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COMPLEX COBORDISM CLASSES OF HOMOGENEOUS SPACES

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ABSTRACT. We consider compact homogeneous spaces G/H of positive Euler characteristic endowed with an invariant almost complex structure J and the canonical action θ of the maximal torus T^k on G/H. We obtain explicit formula for the cobordism class of such manifold through the weights of the action θ at the identity fixed point eH by an action of the quotient group W_G/W_H of the Weyl groups for G and H. In this way we show that the cobordism class for such manifolds can be computed explicitly without information on their cohomology. We also show that formula for cobordism class provides an explicit way for computing the classical Chern numbers for $(G/H, J)$. As a consequence we obtain that the Chern numbers for $(G/H, J)$ can be computed without information on cohomology for G/H. As an application we provide an explicit formula for cobordism classes and characteristic numbers of the flag manifolds $U(n)/T^n$, Grassmann manifolds $G_{n,k} = U(n)/(U(k) \times U(n-k))$ and some particular interesting examples.

1. INTRODUCTION

In this paper we consider the problem of description of complex cobordism classes of homogeneous spaces G/H endowed with an invariant almost complex structure, where G is compact connected Lie group and H is its closed connected subgroup of maximal rank. These spaces are classical manifolds and have a very reach geometric structure from the different points of view and our interest in these manifolds is related to the well known problem in cobordism theory to find the representatives in cobordism classes that have reach geometric structure. Our interest in research of the homogeneous spaces G/H with positive Euler characteristic, is also stimulated by well known relations between the cohomology rings of these spaces and the deep problems in the theory of representations and combinatorics (see, for example [11]). These problems are formulated in terms of different additive basis in cohomology rings for G/H and multiplicative rules related to that basis. We hope the research of the cobordisms of the spaces G/H to give the new relations and bring the new results in that direction.

We use the approach based on the notion of Chern-Dold character originally introduced in [3] and the notion of universal toric genus introduced in [4] and described in details in [6].The universal toric genus can be constructed for any even dimensional manifold M^{2n} with a given torus action and stable complex structure which is equivariant under given torus action. Moreover, if the set of isolated fixed points for this action is finite that the universal toric genus can be localized, which means that it can be explicitly written through the weights and the signs at the fixed points for the representations that gives arise from the given torus action.

The construction of the toric genus is reduced to the computation of Gysin homomorphism of 1 in complex cobordisms for fibration whose fiber is M^{2n} and the base is classifying space of the torus. The problem of the localization of Gysin homomorphism is very known and it was studied by many authors, starting with 60-es of the last century. In [4] and [6] is obtained explicit answer

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for this problem in the terms of the torus action on tangent bundle for M^{2n}. The history of this problem is presented also in these papers.

If consider homogeneous space G/H with $\operatorname{rk} G = \operatorname{rk} H = k$, then we have on it the canonical action θ of the maximal torus T^k for H and G, and any invariant almost complex structure on G/H is compatible with this action. Besides that, all fixed points for the action θ are isolated, so one can apply localization formula to compute universal toric genus for this action and any invariant almost complex structure. Since, in this case, we consider almost complex structures, all fixed points in the localization formula are going to have sign $+1$. We prove that the weights for the action θ at different fixed points can be obtained by an action of the Weyl group W_G up to an action of the Weyl group W_H on the weights for θ at identity fixed point. On this way we get an explicit formula for the cobordism classes of such spaces in terms of the weights at the fixed point eH. This formula also shows that the cobordism class for G/H related to an invariant almost complex structure can be computed without information about cohomology for G/H.

We obtain also the explicit formulas, in terms of the weights at identity fixed point, for the cohomology characteristic numbers for homogeneous spaces of positive Euler characteristic endowed with an invariant almost complex structure. We use further that the cohomology characteristic numbers $s_\omega,\ \omega = (i_1, \dots, i_n)$, and classical Chern numbers $c^\omega = c_1^{i_1} \cdots c_n^{i_n}$ are related by some standard relations from the theory of symmetric polynomials. This fact together with the obtained formulas for the characteristic numbers $s_\omega(\tau(G/H))$ proves that the classical Chern numbers $c^\omega(\tau(G/H))$ for the homogeneous spaces under consideration can be computed without information on their cohomology. It also gives an explicit way for the computation of the classical Chern numbers.

We provide an application of our results by obtaining explicit formula for the cobordism class and top cohomology characteristic number of the flag manifolds $U(n)/T^n$ and Grassmann manifolds $G_{n,k} = U(n)/(U(k) \times U(n-k))$ related to the standard complex structures. We want to emphasize that, our method when applying to the flag manifolds and Grassmann manifolds gives the description of their cobordism classes and characteristic numbers using the technique of *divided difference operators*. Our method also makes possible to compare cobordism classes that correspond to the different invariant almost complex structures on the same homogeneous space. We illustrate that on the space $U(4)/(U(1) \times U(1) \times U(2))$, which is firstly given in [7] as an example of homogeneous space that admits two different invariant complex structure.

This paper comes out from the first part of our work where we mainly considered invariant almost complex structures on homogeneous spaces of positive Euler characteristic. It has continuation which is going to deal with the same questions, but related to the stable complex structures equivariant under given torus action on homogeneous spaces of positive Euler characteristic.

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2. Universal toric genus

We will recall the results from [4], [5] and [6].

2.1. **General setting.** In general setting one considers $2n$-dimensional manifold M^{2n} with a given smooth action θ of the torus T^k. We say that (M^{2n}, θ, c_τ) is *tangentially stable complex* if it admits θ-equivariant stable complex structure c_τ. This means that there exist $l \in \mathbb{N}$ and complex vector bundle ξ such that

$$c_\tau \colon \tau(M^{2n}) \oplus \mathbb{R}^{2(l-n)} \longrightarrow \xi \tag{1}$$

is real isomorphism and the composition

$$r(t)\colon \xi \xrightarrow{c_\tau^{-1}} \tau(M^{2n}) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{d\theta(t)\oplus I} \tau(M^{2n}) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{c_\tau} \xi \tag{2}$$

is a complex transformation for any $t \in T^k$.

If there exists ξ such that $c_\tau\colon \tau(M^{2n}) \longrightarrow \xi$ is an isomorphism, i. e. $l = n$, then (M^{2n}, θ, c_τ) is called *almost complex* T^k-manifold.

Denote by $\Omega_U^*[[u_1, \ldots, u_k]]$ an algebra of formal power series over $\Omega_U^* = U^*(pt)$. It is well known [19] that $U^*(pt) = \Omega_U^* = \mathbb{Z}[y_1, \ldots, y_n, \ldots]$, where $\dim y_n = -2n$. Moreover, as the generators for Ω_U^* over the rationales, or in other words for $\Omega_U^* \otimes \mathbb{Q}$, can be taken the family of cobordism classes $[\mathbb{C}P^n]$ of the complex projective spaces.

When given a θ-equivariant stable complex structure c_τ on M^{2n}, we can always choose θ-equivariant embedding $i\colon M^{2n} \to \mathbb{R}^{2(n+m)}$, where $m > n$, such that c_τ determines, up to natural equivalence, a θ-equivariant complex structure c_ν on the normal bundle $\nu(i)$ of i. Therefore, one can define the universal toric genus for (M^{2n}, θ, c_τ) in complex cobordisms, see [4], [6].

We want to note that, in the case when c_τ is almost complex structure, an universal toric genus for (M^{2n}, θ, c_τ) is completely defined in terms of the action θ on tangent bundle $\tau(M^{2n})$.

The universal toric genus for (M^{2n}, θ, c_τ) could be looked at as an element in algebra $\Omega_U^*[[u_1, \ldots, u_k]]$. It is defined with

$$\Phi(M^{2n}, \theta, c_\tau) = [M^{2n}] + \sum_{|\omega|>0} [G_\omega(M^{2n})] u^\omega \ , \tag{3}$$

where $\omega = (i_1, \ldots, i_k)$ and $u^\omega = u_1^{i_1} \cdots u_k^{i_k}$.

Here by $[M^{2n}]$ is denoted the complex cobordism class of the manifold M^{2n} with stable complex structure c_τ, by $G_\omega(M^{2n})$ is denoted the stable complex manifold obtained as the total space of the fibration $G_\omega \to B_\omega$ with fiber M. The base $B_\omega = \prod_{j=1}^k B_j^{i_j}$, where $B_j^{i_j}$ is Bott tower, i. e. i_j-fold iterated two-sphere bundle over $B_0 = pt$. The base B_ω satisfies $[B_\omega] = 0$, $|\omega| > 0$, where $|\omega| = \sum\limits_{j=1}^{k} i_j$.

2.2. **The action with isolated fixed points.** Assume that the action of T^k on M^{2n} has isolated fixed points. We first introduce, following [4], the general notion of the *sign* at isolated fixed point. Let p be an isolated fixed point. The representation $r_p\colon T^k \to GL(l, \mathbb{C})$ associated to (2) produces the decomposition of the fiber $\xi_p \cong \mathbb{C}^l$ as $\xi_p \cong \mathbb{C}^{l-n} \oplus \mathbb{C}^n$. In this decomposition r_p acts trivially on $\mathbb{C}^{l-n}$ and without trivial summands on $\mathbb{C}^n$. From the other hand the isomorphism $c_{\tau,p}$ from (1) defines an orientation in the tangent space $\tau_p(M)$. This together leads to the following definition.

