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On the Minimal Parabolic System Related to the Monster Simple Group

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1. INTRODUCTION

Almost all of the 26 sporadic finite simple groups possess systems of subgroups which, in several ways, resemble the parabolic systems of the finite simple groups of Lie type. Many of these systems are catalogued in [RSt, RSm]. Those in [RSt] mirror more closely the minimal parabolic systems of a finite simple group of Lie type. In this paper the term minimal parabolic system will be used in a broader sense, to be explained in a moment.

Our interest here is in a very particular kind of (generalized) minimal parabolic system, an example of which occurs in the Monster simple group, and the object of this work is to derive "local" information about such a minimal parabolic system. In order to state these results we need to introduce some definitions and notation.

Definition 1.1. Let G be a (not necessarily finite) group containing a finite p-subgroup S, where p is a prime.

(a) A finite subgroup P of G which contains S is called a *minimal parabolic subgroup* if $N_P(S)$ is contained in a unique maximal subgroup of P and $O_p(P) \neq 1$.

(b) Let $\{P_1, ..., P_n\}$ be a set of minimal parabolic subgroups of G and put $I = \{1, ..., n\}$. Then $\{P_1, ..., P_n\}$ is a *minimal parabolic system of rank n* if

(i) $G = \langle P_i \mid i \in I \rangle \neq \langle P_j \mid j \in J \rangle$ for each proper subset J of I; and

(ii) for $i, j \in I$, P_{ij} is a finite group and $S \in Syl_p P_{ij}$, where P_{ij} denotes $\langle P_i, P_j \rangle$.

We shall be considering minimal parabolic systems in which the P_i are a particular "degenerate" kind of group of Lie type, as specified in

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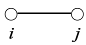
HYPOTHESIS 1.2. *G is a group containing a minimal parabolic system, $p=2$ and for each $i \in I$*

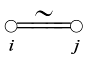
$$P_i/O_2(P_i) \cong SL_2(2)(\cong S_3).$$

By analogy with Dynkin diagrams, we now assign a diagram to a minimal parabolic system $\{P_1, ..., P_n\}$. Assume Hypothesis 1.2 holds. We take as nodes the set I and, putting $\bar{P}_{ij} = P_{ij}/O_2(P_{ij})$, for $i, j \in I$ we join i and j as indicated:

if and only if $\bar{P}_{ij} \cong SL_2(2) \times SL_2(2)$;



if and only if $\bar{P}_{ij} \cong L_3(2)$; and



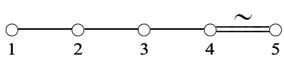
if and only if $\bar{P}_{ij} \cong \hat{S}_6$.

Here $\hat{S}_6$ denotes the group H which is uniquely specified by requiring H' to be the non-split triple cover of A_6, $H/O_3(H') \cong S_6$, and $H' = C_H(O_3(H'))$. The symbol  is chosen because $S_6 \cong Sp_4(2)$.

For $G, S, P_1, ..., P_n$ as in Definition 1.1(b) we put $S_o = core_G\, S$, $P_{i_1 \cdots i_k} = \langle P_{i_1}, ..., P_{i_k} \rangle$, and $S_{i_1 \cdots i_k} = core_{P_{i_1 \cdots i_k}} S$.

The main theorem to be proved is

THEOREM 1.3. *Suppose that Hypothesis 1.2 holds with $n=5$ and that the diagram of $\{P_1, ..., P_5\}$ is*



Further assume that $|S/S_{345}| \neq 2^9$ and $2^{10} \neq |S/S_{2345}| \neq 2^{25}$. Then $|S/S_o| = 2^{46}$ and $O_2(P_{45})/S_o$ has a P_{45}-chief series

$$O_2(P_{45}) = X_0 > X_1 > X_2 > X_3 > X_4 > X_5 > X_6 > X_7 > X_8 > X_9$$
$$> X_{10} > X_{11} > X_{12} > X_{13} = S_o$$

with

$$[X_0 : X_1] = [X_2 : X_3] = [X_6 : X_7] = [X_9 : X_{10}] = 2^6,$$
$$[X_3 : X_4] = [X_7 : X_8] = [X_8 : X_9] = 2^4, \qquad \textit{and}$$
$$[X_1 : X_2] = [X_4 : X_5] = [X_5 : X_6] = [X_{10} : X_{11}]$$
$$= [X_{11} : X_{12}] = [X_{12} : X_{13}] = 2.$$

In the course of establishing Theorem 1.3 many more detailed facts about S/S_o emerge than are stated above. For example, the isomorphism type of the P_{45}-chief factors of $O_2(P_{45})/S_o$ can be easily read as can information about the action of P_{ij} upon $O_2(P_{ij})/S_o$ for various other i, j. The structural data in the conclusion of Theorem 1.3 resembles the situation in the Monster simple group quite closely.

Minimal parabolic systems possessing diagrams of the form ∘——∘$\overset{\sim}{=\!=}$∘ and ∘——∘——∘$\overset{\sim}{=\!=}$∘ have been studied in [R1, R2], respectively. As a consequence of these works, the hypothesis $|S/S_{345}| \neq 2^9$ and $|S/S_{2345}| \neq 2^{25}$ implies that

$$|S/S_{345}| = 2^{10} \qquad \text{and} \qquad |S/S_{2345}| = 2^{21}.$$

A major step in the proof of Theorem 1.3 is the identification of a subgroup $\tilde{S}$ of S which is simultaneously normalized by P_1, P_2, P_3, P_4, and P_5. Then, since $G = \langle P_1, ..., P_5 \rangle$, we have $\tilde{S} \leqslant S_o$ and, in the light of other deductions, it is then straightforward to identify S_o. In locating $\tilde{S}$ our perspective is to start with subgroups "near the top" of S such as $O_2(P_i)$, $O_2(P_{ij})$, $O_2(P_{1234})$, and $O_2(P_{2345})$ and form other subgroups which are the intersections of these groups with certain of their conjugates. The resulting subgroups are then analysed and from them we produce further subgroups by taking suitable intersections. In this manner we proceed down the subgroup lattice of S eventually pinpointing $\tilde{S}$. In our arguments P_{45} frequently plays a prominent role. Indeed the building of a chief series for P_{45} spearheads our assault upon the minimal parabolic system. The arguments we use are piecewise quite elementary, almost self-contained, and are in the spirit of earlier work in [R1, R2].

We now give a more detailed account of this paper.

Section 2 is devoted to establishing notation and assembling results we shall need about minimal parabolic systems with diagrams ∘——∘$\overset{\sim}{=\!=}$∘ and ∘——∘——∘$\overset{\sim}{=\!=}$∘, together with a few facts about the $GF(2)$ representations of $\hat{S}_6$, $L_3(2)$, and $S_3 \times S_3$.

In Section 3 we give certain results which are of use in the manufacture of subgroups of S. Notable among these is the so-called Replication Lemma (Lemma 3.4), which is a crystallization of some ideas in [R2].

Our journey of discovery down the subgroup lattice of S begins in Section 6 where we construct T_5, a subgroup of S which has index 2^{33} in S. This is achieved by using the Replication Lemma in conjunction with certain facts derived in Section 5. Delving further down the subgroup lattice of S we bring to light subgroups T_6 and T_7 where $T_5 > T_6 > T_7$, $[T_5 : T_6] = 2^4$, and $[T_6 : T_7] = 2^2$ or 2^4. The analysis of the possibilities for $[T_6 : T_7]$ turns out to be somewhat lengthy but eventually, in Section 7, we discover that $[T_6 : T_7] = 2^2$ must hold. The subject of Section 8 is T_8, yet another subgroup of S, whose index in T_7 turns out to be equal to 2. Our next step is to pinpoint certain subgroups T_9 and T_{10} of S—it is at this point that the minimal parabolic system fights a strong rear-guard action in an attempt to retain its secrets. The subgroup configuration that we encounter here, since it also appears in other work [R3], has been presented separately in Section 4. It is left to our final section, Section 10, to determine S_o and a P_{45}-chief series for $O_2(P_{45})/S_o$ and so complete the proof of Theorem 1.3.

2. NOTATION AND QUOTED RESULTS

We begin this section with some properties of $L_3(2)$ and $\hat{S}_6$ (for a construction of $\hat{S}_6$ consult Section 2 of [R1]).

LEMMA 2.1. *Suppose G is a finite group with $G/O_2(G)$ isomorphic to either $L_3(2)$ or $\hat{S}_6$, and let $S \in Syl_2\, G$.*

(i) *There is a unique minimal parabolic system $\{X, Y\}$ of rank 2 containing S for which $X/O_2(X) \cong S_3 \cong Y/O_2(Y)$. If $G/O_2(G) \cong L_3(2)$, then $X/O_2(X) \cong S_4 \cong Y/O_2(Y)$ and if $G/O_2(G) \cong \hat{S}_6$, then $X/O_2(X) \cong S_4 \times Z_2 \cong Y/O_2(Y)$.*

(ii) *Let $x \in X \backslash S$ and $y \in Y \backslash S$. Then $G = \langle x, y, O_2 \rangle = \langle x, y, O_2(Y) \rangle$.*

(iii) *Suppose that $G/O_2(G) \cong \hat{S}_6$ and that $\{X, Y\}$ is the minimal parabolic system given in (i). If $x \in X \backslash S$, $y \in Y \backslash S$ where x and y are 2-elements, then $[x, O_2(X)] \nleq O_2(Y)$ and $[y, O_2(Y)] \nleq O_2(X)$.*

Proof. Parts (i) and (ii) follow readily from Lemma 2.1(i), (ii) of [R2], while part (iii) is stated as Lemma 2.2 in [R2].

LEMMA 2.2. *Suppose $G \cong L_3(2)$ and V is an irreducible $GF(2)$ G-module. Then*

(i) dim $V = 1$, 3, *or* 8; *and*

(ii) *if* $\langle \zeta \rangle \in Syl_3 G$ *and* dim $V = 3$, *then* dim $C_V(\zeta) = 1$.

Proof. See Lemma 3.2 of [R2].

LEMMA 2.3. *Suppose $G = G_1 \times G_2$ where $G_i \cong S_3$, $i = 1, 2$, and let $\langle x_i \rangle \in Syl_3\, G_i$ for $i = 1, 2$. Let V be an irreducible $GF(2)$ G-module. Then*

(i) dim $V = 1$, 2, *or* 4:

(ii) *if* dim $V = 4$, *then* $C_V(x_i) = 0$ *for* $i = 1, 2$; *and*

(iii) *if* dim $V = 2$, *then exactly one of* x_1 *and* x_2 *centralizes* V.

Proof. See Lemma 3.1 of [R2].

LEMMA 2.4. *Suppose* $G \cong \hat{S}_6$ *and let* $\{X, Y\}$ *be the minimal parabolic system as given in Lemma* 2.1(i). *Let* $\langle x \rangle \in Syl_3\, X$ *and* $\langle y \rangle \in Syl_3\, Y$. *Suppose* V *is an irreducible* $GF(2)$ G-*module.*

(i) *If* $\dim V \leqslant 6$, *then* $\dim V = 1$, 4, *or* 6.

(ii) *If* $\dim V = 4$, *then* $O_3(G)$ *acts trivially on* V.

