) \,\big|\, \mathcal{F}_t\big) = \mathcal{E}_{T_a \wedge t}(\lambda) \qquad (\mathbf{P}\text{-a.s.}). \tag{9}$$

By the Itô formula

$$d\mathcal{E}_t(\lambda) = \lambda \mathcal{E}_t(\lambda)\, dB_t, \qquad \mathcal{E}_0(\lambda) = 1. \tag{10}$$

From here and (9)

$$\mathbf{E}\big(\mathcal{E}_{T_a}(\lambda) \,\big|\, \mathcal{F}_t\big) = 1 + \lambda \int_0^{T_a \wedge t} e^{\lambda B_s - \lambda^2 s/2}\, dB_s, \tag{11}$$

and, since $\mathcal{E}_{T_a}(\lambda) = e^{\lambda a - \lambda^2 T_a/2}$ ($\mathbf{P}$-a.s.),

$$\mathbf{E}(e^{-\lambda^2 T_a/2} \mid \mathcal{F}_t) = e^{-\lambda a} + \lambda \int_0^{T_a \wedge t} e^{-\lambda(a - B_s) - \lambda^2 s/2}\, dB_s. \tag{12}$$

In particular, setting $t \to \infty$, we obtain that ($\mathbf{P}$-a.s.)

$$e^{-\lambda^2 T_a/2} = e^{-\lambda a} + \lambda \int_0^{T_a} e^{-\lambda(a - B_s) - \lambda^2 s/2}\, dB_s. \tag{13}$$

Let us transform the right-hand sides of (12) and (13). For this goal we set

$$\gamma_a(t) = \frac{d}{dt}\, \mathbf{P}\{T_a \leqq t\}. \tag{14}$$

It is known (see [3], [7]) that

$$\gamma_a(t) = \frac{a}{\sqrt{2\pi t^3}}\, e^{-a^2/2t} \biggl(= -\frac{\partial}{\partial a}\, \varphi_t(a) \biggr). \tag{15}$$

From (9) it follows that

$$\mathbf{E} e^{-\lambda^2 T_a/2} = e^{-\lambda a}. \tag{16}$$

Therefore, for any $c > 0$,

$$\begin{aligned} \lambda e^{-\lambda c} &= -\frac{d}{dc}\, e^{-\lambda c} = -\frac{d}{dc}\, (\mathbf{E} e^{-\lambda^2 T_c/2}) = -\frac{d}{dc} \int_0^\infty e^{-\lambda^2 t/2} \gamma_c(t)\, dt \\ &= -\int_0^\infty e^{-\lambda^2 t/2} \biggl(\frac{\partial}{\partial c}\, \gamma_c(t) \biggr)\, dt = \int_0^\infty e^{-\lambda^2 t/2} \biggl(\frac{\partial^2}{\partial c^2}\, \varphi_t(c) \biggr)\, dt \\ &= 2 \int_0^\infty e^{-\lambda^2 t/2} \biggl(\frac{\partial}{\partial t}\, \varphi_t(c) \biggr)\, dt, \end{aligned} \tag{17}$$

since

$$\frac{\partial \varphi_t(c)}{\partial t} = \frac{1}{2}\, \frac{\partial^2 \varphi_t(c)}{\partial c^2}. \tag{18}$$

From (12), (16), and (17), taking into account that $B_s < a$ on the set $\{\omega\colon\ s < T_a\}$, we find that

$$\begin{aligned} &\mathbf{E}(e^{-\lambda^2 T_a/2} \mid \mathcal{F}_t) \\ &\quad= \mathbf{E} e^{-\lambda^2 T_a/2} + 2 \int_0^{T_a \wedge t} \biggl[\int_0^\infty e^{-\lambda^2 u/2} \biggl(\frac{\partial}{\partial u}\, \varphi_u(a - B_s) \biggr)\, du \biggr]\, e^{-\lambda^2 s/2}\, dB_s \\ &\quad= \mathbf{E} e^{-\lambda^2 T_a/2} + 2 \int_0^{T_a \wedge t} \biggl[\int_0^\infty e^{-\lambda^2 (u+s)/2} \biggl(\frac{\partial}{\partial u}\, \varphi_u(B_s - a) \biggr)\, du \biggr]\, dB_s. \end{aligned} \tag{19}$$