Definition 1. The $\mathrm{sign}(p)$ at isolated fixed point p is $+1$ if the map

$$\tau_p(M^{2n}) \xrightarrow{I\oplus 0} \tau_p(M^{2n}) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{c_{\tau,p}} \xi_p \cong \mathbb{C}^n \oplus \mathbb{C}^{l-n} \xrightarrow{\pi} \mathbb{C}^n \ ,$$

preserves orientation. Otherwise, $\mathrm{sign}(p)$ is -1.

Remark 1. Note that for an almost complex T^k-manifold M^{2n}, it directly follows from the definition that $\mathrm{sign}(p) = +1$ for any isolated fixed point.

If an action θ of T^k on M^{2n} has only isolated fixed points, then it is proved that toric genus for M^{2n} can be completely described using just local data at the fixed points, [4], [6].

Namely, let p again be an isolated fixed point. Then the non trivial summand of r_p from (2) gives rise to the tangential representation of T^k in $GL(n,\mathbb{C})$. This representation decomposes into n non-trivial one-dimensional representations of $r_{p,1}\oplus\ldots\oplus r_{p,n}$ of T^k. Each of the representations $r_{p,j}$ can be written as

$$r_{p,j}(e^{2\pi i x_1},\ldots,e^{2\pi i x_k})v = e^{2\pi i\langle\Lambda_j(p),\mathbf{x}\rangle}v\ ,$$

for some $\Lambda_j(p) = (\Lambda_j^1(p),\ldots,\Lambda_j^k(p)) \in \mathbb{Z}^k$, where $\mathbf{x} = (x_1,\ldots,x_k) \in \mathbb{R}^k$ and $\langle\Lambda_j(p),\mathbf{x}\rangle = \sum\limits_{l=1}^{k}\Lambda_j^l(p)x_l$. The sequence $\{\Lambda_1(p),\ldots,\Lambda_n(p)\}$ is called *the weight vector* for representation r_p in the fixed point p.

Let $F(u,v) = u+v+\sum\alpha_{ij}u^iv^j$ be *the formal group for complex cobordism* [20]. The corresponding power system $\{[w](u) \in \Omega^*[[u]] : w \in \mathbb{Z}\}$ is uniquely defined with $[0](u) = 0$ and $[w](u) = F(u,[w-1])(u)$, for $w \in \mathbb{Z}$. For $\mathbf{w} = (w_1,\ldots,w_k) \in \mathbb{Z}^k$ and $\mathbf{u} = (u_1,\ldots,u_k)$ one defines $[\mathbf{w}](\mathbf{u})$ inductively with $[\mathbf{w}](\mathbf{u}) = [w](u)$ for $k = 1$ and

$$[\mathbf{w}](\mathbf{u}) = F_{q=1}^{k}[w_q](u_q) = F(F_{q=1}^{k-1}[w_q](u_q),[w_k](u_k))\ ,$$

for $k \geqslant 2$. Then for toric genus of the action θ with isolated fixed points the following localization formula holds, which is first formulated in [4] and proved in details in [6].

Theorem 1. *If the action θ has a finite set P of isolated fixed points then*

$$\Phi(M^{2n},\theta,c_\tau) = \sum_{p\in P}\mathrm{sign}(p)\prod_{j=1}^{n}\frac{1}{[\Lambda_j(p)](\mathbf{u})} \tag{4}$$

and it is equal to $[M^{2n}] + \mathfrak{L}(\mathbf{u})$, where $\mathfrak{L}(\mathbf{u}) \in \Omega_U^*[[u_1,\ldots,u_k]]$ and $\mathfrak{L}(\mathbf{0}) = 0$.

2.3. **Chern-Dold character.** In review of the basic definitions and results on Chern character we follow [3].

Let U^* be the theory of unitary cobordisms.

Definition 2. The Chern-Dold character for a topological space X in the theory of unitary cobordisms U^* is a ring homomorphism

$$ch_U : U^*(X) \to H^*(X,\Omega_U^*\otimes\mathbb{Q})\ . \tag{5}$$

Recall that the Chern-Dold character as a multiplicative transformation of cohomology theories is uniquely defined by the condition that for $X = (pt)$ it gives canonical inclusion $\Omega_U^* \to \Omega_U^*\otimes\mathbb{Q}$.

The Chern-Dold character splits into composition

$$ch_U : U^*(X) \to H^*(X,\Omega_U^*(\mathbb{Z})) \to H^*(X,\Omega_U^*\otimes\mathbb{Q})\ . \tag{6}$$

The ring $\Omega_U^*(\mathbb{Z}))$ in (6) is firstly described in [3]. It is a subring of $\Omega_U^*\otimes\mathbb{Q}$ generated by the elements from $\Omega_U^{-2n}\otimes\mathbb{Q}$ having integers Chern numbers. It is equal to

$$\Omega_U^*(\mathbb{Z}) = \mathbb{Z}[b_1,\ldots,b_n,\ldots]\ ,$$

where $b_n = \frac{[\mathbb{C}P^n]}{n+1}$.

Then the Chern character leaves $[M^{2n}]$ invariant, i. e.

$$ch_U([M^{2n}]) = [M^{2n}]\ , \tag{7}$$

and ch_U is the homomorphism of Ω_U^*-modules

It follows from the its description [3] that the Chern-Dold character $ch_U : U^*(X) \to H^*(X, \Omega_U^*(\mathbb{Z}))$ as a multiplicative transformation of the cohomology theories is given by the series

$$ch_U u = h(x) = \frac{x}{f(x)}, \quad \text{where} \quad f(x) = 1 + \sum_{i=1}^{\infty} a_i x^i \quad \text{and} \quad a_i \in \Omega_U^{-2i}(\mathbb{Z}) \ .$$

Here $u = c_1^U(\eta) \in U^2(\mathbb{C}P^\infty),\ x = c_1^H(\eta) \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$ denote the first Chern classes of the universal complex line bundle $\eta \to \mathbb{C}P^\infty$.

From the construction of Chern-Dold character follows also the equality

$$ch_U([M^{2n}]) = [M^{2n}] = \sum_{\|\omega\|=n} s_\omega(\tau(M^{2n}))a^\omega \ , \tag{8}$$

where $\omega = (i_1, \dots, i_n),\ \|\omega\| = \sum_{l=1}^n l \cdot i_l$ and $a^\omega = a_1^{i_1} \cdots a_n^{i_n}$. Here the numbers $s_\omega(\tau(M^{2n})), \|\omega\| = n$ are *the cohomology characteristic numbers* of M^{2n}.

If on M^{2n} is given torus action θ of T^k and stable complex structure c_τ which is θ-equivariant, then the Chern character of its toric genus is

$$ch_U(\Phi(M^{2n}, \theta, c_\tau)) = [M^{2n}] + \sum_{\|\omega\|>0} [G_\omega(M^{2n})](ch_U \mathbf{u})^\omega \ , \tag{9}$$

where $ch_U \mathbf{u} = (ch_U u_1, \dots, ch_U u_k)$ and $ch_U u_i = \frac{x_i}{f(x_i)}$.

We have that $F(u, v) = g^{-1}(g(u) + g(v))$, where $g(u) = u + \sum\limits_{n>0} \frac{[\mathbb{C}P^n]}{n+1} u^{n+1}$ (see [20]) is *the logarithm of the formal group* $F(u, v)$ and $g^{-1}(u)$ is the function inverse to the series $g(u)$. Using that $ch_U g(u) = x$ (see [3]), we obtain $ch_U F(u_1, u_2) = h(x_1 + x_2)$ and therefore

$$ch_U[\Lambda_j(p)](u) = \frac{\langle \Lambda_j(p), \mathbf{x} \rangle}{f(\langle \Lambda_j(p), \mathbf{x} \rangle)} \ .$$

Applying these results to the theorem (1) we get

$$ch_U(\Phi(M^{2n}, \theta, c_\tau)) = \sum_{p \in P} \operatorname{sign}(p) \prod_{j=1}^n \frac{f(\langle \Lambda_j(p), \mathbf{x} \rangle)}{\langle \Lambda_j(p), \mathbf{x} \rangle} \ . \tag{10}$$

The formulas (9) and (10) gives that

$$\sum_{p \in P} \operatorname{sign}(p) \prod_{j=1}^n \frac{f(\langle \Lambda_j(p), \mathbf{x} \rangle)}{\langle \Lambda_j(p), \mathbf{x} \rangle} = [M^{2n}] + \sum_{\|\omega\|>0} [G_\omega(M^{2n})](ch_U \mathbf{u})^\omega \ . \tag{11}$$

If in the left hand side of this equation we put $t\mathbf{x}$ instead of $\mathbf{x}$ and then multiplying it with t^n we obtain the following result.

Proposition 1. *The coefficient for t^n in the series in t*

$$\sum_{p \in P} \operatorname{sign}(p) \prod_{j=1}^n \frac{f(t\langle \Lambda_j(p), \mathbf{x} \rangle)}{\langle \Lambda_j(p), \mathbf{x} \rangle}$$

represents the complex cobordism class $[M^{2n}]$.