(iii) *If* $\dim V = 4$, *then exactly one of the following holds*

$$|C_V(x)| = 2^2, \qquad |C_V(y)| = 1$$

and

$$|C_V(x)| = 1, \qquad |C_V(y)| = 2^2.$$

(iv) *If* $\dim V = 6$, *then* $V = [V, O_3(G)]$.

(v) *If* $\dim V = 6$, *then* $|C_V(x)| = |C_V(y)| = 2^2$.

Proof. For parts (i), (iii), and (v) see, respectively, Lemma 3.4(i), (ii) and Lemma 3.5(i) of [R2].

(ii) From the fact that $3^3 \nmid |GL_4(2)|$ and the structure of $\hat{S}_6$, we see that $\ker V \geqslant O_3(G)$. So (ii) holds.

(iv) If $V \neq [V, O_3(G)]$, then $[V, O_3(G)] = 1$ by the irreducibility of V. Then V is an irreducible $GF(2)$-module for S_6 which, since $\dim V = 6$, is impossible. Therefore $V = [V, O_3(G)]$.

LEMMA 2.5. *Suppose* $G/O_2(G) \cong L_3(2)$ *and* $\{X, Y\}$ *is the minimal parabolic system as given in Lemma* 2.1(i). *If* G *is an operator group on a finite group* F *and* $\langle x \rangle \in Syl_3\, X$, $\langle y \rangle \in Syl_3\, Y$, *then* $|C_F(x)| = |C_F(y)|$.

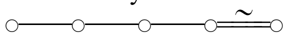
Proof. This follows from the observation that $\langle x \rangle$ and $\langle y \rangle$ are conjugate subgroups of G.

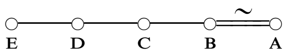
LEMMA 2.6. *Suppose* G *is a finite group with* $G/O_2(G) \cong \hat{S}_6$ *and let* $\{X, Y\}$ *be the minimal parabolic system as given in Lemma* 2.1(i). *Let* $\langle x \rangle \in Syl_3\, X$ *and* $\langle y \rangle \in Syl_3\, Y$ *and assume* G *is an operator group on a group* F.

(i) *If* $|F| = 2^5$, *then* $|C_F(x)| = 2 = |C_F(y)|$ *cannot hold.*

(ii) *If* $|F| = 2^7$, *then* $|C_F(x)| \neq 2 \neq |C_F(y)|$.

Proof. See Lemma 3.6 of [R2].

We pause from our cataloguing of results we shall need later in this paper so as to introduce some notation. Suppose, for the moment, that G is a group possessing a minimal parabolic system which satisfies Hypothesis 1.2 and whose diagram is . We use $\{A, B, C, D, E\}$ to denote this parabolic system with $A \cap B \cap C \cap D \cap E = S$ and such that



We now list a number of subgroups of G which will occupy our attention in later sections,

$$J := \langle B, C \rangle, \qquad K := \langle B, A \rangle, \qquad L := \langle C, A \rangle,$$

$$M := O_2(J), \qquad N := O_2(K), \qquad P := O_2(L)$$

$$U := O_2(\langle B, D \rangle), \qquad V := O_2(\langle C, D \rangle), \qquad W := O_2(\langle A, D \rangle).$$

Let N_o be such that $N < N_o \leqslant S$ and $N_o/N = Z(B/N)$ (by Lemma 2.1(i) we have $B/N \cong S_4 \times Z_2$). Then we further introduce

$$H := M \cap N, \qquad H_1 := M \cap V, \qquad H_o := M \cap N_o$$

$$T_1 := core_{\langle A, B, C \rangle} S, \qquad T_3 := core_{\langle A, B, C, D \rangle} S,$$

$$Z := core_{\langle B, C, D, E \rangle} S, \qquad H_2 := core_{\langle B, C, D \rangle} S.$$

Clearly we have $T_1 \geqslant T_3$, $H_2 \geqslant Z$, and $V \cap U \cap M \geqslant Z$. Since M, N, and N_o are normal in B, H and H_o are normal in B too. Likewise we note that $H_1 \trianglelefteq C$.

We may display some of the above notation in the form shown in Fig. 1.

Since $\langle B, D \rangle / U \cong S_3 \times S_3$, $|O_2(B)/U| = 2$ and hence $B/U \leqslant C_{\langle B, D \rangle}(O_2(B)/U)$ from which we deduce that B/U contains one of the S_3 direct factors (of $\langle B, D \rangle / U$) as a subgroup of index 2. Let B^o_D be such that $U < B^o_D < B$ and B^o_D/U is an S_3 direct factor of $\langle B, D \rangle / U$. Similarly, we define D^o_B to be such that $U < D^o_B < D$ and D^o_B/U is an S_3 direct factor of $\langle B, D \rangle / U$. (Because $B \cap D = S$, D^o_B/U contains the other S_3 direct factor.) Clearly we have that both B^o_D and D^o_B are normal subgroups of $\langle B, D \rangle$, $[B^o_D, D^o_B] \leqslant U$, $[B : B^o_D] = [D : D^o_B] = 2$, and $B^o_D/U \cong S_3 \cong D^o_B/U$. Since we also have $\langle A, C \rangle / P \cong \langle A, D \rangle / W \cong \langle C, E \rangle / O_2(\langle C, E \rangle) \cong \langle B, E \rangle / O_2(\langle B, E \rangle) \cong \langle A, E \rangle / O_2(\langle A, E \rangle) \cong S_3 \times S_3$, we may define analogous subgroups C^o_A, A^o_C, A^o_D, D^o_A, C^o_E, E^o_C, B^o_E, E^o_B, A^o_E, E^o_A of (respectively) C, A, A, D, C, E, B, E, A, E. Just as before we have that $C^o_A \trianglelefteq L$, $A^o_C \trianglelefteq L$, $A^o_D \trianglelefteq \langle A, D \rangle$, $D^o_A \trianglelefteq \langle A, D \rangle$, $C^o_E \trianglelefteq \langle C, E \rangle$, $E^o_C \trianglelefteq \langle C, E \rangle$, $B^o_E \trianglelefteq \langle B, E \rangle$, $E^o_B \trianglelefteq \langle B, E \rangle$,

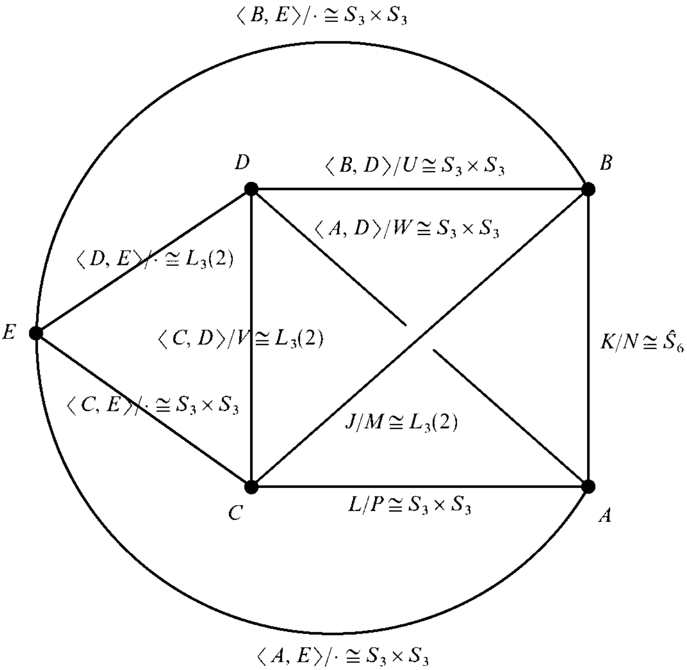


FIGURE 1

$A^o_E \trianglelefteq \langle A, E\rangle$, and $E^o_A \trianglelefteq \langle A, E\rangle$. Further, the following hold: $P = C^o_A \cap A^o_C$, $W = A^o_D \cap D^o_A$, $O_2(\langle C, E\rangle) = C^o_E \cap E^o_C$, $O_2(\langle B, E\rangle) = B^o_E \cap E^o_B$, $O_2(\langle A, E\rangle) = A^o_E \cap E^o_A$, and $[C^o_A, A^o_C] \leqslant P$, $[A^o_D, D^o_A] \leqslant W$, $[C^o_E, E^o_C] \leqslant O_2(\langle C, E\rangle)$, $[B^o_E, E^o_B] \leqslant O_2(\langle B, E\rangle)$, $[A^o_E, E^o_A] \leqslant O_2(\langle A, E\rangle)$. If we encounter a parabolic system denoted by $\{P_1, ..., P_n\}$ where $\langle P_i, P_j\rangle / O_2(\langle P_i, P_j\rangle) \cong S_3 \times S_3$, $P^o_{i_{P_j}}$ and $P^o_{j_{P_i}}$ will have the obvious meaning.

We take x_A (respectively x_B, x_C, x_D, x_E) to be some fixed element of A (respectively B, C, D, E) such that $\langle x_A\rangle \in Syl_3\, A$ (respectively $\langle x_B\rangle \in Syl_3\, B$, $\langle x_C\rangle \in Syl_3\, C$, $\langle x_D\rangle \in Syl_3\, D$, $\langle x_E\rangle \in Syl_3\, E$).

LEMMA 2.7. *Let G be a group possessing a minimal parabolic system which satisfies Hypothesis* 1.2 *and has* ○——○——○——○$\overset{\sim}{=\!=}$○ *as its diagram. Then* (*using the above notation*) $A/N \cong B/N \cong S_4 \times Z_2$ *and* $B/M \cong C/M \cong C/V \cong D/V \cong D/O_2(\langle D, E\rangle) \cong E/O_2(\langle D, E\rangle) \cong S_4$.

Proof. Since $\langle A, B\rangle/N \cong \hat{S}_6$ and $\langle B, C\rangle/M \cong \langle C, D\rangle/V \cong \langle D, E\rangle/O_2(\langle D, E\rangle) \cong L_3(2)$, the lemma follows from Lemma 2.1(i).

LEMMA 2.8. *Assume that $|S/T_1| = 2^{10}$ holds. Then we have the lattice of subgroups of S/T_1 shown in Fig.* 2.