From here, for "good" enough and slowly growing functions $f = f(x)$ we obtain (see details in the second part of the paper) that (**P**-a.s.)

$$\mathbf{E}\big(f(T_a)\,\big|\,\mathcal{F}_t\big) = \mathbf{E}f(T_a) + 2\int_0^{T_a\wedge t}\left[\int_0^\infty f(u+s)\left(\frac{\partial}{\partial u}\,\varphi_u(B_s-a)\right)du\right]dB_s. \tag{20}$$

In particular, for $f(t) = I(t<T)$, we arrive at the following:

$$\begin{aligned} I(T_a<T) &= \mathbf{P}\{T_a<T\} + 2\int_0^{T_a}\left[\int_0^\infty I(u+s<T)\left(\frac{\partial}{\partial u}\,\varphi_u(B_s-a)\right)du\right]dB_s \\ &= \mathbf{P}\{T_a<T\} + 2\int_0^{T_a}\left[\int_0^{T-s}\frac{\partial}{\partial u}\,\varphi_u(B_s-a)\,du\right]dB_s \\ &= \mathbf{P}\{T_a<T\} + 2\int_0^{T_a\wedge t}\varphi_{T-s}(B_s-a)\,dB_s, \end{aligned} \tag{21}$$

which proves the required representation. Lemma 1 is proved.

2.2. From representation (6) and from (5) it is seen that

$$\begin{aligned} S_T = \max_{t\leqq T} B_t &= \int_0^\infty I(T_a<T)\,da \\ &= \int_0^\infty \mathbf{P}\{T_a<T\}\,da + 2\int_0^\infty\left[\int_0^{T_a\wedge T}\varphi_{T-s}(B_s-a)\,dB_s\right]da \\ &= \int_0^\infty \mathbf{P}\Big(\max_{s\leqq T} B_s > a\Big)\,da + 2\int_0^\infty\int_0^\infty I(s<T_a\wedge T)\,\varphi_{T-s}(B_s-a)\,dB_s\,da \\ &= \mathbf{E}\max_{s\leqq T} B_s + 2\int_0^T\left[\int_0^\infty I(s<T_a)\,\varphi_{T-s}(B_s-s)\,da\right]dB_s \\ &= \mathbf{E}S_T + 2\int_0^T\left[\int_0^\infty \varphi_{T-u}(B_u-a)\,I(S_u<a)\,da\right]dB_u. \end{aligned} \tag{22}$$

Here

$$\begin{aligned} \int_0^\infty \varphi_{T-u}(B_u-a)\,I(S_u<a)\,da &= \int_{S_u}^\infty \varphi_{T-u}(B_u-a)\,da \\ &= \int_{S_u}^\infty \frac{1}{\sqrt{2\pi(T-u)}}\,\exp\left(-\frac{(B_u-a)^2}{2(T-u)}\right)da \\ &= \frac{1}{\sqrt{2\pi}}\int_{(S_u-B_u)/\sqrt{T-u}} e^{-a^2/2}\,da = 1-\Phi\left(\frac{S_u-B_u}{\sqrt{T-u}}\right). \end{aligned} \tag{23}$$

From (22) and (23) we receive representation (4).

COROLLARY. *Let $M_T = \min_{t\leqq T} B_t$. Then*

$$M_T = \mathbf{E}M_T - 2\int_0^T\left[1-\Phi\left(\frac{B_t-M_t}{\sqrt{T-t}}\right)\right]dB_t. \tag{24}$$

For the proof it is enough only to notice that if $M_t = M_t(B)$ $(=\min_{u\leqq t} B_u)$ and $S_t = S_t(B)(=\max_{u\leqq t} B_u)$, then $M_t(B) = -S_t(-B)$, and (24) follows from (4).

2.3. *Proof of Theorem* 1. *The second way.* In the proof of (4) given above, formulas (5) and (6) for the indicator $I(T_a<T)$ played the key role.

The way of proving representations of type (4) given below consists of the direct investigation of associated martingales with the following application of the Itô formula. In addition we shall consider not only the process of the Brownian motion $B = (B_t)_{t \geqq 0}$, which is of interest to us, but also the more general processes, namely Lévy processes.