Proposition 2. *The coefficient for t^l in the series in t*

$$\sum_{p\in P}\operatorname{sign}(p)\prod_{j=1}^{n}\frac{f(t\langle\Lambda_j(p),\mathbf{x}\rangle)}{\langle\Lambda_j(p),\mathbf{x}\rangle}$$

is equal to zero for $0\leqslant l\leqslant n-1$.

3. Torus action on homogeneous spaces with positive Euler characteristic.

Let G/H be a compact homogeneous space of positive Euler characteristic. It means that G is a compact connected Lie group and H its connected closed subgroup, such that $\operatorname{rk}G=\operatorname{rk}H$. Let T be the maximal common torus for G and H. There is canonical action θ of T on G/H given by $t(gH)=(tg)H$, where $t\in T$ and $gH\in G/H$. Denote by $N_G(T)$ the normalizer of the torus T in G. Then $W_G=N_G(T)/T$ is the Weyl group for G. For the set of the fixed points for the action θ we prove the following.

Proposition 3. *The set of fixed points under the canonical action θ of T on G/H is given by $(N_G(T))\cdot H$.*

Proof. It is easily to see that gH is fixed point for θ for any $g\in N_G(T)$. If gH is the fixed point under the canonical action of T on G/H then $t(gH)=gH$ for all $t\in T$. It follows that $g^{-1}tg\in H$ for all $t\in T$, i. e. $g^{-1}Tg\subset H$. This gives that $g^{-1}Tg$ is a maximal torus in H and, since any two maximal toruses in H are conjugate, it follows that $g^{-1}Tg=h^{-1}Th$ for some $h\in H$. Thus, $(gh)^{-1}T(gh)=T$ what means that $gh\in N_G(T)$. But, $(gh)H=gH$, what proves the statement. □

Since $T\subset N_G(T)$ leaves H fixed, the following Lemma is direct implication of the Proposition 3.

Lemma 1. *The set of fixed points under the canonical action θ of T on G/H is given by $W_G\cdot H$.*

Regarding the number of fixed points, it holds the following.

Lemma 2. *The number of fixed points under the canonical action θ of T on G/H is equal to the Euler characteristic $\chi(G/H)$.*

Proof. Let $g,g^{'}\in N_G(T)$ are representatives of the same fixed point. Then $g^{'}g^{-1}\in H$ and $g^{-1}Tg=T=(g^{'})^{-1}Tg^{'}$, what gives that $g^{'}g^{-1}Tg(g^{'})^{-1}=T$ and, thus, $g^{'}g^{-1}\in N_H(T)$. This implies that the number of fixed points is equal to

$$\Big\|\frac{N_G(T)}{N_H(T)}\Big\|=\frac{\|\frac{N_G(T)}{T}\|}{\|\frac{N_H(T)}{T}\|}=\frac{\|W_G\|}{\|W_H\|}=\chi(G/H)\ . \tag{12}$$

The last equality is classical result related to equal ranks homogeneous spaces, see [21]. □

Remark 2. The proof of the Lemma 2 gives that the set of fixed points under the canonical action θ of T on G/H can be obtained as an orbit of eH by an action of the Weyl group W_G up to an action of the Weyl group W_H.

4. THE WEIGHTS AT THE FIXED POINTS.

Denote by $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{t}$ the Lie algebras for G, H respectively and $T = T^k$, where $k = \operatorname{rk} G = \operatorname{rk} H$. Let $\alpha_1, \ldots, \alpha_m$ be the roots for $\mathfrak{g}$ related to $\mathfrak{t}$, where $\dim G = 2m + k$. Recall that the roots for $\mathfrak{g}$ related to $\mathfrak{t}$ are the weights for the adjoint representation Ad_T of T which is given with $\mathrm{Ad}_T(t) = d_e \operatorname{ad}(t)$, where $\operatorname{ad}(t)$ is inner automorphism of G defined by the element $t \in T$. One can always choose the roots for G such that $\alpha_{n+1}, \ldots, \alpha_m$ gives the roots for $\mathfrak{h}$ related to $\mathfrak{t}$, where $\dim H = 2(m-n) + k$. The roots $\alpha_1, \ldots, \alpha_n$ are called the *complementary* roots for $\mathfrak{g}$ related to $\mathfrak{h}$. Using root decomposition for $\mathfrak{g}$ and $\mathfrak{h}$ it follows that $T_e(G/H) \cong \mathfrak{g}^{\mathbb{C}}_{\alpha_1} \oplus \ldots \oplus \mathfrak{g}^{\mathbb{C}}_{\alpha_n}$, where by $\mathfrak{g}_{\alpha_i}$ is denoted the root subspace defined with the root α_i and $T_e(G/H)$ is the tangent space for G/H at the $e \cdot H$. It is obvious that $\dim_{\mathbb{R}} G/H = 2n$.

4.1. **Description of the invariant almost complex structures.** Assume we are given an invariant almost complex structure J on G/H. This means that J is invariant under the canonical action of G on G/H. Then according to the paper [7], we can say the following.

- Since J is invariant it commutes with adjoint representation Ad_T of the torus T. This implies that J induces the complex structure on each complementary root subspace $\mathfrak{g}_{\alpha_1}, \ldots, \mathfrak{g}_{\alpha_n}$. Therefore, J can be completely described by the root system $\varepsilon_1\alpha_1, \ldots, \varepsilon_n\alpha_n$, where we take $\varepsilon_i = \pm 1$ depending if J and adjoint representation Ad_T define the same orientation on $\mathfrak{g}_{\alpha_i}$ or not, where $1 \leqslant i \leqslant n$. The roots $\varepsilon_k\alpha_k$ are called *the roots of the almost complex structure* J.
- If we assume J to be integrable, it follows that it can be chosen an ordering on the canonical coordinates of $\mathfrak{t}$ such that the roots $\varepsilon_1\alpha_1, \ldots, \varepsilon_n\alpha_n$ which define J make the closed system of positive roots.

Let us assume that G/H admits an invariant almost complex structure. Consider an isotropy representation I_e of H in $T_e(G/H)$ and let it decomposes into s *real irreducible representations* $I_e = I_e^1 + \ldots + I_e^s$. Then it is proved in [7] that G/H admits exactly 2^s invariant almost complex structures. Because of completeness we recall the proof of this fact shortly here. Consider the decomposition of $T_e(G/H)$

$$T_e(G/H) = \mathfrak{I}_1 \oplus \ldots \oplus \mathfrak{I}_s$$

such that the restriction of I_e on $\mathfrak{I}_i$ is I_e^i. The subspaces $\mathfrak{I}_1, \ldots, \mathfrak{I}_s$ are invariant under T and therefore each of them is the sum of some root subspaces, i.e. $\mathfrak{I}_i = \mathfrak{g}_{\alpha_{i_1}} \oplus \ldots \oplus \mathfrak{g}_{\alpha_{i_j}}$, for some complementary roots $\alpha_{i_1}, \ldots, \alpha_{i_j}$. Any linear transformation that commutes with I_e leaves each of $\mathfrak{I}_i$ invariant. Since, by assumption G/H admits an invariant almost complex structure, we have at least one linear transformation without real eigenvalue that commutes with I_e. This implies that the commuting field for each of I_e^i is the field of complex numbers and, thus, on each $\mathfrak{I}_i$ we have exactly two invariant complex structures.

Remark 3. Note that this consideration shows that the numbers $\varepsilon_1, \ldots, \varepsilon_n$ that define an invariant almost complex structure may not vary independently.

Remark 4. In this paper we consider almost complex structures on G/H that are invariant under the canonical action of the group G, what, as we remarked, imposes some relations on $\varepsilon_1, \ldots, \varepsilon_n$. If we do not require G-invariance, but just T-invariance, we will have more degrees of freedom on $\varepsilon_1, \ldots, \varepsilon_n$. This paper is going to have continuation, where, among the other, the case of T-invariant structures will be studied.

Example 1. Since the isotropy representation for $\mathbb{C}P^n$ is irreducible over reals, it follows that on $\mathbb{C}P^n$ we have only two invariant almost complex structures, which are actually the standard complex structure and its conjugate.

Example 2. The flag manifold $U(n)/T^n$ admits 2^m invariant almost complex structure, where $m = \frac{n(n-1)}{2}$. By [7] only two of them, conjugate to each other, are integrable.

Example 3. As we already mentioned, the 10-dimensional manifold $M^{10} = U(4)/(U(1) \times U(1) \times U(2))$ is the first example of homogeneous space, where we have an existence of two non-equivalent invariant complex structures, see [7]. We will in the last section of this paper also describe cobordism class of M^{10} for these structures.

4.2. **The weights at the fixed points.** We fix now an invariant almost complex structure J on G/H and we want to describe the weights of the canonical action θ of T on G/H at the fixed points of this action. If gH is the fixed point for the action θ, then we obtain a linear map $d_g\theta(t)\colon T_g(G/H) \to T_g(G/H)$ for all $t \in T$. Therefore, this action gives rise to the complex representation $d_g\theta$ of T in $(T_g(G/H), J)$.

The weights for this representation at identity fixed point are described in [7].