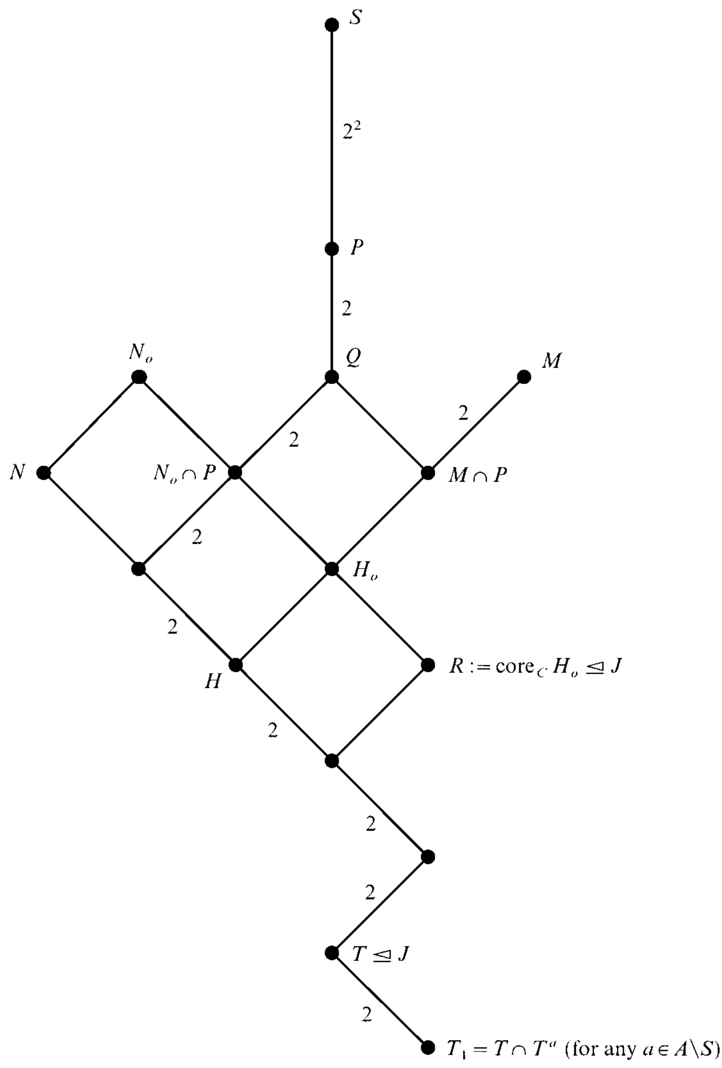


FIGURE 2

Further, the following hold.

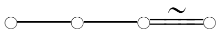
(i) *M/R and R/T are both J-chief factors* (*each of order* 2^3);

(ii) *N/T_1 is a chief factor for K of order* 2^6;

(iii) $O_2(B) = MN$; *and*

(iv) *for any $c \in C \backslash S$, $H_o \cap H_o^c = R$.*

Proof. For the subgroup lattice see Fig. 1 and (3.22) of [R1] and for part (i) consult (3.10)(i) and (3.18) of [R1]. The main theorem of [R1] establishes (ii). Parts (iii) and (iv) are proved in (3.3)(i) and (3.11) of [R1], respectively.

Our next result is an omnibus lemma on a certain parabolic system with diagram .

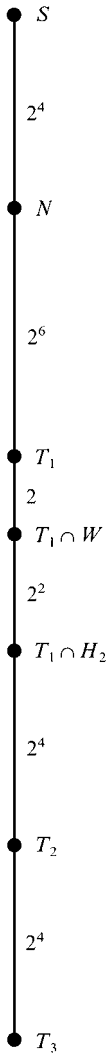


FIGURE 3

LEMMA 2.9. *Assume that* $|S/T_3| = 2^{21}$ *holds. Then we have the lattice of subgroups of* S/T_3 *shown in Fig.* 3.

Further, the following hold.

(i) $T_1 > T_1 \cap W > T_2 > T_3$ *is a K-chief series for* T_1/T_3;

(ii) $T_1/T_1 \cap H_2$ *is a chief factor for J of order* 2^3;

(iii) $T_1 \cap W$ *contains subgroups* Y_1 *and* Y_2 *with* $Y_i \trianglelefteq L$ *for* $i = 1, 2$ *and* $T_1 \cap W > Y_1 > T_2 > Y_2$. *Moreover* $[Y_1 : T_2] = [T_2 : Y_2] = 2^2$ *and* Y_1/Y_2 *is an L-chief factor*;

(iv) T_2/T_3 *does not possess a B-invariant subgroup of index* 2^2;

(v) $T_2 \trianglelefteq \langle A, B, D \rangle$;

(vi) *for* $c \in C \backslash S$, $T_2 \cap T_2^c \trianglelefteq \langle A, C, D \rangle$ (*in fact* $T_2 \cap T_2^c = Y_2$ *where* Y_2 *is as given in* (iii));

(vii) $N \cap W \trianglelefteq \langle A, B, D \rangle$;

(viii) $H_2 = H_1 \cap H_1^d$ *for any* $d \in D \backslash S$ *and* $H_2 < H_1$;

(ix) $M \cap N \cap P \cap U \cap V \cap W = H \cap H_1$;

(x) $(H_o \cap H_1) H_2 = H_1$;

(xi) $[S : H_2 \cap R] = 2^9$ *and* $H_2 \cap R \trianglelefteq J$;

(xii) $[M : T \cap H_2] = 2^9$;

(xiii) $[S : H_o \cap H_1] = 2^7$; *and*

(xiv) $[S : N \cap V \cap W] = 2^7$.

Proof. The subgroup lattice and part (i) are mostly presented in the main theorem of [R2] (see especially the end of Section 7 of [R2]). That $[T_1 : T_1 \cap W] = 2$ is given in [R2, Lemma 5.1], $T_1 \cap W \geqslant T_1 \cap H_2$ holds since $W \geqslant H_2$ and $T_1 \cap H_2 \geqslant T_2$ follows from [R2, Theorem 5.8(iii)].

For parts (ii), (v), (vii), (ix), (xiii), and (xiv) consult (respectively) Lemmas 5.4(iii), 5.8(i), 4.8(i), 4.1(v), 4.4(iii), and 5.3(iv) of [R2].

(iii), (vi) (Beware that the T_3 in [R2] is different from our present T_3.) Since $T_1 \cap H_2 \trianglelefteq C$, $T_2 T_2^c \leqslant T_1 \cap H_2 (\leqslant T_1 \cap W)$ for any $c \in C$ and hence (iv) follows using [R2, Lemma 6.7(iv)]. Combining Lemmas 6.4 and 6.2(i) of [R2] yields (vi).

(iv) By [R2, Lemma 6.8(a)], T_2 contains a subgroup of index 2 which contains T_3 and is normal in B. Since T_2/T_3 is an irreducible 4-dimension $GF(2)\,\hat{S}_6$-module, by properties of such modules we obtain (iv).

For part (viii) see [R2, Lemma 2.5(i)] noting that $[S : H_2] = 2^6$ and $[S : H_1] = 2^5$ implies $H_2 < H_1$.

(x) Inspection of the lattice in [R2, Lemma 4.4] gives $H_0 \cap H_1 \not\leq H_2$.

So, since $[H_1 : H_2] = 2$, we obtain $H_1 = (H_o \cap H_1) H_2$.

(xi), (xii) Consulting the lattice in [R2, Lemma 5.4] yields $[S : H_2 \cap R] = 2^9$ and $[M : T \cap H_2] = 2^9$. Since $H_2 \trianglelefteq J$ and $R \trianglelefteq J$ (see Lemma 2.8), we clearly have $H_2 \cap R \trianglelefteq J$.

The proof of the lemma is complete.

THEOREM 2.10 (Timmesfeld). *Suppose G is a group containing a minimal parabolic system* $\{P_1, ..., P_n\}$, *which satisfies Hypothesis* 1.2. *Further suppose that* $\{P_1, ..., P_n\}$ *has diagram*

○——○——○ ··· ○——○.

Then $G/core_G\,(\bigcap_{i=1}^{n} P_i) \cong L_{n+1}(2)$ *and* $S \in Syl_2\, G$.

Proof. Putting $S = \bigcap_{i=1}^{n} P_i$ we define a chamber system $\mathscr{C}$ as follows. The chambers will be the right cosets of S in G with two chambers Sx and Sy being i adjacent if $xy^{-1} \in P_i$, $i \in \{1, ..., n\}$. By right multiplication G acts upon $\mathscr{C}$ and $\bar{G} = G/core_G\, S$ acts faithfully upon $\mathscr{C}$. Since $G = \langle P_1, ..., P_n \rangle$, G acts transitively upon $\mathscr{C}$ and from the diagram of the minimal parabolic system each rank 2 cell of $\mathscr{C}$ is either a generalized digon or a finite projective plane of order 2. Appealing to a result of Timmesfeld [T, Theorem 3] yields that $\mathscr{C}$ is the chamber system of a building of type A_n and $\bar{G}$ is an extension of a Chevalley group of type $A_n(2^m)$ by diagonal or field automorphisms. (Note that $\bar{G} \cong A_7$ is not possible by the structure of the P_{ij} as A_7 has no subgroups isomorphic to $S_3 \times S_3$.) If $\bar{B}$ is a Sylow 2-normalizer in $\bar{G}$, then the number of chambers in $\mathscr{C}$ is $[\bar{G} : \bar{B}]$ which is also equal to $[\bar{G} : \bar{S}]$. Hence $N_{\bar{G}}(\bar{S}) \in Syl_2\, \bar{G}$. By a Frattini argument and the fact that $\bar{P}_i / O_2(\bar{P}_i) \cong SL_2(2)$ it follows that $m = 1$, and therefore $\bar{G} \cong L_{n+1}(2)$. This proves the theorem.

One final piece of notation is $F_1 \cong_X F_2$ which means that F_1 and F_2 are both X-operator groups which are isomorphic as X-operator groups.

Our remaining notation is standard, for which we refer the reader to either [G] or [S].

3. THE REPLICATION LEMMA

In this section we consider minimal parabolic systems which are more general than the one featured in our main theorem. Lemma 3.4 is referred to as the Replication Lemma, since its application to the subgroup lattice of S results in the creation of chief factors contained in S which resemble certain chief factors higher up the subgroup lattice of S. The Replication Lemma plays an important part in our arguments. It, in conjunction with other results such as Lemma 3.2, enables us to penetrate the subgroup lattice of S.

So, in this section G is assumed to be a (not necessarily finite) group possessing a minimal parabolic system $\{P_1, ..., P_n\}$ which satisfies Hypothesis 1.2. Puts $\mathscr{I} = \{1, ..., n\}$, and let S be a 2-subgroup of G such that $S \in Syl_2\, P_{ij}$ for all $i, j \in \mathscr{I}$.

LEMMA 3.1. *Let $\{i, j, k\} \subseteq \mathscr{I}$ be such that i, j and k are distinct and suppose $\bar{P}_{ik} \cong S_3 \times S_3 \cong \bar{P}_{ij}$.*

(i) $O_2(P_{ij}) \cap O_2(P_{jk}) = O_2(P_{ik}) \cap O_2(P_{jk}) \trianglelefteq P_{jk}$.

(ii) *If $P^o_{k_{P_i}}$ covers $P_k/O_2(P_{jk})$ and $P^o_{j_{P_i}}$ covers $P_j/O_2(P_{jk})$, then*

$$O_2(P_{ij}) \cap O_2(P_{jk}) = O_2(P_{ik}) \cap O_2(P_{jk}) \trianglelefteq P_{ijk}.$$

Proof. (i) Clearly we have

$$O_2(P_{ij}) \cap O_2(P_{jk}) \leqslant O_2(P_i) \cap O_2(P_k).$$

Since $\bar{P}_{ik} \cong S_3 \times S_3$, $O_2(P_i) \cap O_2(P_k) \leqslant O_2(P_{ik})$ and hence

$$O_2(P_{ij}) \cap O_2(P_{jk}) \leqslant O_2(P_{ik}) \cap O_2(P_{jk}).$$

A similar argument shows that

$$O_2(P_{ik}) \cap O_2(P_{jk}) \leqslant O_2(P_{ij}) \cap O_2(P_{jk}).$$

Hence $O_2(P_{ij}) \cap O_2(P_{jk}) = O_2(P_{ik}) \cap O_2(P_{jk})$. Now P_j normalizes $O_2(P_{ij}) \cap O_2(P_{jk})$ and P_k normalizes $O_2(P_{ik}) \cap O_2(P_{jk})$ and so, since $P_{jk} = \langle P_j, P_k \rangle$, this proves (i).