LEMMA 2. *Let $X = (X_t)_{t \geqq 0}$ be the Lévy process with $\mathbf{E} S_T < \infty$, where $T < \infty$ and $S_t = \sup_{u \leqq t} X_u$. Let $F_{T-t}(s) = \mathbf{P}\{S_{T-t} \leqq s\}$ and $\mathcal{F}_t^X = \sigma(X_u, u \leqq t)$. Then, for $t \leqq T$,*

$$\mathbf{E}\big(S_T \mid \mathcal{F}_t^X\big) = S_t + \int_{S_t - X_t}^{\infty} \big(1 - F_{T-t}(u)\big)\, du. \tag{25}$$

The proof, in general, follows from section 3 of [6]. From the properties of the Lévy process it follows that

$$\begin{aligned} \mathbf{E}\big(S_T \mid \mathcal{F}_t^X\big) &= \mathbf{E}\bigg(\max\Big(S_t, \sup_{t \leqq u \leqq T} X_u\Big) \,\bigg|\, \mathcal{F}_t^X\bigg) = S_t + \mathbf{E}\bigg(\Big(\sup_{t \leqq u \leqq T} X_u - S_t\Big)^+ \,\bigg|\, \mathcal{F}_t^X\bigg) \\ &= S_t + \mathbf{E}\bigg(\Big[\sup_{t \leqq u \leqq T} (X_u - X_t) - (S_t - X_t)\Big]^+ \,\bigg|\, \mathcal{F}_t^X\bigg) \\ &= S_t + \mathbf{E}\big(S_{T-t} - (s - x)\big)^+ \big|_{x = X_t, s = S_t}. \end{aligned} \tag{26}$$

Using the relation $\mathbf{E}(\xi - c)^+ = \int_c^\infty \mathbf{P}\{\xi > z\}\, dz$, from (26) we find the required representation (25).

Let us note that the right-hand side of (25) has the form $f(t, X_t, S_t)$, where

$$f(t, x, s) = s + \int_{s-x}^{\infty} \big(1 - F_T - t(u)\big)\, du. \tag{27}$$

In particular, if $X_t = B_t^\mu$, where $B_t^\mu = \mu t + B_t$ (the Brownian motion with a drift), the application of the Itô formula to the *continuous* martingale $f(t, B_t^\mu, S_t^\mu)$, $t \leqq T$, with $S_t^\mu = \max_{u \leqq t} B_u^\mu$ gives the following representation:

$$\mathbf{E}\big(S_T^\mu \mid \mathcal{F}_t^B\big) = \mathbf{E} S_T^\mu + \int_0^t \frac{\partial f}{\partial x}\,(s, B_s^\mu, S_s^\mu)\, dB_s, \tag{28}$$

where $\partial f / \partial x$ is a partial derivative of the function $f = f(t, x, s)$.

From (27) and (28) we obtain

$$\mathbf{E}\big(S_T^\mu \mid \mathcal{F}_t^B\big) = \mathbf{E} S_T^\mu + \int_0^t \big[1 - F_{T-u}(S_u^\mu - B_u^\mu)\big]\, dB_u. \tag{29}$$

In the case of the Brownian motion with a drift (see [3], [7])

$$F_{T-u}(s) = \mathbf{P}\{S_{T-u} \leqq s\} = \Phi\bigg(\frac{s - \mu(T-u)}{\sqrt{T-u}}\bigg) - e^{2\mu s}\Phi\bigg(\frac{-s - \mu(T-u)}{\sqrt{T-u}}\bigg).$$

Thus

$$\mathbf{E}\big(S_T^\mu \mid \mathcal{F}_t^B\big) = \mathbf{E} S_T^\mu + \int_0^t H_u^\mu\, dB_u, \tag{30}$$

where

$$H_u^\mu = 1 - \Phi\biggl(\frac{(S_u^\mu - B_u^\mu) - \mu(T-u)}{\sqrt{T-u}}\biggr) + \exp\bigl\{2\mu(S_u^\mu - B_u^\mu)\bigr\}\,\Phi\biggl(\frac{-(S_u^\mu - B_u^\mu) - \mu(T-u)}{\sqrt{T-u}}\biggr). \tag{31}$$

If $\mu = 0$ from (30) and (31), we obtain representation (4).