Lemma 3. *The weights for the representation $d_e\theta$ of T in $(T_e(G/H), J)$ are given by the roots of an invariant almost complex structure J.*

Proof. Let us, because of clearness, recall the proof. The inner automorphism $\mathrm{ad}(t)$, for $t \in T$ induces the map $\overline{\mathrm{ad}}(t) : G/H \to G/H$ given with $\overline{\mathrm{ad}}(t)(gH) = t(gH)t^{-1} = (tg)H$. Therefore, $\theta(t) = \overline{\mathrm{ad}}(t)$ and, thus, $d_e\theta(t) = d_e\overline{\mathrm{ad}}(t)$ for any $t \in T$. This directly gives that the weights for $d_e\theta$ in $(T_e(G/H), J)$ are the roots that define J. □

For an arbitrary fixed point we prove the following.

Theorem 2. *Let gH be the fixed point for the canonical action θ of T on G/H. The weights of the induced representation $d_g\theta$ of T in $(T_g(G/H), J)$ can be obtained from the weights of the representation $d_e\theta$ of T in $(T_e(G/H), J)$ by the action of the Weyl group W_G up to the action of the Weyl group W_H.*

Proof. Note that Lemma 1 gives that an arbitrary fixed point can be written as $\mathrm{w}H$ for some $\mathrm{w} \in W_G/W_H$. Fix $\mathrm{w} \in W_G/W_H$ and denote by $l(\mathrm{w})$ the action of w on G/H, given by $l(\mathrm{w})gH = (\mathrm{w}g)H$ and by $\mathrm{ad}(\mathrm{w})$ the inner automorphism of G given by w.

We observe that $\theta \circ \mathrm{ad}(\mathrm{w}) = \mathrm{ad}(\mathrm{w}) \circ \theta$. Then $d_e\theta \circ d_e\,\mathrm{ad}(\mathrm{w}) = d_e\,\mathrm{ad}(\mathrm{w}) \circ d_e\theta$. This implies that the weights for $d_e\theta \circ d_e\,\mathrm{ad}(\mathrm{w})$ we get by the action of $d_e\,\mathrm{ad}(\mathrm{w})$ on the weights for $d_e\theta$. From the other hand $\theta(\mathrm{ad}(\mathrm{w})t)gH = (\mathrm{w}^{-1}t\mathrm{w}g)H = (l(\mathrm{w}^{-1}) \circ \theta(t) \circ l(\mathrm{w}))gH$ what implies that $d_e(\theta \circ \mathrm{ad}(\mathrm{w})) = d_{\mathrm{w}}l(\mathrm{w}^{-1}) \circ d_{\mathrm{w}}\theta \circ d_el(\mathrm{w})$. This gives that if, using the map $d_{\mathrm{w}}l_{\mathrm{w}^{-1}}$, we lift the weights for $d_{\mathrm{w}}\theta$ from $T_{\mathrm{w}}(G/H)$ to $T_e(G/H)$, we get that they coincide with the weights for $d_e\theta \circ d_e\,\mathrm{ad}(\mathrm{w})$. Therefore, the weights for $d_{\mathrm{w}}\theta$ we can get by the an action of the element w on the weights for $d_e\theta$. □

5. The cobordism classes of homogeneous spaces with positive Euler characteristic

Theorem 3. *Let G/H be homogeneous space of compact connected Lie group such that $\mathrm{rk}\,G = \mathrm{rk}\,H = k$ and $\dim G/H = 2n$ and consider the canonical action θ of maximal torus $T = T^k$ for*

G and H on G/H. Assume we are given an invariant almost complex structure J on G/H. Let $\Lambda_j = \varepsilon_j\alpha_j$, $1 \leqslant j \leqslant n$, where $\varepsilon_1\alpha_1, \ldots, \varepsilon_n\alpha_n$ are the complementary roots of G related to H which define an invariant almost complex structure J. Then the toric genus for $(G/H, J)$ is given with

$$\Phi(G/H, J) = \sum_{\mathrm{w}\in W_G/W_H} \prod_{j=1}^{n} \frac{1}{[\mathrm{w}(\Lambda_j)](\mathbf{u})} \ . \tag{13}$$

Proof. Rewriting the Theorem 1, since all fixed points have sign $+1$, we get that the toric genus for $(G/H, J)$ is

$$\Phi(G/H, J) = \sum_{p\in P} \prod_{j=1}^{n} \frac{1}{[\Lambda_j(p)](\mathbf{u})} \ , \tag{14}$$

where P is the set of isolated fixed points and $(\Lambda_1(p), \ldots, \Lambda_n(p))$ is the weight vector of the representation for T in $T_p(G/H)$ associated to an action θ. By Theorem 2, the set of fixed points P coincides with the orbit of the action of W_G/W_H on eH and also by Theorem 2 the set of weight vectors at fixed points coincides with the orbit of the action of W_G/W_H on the weight vector Λ at eH. The result follows if we put this data into formula (14). □

Corollary 1. *The Chern character of the toric genus for homogeneous space $(G/H, J)$ is given with*

$$ch_U\Phi(G/H, J) = \sum_{\mathrm{w}\in W_G/W_H} \prod_{j=1}^{n} \frac{f(\langle \mathrm{w}(\Lambda_j), \mathbf{x}\rangle)}{\langle \mathrm{w}(\Lambda_j), \mathbf{x}\rangle} \ , \tag{15}$$

where $f(t) = 1 + \sum\limits_{i\geqslant 1} a_i t^i$ for $a_i \in \Omega_U^{-2i}(\mathbb{Z})$, $\mathbf{x} = (x_1, \ldots, x_k)$ and by $\langle \Lambda_j, \mathbf{x}\rangle = \sum\limits_{l=1}^{k} \Lambda_j^l x_l$ is denoted the weight vector Λ_j of T^k-representation at $e \cdot H$.

Corollary 2. *The cobordism class for $(G/H, J)$ is given as the coefficient for t^n in the series in t*

$$\sum_{\mathrm{w}\in W_G/W_H} \prod_{j=1}^{n} \frac{f(t\langle \mathrm{w}(\Lambda_j), \mathbf{x}\rangle)}{\langle \mathrm{w}(\Lambda_j), \mathbf{x}\rangle} \ . \tag{16}$$

Remark 5. Since the weights of different invariant almost complex structures on the fixed homogeneous space differ only by sign, the Corollary 2 provides the way for comparing cobordism classes of two such structures without having their cobordism classes explicitly computed.

6. Characteristic numbers of homogeneous spaces with positive Euler characteristic.

6.1. **Generally about stable complex manifolds.** Let M^{2n} be tangentially stable complex manifold whose given action θ of the torus T^k on M^{2n} has only isolated fixed points. Denote by P the set of fixed points for θ and set $t_j(p) = \langle \Lambda_j(p), \mathbf{x}\rangle$, where $\{\Lambda_j(p),\ j = 1, \ldots, n\}$ are the weight vectors of the representation of T^k at a fixed point p given by the action θ and $\mathbf{x} = (x_1, \ldots x_k)$.

Set

$$\prod_{i=1}^{n} f(t_i) = 1 + \sum f_\omega(t_1, \ldots, t_n)a^\omega \ . \tag{17}$$

Using this notation the Proposition 2 could be formulated in the following way.

Proposition 4. *For any ω with $0 \leqslant \|\omega\| \leqslant (n-1)$ we have that*

$$\sum_{p\in P} \mathrm{sign}(p) \cdot \frac{f_\omega(t_1(p),\dots,t_n(p))}{t_1(p)\cdots t_n(p)} = 0 \ . \tag{18}$$

Note that the Proposition 4 gives the strong constraints on the set of signs $\{\mathrm{sign}(p)\}$ and the set of weights $\{\Lambda_j(p)\}$ at fixed points for some tangentially stable complex manifold. For example $\|\omega\| = 0$ and $\|\omega\| = 1$ gives that the signs and the weights at fixed points have to satisfy the following relations.

Corollary 3.

$$\sum_{p\in P} \mathrm{sign}(p) \cdot \frac{1}{t_1(p)\cdots t_n(p)} = 0 \ , \tag{19}$$

$$\sum_{p\in P} \mathrm{sign}(p) \cdot \frac{\sum\limits_{i=1}^{n} t_j(p)}{t_1(p)\cdots t_n(p)} = 0 \ . \tag{20}$$

As we already mentioned in (8) the cobordism class for M^{2n} can be represented as

$$[M^{2n}] = \sum_{\|\omega\|=n} s_\omega(\tau(M^{2n}))a^\omega \ , \tag{21}$$

where $\omega = (i_1,\dots,i_n)$, $\|\omega\| = \sum_{l=1}^{n} l\cdot i_l$ and $a^\omega = a_1^{i_1}\cdots a_n^{i_n}$.

If the given action θ of T^k on M^{2n} is with isolated fixed points, the coefficients $s_\omega(\tau(M^{2n}))$ can be explicitly described using Proposition 1 and expression (17).