(ii) Since $[P_k : P^o_{k_{P_i}}] = 2$ and $P^o_{k_{P_i}}$ covers $P_k/O_2(P_{jk})$, we have that

(3.1.1) $$P^o_{k_{P_i}} \cap O_2(P_{jk}) \text{ has index 2 in } O_2(P_{jk}).$$

Note that

$$P^o_{k_{P_i}} \cap O_2(P_{jk}) \leqslant O_2(P^o_{k_{P_i}})$$

and so, since $O_2(P^o_{k_{P_i}}) = O_2(P_{ik})$, we see that

(3.1.2) $$P^o_{k_{P_i}} \cap O_2(P_{jk}) \leqslant O_2(P_{ik}) \cap O_2(P_{jk}).$$

Put $P^o_{jk} = \langle P^o_{j_{P_i}}, P^o_{k_{P_i}} \rangle$. From our assumption P^o_{jk} covers $P_{jk}/O_2(P_{jk})$. Also observe that P_i normalizes P^o_{jk}. By the definition of a minimal parabolic system and Hypothesis 1.2,

$$P_i \cap P_{jk} = S. \tag{3.1.3}$$

Suppose $P^o_{jk} = P_{jk}$. Then P_i normalizes P_{jk} whence (3.1.3) yields $S \trianglelefteq P_i$, a contradiction. Therefore $P^o_{jk} \neq P_{jk}$. So (3.1.1), (3.1.2), and P^o_{jk} covering $P_{jk}/O_2(P_{jk})$ imply that

$$O_2(P_{ik}) \cap O_2(P_{jk}) = P^o_{jk} \cap O_2(P_{jk}) = O_2(P^o_{jk}).$$

Because P_i normalizes P^o_{jk}, P_i must also normalize $O_2(P^o_{jk})$ which then yields (ii).

LEMMA 3.2. *Let $i, j \in \mathscr{I}$ be such that $P_{ij}/O_2(P_{ij}) \cong L_3(2)$.*

(i) *There exists $x_i \in P_i \backslash S$ and $x_j \in P_j \backslash S$ such that x_i^2, x_j^2, $(x_i x_j)^3 \in O_2(P_{ij})$.*

(ii) *Let x_i and x_j be as in part* (i). *Assume that X is a normal 2-subgroup of P_i and that $Y = X \cap X^{x_j} \trianglelefteq S$. Then $Y \cap Y^{x_i} \trianglelefteq P_{ij}$.*

Proof. (i) See (2.1) (ii) of [R1].

(ii) Since $Y \trianglelefteq S$ and $x_i^2 \in O_2(P_{ij}) \leqslant S$, we note that $Y \cap Y^{x_i}$ is normalized by $\langle O_2(P_i), x_i \rangle$. Now

$$\begin{aligned} Y \cap Y^{x_i} &= (X \cap X^{x_j}) \cap (X \cap X^{x_j})^{x_i} \\ &= X \cap X^{x_j} \cap X^{x_j x_i}, \end{aligned}$$

because $X \trianglelefteq P_i$. Hence

$$\begin{aligned} (Y \cap Y^{x_i})^{x_j} &= X^{x_j} \cap X^{x_j^2} \cap X^{x_j x_i x_j} \\ &= X \cap X^{x_j} \cap X^{x_j x_i x_j}. \end{aligned}$$

Because x_i^2, $x_j^2 \in O_2(P_{ij})$, $X \trianglelefteq P_i$, and $x_j x_i x_j = x_i^{-1} x_j^{-1} x_i^{-1} \pmod{O_2(P_{ij})}$, we see that $X^{x_j x_i x_j} = X^{x_j x_i}$, whence $\langle O_2(P_i), x_i, x_j \rangle$ normalizes $Y \cap Y^{x_i}$. Using Lemma 2.1(ii) gives $\langle O_2(P_i), x_i, x_j \rangle = P_{ij}$, whence (ii) holds.

LEMMA 3.3. *Let $i, j \in \mathscr{I}$ be such that $P_{ij}/O_2(P_{ij}) \cong \hat{S}_6$.*

(i) *There exists $x_o \in P_i \backslash S$ and y_1, $y_2 \in P_j \backslash S$ such that x_o^2, y_1^2, y_2^2, $(x_o y_1)^5$, $(x_o y_2)^5 \in O_2(P_{ij})$, and $y_1 y_2 \notin S$.*

(ii) *There exists $y_o \in P_j \backslash S$ and x_i, $x_2 \in P_i \backslash S$ such that y_o^2, x_1^2, x_2^2, $(y_o x_1)^5$, $(y_o x_2)^5 \in O_2(P_{ij})$, and $x_1 x_2 \notin S$.*

(iii) $\langle x_o, y_1, [O_2(X), x_o]\rangle$ *contains a Sylow* 3-*subgroup of* Y *and* $\langle y_o, x_1, [O_2(Y), y_o]\rangle$ *contains a Sylow* 3-*subgroup of* X.

Proof. From [R1, (2.2); R2, Lemma 2.3], we obtain parts (i) and (ii). Part (iii) results from [R2, Lemma 2.3; R1, (2.4)] (note that $x_o = x_1 = a$ and $y_o = y_1 = b$ where a and b are as in [R1, (2.3)]).

LEMMA 3.4 (Replication Lemma). *Let* $k \in \mathcal{I}$ *and let* $\emptyset \neq \mathcal{J} \subseteq \mathcal{I}$ *be such that* $\bar{P}_{kj} \cong S_3 \times S_3$ *for all* $j \in \mathcal{J}$. *Put* $P_{\mathcal{J}} = \langle P_j \mid j \in \mathcal{J}\rangle$ *and* $P^o_{\mathcal{J}} = \langle P^o_{j_{P_k}} \mid j \in \mathcal{J}\rangle$. *Suppose* $X \trianglelefteq P_{\mathcal{J}}$ *with* $X \leqslant S$ *and let* $g \in P_k \backslash S$. *Then, putting* $X_k = O_2(P_k) \cap X$, *we have that the following hold.*

(i) $X \cap X^g$, X^g, X_k, X^g_k, *and* XX^g_k *are all normalized by* $P^o_{\mathcal{J}}$.

(ii) *Either* $X_k \trianglelefteq P_k$ (*and in which case* $[X : X \cap X^g] \leqslant 2$) *or* $XX^g_k > X$.

(iii) *If* $P^o_{\mathcal{J}}$ *covers* $P_{\mathcal{J}}/K_{\mathcal{J}}$ (*where* $K_{\mathcal{J}} = \mathrm{core}_{P_{\mathcal{J}}} S$), *then* $XX^g_k \leqslant K_{\mathcal{J}}$.

(iv) $X_k \cap X^g_k = X \cap X^g_k$.

(v) *Suppose* $g^2 \in O_2(P_k)$ *and let* $\hat{\mathcal{J}}$ *be a non-empty subset of* $\mathcal{J}$. *Set* $P^o_{\hat{\mathcal{J}}} = \langle P^o_{j_{P_k}} \mid j \in \hat{\mathcal{J}}\rangle$. *Suppose* Y *is a subgroup with* $X \leqslant Y \leqslant XX^g_k$. *Then* Y *is normalized by* $P^o_{\hat{\mathcal{J}}}$ *if and only if* $Y^{g^{-1}} \cap X_k$ *is normalized by* $P^o_{\hat{\mathcal{J}}}$. *In particular, if*

$$X = Y_1 < Y_2 \cdots < Y_{n-1} < Y_n = XX^g_k$$

is a $P^o_{\hat{\mathcal{J}}}$-*composition series of* XX^g_k/X, *then*

$$X_k \cap X^g_k < Y_2^{g^{-1}} \cap X_k < \cdots < Y_{n-1}^{g^{-1}} \cap X_k < X_k$$

is a $P^o_{\hat{\mathcal{J}}}$-*composition series of* $X_k/X_k \cap X^g_k$.

(vi) $X \cap X^g/X_k \cap X^g_k$, *which admits* $P^o_{\mathcal{J}}$ *has order at most two.*

Proof. Note that g normalizes $P^o_{\mathcal{J}}$ since $P^o_{j_{P_k}} \trianglelefteq P_{jk}$ for all $j \in \mathcal{J}$, and so $P^o_{\mathcal{J}}$ normalizes X^g. Also observe that $\langle X, X^g\rangle \leqslant P_k$ and, as $P^o_{\mathcal{J}}$ normalizes $\langle X, X^g\rangle$, that $P^o_{\mathcal{J}}$ normalizes $O_2(\langle X, X^g\rangle)$. Evidently $X_k \leqslant O_2(\langle X, X^g\rangle) \cap X$ and, since $X^g_k \leqslant O_2(P_k) \leqslant S$, $\langle X, X^g_k\rangle = XX^g_k$. If $X_k = X$ then (i) clearly holds. If $X_k \neq X$, then, since $g \notin S$ and $P_k/O_2(P_k) \cong S_3$, $\langle X, X^g\rangle$ covers $P_k/O_2(P_k)$, whence $O_2(\langle X, X^g\rangle) \leqslant O_2(P_k)$. Thus $X_k = O_2(\langle X, X^g\rangle) \cap X$ and so $P^o_{\mathcal{J}}$ normalizes X_k, which implies that (i) holds in this case also.

(ii) First observe that $X_k \trianglelefteq S$. Now assume $XX^g_k = X$ holds. Then $X^g_k \leqslant X$ and since $X^g_k \leqslant O_2(P_k)$ (because $X_k \leqslant O_2(P_k)$) we obtain $X^g_k \leqslant X \cap O_2(P_k) = X_k$. Hence $X^g_k = X_k$ and so $X_k \trianglelefteq \langle S, g\rangle = P_k$. Therefore either $XX_k > X$ or $X_k \trianglelefteq P_k$ holds.

(iii) From (i), XX_k^g is a 2-subgroup of S which is normalized by $P^o_{\mathscr{J}}$ and hence $P^o_{\mathscr{J}}$ covering $P_{\mathscr{J}}/K_{\mathscr{J}}$ forces $XX_k^g \leqslant K_{\mathscr{J}}$.

(iv) Now

$$\begin{aligned} X_k \cap X_k^g &= (O_2(P_k) \cap X) \cap (O_2(P_k) \cap X)^g \\ &= O_2(P_k) \cap X \cap X^g \\ &= X \cap (O_2(P_k) \cap X)^g \\ &= X \cap X_k^g, \end{aligned}$$

which proves (iv).