3. Case $S_{T_{-a}} = \max_{t \leqq T_{-a}} B_t$, $a > 0$.

3.1. Since $\mathbf{E}S_{T_{-a}} = \infty$, direct use of the associated martingale becomes impossible. Therefore we shall consider a "good" enough nonnegative function $f = f(x)$ such that $\mathbf{E}f(S_{T_{-a}}) < \infty$ (see further conditions in Theorem 2).

Denote $F = f(S_{T_{-a}})$ and let $(F_t)_{t \geqq 0}$ be the Lévy martingale associated with F, $F_t = \mathbf{E}(F \,|\, \mathcal{F}_t^B)$.

Fix now some $t \geqq 0$. Then on the set $\{t < T_{-a}\}$

$$F_t = \mathbf{E}\bigl(f(S_{T_{-a}}) \,\big|\, \mathcal{F}_t^B\bigr) = \mathbf{E}\biggl[f\biggl(\max\biggl(S_t, \sup_{t \leqq u \leqq T_{-a}} B_u\biggr)\biggr) \,\bigg|\, \mathcal{F}_t^B\biggr].$$

If we put $\widehat{B}_u = B_{u+t} - B_t$, $u \geqq 0$, and $\widehat{T}_{-b} = \inf\{u > 0\colon\ \widehat{B}_u = -b\}$, $b > 0$, then we find that identically

$$\begin{aligned} \sup_{t \leqq u \leqq T_{-a}} B_u &= B_t + \sup_{t \leqq u \leqq T_{-a}} (B_u - B_t) = B_t + \sup_{0 \leqq v \leqq T_{-a}+t} (B_{v+t} - B_t) \\ &= B_t + \sup_{0 \leqq u \leqq \widehat{T}_{-(a+B_t)}} (B_{u+t} - B_t), \end{aligned} \tag{32}$$

where the last equality follows from the fact that on the set $\{t < T_{-a}\}$

$$\begin{aligned} \widehat{T}_{-(a+B_t)} &= \inf\bigl\{u \geqq 0\colon\ \widehat{B}_u = -(a+B_t)\bigr\} = \inf\bigl\{u \geqq 0\colon\ B_{u+t} - B_t = -(a+B_t)\bigr\} \\ &= \inf\{u \geqq 0\colon\ B_{u+t} = -a\} = T_{-a} - t. \end{aligned}$$

From (32) it follows that on $\{t < T_{-a}\}$

$$\sup_{t \leqq u \leqq T_{-a}} B_u = B_t + \widehat{S}_{\widehat{T}_{-(a+B_t)}},$$

where $\widehat{S}_t = \sup_{u \leqq t} \widehat{B}_u$, and, thus, on this set

$$F_t = \mathbf{E}\Bigl[f\Bigl(\max\Bigl(S_t, B_t + \widehat{S}_{\widehat{T}_{-(a+B_t)}}\Bigr)\Bigr) \,\Big|\, \mathcal{F}_t\Bigr]. \tag{33}$$

3.2. For further simplification of the right-hand side of equality (33) we need the following lemma (known to specialists), given here together with its proof, for completeness of the statement.

Lemma 3. *Let $M = (M_t)_{t \geqq 0}$ be a continuous nonnegative martingale ($M \in \mathcal{M}^{\mathrm{c}}$, $M \geqq 0$) with $M_0 = b > 0$ and $M_\infty \equiv \lim_{t\to\infty} M_t = 0$. Then*

$$\sup_{t \geqq 0} M_t \overset{\text{law}}{=} \frac{b}{U},$$

where $U = U[0,1]$ *is a random variable uniformly distributed on* $[0,1]$, *i.e.*,

$$\mathbf{P}\Big\{\sup_{t\geqq 0} M_t \leqq b+x\Big\} = \frac{x}{b+x}, \qquad x \geqq 0. \tag{34}$$