Theorem 4. *Let M^{2n} be tangentially stable complex manifold whose given action θ of the T^k have only isolated fixed points. Denote by P the set of fixed points for θ and set $t_j(p) = \langle \Lambda_j(p), \mathbf{x}\rangle$, where $\Lambda_j(p)$ are the weight vectors of the representation of T^k given by the action θ and $\mathbf{x} = (x_1,\dots x_k)$. Then for $\|\omega\| = n$*

$$s_\omega(\tau(M^{2n})) = \sum_{p\in P} \mathrm{sign}(p) \cdot \frac{f_\omega(t_1(p),\dots,t_n(p))}{t_1(p)\cdots t_n(p)} \ . \tag{22}$$

Example 4.

$$s_{(n,\dots,0)}(\tau(M^{2n})) = \sum_{p\in P} \mathrm{sign}(p) \ .$$

Example 5.

$$s_{(0,\dots,1)}(\tau(M^{2n})) = s_n(M^{2n}) = \sum_{p\in P} \mathrm{sign}(p) \frac{\sum\limits_{j=1}^{n} t_j^n(p)}{t_1(p)\cdots t_n(p)} \ .$$

Remark 6. Note that the left hand side of (22) in the Theorem 4 is an integer number $s_\omega(\tau(M^{2n}))$ while the right hand side is a rational function in variables $x_1,\dots,x_k$. So this theorem imposes strong restrictions on the sets of signs $\{\mathrm{sign}(p)\}$ and weight vectors $\{\Lambda_j(p)\}$ of the fixed points.

6.2. **Homogeneous spaces of positive Euler characteristic and with invariant almost complex structure.** Let us assume M^{2n} to be homogeneous space G/H of positive Euler characteristic with canonical action of a maximal torus and endowed with an invariant almost complex structure J. All fixed points have sign $+1$ and taking into account the Theorem 2, the Proposition 4 gives that the weights at the fixed points have to satisfy the following relations.

Corollary 4. *For any ω with $0 \leqslant \|\omega\| \leqslant (n-1)$ where $2n = \dim G/H$ we have that*

$$\sum_{\mathrm{w} \in W_G/W_H} \mathrm{w}\Big(\frac{f_\omega(t_1,\dots,t_n)}{t_1 \cdots t_n}\Big) = 0 \ , \tag{23}$$

where $t_j = \langle \Lambda_j, \mathbf{x}\rangle$ and $\Lambda_j,\ 1 \leqslant j \leqslant n$, are the weights at the fixed point $e \cdot H$.

In the same way, the Theorem 4 implies that

Theorem 5. *For $M^{2n} = G/H$ and $t_j = \langle \Lambda_j, \mathbf{x}\rangle$, where $\langle \Lambda_j, \mathbf{x}\rangle = \sum\limits_{l=1}^{k} \Lambda_j^l x_l,\ \mathbf{x} = (x_1,\dots,x_k),\ k = \mathrm{rk}\, G = \mathrm{rk}\, H$, we have*

$$s_\omega(\tau(M^{2n})) = \sum_{\mathrm{w} \in W_G/W_H} \mathrm{w}\Big(\frac{f_\omega(t_1,\dots,t_n)}{t_1 \cdots t_n}\Big) \tag{24}$$

for any ω such that $\|\omega\| = n$.

Example 6.

$$s_{(n,\dots,0)}(G/H, J) = \|W_G/W_H\| = \chi(G/H)$$

and, therefore, $s_{(n,\dots,0)}(G/H, J)$ does not depend on invariant almost complex structure J.

Corollary 5.

$$s_{(0,\dots,1)}(G/H, J) = s_n(G/H, J)) = \sum_{\mathrm{w} \in W_G/W_H} \mathrm{w}\Big(\frac{\sum\limits_{j=1}^{n} t_j^n}{t_1 \cdots t_n}\Big) \ .$$

Example 7. In the case $\mathbb{C}P^n = G/H$ where $G = U(n+1),\ H = U(1) \times U(n)$ we have action of T^{n+1} and related to the standard complex structure the weights are given with $\langle \Lambda_j, \mathbf{x}\rangle = x_j - x_{n+1},\ j = 1,\dots,n$ and $W_G/W_H = \mathbb{Z}_{n+1}$ is cyclic group. So

$$s_n(\mathbb{C}P^n) = \sum_{i=1}^{n+1} \frac{\sum\limits_{j \neq i} (x_i - x_j)^n}{\prod\limits_{j \neq i} (x_i - x_j)} = n+1 \ . \tag{25}$$

Example 8. Let us consider Grassmann manifold $G_{q+2,2} = G/H$ where $G = U(q+2),\ H = U(2) \times U(q)$. We have here the canonical action of the torus T^{q+2}. The weights for this action at identity point related to the standard complex structure are given with $\langle \Lambda_{ij}, \mathbf{x}\rangle = x_i - x_j$, where $i = 1, 2$ and $3 \leqslant j \leqslant q+2$. There are $\|W_{U(q+2)}/W_{U(2)\times U(q)}\| = \frac{(q+2)(q+1)}{2}$ fixed points for this action. Therefore

$$s_{2q}(G_{q+2,2}) = \sum_{\mathrm{w}\in W_{U(q+2)}/W_{U(2)\times U(q)}} \mathrm{w}\Big(\frac{\sum\limits_{j=3}^{q+2}\big((x_1-x_j)^{2q}+(x_2-x_j)^{2q}\big)}{\prod\limits_{j=3}^{q+2}(x_1-x_j)(x_2-x_j)}\Big)\ . \tag{26}$$

The action of the group $W_{U(q+2)}/W_{U(2)\times U(q)}$ on the weights at the identity point in formula (26) is given by the following permutations $\mathrm{w}=\mathrm{w}_{kl}$:

$$\begin{aligned}\mathrm{w}_{00} &= Id\ ,\\ \mathrm{w}_{k0}(1)=k,\ \mathrm{w}_{k0}(k)&=1,\ \text{where } 3\leqslant k\leqslant q+2\ ,\\ \mathrm{w}_{0l}(2)=l,\ \ \mathrm{w}_{0l}(l)\ &=2,\ \text{where } 3\leqslant l\leqslant q+2\ ,\end{aligned}$$

$$\mathrm{w}_{kl}(1)=k,\ \mathrm{w}_{kl}(k)=1,\ \mathrm{w}_{kl}(2)=l,\ \mathrm{w}_{kl}(l)=2\ \text{ for } 3\leqslant k\leqslant q+1,\ k+1\leqslant l\leqslant q+2\ .$$

As we remarked before (see Remark 6), the expression on the right hand side in (26) is an integer, so we can get a value for s_{2q} by choosing the appropriate values for the vector $(x_1,\dots,x_{q+2})$. For example, if we take $q=2$ and $(x_1,x_2,x_3,x_4)=(1,2,3,4)$ the straightforward application of formula (26) will give that $s_4(G_{4,2})=-20$.

Example 9. In the case $G_{q+l,l}=G/H$ where $G=U(q+l),\ H=U(q)\times U(l)$ we have

$$s_{lq}(G_{q+l,l}) = \sum_{\sigma\in S_{q+l}/(S_q\times S_l)} \sigma\Big(\frac{\sum(x_i-x_j)^{lq}}{\prod(x_i-x_j)}\Big)\ , \tag{27}$$

where $1\leqslant i\leqslant q,\ (q+1)\leqslant j\leqslant (q+l)$ and S_{q+l} is the symmetric group.

We consider later, in the Section 7, the case of this Grassmann manifold in more details.

6.2.1. *Chern numbers.* We want to deduce an explicit relations between cohomology characteristic numbers s_ω and classical Chern numbers for an invariant almost complex structure on G/H.

Proposition 5. *The number $s_\omega(\tau(M^{2n}))$, where $\omega=(i_1,\dots,i_n),\ \|\omega\|=n$, is the characteristic number that corresponds to the characteristic class given by the orbit of the monomial*

$$(u_1\cdots u_{i_1})(u_{i_1+1}^2\cdots u_{i_1+i_2}^2)\cdots(u_{i_1+\dots i_{n-1}+1}^n\cdots u_{i_1+\dots i_n}^n)\ .$$

Remark 7. Let $\xi=(j_1,\dots,j_n)$ and $\mathbf{u}^\xi=u_1^{j_1}\cdots u_n^{j_n}$. The orbit of the monomial $\mathbf{u}^\xi$ is defined with

$$O(\mathbf{u}^\xi)=\sum \mathbf{u}^{\xi'}\ ,$$

where the sum is over the orbit $\{\xi'=\sigma\xi,\ \sigma\in S_n\}$ of the vector $\xi\in\mathbb{Z}^n$ under the symmetric group S_n acting by permutation of coordinates of ξ.

Example 10. If we take $\omega=(n,\dots,0)$ we need to compute the coefficient for a_1^n and it is given as an orbit $O(u_1\cdots u_n)$ and that is elementary symmetric function σ_n. If we take $\omega=(0,\dots,1)$ then we should compute the coefficient for a_n and it is given with $O(u_1^n)=\sum_{j=1}^n u_j^n$, what is Newton polynomial.

It is well known fact from the algebra of symmetric functions that the orbits of monomials give the additive basis for the algebra of symmetric functions. Therefore, any orbit of monomial can be expressed through elementary symmetric functions and vice versa. It gives a relation between the characteristic numbers s_ω in terms of Chern characteristic numbers $c^\omega = c_1^{i_1} \cdots c_n^{i_n}$ for an almost complex homogeneous space $(G/H, J)$.