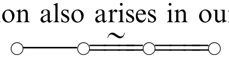
(v) Since X and X_k^g are both normalized by $P^o_{\hat{\mathscr{J}}}$ the operator isomorphism theorem (see [S, p. 116]) and (iv) imply

$$XX_k^g/X \underset{P^o_{\hat{\mathscr{J}}}}{\cong} X_k^g/X \cap X_k^g = X_k^g/X_k \cap X_k^g.$$

So Y is normalized by $P^o_{\hat{\mathscr{J}}}$ if and only if Y/X is normalized by $P^o_{\hat{\mathscr{J}}}$ if and only if $Y \cap X_k^g/X_k \cap X_k^g$ is normalized by $P^o_{\hat{\mathscr{J}}}$. Because $g^2 \in S$ and $X_k \trianglelefteq S$, $X_k \cap X_k^g$ is normalized by g. So $X_k/X_k \cap X_k^g$ and $X_k^g/X_k \cap X_k^g$ are isomorphic by conjugating by g^{-1}. Thus, as g normalizes $P^o_{\hat{\mathscr{J}}}$, we deduce that $Y \cap X_k^g/X_k \cap X_k^g$ is normalized by $P^o_{\hat{\mathscr{J}}}$ if and only if $Y^{g^{-1}} \cap X_k/X_k \cap X_k^g$ is normalized by $P^o_{\hat{\mathscr{J}}}$. Therefore Y is normalized by $P^o_{\hat{\mathscr{J}}}$ if and only if $Y^{g^{-1}} \cap X_k$ is normalized by $P^o_{\hat{\mathscr{J}}}$. The remainder of (v) now follows.

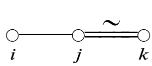
(vi) Because $X \cap X^g$ and X_k^g are subgroups of X^g and $[X^g : X_k^g] \leqslant 2$, $X \cap X^g \cap X_k^g = X \cap X_k^g$ has index at most two in $X \cap X^g$. Thus, using (iv), $[X \cap X^g : X_k \cap X_k^g] \leqslant 2$ and we have proved (vi).

4. A SUBGROUP CONFIGURATION

The subject of this section is a particular configuration of subgroups we shall eventually encounter in the course of proving Theorem 1.3. Since this configuration also arises in our work on minimal parabolic systems whose diagram is  [R3] we treat it separately.

The situation we shall study is described in

HYPOTHESIS 4.1. *Let G be a group containing a minimal parabolic system $\{P_1, ..., P_n\}$, which satisfies Hypothesis* 1.2. (*So $S = P_1 \cap \cdots \cap P_n$.*) *Suppose $\{i, j, k\}$ is a three-element subset of $I = \{1, ..., n\}$ whose subdiagram is*



.

Further, assume that $|S/\mathrm{core}_{P_{ijk}} S| \neq 2^9$ *and that* $\mathrm{core}_{P_{ijk}}(S) \geqslant X_1 > X_2 > X_3 > X_4$ *are four subgroups of S satisfying*

(i) $[X_i : X_{i+1}] = 2$ *for* $i = 1, 2, 3$;

(ii) $X_1 \trianglelefteq P_{jk}$, $X_2 \trianglelefteq P_j$, $X_3 \trianglelefteq P_k$, $X_4 \trianglelefteq P_{ij}$;

(iii) $X_2 \ntrianglelefteq P_k$; *and*

(iv) $X_3 \ntrianglelefteq P_j$.

Define X_5 to be $\mathrm{core}_{P_k} X_4$.

Our efforts in the present section are directed towards establishing

THEOREM 4.2. *Assume Hypothesis* 4.1 *holds. Then one of the following occurs*:

(a) $X_4 = X_5$ (*and so* $X_4 \trianglelefteq P_{ijk}$);

(b) $[X_4 : X_5] = 2$ *and* $X_5 \trianglelefteq P_{jk}$; *and*

(c) $[X_4 : X_5] = 2^2$ *and for any* $x \in P_j \backslash S$ $X_5 \cap X_5^X \trianglelefteq P_{jk}$ *with* $[X_5 : X_5 \cap X_5^X] = 2$.

In the final stages of the proof of Theorem 4.2 we need to call on results from [R1]. So as to allow an easy transition to [R1] and also to banish some subscripts we modify the notation in Hypothesis 4.1 as follows. For P_k, P_j, P_i we write A, B, C (respectively) and we will use notation already established in Section 2 for $\{A, B, C\}$. We thus have the lattice in Fig. 4 with $|S/T_1| \neq 2^9$, $X_2 \ntrianglelefteq A$, and $X_3 \ntrianglelefteq B$.

For the rest of this section Hypothesis 4.1 is assumed to hold sway.

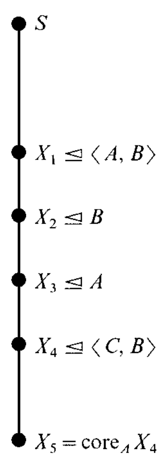


FIGURE 4

Before starting the proof of Theorem 4.2 we must make some preparatory observations concerning Hypothesis 4.1. These form the content of Lemmas 4.3, 4.4, and 4.5.

One further piece of notation. For $a \in A \backslash S$ and $b \in B \backslash S$ we set

$$X_4(a, b) = (X_4 \cap X_4^a) \cap (X_4 \cap X_4^a)^b.$$

LEMMA 4.3. (i) *$X_2 \cap X_2^a = X_3$ for any $a \in A \backslash S$.*

(ii) *$X_4 = X_2 \cap X_2^a \cap X_2^{ab}$ for any $a \in A \backslash S$ and for any $b \in B \backslash S$.*

(iii) *If $a^2, b^2, (ba)^5 \in N$, then $\langle a, b, N_o, [O_2(A), a] \rangle$ normalizes $X_4(a, b)$.*

Proof. (i) Let $a \in A \backslash S$. Since $[X_2 : X_3] = 2$ and $X_3 \trianglelefteq A$, we have either $X_2 \cap X_2^a = X_2$ or X_3. The first possibility, since $X_2 \trianglelefteq B$, would yield $X_2 \trianglelefteq \langle S, a \rangle = A$, contrary to our supposition that $X_2 \ntrianglelefteq A$. Therefore $X_2 \cap X_2^a = X_3$.

(ii) Let $a \in A \backslash S$ and $b \in B \backslash S$. Since $X_3 \ntrianglelefteq B$, $X_4 \trianglelefteq B$, and $[X_3 : X_4] = 2$, the argument in (i) also shows that $X_4 = X_3 \cap X_3^b$. Hence, using (i) and $X_2 \trianglelefteq B$,

$$\begin{aligned} X_4 &= X_3 \cap X_3^b \\ &= (X_2 \cap X_2^a) \cap (X_2 \cap X_2^a)^b \\ &= X_2 \cap X_2^a \cap X_2^{ab}, \end{aligned}$$

as required

(iii) Using part (ii) we substitute for X_4 and obtain

$$\begin{aligned} X_4(a, b) = {} & (X_2 \cap X_2^a \cap X_2^{ab} \cap X_2^a \cap X_2^{a^2} \cap X_2^{aba}) \\ & \cap (X_2^b \cap X_2^{ab} \cap X_2^{ab^2} \cap X_2^{ab} \cap X_2^{a^2b} \cap X_2^{abab}). \end{aligned}$$

Note that $X_2 \leqslant X_1 \leqslant N$ implies $X_2^a \trianglelefteq N$ whence $b^2 \in N$ gives $X_2^{ab^2} = X_2^a$. Since we also have $X_2 \trianglelefteq B$ and $a^2 \in N$, the above becomes

$$\begin{aligned} X_4(a, b) &= X_2 \cap X_2^a \cap X_2^{ab} \cap X_2^{aba} \cap X_2^{abab} \\ &= X_4 \cap X_4^a \cap X_2^{abab}. \end{aligned} \tag{4.3.1}$$

We now consider

$$\begin{aligned} X_4(a, b)^a &= (X_4 \cap X_4^a)^a \cap X_2^{ababa} \\ &= (X_4 \cap X_4^a) \cap X_2^{ababa}, \end{aligned}$$

since $a^2 \in N$ and $X_4 \trianglelefteq S$. By hypothesis $(ba)^5 \in N$ and so $X_2 \trianglelefteq B$ yields

$$X_2^{ababa} = X_2^{(ba)^3} = X^{a^{-1}b^{-1}a^{-1}b^{-1}}$$
$$= X_2^{ab^{-1}a^{-1}b^{-1}}$$

as $a^2 \in N$. Now $X_2 \trianglelefteq S$ implies $X_2^a \trianglelefteq N$, whence $X_2^{ab^{-1}} = X_2^{ab}$. Similarly, $X_2^{ab} \trianglelefteq N$ and $X_2^{aba} \trianglelefteq N$ yield that

$$X_2^{ababa} = X_2^{ab^{-1}a^{-1}b^{-1}} = X_2^{abab}.$$

Consequently

$$X_4(a, b)^a = (X_4 \cap X_4^a) \cap X_2^{abab} = X_4(a, b)$$

by (4.3.1). Thus a normalizes $X_4(a, b)$.

Observing that $X_4 \cap X_4^a \trianglelefteq O_2(A)$ we see that b must also normalize $X_4(a, b)$, since $b^2 \in N \leqslant O_2(A)$. Moreover, from $N_o \leqslant O_2(A)$ and $N_o \trianglelefteq B$ it follows that N_o normalizes $X_4(a, b)$. Clearly, as a normalizes $X_4(a, b)$, we have that a normalizes $N_{O_2(A)}(X_4(a, b))$. Hence $N_{O_2(A)}(X_4(a, b)) > N_o$, for $N_{O_2(A)}(X_4(a, b)) = N_o$ implies $N_o \trianglelefteq \langle S, a\rangle = A$, which contradicts $\langle A, B\rangle/N \cong \hat{S}_6$. Thus

$$[O_2(A) : N_{O_2(A)}(X_4(a, b))] \leqslant 2$$

and so $[O_2(A), a]$ normalizes $X_4(a, b)$.

This completes the proof that $\langle b, a, N_o, [O_2(A), a]\rangle$ normalizes $X_4(a, b)$.

Now let $\beta \in B\backslash S$ and $\alpha_1, \alpha_2 \in A\backslash S$ be chosen so that

$$\beta^2, \alpha_1^2, \alpha_2^2, (\beta\alpha_1)^5, (\beta\alpha_2)^5 \in N$$

and

$$\alpha_1\alpha_2^{-1} \notin S.$$

Since $\langle A, B\rangle/N \cong \hat{S}_6$, Lemma 3.3 guarantees that such β, α_1, α_2 may be found (note that $\alpha_1\alpha_2^{-1}N = \alpha_1\alpha_2 N \not\subseteq S$ implies $\alpha_1\alpha_2^{-1} \notin S$). We keep β, α_1, and α_2 fixed for the rest of Section 4 and define

$$X_6 = X_5 \cap X_5^{\beta}.$$

Clearly $\langle O_2(B), \beta\rangle$ normalizes X_6.

LEMMA 4.4. (i) $X_5 = (X_4 \cap X_4^{\alpha_1}) \cap (X_4 \cap X_4^{\alpha_2})$.

(ii) $X_5 \trianglelefteq \langle A, C\rangle$.

(iii) $X_6 = X_4(\alpha_1, \beta) \cap X_4(\alpha_2, \beta)$.

Proof. (i) If $N_A(X_4)=A$, then (i) is clear. Otherwise we have $N_A(X_4)=S$ and then, as $\alpha_1\alpha_2^{-1}\notin S$, the three A-conjugates of X_4 are X_4, $X_4^{\alpha_1}$, and $X_4^{\alpha_2}$, which also gives (i).