Proof. Let $T_{b+x} = \inf\{t \geqq 0\colon\ M_t = b+x\}$ and $\widetilde{M}_t = M_{t\wedge T_{b+x}}$. The martingale $\widetilde{M} = (\widetilde{M}_t)_{t\geqq 0}$ is bounded from above and from below ($0 \leqq \widetilde{M}_t \leqq b+x$). Therefore according to the Doob stopping theorem, $\mathbf{E}\widetilde{M}_0 = \mathbf{E}\widetilde{M}_{T_{b+x}}$, and thus

$$\begin{aligned} b = \mathbf{E}\widetilde{M}_0 &= \mathbf{E}\widetilde{M}_{T_{b+x}} = \mathbf{E}M_{T_{b+x}} = \mathbf{E}M_{T_{b+x}} I(T_{b+x}<\infty) \\ &= (b+x)\,\mathbf{P}\{T_{b+x}<\infty\} = (b+x)\,\mathbf{P}\Big\{\sup_{t\geqq 0} M_t \geqq b+x\Big\}, \end{aligned}$$

which proves (34).

Applying this lemma to $M_t = b + \widehat{B}_{t\wedge \widehat{T}_{-b}}$, we see that

$$\mathbf{P}\Big\{\sup_{t\leqq \widehat{T}_{-b}} \widehat{B}_t > x\Big\} = \mathbf{P}\Big\{\sup_{t\geqq 0} M_t > b+x\Big\} = \frac{b}{b+x}$$

or

$$\mathbf{P}\Big\{\sup_{t\leqq \widehat{T}_{-b}} \widehat{B}_t \in dx\Big\} = \frac{b\,dx}{(b+x)^2}, \qquad x \geqq 0. \tag{35}$$

From (33) and (35), and since the Brownian motion $\widehat{B} = (\widehat{B}_u)_{u\geqq 0}$ is independent of $\mathcal{F}_t$, we find that on the set $\{t \leqq T_{-a}\}$

$$F_t = \int_0^\infty f\big(\max(S_t, B_t + x)\big)\,\frac{b\,dx}{(b+x)^2}, \qquad b = a + B_t. \tag{36}$$

If $B_t + x \leqq S_t$ (i.e., $x \leqq S_t - B_t$), then

$$\begin{aligned} F_t &= \int_0^{S_t - B_t} f(S_t)\,\frac{b\,dx}{(b+x)^2} + \int_{S_t-B_t}^\infty f(B_t+x)\,\frac{b\,dx}{(b+x)^2} \\ &= f(S_t)\,\frac{S_t - B_t}{a+S_t} + \int_0^\infty f(S_t+y)\,\frac{b\,dy}{(a+S_t+y)^2} \\ &= f(S_t)\,\frac{S_t - B_t}{a+S_t} + (a+B_t)\int_0^\infty f(S_t+y)\,\frac{dy}{(a+S_t+y)^2} \\ &= \bigg[f(S_t)\,\frac{S_t}{a+S_t} + a\int_0^\infty \frac{f(S_t+y)}{(a+S_t+y)^2}\,dy\bigg] \\ &\quad + B_t\bigg[-\frac{f(S_t)}{a+S_t} + \int_0^\infty \frac{f(S_t+y)}{(a+S_t+y)^2}\,dy\bigg] \equiv g(S_t) + B_t h(S_t), \end{aligned} \tag{37}$$

where

$$\begin{aligned} g(\sigma) &= \frac{\sigma f(\sigma)}{a+\sigma} + a\int_0^\infty \frac{f(\sigma+y)}{(a+\sigma+y)^2}\,dy, \\ h(\sigma) &= -\frac{f(\sigma)}{a+\sigma} + \int_0^\infty \frac{f(\sigma+y)}{(a+\sigma+y)^2}\,dy = \int_0^\infty \big[f(\sigma+y) - f(\sigma)\big]\,\frac{dy}{(a+\sigma+y)^2}. \end{aligned}$$