Theorem 6. *Let $\omega = (i_1, \dots, i_n)$, $\|\omega\| = n$, and assume that the orbit of the monomial*

$$(u_1 \cdots u_{i_1})(u_{i_1+1}^2 \cdots u_{i_1+i_2}^2) \cdots (u_{i_1+\dots i_{n-1}+1}^n \cdots u_{i_1+\dots i_n}^n)$$

is expressed through the elementary symmetric function as

$$O((u_1 \cdots u_{i_1})(u_{i_1+1}^2 \cdots u_{i_1+i_2}^2) \cdots (u_{i_1+\dots i_{n-1}+1}^n \cdots u_{i_1+\dots i_n}^n)) = \\ = \sum_{\|\xi\|=n} \beta_\xi \sigma_1^{l_1} \cdots \sigma_n^{l_n} \tag{28}$$

for some $\beta_\xi \in \mathbb{Z}$ and $\|\xi\| = \sum_{j=1}^n j \cdot l_j$, where $\xi = (l_1, \dots, l_n)$. Then it holds

$$s_\omega(G/H, J) = \sum_{\mathrm{w} \in W_G/W_H} \mathrm{w}\Big(\frac{f_\omega(t_1, \dots, t_n)}{t_1 \dots t_n}\Big) = \sum_{\|\xi\|=n} \beta_\xi c_1^{l_1} \cdots c_n^{l_n} \ , \tag{29}$$

where c_i are the Chern classes for the tangent bundle of $(G/H, J)$.

Remark 8. Let $p(n)$ denote the number of partitions of the number n. By varying ω, the equation (29) gives the system of $p(n)$ linear equations in Chern numbers whose determinant is, by (28), non-zero. Therefore, it provides the explicit formulas for the computation of Chern numbers.

Remark 9. We want to point that relation (29) in the Theorem 6 together with Theorem 5 proves that the Chern numbers for $(G/H, J)$ can be computed without having any information on cohomology for G/H.

Example 11. We provide the direct application of the Theorem 6 following Example 10. It is straightforward to see that $s_{(n,\dots,0)}(G/H) = c_n(G/H)$ for any invariant almost complex structure. This together with Example 6 gives that $c_n(G/H) = \chi(G/H)$.

We want to add that it is given in [18] a description of the numbers s_I that correspond to our characteristic numbers s_ω, but the numerations I and ω are different. To the partition $i \in I$ correspond the n-tuple $\omega = (i_1, \dots, i_n)$ such that i_k is equal to the number of appearances of the number k in the partition i.

7. Some applications.

7.1. **Flag manifolds $U(n)/T^n$.** We consider invariant complex structure on $U(n)/T^n$. Recall [1] that the Weyl group $W_{U(n)}$ is symmetric group and it permutes the coordinate $x_1, \dots, x_n$ on Lie algebra $\mathfrak{t}^n$ for T^n. The canonical action of the torus T^n on this manifold has $\|W_{U(n)}\| = \chi(U(n)/T^n) = n!$ fixed points and its weights at identity point are given by the roots of U(n).

We first consider the case $n = 3$ and apply our results to explicitly compute cobordism class and Chern numbers for $U(3)/T^3$. The roots for U(3) are $x_1 - x_2$, $x_1 - x_3$ and $x_2 - x_3$. Therefore the cobordism class for $U(3)/T^3$ is given as the coefficient for t^3 in the polynomial

$$[U(3)/T^3] = \sum_{\sigma \in S_3} \sigma\Big(\frac{f(t(x_1 - x_2))f(t(x_1 - x_3))f(t(x_2 - x_3))}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}\Big) \ ,$$

where $f(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3$, what implies

$$[U(3)/T^3] = 6(a_1^3 + a_1 a_2 - a_3) \ .$$

This gives that the characteristic numbers s_ω for $U(3)/T^3$ are

$$s_{(3,0,0)} = 6, \quad s_{(1,1,0)} = 6, \quad s_{(0,0,1)} = -6 \ .$$

By the Theorem 6 we have the following relations between the characteristic numbers s_ω and the Chern numbers c^ω

$$c_3 = 6, \ c_1 c_2 - 3c_3 = 6, \ c_1^3 - 3c_1 c_2 + 3c_3 = -6, \ \text{ what gives } \ c_1 c_2 = 24, \ c_1^3 = 48 \ .$$

To simplify the notations we take further $\Delta_n = \prod\limits_{1 \leqslant i < j \leqslant n} (x_i - x_j)$.

Theorem 7. *The cobordism class for the flag manifold $U(n)/T^n$ is given as the coefficient for $t^{\frac{n(n-1)}{2}}$ in the series in t*

$$\frac{1}{\Delta_n} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \sigma \Big(\prod_{1 \leqslant i < j \leqslant n} f(t(x_i - x_j)) \Big) \ , \tag{30}$$

where $f(t) = 1 + \sum\limits_{i \geqslant 1} a_i t^i$ and $\mathrm{sign}(\sigma)$ is the sign ± 1 of the permutation σ.

7.1.1. *Using of divided difference operators.* Consider the ring of the symmetric polynomials $\mathrm{Sym}_n \subset \mathbb{Z}[x_1, \ldots, x_n]$. There is a linear operator (see [17])

$$L : \mathbb{Z}[x_1, \ldots, x_n] \longrightarrow \mathrm{Sym}_n : \ L\mathbf{x}^\xi = \frac{1}{\Delta_n} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \sigma(\mathbf{x}^\xi) \ ,$$

where $\xi = (j_1, \ldots, j_n)$ and $\mathbf{x}^\xi = x_1^{j_1}, \ldots, x_n^{j_n}$.

It follows from the definition of Schur polynomials $\mathrm{Sh}_\lambda(x_1, \ldots, x_n)$ where $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$ (see [17]), that

$$L\mathbf{x}^{\lambda + \delta} = \mathrm{Sh}_\lambda(x_1, \ldots, x_n) \ ,$$

where $\delta = (n-1, n-2, \ldots, 1, 0)$ and $L\mathbf{x}^\delta = 1$. Moreover, the operator L have the following properties:

- $L\mathbf{x}^\xi = 0$, if $j_1 \geqslant j_2 \geqslant \cdots \geqslant j_n \geqslant 0$ and $\xi \neq \lambda + \delta$ for some $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$;
- $L\mathbf{x}^\xi = \mathrm{sign}(\sigma) L\sigma(\mathbf{x}^\xi)$, where $\xi = (j_1, \ldots, j_n)$ and $\sigma(\mathbf{x}^\xi) = \mathbf{x}^{\xi'}$, where $\sigma \in S_n$ and $\xi' = (j_1' \geqslant j_2' \geqslant \cdots \geqslant j_n' \geqslant 0)$;
- L is a homomorphism of Sym_n-modules.

We have

$$\prod_{1 \leqslant i < j \leqslant n} f(t(x_i - x_j)) = 1 + \sum_{|\xi| > 0} P_\xi(a_1, \ldots, a_n, \ldots) t^{|\xi|} \mathbf{x}^\xi \ , \tag{31}$$

where $|\xi| = \sum\limits_{q=1}^{n} j_q$.

Corollary 6. *Set* $m = \frac{n(n-1)}{2}$. *The cobordism class for the flag manifold* $U(n)/T^n$ *is given by the formula*

$$[U(n)/T^n] = \sum_{|\xi|=m} P_\xi(a_1, \dots, a_n, \dots) L\mathbf{x}^\xi \ . \tag{32}$$

Remark 10. As we will show in Corollary 8 below, polynomials P_ξ in the formula (32) appears to be polynomials only in variables $a_1, \dots, a_{2n-3}$.

The characteristic number s_m for $U(n)/T^n$ is given as

$$s_m(U(n)/T^n) = \sum_{1 \leqslant i < j \leqslant n} L(x_i - x_j)^m \ . \tag{33}$$

Remark 11. The first property of the operator L gives that for any ξ such that $|\xi| = m$, we will have $L\mathbf{x}^\xi = 0$, whenever $\mathbf{x}^\xi \neq \sigma(\mathbf{x}^\delta)$ for every $\sigma \in S_n$. In other words, in order to have $L\mathbf{x}^\xi \neq 0$, we need $\mathbf{x}^\xi$ to contain $n-1$ variables and with different degrees.

Remark 11 together with (33) and Corollary 6, implies the following:

Corollary 7. $s_1(U(2)/T^2) = 2;\ s_3(U(3)/T^3) = -6$ *and*

$$s_m(U(n)/T^n) = 0 \ , \tag{34}$$

where $m = \frac{n(n-1)}{2}$ *and* $n > 3$.

We can push up this further. Denote by $(u_1, \dots, u_m) = \big((x_i - x_j),\ i < j\big)$, where $m = \frac{n(n-1)}{2}$. Then for $\omega = (i_1, \dots, i_m),\ \|\omega\| = m$ we have that

$$s_\omega(U(n)/T^n) = \sum L\Big((u_1 \cdots u_{i_1})(u_{i_1+1}^2 \cdots u_{i_1+i_2}^2) \cdots\Big) = \sum Lx^\xi \ , \tag{35}$$

where $x^\xi = \sigma(x_1^{n-1} x_2^{n-2} \cdots x_{n-1})$ for some $\sigma \in S_n$.