(ii) From $X_4 \trianglelefteq C$ and $\langle C, A\rangle/P \cong S_3\times S_3$, Lemma 3.4(i) implies that C_A^O normalizes $X_4\cap X_4^{\alpha_i}$ for $i=1, 2$. Therefore, using (i), we obtain

$$X_5 \trianglelefteq \langle A, C_A^O\rangle = \langle A, C\rangle.$$

(iii) Combining the definition of X_6 and (i) gives

$$\begin{aligned} X_6 &= X_5\cap X_5^\beta \\ &= (X_4\cap X_4^{\alpha_1})\cap(X_4\cap X_4^{\alpha_2})\cap(X_4\cap X_4^{\alpha_1})^\beta\cap(X_4\cap X_4^{\alpha_2})^\beta \\ &= X_4(\alpha_1,\beta)\cap X_4(\alpha_2,\beta), \end{aligned}$$

as required.

LEMMA 4.5. *One of the following must hold*:

(a) $X_4=X_5$;

(b) $[X_4:X_5]=2$ *and* $X_5\trianglelefteq\langle A, B\rangle$; *and*

(c) $[X_4:X_5]=2^2$.

Proof. Since $[X_3:X_4]=2$ and $X_3\trianglelefteq A$, $[X_4:X_4\cap X_4^{\alpha_i}]\leqslant 2$ for $i=1, 2$. Therefore $[X_4:X_5]=1, 2$ or 2^2 by Lemma 4.4.

Suppose $[X_4:X_5]=2$ holds. Then

$$X_5 = X_4\cap X_4^{\alpha_1}.$$

Hence, by the definition of $X_4(\alpha_1,\beta)$,

(4.5.1) $$X_4(\alpha_1,\beta)=X_5\cap X_5^\beta.$$

Because $X_5\trianglelefteq S$, $O_2(B)$ must normalize $X_4(\alpha_1,\beta)$. Thus, using Lemma 4.3(iii) and Lemma 2(ii) gives that $\langle A, B\rangle=\langle O_2(B),\beta,\alpha_1\rangle$ normalizes $X_4(\alpha_1,\beta)$. From $[X_4:X_5]=2$ and $X_4\trianglelefteq B$ we have $[X_5:X_4(\alpha_1,\beta)]\leqslant 2$.

Suppose $[X_5:X_4(\alpha_1,\beta)]=2$ holds. Then $\bar{X}_1=X_1/X_4(\alpha_1,\beta)$ has order 2^5 and admits $\langle A, B\rangle$. We now show that

(4.5.2) $$|C_{\bar{X}_1}(x_A)|=2=|C_{\bar{X}_1}(x_B)|.$$

We have that X_1/X_4 and $X_4/X_4(\alpha_1, \beta)$ both admit B. Since $X_3 \ntrianglelefteq B$ and $|X_1/X_4| = 2^3$ by Hypothesis 4.1, it follows that $|C_{X_1/X_4}(x_B)| = 2$. If x_B were to centralize $X_4/X_4(\alpha_1, \beta)$, then, as $X_4 > X_5 > X_4(\alpha_1, \beta)$, we obtain

$$X_5 \trianglelefteq \langle S, x_B \rangle = B.$$

But then $X_4(\alpha_1, \beta) = X_5$ by (4.5.1), a contradiction. So, since $|X_4/X_4(\alpha_1, \beta)| = 2^2$, this gives $|C_{X_4/X_4(\alpha_1, \beta)}(x_B)| = 1$. Therefore $|C_{\bar{X}_1}(x_B)| = 2$.

Now X_1/X_3 and $X_3/X_4(\alpha_1, \beta)$ both admit A. By Hypothesis 4.1, $|X_1/X_3| = 2^2$ and $X_2 \ntrianglelefteq A$. Hence $|C_{X_1/X_3}(x_A)| = 1$. Because

$$X_3 > X_5 = core_A\, X_4 > X_4(\alpha_1, \beta) \qquad \text{and} \qquad X_4 > X_5,$$

x_A cannot centralize $X_3/X_4(\alpha_1, \beta)$. Therefore, as $|X_3/X_4(\alpha_1, \beta)| = 2^3$, we infer that $|C_{X_3/X_4(\alpha_1, \beta)}(x_A)| = 2$ whence $|C_{\bar{X}_1}(x_A)| = 2$. So we have established (4.5.2).

By Lemma 2.6(i), (4.5.2) is impossible and so we conclude that $[X_5 : X_4(\alpha_1, \beta)] = 2$ cannot occur. Hence $X_5 = X_4(\alpha_1, \beta)$. So, when $[X_4 : X_5] = 2$, we have shown that $X_5 \trianglelefteq \langle A, B \rangle$. Thus we obtain (b) and the proof of Lemma 4.5 is complete.

We now begin the

Proof of Theorem 4.2. As a consequence of Lemma 4.5 we only need to study the situation $[X_4 : X_5] = 2^2$ and show that (c) of Theorem 4.2 holds.

First we pinpoint certain subgroup indices.

(4.2.1) For $i = 1, 2$ $\quad [X_4 \cap X_4^{\alpha_i} : X_4(\alpha_i, \beta)] = 2.$

Since $X_4 \trianglelefteq B$ and $[X_4 : X_4 \cap X_4^{\alpha_i}] = 2$, we clearly have $[X_4 \cap X_4^{\alpha_i} : X_4(\alpha_i, \beta)] \leqslant 2$. If $X_4 \cap X_4^{\alpha_i} = X_4(\alpha_i, \beta)$, then

$$X_4 \cap X_4^{\alpha_i} \trianglelefteq \langle O_2(A), \beta, \alpha_i \rangle = \langle A, B \rangle$$

by Lemma 4.3(iii) and Lemma 2.1(ii). In particular,

$$X_4 \cap X_4^{\alpha_i} \leqslant core_A X_4 = X_5$$

whereas $[X_4 : X_5] = 2^2$. Therefore $X_4 \cap X_4^{\alpha_i} \neq X_4(\alpha_i, \beta)$ and we have (4.2.1).

(4.2.2) For $i = 1, 2$ $\quad [X_5 : X_5 \cap X_4(\alpha_i, \beta)] = 2$

Since $[X_4 : X_5] = 2^2$, we have $[X_4 \cap X_4^{\alpha_i} : X_5] = 2$ and so, appealing to (4.2.1),

$$[X_5 : X_5 \cap X_4(\alpha_i, \beta)] \leqslant 2.$$

If $X_5 = X_5 \cap X_4(\alpha_i, \beta)$, then using Lemma 4.3(iii) gives

$$X_5 \trianglelefteq \langle A, \beta \rangle = \langle A, B \rangle.$$

So $\bar{X}_1 = X_1/X_5$, which has order 2^5, admits $\langle A, B \rangle$. Since $X_2 \ntrianglelefteq A$ by Hypothesis 4.1 and $X_4 \ntrianglelefteq A$ (as $X_4 \neq X_5$), we conclude that $|C_{\bar{X}_1}(x_A)| = 2$. Because $X_3 \ntrianglelefteq B$, we have that $|C_{X_1/X_4}(x_B)| = 2$. If x_B were to centralize X_4/X_5 then we obtain

$$X_4 \cap X_4^{\alpha_i} \trianglelefteq \langle O_2(A), \alpha_i, x_B \rangle = \langle A, B \rangle$$

by Lemma 2.1(ii). This contradicts our supposition that $[X_4 : X_5] = 2^2$. Thus $|C_{X_4/X_5}(x_B)| = 1$ and so we also have $|C_{\bar{X}_1}(x_B)| = 2$. Now Lemma 2.6(i) yields a contradiction and hence $X_5 \neq X_5 \cap X_4(\alpha_i, \beta)$. This proves (4.2.2).

Suppose $[X_5 : X_6] \leqslant 2$. Then $X_4(\alpha_1, \beta) \geqslant X_6$ (by Lemma 4.4(iii)) and (4.2.2) imply that

$$X_6 = X_5 \cap X_4(\alpha_1, \beta).$$

Hence, using Lemma 4.3(iii), α_1 normalizes X_6; hence

$$X_6 \trianglelefteq \langle O_2(B), \beta, \alpha_1 \rangle = \langle A, B \rangle.$$

Thus (c) of Theorem 4.2 holds. Noting that $[X_5 : X_6] \leqslant 2^2$ by Lemma 4.4(iii) and (4.2.2), the proof of the theorem will be complete when we have ruled out the possibility $[X_5 : X_6] = 2^2$. From now on we assume $[X_5 : X_6] = 2^2$. So, assembling the information in (4.2.1), (4.2.2), and Lemma 4.4, we have the subgroup lattice in Fig. 5.

(4.2.3) $O_2(A)$ centralizes X_4/X_5.

We have $[X_3, O_2(A)] \leqslant X_4$ because $[X_3 : X_4] = 2$ and both X_3 and X_4 are normal subgroups of S. Since $X_3 \trianglelefteq A$ by Hypothesis 4.1, $[X_3, O_2(A)] \trianglelefteq A$ also. Thus

$$[X_3, O_2(A)] \leqslant \mathit{core}_A\, X_4 = X_5.$$

Consequently

$$[X_4, O_2(A)] \leqslant [X_3, O_2(A)] \leqslant X_5,$$

which proves (4.2.3).

(4.2.4) N centralizes X_4/X_6.

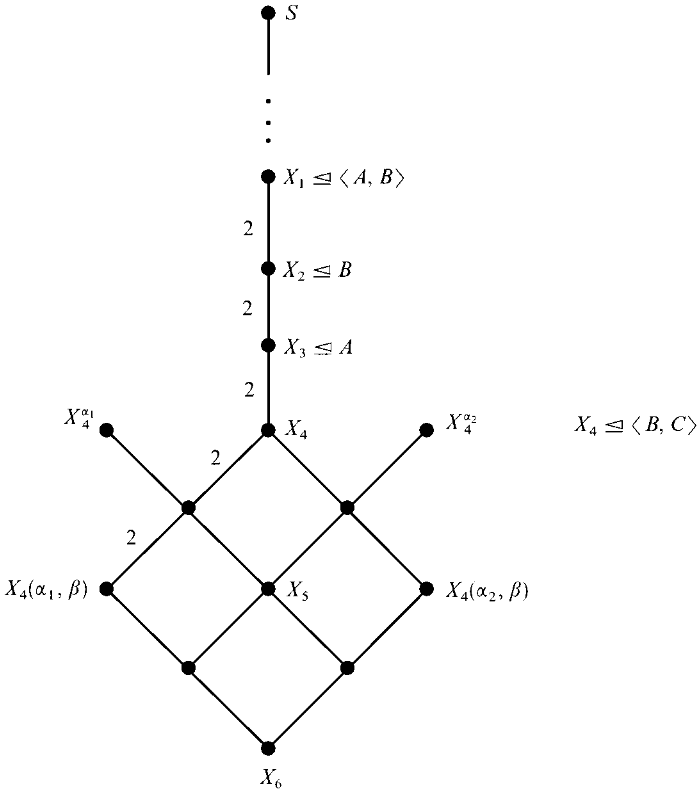


FIGURE 5

Clearly we have

$$[X_4, N] \leqslant [X_4, O_2(A)] \leqslant X_5$$

by (4.2.3). Now $X_4 \trianglelefteq B$ by Hypothesis 4.1 and so $[X_4, N] \trianglelefteq B$. Therefore

$$[X_4, N] \leqslant X_5 \cap X_5^{\beta} = X_6$$

whence (4.2.4) holds

(4.2.5) For $i=1, 2$ $\langle \alpha_i, N_o, [O_2(A), \alpha_i]\rangle$ normalizes $X_4(\alpha_i, \beta) \cap X_5$.