The process $S=(S_t)_{t\geqq 0}$ is a bounded variation process. Therefore from (37), by the Itô formula we find that on the set $\{t<T_{-a}\}$

$$dF_t=\big[g'(S_t)+B_t h'(S_t)\big]\,dS_t+h(S_t)\,dB_t. \tag{38}$$

However, $B_t h'(S_t)\,dS_t=S_t h'(S_t)\,dS_t$ (in the sense of satisfying the appropriate integral relation). From here and (38), taking account of the easily checked equality $g'(\sigma)+\sigma h'(\sigma)=0$, we obtain that on the set $\{t\leqq T_{-a}\}$

$$F_t=F_0+\int_0^t h(S_u)\,dB_u=g(0)+\int_0^t h(S_u)\,dB_u. \tag{39}$$

Remark. One may anticipate a "disappearance"of the first summand in the right-hand side of (38) and not applying the equality $g'(\sigma)+\sigma h'(\sigma)=0$, since the process $(F_t)_{t\geqq 0}$ should be a martingale, and a stochastic differential with respect to dB_t in the integral form gives a local martingale.

From (39), by limiting pass $t\uparrow T_{-a}$ we arrive at the following statement.

THEOREM 2. *Let the function $f=f(x)$, $x\geqq 0$, be such that for $a>0$, $\sigma\geqq 0$*

$$\int_0^\infty \frac{|f(\sigma+y)|}{(a+\sigma+y)^2}\,dy<\infty$$

and

$$\int_0^{T_{-a}} h^2(S_u)\,du<\infty \qquad (\mathbf{P}\text{-}a.s.).$$

Then the following stochastic integral representation is valid:

$$f\Big(\max_{u\leqq T_{-a}} B_u\Big)=\mathbf{E}f\Big(\max_{u\leqq T_{-a}} B_u\Big)+\int_0^{T_{-a}} h(S_u)\,dB_u, \tag{40}$$

where

$$\mathbf{E}f\Big(\max_{u\leqq T_{-a}} B_u\Big)=\int_0^\infty \frac{af(y)}{(a+y)^2}\,dy. \tag{41}$$

COROLLARY. *From* (40)

$$I\Big(\max_{u\leqq T_{-a}} B_u>z\Big)=\frac{a}{a+z}+\int_0^{T_{-a}}\frac{I(S_u<z)}{a+z}\,dB_u. \tag{42}$$

It is interesting to note that (42) is equivalent to

$$(a+z)\,I(S_{T_{-a}}>z)=a+B_{T_{-a}\wedge T_z}. \tag{43}$$

This equality is obvious. Indeed, on the set $\{S_{T_{-a}}>z\}$ the left-hand side is equal to $a+z$, and the right-hand side is also equal to $a+z$, since $B_{T_{-a}\wedge T_z}=z$ on the set $\{S_{T_{-a}}>z\}$. If $S_{T_{-a}}\leqq z$, then both sides in (43) are equal to zero.

3.3. Let us return to representation (42). Since $\int_0^{T_{-a}} dB_u=-a$ ($\mathbf{P}$-a.s.), relation (42) can be rewritten in the form

$$I\Big(\max_{u\leqq T_{-a}} B_u>z\Big)=-\int_0^{T_{-a}}\frac{I(z<S_u)}{a+z}\,dB_u.$$

From here, using the Fubini theorem for a stochastic integral, we find

$$\begin{aligned} S_{T_{-a}} &= \max_{u \leqq T_{-a}} B_u = \int_0^\infty I(S_{T_{-a}} > z)\, dz = -\int_0^\infty \left[\int_0^{T_{-a}} \frac{I(z < S_u)}{a+z}\, dB_u \right] dz \\ &= -\int_0^{T_{-a}} \left[\int_0^\infty \frac{I(z < S_u)}{a+z}\, dz \right] dB_u = -\int_0^{T_{-a}} \left[\int_0^{S_u} \frac{dz}{a+z} \right] dB_u \\ &= -\int_0^{T_{-a}} \log \frac{a+S_u}{a}\, dB_u. \end{aligned}$$

Hence, *for* $S_{T_{-a}} = \max_{u \leqq T_{-a}} B_u$ *the following stochastic integral representation:*

$$S_{T_{-a}} = -\int_0^{T_{-a}} \log\left(1 + \frac{S_u}{a}\right) dB_u \tag{44}$$

is valid. As was already noted, $\mathbf{E} \max_{u \leqq T_{-a}} B_u = \infty$. Thus (44) gives an example of the stochastic integral representation of a functional of Brownian motion with infinite mathematical expectation.

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