Therefore, if $\xi = (j_1, \dots, j_n)$, then $\max\limits_{p_1,\dots,p_s} (j_{p_1} + \cdots + j_{p_s}) = s\Big(n - \frac{s+1}{2}\Big),\ 1 \leqslant s \leqslant n$. In particular, it holds that $\max\limits_{p_1,p_2}(j_{p_1} + j_{p_2}) = 2n - 3$.

Corollary 8. *Let* $\omega = (i_1, \dots, i_m)$ *such that* $i_k \neq 0$ *for some* $k > 2n-3$, *then*

$$s_\omega(U(n)/T^n) = 0 \ .$$

If $\omega = (i_1, \dots, i_k),\ \|\omega\| = m$, does not satisfy Corollary 8, but $i_{k_1}, \dots, i_{k_l} \neq 0$ for some $k_1, \dots, k_l$ then we have that $k_p = 2(n-1) - q_p$, for $q_p \geqslant 1,\ 1 \leqslant p \leqslant l$. In this case we can say the following.

Corollary 9. *If* $n \geqslant 2l$ *and* $\sum\limits_{p=1}^{l} q_p < l(2l-1)$ *then*

$$s_\omega(U(n)/T^n) = 0 \ .$$

Remark 12. From the second property of the operator L we obtain that $LP(x_1, \dots, x_n) = 0$, whenever $\sigma(P(x_1, \dots, x_n)) = \varepsilon P(x_1, \dots, x_n)$ for a permutation $\sigma \in S_n$, where $\varepsilon = \pm 1$ and $\varepsilon \cdot \mathrm{sign}(\sigma) = -1$. This, in particular, gives that $L(P(x_1, \dots, x_n) + \sigma_{ij}(P(x_1, \dots, x_n))) = 0$ for any transposition σ_{ij} of x_i and x_j, where $1 \leqslant i < j \leqslant n$.

Using Remark 12 we can compute some more characteristic numbers of flag manifolds.

Corollary 10. *Let $n = 4q$ or $4q+1$ and $\omega = (i_1, \ldots, i_m)$, $\|\omega\| = m$, where $i_{2l-1} = 0$ for $l = 1, \ldots, \frac{m}{2}$. Then $s_\omega(U(n)/T^n) = 0$.*

Since $\sigma_{12}((x_1 - x_2)^{2l} \prod_{\substack{1 \leqslant i < j \leqslant n \\ (i,j) \neq (1,2)}} f(t(x_i - x_j))) = (x_1 - x_2)^{2l} \prod_{\substack{1 \leqslant i < j \leqslant n \\ (i,j) \neq (1,2)}} f(t(x_i - x_j))$ we have, also because of Remark 12, that

$$L\Big(\prod_{1 \leqslant i < j \leqslant n} f(t(x_i - x_j)\Big) = L\Big(\widetilde{f}(t(x_1 - x_2)) \prod_{\substack{1 \leqslant i < j \leqslant n \\ (i,j) \neq (1,2)}} f(t(x_i - x_j))\Big) \, ,$$

where $\widetilde{f}(t) = \sum_{l \geqslant 1} a_{2l-1} t^{2l-1}$. Using this property of L once more we obtain

Theorem 8. *For $n \geqslant 4$ the cobordism class for the flag manifold $U(n)/T^n$ is given as the coefficient for $t^{\frac{n(n-1)}{2}}$ in the series in t*

$$L\Big(\widetilde{f}(t(x_1 - x_2)) \widetilde{f}(t(x_{n-1} - x_n)) \prod_{\substack{1 \leqslant i < j \leqslant n \\ (i,j) \neq (1,2), (n-1,n)}} f(t(x_i - x_j))\Big) \, . \tag{36}$$

Remark 13. The Corollary 10 implies that if $s_\omega \neq 0$ for some $\omega = (i_1, \ldots, i_m)$, then for some $1 \leqslant l \leqslant \frac{m}{2}$ it has to be $i_{2l-1} \neq 0$. The Theorem 8 gives stronger results that, for $n \geqslant 4$ in polynomials P_ξ in (32) each monom contains the product of at least two elements of the form a_{2i-1}.

The Theorem 8 provide a way for direct computation of the number s_ω, for $\omega = (i_1, \ldots, i_m)$ such that $\|\omega\| = 2$, where $\|\omega\| = i_1 + \ldots + i_m$. For $n > 5$ we have that $s_\omega(U(n)/T^n) = 0$ for such ω. For $n = 4$ and $n = 5$ these numbers can be computed very straightforward as the next example shows.

Example 12. We provide the computation of the characteristic number $s_{(1,0,0,0,1,0)}$ for $U(4)/T^4$. From the formula (36) we obtain immediately:

$$\begin{aligned} s_{(1,0,0,0,1,0)}(U(4)/T^4) &= L\Big((x_1 - x_2)(x_3 - x_4)^5 + (x_1 - x_2)^5 (x_3 - x_4)\Big) = \\ &= 10 L\Big((x_1 - x_2)(x_3 - x_4)(x_1^2 x_2^2 + x_3^2 x_4^2)\Big) = \\ &= 20 L\Big(x_1^3 x_2^2 (x_3 - x_4) + (x_1 - x_2) x_3^3 x_4^2)\Big) = \\ &= 40 L\Big(x_1^3 x_2^2 x_3 + x_1 x_3^3 x_4^2\Big) = 80 \, . \end{aligned}$$

Remark 14. We want to emphasize that the formula (36) gives the description of the cobordism classes of the flag manifolds in terms of *divided difference operators*. The divided difference operators are defined with (see [2])

$$\partial_{ij} P(x_1, \ldots, x_n) = \frac{1}{x_i - x_j} \Big(P(x_1, \ldots, x_n) - \sigma_{ij} P(x_1, \ldots, x_n)\Big),$$

where $i < j$. Put $\sigma_{i,i+1} = \sigma_i$, $\partial_{i,i+1} = \partial_i$, $1 \leqslant i \leqslant n-1$. We can wright down operator L as the following composition (see [11, 16])

$$L = (\partial_1 \partial_2 \cdots \partial_{n-1})(\partial_1 \partial_2 \cdots \partial_{n-2}) \cdots (\partial_1 \partial_2) \partial_1 \, .$$

Denote by w_0 the permutation $(n, n-1, \dots, 1)$. Wright down a permutation $\mathrm{w} \in S_n$ in the form $\mathrm{w} = \mathrm{w}_0\sigma_{i_1}\cdots\sigma_{i_p}$ and set $\nabla_{\mathrm{w}} = \partial_{i_p}\cdots\partial_{i_1}$. It is natural to set $\nabla_{\mathrm{w}_0} = I$ — identity operator. The space of operators ∇_{w} is dual to the space of the Schubert polynomials $\mathfrak{G}_{\mathrm{w}} = \mathfrak{G}_{\mathrm{w}}(x_1, \dots, x_n)$, since it follows from their definition that $\mathfrak{G}_{\mathrm{w}} = \nabla_{\mathrm{w}}\mathbf{x}^{\delta}$. Note that $\mathfrak{G}_{\mathrm{w}_0} = \mathbf{x}^{\delta}$. For the identity permutation $e = (1, 2, \dots, n)$ we have $e = \mathrm{w}_0 \cdot \mathrm{w}_0^{-1}$. So $\nabla_e = L$ and $\mathfrak{G}_e = \nabla_e\mathbf{x}^{\delta} = 1$.

Schubert polynomials were introduced in [2] and in [10] in context of an arbitrary root systems. The main reference on algebras of operators ∇_{w} and Schubert polynomials $\mathfrak{G}_{\mathrm{w}}$ is [16].

The description of the cohomology rings of the flag manifolds $U(n)/T^n$and Grassmann manifolds $G_{n,k} = U(n)/(U(k) \times U(n-k))$ in the terms of Schubert polynomials is given in [11].

The description of the *complex cobordism ring* of the flag manifolds G/T, for G compact, connected Lie group and T its maximal torus, in the terms of the Schubert polynomials calculus is given in [8, 9].

7.2. **Grassmann manifolds.** As a next application we will compute cobordism class, characteristic numbers s_ω and, consequently, Chern numbers for invariant complex structure on Grassmannian $G_{4,2} = U(4)/(U(2) \times U(2)) = SU(4)/S(U(2) \times U(2))$. Note that, it follows by [7] that, up to equivalence, $G_{4,2}$ has one invariant complex structure J. The corresponding Lie algebra description for $G_{4,2}$ is $A_3/(\mathfrak{t}^1 \oplus A_1 \oplus A_1)$.

The number of the fixed points under the canonical action of T^3 on $G_{4,2}$ is, by Theorem 2, equal to 6. Let x_1, x_2, x_3, x_4 be canonical coordinates on maximal abelian algebra for A_3. Then x_1, x_2 and x_3, x_4 represents canonical coordinates for $A_1 \oplus A_1$. The weights of this action at identity point $(T_e(G_{4,2}), J)$are given by the positive complementary roots $x_1 - x_3, x_1 - x_4, x_2 - x_3, x_2 - x_4$ for A_3 related to $A_1 \oplus A_1$ that define J.