This follows since $X_5 \trianglelefteq A$ and $\langle \alpha_i, N_o, [0_2(A), \alpha_i]\rangle$ normalizes $X_4(\alpha_i, \beta)$ by Lemma 4.3(iii).

Observe, since $X_5 \trianglelefteq S$, that X_5/X_6 admits $O_2(B)$. Put $C_{O_2(B)}(X_5/X_6) = O_2(B)^*$. From (4.2.4) we have that $N \leqslant O_2(B)^*$ and, since $|X_5/X_6| = 2^2$, $[O_2(B) : O_2(B)^*] \leqslant 2$.

One of the following holds:

(4.2.6) (a) $X_6 \trianglelefteq \langle A, B \rangle$

(b) $X_6 \trianglelefteq B$ and $O_2(A)$ centralizes X_5/X_6.

First we consider the possibility that $O_2(B)^* \nleqslant O_2(A)$. Let $z \in O_2(B)^* \backslash O_2(A)$ and let $i \in \{1, 2\}$. Then $\langle z, \alpha_i, N_o, [O_2(A), \alpha_i] \rangle$ normalizes $X_4(\alpha_i, \beta) \cap X_5$, using (4.2.5). Since $z \in S \backslash O_2(A)$ and $\alpha_i \in A \backslash S$, $\langle z, \alpha_i \rangle$ will cover $A \backslash O_2(A)$. Using Lemma 2.1(iii) we infer that

$$[A : N_A(X_4(\alpha_i, \beta) \cap X_5)] \leqslant 2,$$

whence $O^2(A)$ normalizes $X_4(\alpha_i, \beta) \cap X_5$ for $i = 1, 2$. Hence, by Lemma 4.4(iii), $O^2(A)$ normalizes X_6. Thus

$$X_6 \trianglelefteq \langle O^2(A), \beta, O_2(B) \rangle = \langle A, B \rangle$$

by Lemma 2.1(ii), and so (a) holds.

We now turn to the case $O_2(B)^* \leqslant O_2(A)$. Then $O_2(B)^* = O_2(B) \cap O_2(A)$. From Lemma 2.1(iii), $[O_2(A), \alpha_i] \nleqslant O_2(B) \cap O_2(A)$. Hence $O_2(A) = \langle O_2(B) \cap O_2(A), [O_2(A), \alpha_i] \rangle$ normalizes $X_4(\alpha_i, \beta) \cap X_5$ for $i = 1, 2$ by (4.2.5). Therefore, by Lemma 4.4(iii), $O_2(A)$ normalizes

$$(X_4(\alpha_1, \beta) \cap X_5) \cap (X_4(\alpha_2, \beta) \cap X_5) = X_6,$$

which implies

$$X_6 \trianglelefteq \langle O_2(A), O_2(B), \beta \rangle = B.$$

Since we also have $[O_2(A), X_5] \leqslant X_4(\alpha_i, \beta) \cap X_5$ for $i = 1, 2$, we obtain $[O_2(A), X_5] \leqslant X_6$. So the case $O_2(B)^* \leqslant O_2(A)$ leads to alternative (b) of (4.2.6) and this completes the proof of (4.2.6).

So, by (4.2.6), we know that $X_6 \trianglelefteq B$. This turns out to be a crucial observation as it allows us to construct (in (4.2.8)) a further subgroup of S which enables us to tighten our grip on this configuration.

(4.2.7) (i) X_4 contains no proper B-invariant subgroups which contain X_5.

(ii) For any $b \in B \backslash S$, $X_6 = X_5 \cap X_5^b$.

If (i) were false, then we would have $[X_5 X_5^\beta : X_5] \leqslant 2$ contrary to $[X_5 : X_6] = 2^2$. So (i) holds.

Let $b \in B \backslash S$. Because each right coset of S in B contains an element ρ such that $\rho^2 \in S$ and $X_5 \trianglelefteq S$, we may suppose that $b^2 \in S$. Hence b normalizes $X_5 X_5^b$ and, since $X_5 \leqslant X_5 X_5^b \leqslant X_4$, (4.2.3) yields

$$X_5 X_5^b \trianglelefteq \langle O_2(A), b \rangle = B.$$

Consequently $X_5 X_5^b = X_4$ by part (i) whence $[X_5 : X_5 \cap X_5^b] = 2^2$. Since $X_6 \trianglelefteq B$ by (4.2.6) and $[X_5 : X_6] = 2^2$, we obtain $X_5 \cap X_5^b = X_6$, as desired.

By Lemma 3.2(i) we can find $c^* \in C \backslash S$ and $b^* \in B \backslash S$ such that b^{*2}, c^{*2}, $(c^* b^*)^3 \in M$. Put $X_7 = X_6 \cap X_6^{c^*}$.

(4.2.8) $$X_7 \trianglelefteq \langle B, C \rangle.$$

Combining (4.2.6) and (4.2.7)(ii) we see that $X_6 = X_5 \cap X_5^{b^*} \trianglelefteq S$. By Lemma 4.4(ii), $X_5 \trianglelefteq C$ and so appealing to Lemma 3.2(ii) gives (4.2.8).

Put $\bar{X}_4 = X_4 / X_6$ and $\tilde{X}_4 = X_4 / X_7$. Note that $|\bar{X}_4| = 2^4$ and, by (4.2.6), B acts on $\bar{X}_4$. From (4.2.8), $\langle B, C \rangle$ acts on $\tilde{X}_4$. Moreover, $X_5 \trianglelefteq C$ and $[X_5 : X_6] = 2^2$ yield that $2^4 \leqslant |\tilde{X}_4| \leqslant 2^6$. Our attention now focuses on $\bar{X}_4$ and $\tilde{X}_4$.

(4.2.9) $$C_{\bar{X}_4}(x_B) = 1.$$

Together Lemma 3.3(iii) and Lemma 4.3 imply there exists $\langle x \rangle \in Syl_3 B$ such that x normalizes $X_4(\alpha_i, \beta)$ for some $i \in \{1, 2\}$. Recalling that $X_4 \trianglelefteq B$ and $X_6 \trianglelefteq B$, we have $\langle x \rangle$ acting upon $X_4 / X_4(\alpha_i, \beta)$ and $X_4(\alpha_i, \beta)/X_6$.

If x were to centralize $X_4/X_4(\alpha_i, \beta)$, then x would normalize $X_4 \cap X_4^{\alpha_i}$ whence

$$X_4 \cap X_4^{\alpha_i} \trianglelefteq \langle O_2(A), \alpha_i, x \rangle = \langle A, B \rangle,$$

by Lemma 2.1(ii). This contradicts $[X_4 : X_5] = 2^2$ and so x does not centralize $X_4/X_4(\alpha_i, \beta)$. If x were to centralize $X_4(\alpha_i, \beta)/X_6$, then x would normalize $X_4(\alpha_i, \beta) \cap X_5$. But then

$$X_5 \cap X_5^x \geqslant X_4(\alpha_i, \beta) \cap X_5 > X_6,$$

contradicting (4.2.7)(ii). Thus x doesn't centralize $X_4(\alpha_i, \beta)/X_6$ and so we conclude that $C_{\bar{X}_4}(x) = 1$. Since $\langle x \rangle$ and $\langle x_B \rangle$ are conjugate in B, we have (4.2.9).

(4.2.10) $X_6 \ntrianglelefteq \langle A, B \rangle$ (and so, by (4.2.6), $O_2(A)$ centralizes X_5/X_6).

Suppose $X_6 \trianglelefteq \langle A, B \rangle$ holds. Then $\langle A, B \rangle$ acts on X_1/X_6, which has order 2^7. Since $X_3 \ntrianglelefteq B$ and $|X_1/X_4| = 2^3$, we have $|C_{X_1/X_4}(x_B)| = 2$. Thus $|C_{X_1/X_6}(x_B)| = 2$ by (4.2.9). This is impossible by Lemma 2.6(ii) and so we deduce that $X_6 \ntrianglelefteq \langle A, B \rangle$.

Using (4.2.9) once again we pin down $\tilde{X}_4$ in our next statement.

(4.2.11) $|\tilde{X}_4| = 2^6$ and, in a chief series for $\langle B, C\rangle$,

$$\tilde{X}_4 \text{ has two chief factors each of order } 2^3.$$

In view of (4.2.9) we have $C_{\tilde{X}_4}(x_B) \leqslant X_6/X_7$. Now combining $|\tilde{X}_4| = 2^4\,|X_6/X_7|$ with Lemma 2.2 yields (4.2.11).

Let $X_7 < X < X_4$ be such that $1 < \tilde{X} < \tilde{X}_4$ is a $\langle B, C\rangle$ chief series for $\tilde{X}_4$. Note that $|X_4/X| = |X/X_7| = 2^3$ by (4.2.11).

(4.2.12) $$|\tilde{X} \cap \tilde{X}_6| = 2.$$

Since $|\tilde{X}| = 2^3$ and, by (4.2.11), $|\tilde{X}_6| = 2^2$, we have $2^3 \leqslant |\tilde{X}\tilde{X}_6| \leqslant 2^5$. Because $|\bar{X}_4| = 2^4$ and, by (4.2.9), $C_{\bar{X}_4}(x_B) = 1$ every proper non-trivial B-invariant subgroup of $\bar{X}_4$ must have order 2^2. By (4.2.6), $\tilde{X}\tilde{X}_6$ is B-invariant and so $\bar{X}_4 \cong_B \tilde{X}_4/\tilde{X}_6$ yields that $|\tilde{X}\tilde{X}_6/\tilde{X}_6| = 2^2$. Hence $|\tilde{X} \cap \tilde{X}_6| = 2$ and we have (4.2.12).

(4.2.13) $$[N_o, X_4] \leqslant X_6.$$

Since $N_o \leqslant O_2(A)$, an appeal to (4.2.10) yields that N_o centralizes X_5/X_6. So $\bar{X}_5 \leqslant C_{\bar{X}_4}(N_o)$. Note that, as $N_o \trianglelefteq B$, $C_{\bar{X}_4}(N_o)$ is B-invariant. So if $X_5 = C_{\bar{X}_4}(N_o)$, then $X_5 \trianglelefteq B$ whence $X_5 = X_6$, whereas $[X_5 : X_6] = 2$.

Therefore $\bar{X}_5 < C_{\bar{X}_4}(N_o)$. Now (4.2.7)(i) forces $C_{\bar{X}_4}(N_o) = \bar{X}_4$. So $[N_o, X_4] \leqslant X_6$, as required.