The Weyl group $W_{U(4)}$ is symmetric group of permutation on coordinates $x_1, \dots, x_4$ and the Weyl group $W_{U(2)\times U(2)} = W_{U(2)} \times W_{U(2)}$ is the product of symmetric groups on coordinates x_1, x_2 and x_3, x_4 respectively. Let $\mathrm{w}_j \in W_{U(4)}/W_{U(2)\times U(2)}$. Corollary 2 gives that the cobordism class $[G_{4,2}]$ is the coefficient for t^4 in polynomial

$$\sum_{j=1}^{6} \mathrm{w}_j\Big(\frac{f(t(x_1-x_3))f(t(x_1-x_4))f(t(x_2-x_3))f(t(x_2-x_4))}{(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)}\Big) = \tag{37}$$
$$= \frac{1}{4}L\Big((x_1-x_2)(x_3-x_4)f(t(x_1-x_3))f(t(x_1-x_4))f(t(x_2-x_3))f(t(x_2-x_4))\Big)\ ,$$

where $f(t) = 1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4$.

Expanding formula (37) we get that

$$[G_{4,2}] = 2(3a_1^4 + 12a_1^2a_2 + 7a_2^2 + 2a_1a_3 - 10a_4)\ .$$

The characteristic numbers s_ω can be read off form this formula:

$$s_{(4,0,0,0)} = 6, \quad s_{(2,1,0,0)} = 24, \quad s_{(0,2,0,0)} = 14, \quad s_{(1,0,1,0)} = 4, \quad s_{(0,0,0,1)} = -20\ .$$

The coefficients β_ξ from the Theorem 6 can be explicitly computed and for 8-dimensional manifold give the following relation between characteristic numbers s_ω and Chern numbers:

$$s_{(0,0,0,1)} = c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3 - 4c_4, \quad s_{(2,1,0,0)} = c_1c_3 - 4c_4\ ,$$

$$s_{(0,2,0,0)} = c_2^2 - 2c_1c_3 + 2c_4, \quad s_{(1,0,1,0)} = c_1^2c_2 - c_1c_3 + 4c_4 - 2c_2^2, \quad s_{(4,0,0,0)} = c_4\ .$$

We deduce that the Chern numbers for $(G_{4,2}, J)$ are

$$c_4 = 6, \quad c_1c_3 = 48, \quad c_2^2 = 98, \quad c_1^2c_2 = 224, \quad c_1^4 = 512 \ .$$

The given example generalizes as follows. Denote by $\Delta_{p,q} = \prod_{p\leqslant i<j\leqslant q}(x_i - x_j)$, then $\Delta_n = \Delta_{1,n}$.

Theorem 9. *The cobordism class for Grassmann manifold $G_{q+l,l}$ is given as the coefficient for t^{lq} in the series in t*

$$\sum_{\sigma\in S_{q+l}/S_q\times S_l} \sigma\Big(\prod \frac{f(t(x_i-x_j))}{(x_i-x_j)}\Big) = \frac{1}{q!l!}L\Big(\Delta_q\Delta_{q+1,q+l}\prod f(t(x_i-x_j))\Big) \ , \tag{38}$$

where $1\leqslant i\leqslant q,\ (q+1)\leqslant j\leqslant (q+l)$ and S_{q+l} is the symmetric group.

7.3. **Homogeneous space** $SU(4)/S(U(1)\times U(1)\times U(2))$**.** Following [7] and [15] we know that 10-dimensional space $M^{10} = SU(4)/S(U(1)\times U(1)\times U(2))$ admits, up to equivalence, two invariant complex structure J_1 and J_2 and one non-integrable invariant almost complex structure J_3. We provide here the description of cobordism classes for all of three invariant almost complex structures. The Chern numbers for all the invariant almost complex structures are known and they have been completely computed in [15] through multiplication in cohomology. We provide also their computation using our method.

The corresponding Lie algebra description for M^{10} is $A_3/(\mathfrak{t}^2\oplus A_1)$. Let x_1, x_2, x_3, x_4 be canonical coordinates on maximal Abelian subalgebra for A_3. Then x_1, x_2 represent canonical coordinates for A_1. The number of fixed points under the canonical action of T^3 on M^{10} is, by Theorem 2, equal to 12.

7.3.1. *The invariant complex structure J_1.* The weights of the action of T^3 on M^{10} at identity point related to J_1 are given by the complementary roots $x_1-x_3, x_1-x_4, x_2-x_3, x_2-x_4, x_3-x_4$ for A_3 related to A_1, (see [7], [15]). The cobordism class $[M^1, J_1]$ is, by Corollary 2, given as the coefficient for t^5 in polynomial

$$\sum_{j=1}^{12} \mathrm{w}_j\Big(\frac{f(t(x_1-x_3))f(t(x_1-x_4))f(t(x_2-x_3))f(t(x_2-x_4))f(t(x_3-x_4))}{(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4)}\Big) \ ,$$

where $\mathrm{w}_j \in W_{U(4)}/W_{U(2)}$ and $f(t) = 1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5$.

Therefore we get that

$$[M^{10}, J_1] = 4(3a_1^5 + 12a_1^3a_2 + 7a_1a_2^2 - 5a_1^2a_3 - 2a_2a_3 - 10a_1a_4 + 5a_5) \ .$$

Then Theorem 6 gives the following relations between characteristic numbers s_ω and Chern numbers for (M^{10}, J_1).

$$s_{(0,0,0,0,1)} = 20 = c_1^5 - 5c_1^3c_2 + 5c_1^2c_3 + 5c_1c_2^2 - 5c_1c_4 - 5c_2c_3 + 5c_5 \ ,$$
$$s_{(1,2,0,0,0)} = 28 = c_2c_3 - 3c_1c_4 + 5c_5, \quad s_{(2,0,1,0,0)} = -20 = c_1^2c_3 - c_1c_4 - 2c_2c_3 + 5c_5 \ ,$$
$$s_{(0,1,1,0,0)} = -8 = -2c_1^2c_3 + c_1c_2^2 - c_2c_3 + 5c_1c_4 - 5c_5, \quad s_{(3,1,0,0,0)} = 48 = c_1c_4 - 5c_5 \ ,$$
$$s_{(1,0,0,1,0)} = -40 = c_1^3c_2 - c_1^2c_3 - 3c_1c_2^2 + c_1c_4 + 5c_2c_3 - 5c_5, \quad s_{(5,0,0,0,0)} = 12 = c_5 \ .$$

This implies that the Chern numbers for (M^{10}, J_1) are as follows:

$$c_5 = 12, \quad c_1c_4 = 108, \quad c_2c_3 = 292, \quad c_1^2c_3 = 612, \quad c_1c_2^2 = 1028, \quad c_1^3c_2 = 2148, \quad c_1^5 = 4500 \ .$$

7.3.2. *The invariant complex structure* J_2. The weights of the action of T^3 on M^{10} at identity point related to J_2 are given by the positive complementary roots $x_4-x_1, x_4-x_2, x_4-x_3, x_1-x_3, x_2-x_3$ for A_3 related to A_1, (see [7], [15]). The cobordism class $[M^1, J_2]$ is, by Corollary 2, given as the coefficient for t^5 in polynomial

$$\sum_{j=1}^{12} \mathrm{w}_j\Big(\frac{f(t(x_4-x_1))f(t(x_4-x_2))f(t(x_4-x_3))f(t(x_1-x_3))f(t(x_2-x_3))}{(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_1-x_3)(x_2-x_3)}\Big) ,$$

where $\mathrm{w}_j \in W_{U(4)}/W_{U(2)}$ and $f(t) = 1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5$.

Therefore we get that

$$[M^{10}, J_2] = 4(3a_1^5 + 12a_1^3a_2 + 7a_1a_2^2 - 5a_1^2a_3 + 8a_2a_3 - 10a_1a_4 - 5a_5) .$$

Applying the same procedure as for above we get that the Chern numbers for (M^{10}, J_2) are:

$$c_5 = 12, \quad c_1c_4 = 108, \quad c_2c_3 = 292, \quad c_1^2c_3 = 612, \quad c_1c_2^2 = 1068, \quad c_1^3c_2 = 2268, \quad c_1^5 = 4860 .$$

7.3.3. *The invariant almost complex structure* J_3. The weights for the action of T^3 on M^{10} at identity point related to J_3 are given by complementary roots $x_1 - x_3, x_2 - x_3, x_4 - x_1, x_4 - x_2, x_3 - x_4$, (see [15]). Using Corollary 2 we get that the cobordism class for (M^{10}, J_3) is

$$[M^{10}, J_3] = 4(3a_1^5 - 12a_1^3a_2 + 7a_1a_2^2 + 15a_1^2a_3 - 12a_2a_3 - 10a_1a_4 + 15a_5) .$$

The characteristic numbers for (M^{10}, J_3) are given as coefficients in its cobordism class, what, as above, together with Theorem 6 gives that the Chern numbers for (M^{10}, J_3) are as follows:

$$c_5 = 12, \quad c_1c_4 = 12, \quad c_2c_3 = 4, \quad c_1^2c_3 = 20, \quad c_1c_2^2 = -4, \quad c_1^3c_2 = -4, \quad c_1^5 = -20 .$$

Remark 15. Further work on the studying of Chern numbers and the geometry for the generalizations of this example is done in [12] and in [15].

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