Since $\tilde{X}_4/\tilde{X}$ is a chief factor for $\langle B, C\rangle$, we have

$$[M, \tilde{X}_4] \leqslant \tilde{X},$$

whence, by (4.2.13),

(4.2.14) $$[H_o, \tilde{X}_4] = [M \cap N_o, \tilde{X}_4] \leqslant \tilde{X} \cap \tilde{X}_6.$$

We make further connections between $\tilde{X}_4$ and the "top" of M beginning with

(4.2.15) $$[M, \tilde{X}_4] \neq 1.$$

Suppose $[M, \tilde{X}_4] = 1$ holds and argue for a contradiction. Then M centralizes X_5/X_6 and hence (4.2.10) implies that $S = O_2(A)\,M$ also centralizes X_5/X_6. Consequently

$$X_4(\alpha_i, \beta) \cap X_5 \trianglelefteq \langle S, \alpha_i\rangle = A$$

for $i=1,2$ by (4.2.5). Thus $X_6 \trianglelefteq A$ by Lemma 4.4(iii). But then $X_6 \trianglelefteq \langle A, B\rangle$, contrary to (4.2.10). This completes the proof of (4.2.15).

(4.2.16) $$R = C_M(\tilde{X}_4).$$

Let c be any element in $C\backslash S$. Then

$$(\tilde{X} \cap \tilde{X}_6) \cap (\tilde{X} \cap \tilde{X}_6)^c = 1.$$

If this were not the case, then as $|\tilde{X} \cap \tilde{X}_6| = 2$ by (4.2.12), we would obtain

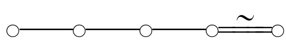
$$\tilde{X} \cap \tilde{X}_6 \trianglelefteq \langle B, c\rangle = \langle B, C\rangle,$$

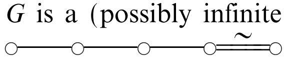
contrary to $\tilde{X}$ being an $\langle B, C\rangle$-chief factor of order 2^3. Since $\tilde{X}_4^c = \tilde{X}_4$, (4.2.14) yields

$$[H_o \cap H_o^c, \tilde{X}_4] \leqslant (\tilde{X} \cap \tilde{X}_6) \cap (\tilde{X} \cap \tilde{X}_6)^c = 1.$$

From Lemma 2.8(iv), $R = H_o \cap H_o^c$ and so $R \leqslant C_M(\tilde{X}_4)$. Noting that $C_M(\tilde{X}_4) \trianglelefteq \langle B, C\rangle$, M/R being a chief factor for $\langle B, C\rangle$ and (4.2.15) yield that $R = C_M(\tilde{X}_4)$.

Combining (4.2.12) and (4.2.14) gives $|[H_o, \tilde{X}_4]| \leqslant 2$. Recalling that $[H_o : R] = 2$, (4.2.16) implies that $|[H_o, \tilde{X}_4]| = 2$. We claim that $X_4 \leqslant R$. Since $X_1 \trianglelefteq \langle A, B\rangle$ and $X_4 \trianglelefteq \langle B, C\rangle$, clearly $X_4 \leqslant M \cap N = H$. Thus $X_4 R \leqslant H_o$ and so, as M/R is an $\langle B, C\rangle$-chief factor, we must have $X_4 \leqslant R$. Hence $\tilde{X}_4$ is abelian by (4.2.16). Therefore, as H_o acts as an involution upon $\tilde{X}_4$ with $|[H_o, \tilde{X}]| = 2$, $[\tilde{X}_4 : C_{\tilde{X}_4}(H_o)] = 2$. Further, we have that $C_{\tilde{X}_4}(H_o)$ is B-invariant since $H_o \trianglelefteq B$ and, since $H_o \leqslant M$, that $\tilde{X} \leqslant C_{\tilde{X}_4}(H_o)$. Since $|\tilde{X} \cap \tilde{X}_6| = 2$ by (4.2.12) and $|\tilde{X}_6| = 2^2$, $\tilde{X}\tilde{X}_6$ has order 2^4 and is also B-invariant. So $\tilde{X}_4$ has two B-invariant subgroups of order 2^4 and 2^5 both containing $\tilde{X}$. This is impossible since $\tilde{X}_4/\tilde{X}$ is a chief factor for $\langle B, C\rangle$ of order 2^3. So we have finally eliminated the case $[X_5 : X_6] = 2^2$ and hence, as noted earlier, this completes the proof of Theorem 4.2.

5. PRELIMINARY OBSERVATIONS ON 

For the remainder of this paper G is a (possibly infinite) group which possesses a parabolic system having  as its diagram and satisfies the hypotheses of Theorem 1.3. Consequently $|S/T_1| = 2^{10}$ by [R1] and $|S/T_3| = 2^{21}$ by [R2]. Thus results such as Lemmas 2.8 and 2.9 are available to us. We now employ the notation that was introduced in Section 2. In particular, $\{A, B, C, D, E\}$ denotes the parabolic system contained in G and $A \cap B \cap C \cap D \cap E = S$.

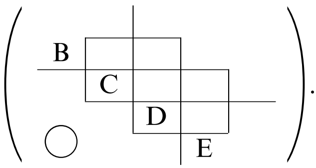
Since $\langle B, C, D, E\rangle$ has a parabolic system whose diagram is ○——○——○——○, Theorem 2.10 yields

LEMMA 5.1. (i) $\langle B, C, D, E\rangle/Z \cong L_5(2)$

(ii) $[S : Z] = 2^{10}$.

Since we wish to exploit the information contained in Lemma 5.1, it is convenient to set up further notation as follows. In view of $L_5(2)$ possessing a graph automorphism, we may suppose that (mod Z), S, B, C, D, E take the form

$$S = \begin{pmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ \bigcirc & & & 1 & * \\ & & & & 1 \end{pmatrix}$$



Our next observation will be used in Lemmas 5.3–5.7.

LEMMA 5.2. $M = ZR$.

Proof. Clearly we have

$$Z \leqslant core_{\langle B, C, D\rangle} S = H_2;$$

and we also have that

$$H_2/Z = \begin{pmatrix} 1 & & & & * \\ & 1 & \bigcirc & & * \\ & & 1 & & * \\ & & & 1 & * \\ \bigcirc & & & & 1 \end{pmatrix}.$$

Hence $C_{H_2/Z}(J) = 1$.

We claim that $Z \nleqslant R$, for $Z \leqslant R$ implies $Z \leqslant H_2 \cap R$ and then Lemmas 2.9(xi) and 5.1(ii) give $[H_2 \cap R : Z] = 2$. But then $H_2 \cap R \trianglelefteq J$ (see Lemma 2.9(xi)) gives $C_{H_2/Z}(J) \neq 1$, a contradiction. So $Z \nleqslant R$ and thus $R < ZR \leqslant M$. Since, by Lemma 2.8(i), M/R is a J-chief factor we deduce that $M = ZR$.

LEMMA 5.3. $NO_2(\langle E, B\rangle) = O_2(B)$ *and* $NO_2(\langle E, A\rangle) = O_2(A)$.

Proof. Suppose the lemma is false and argue for a contradiction. Since $[O_2(B) : O_2(\langle E, B\rangle)] = [O_2(A) : O_2(\langle E, A\rangle)] = 2$, at least one of

$N \leqslant O_2(\langle E, B\rangle)$ and $N \leqslant O_2(\langle E, A\rangle)$ must hold. From Lemma 3.1(i), $N \cap O_2(\langle E, B\rangle) = N \cap O_2(\langle E, A\rangle)$ and so we have

$$(5.3.1) \qquad N \leqslant O_2(\langle E, B\rangle),$$

$$(5.3.2) \qquad TZ \neq M.$$

Suppose $TZ = M$ were to hold. By Lemma 2.8(iii), $O_2(B) = MN$ and $T \leqslant N$. Hence, using (5.3.1) and the fact that $Z \leqslant O_2(\langle E, B\rangle)$, we see that

$$O_2(B) = MN = TZN = ZN \leqslant O_2(\langle E, B\rangle).$$

However, $[S : O_2(B)] = 2$ and $[S : O_2(\langle E, B\rangle)] = 2^2$, and so we conclude that $TZ \neq M$. This proves (5.3.2).

Using (5.3.2) we can now establish that

$$(5.3.3) \qquad Z \cap R \leqslant T.$$

Suppose $Z \cap R \nleqslant T$ were to hold. Since $(Z \cap R)\, T \trianglelefteq J$ and R/T is a J-chief factor, this forces $(Z \cap R)\, T = R$. Using Lemma 5.2 this gives

$$M = ZR = Z(Z \cap R)\, T = ZT,$$

which contradicts (5.3.2). Therefore $Z \cap R \leqslant T$ must hold.

Hence, by (5.3.3), we have $Z \cap R \leqslant T \cap H_2 (\leqslant R \cap H_2)$. Now $[M : Z] = 2^7$, $[M : R] = 2^3$, and Lemma 5.2 imply that $[M : Z \cap R] = 2^{10}$. From Lemma 2.9(xii),$[M : T \cap H_2] = 2^9$ and thus $[T \cap H_2 : Z \cap R] = 2$. Consequently, as J normalizes $Z \cap R$, $T \cap H_2$, and $R \cap H_2$,

$$C_{R \cap H_2/Z \cap R}(J) \neq 1.$$

However, from $M = ZR$ we obtain $H_2 = Z(H_2 \cap R)$ whence

$$H_2/Z \underset{J}{\cong} H_2 \cap R/Z \cap R,$$

yielding $C_{H_2/Z}(J) \neq 1$. This contradicts our previous observation in Lemma 5.2. With this contradiction the proof of Lemma 5.3 is complete.

LEMMA 5.4. (i) *$N \cap O_2(\langle E, B\rangle) = N \cap O_2(\langle E, A\rangle)$ has index 2 in N*

(ii) $[N : N \cap W] = 2$

(iii) $[O_2(\langle D, E\rangle) : W \cap O_2(\langle D, E\rangle)] = 2.$

Proof. (i) This follows from $[O_2(B) : O_2(\langle E, B\rangle)] = 2$ and Lemmas 3.1(i) and 5.3. For part (ii) see [R2, Lemma 4.6(i)].

Because D acts irreducibly upon $O_2(D)/O_2(\langle D, E\rangle)$ and $O_2(D) \geqslant O_2(\langle D, E\rangle)\, W > O_2(\langle D, E\rangle)$, it follows that $O_2(D) = O_2(\langle D, E\rangle)\, W$. Hence, as $[O_2(D) : W] = 2$, we obtain (iii).

LEMMA 5.5. (i) *B^o_D and B^o_E cover B/N*

(ii) *A^o_C, A^o_D, and A^o_E covers A/N*

(iii) *C^o_E covers C/V*

(iv) *D^o_B covers D/V*

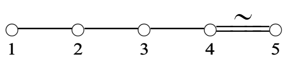
(v) *B^o_E covers B/M*

(vi) *C^o_A covers C/M*

(vii) *E^o_C covers $E/O_2(\langle E, D\rangle)$.*

Proof. From Lemma 5.3 we have $O_2(B) = NO_2(\langle E, B\rangle)$ and $O_2(A) = NO_2(\langle E, A\rangle)$. Since $\langle C, D\rangle/V \cong L_3(2)$, it follows (see Lemma 5.4(iii)) that $O_2(\langle E, C\rangle)\, V = O_2(C)$ and $VU = O_2(D)$. Similarly since $J/M \cong L_3(2) \cong \langle E, D\rangle/O_2(\langle E, D\rangle)$, we obtain $O_2(\langle E, B\rangle)\, M = O_2(B)$, $PM = O_2(C)$, and $O_2(\langle E, D\rangle)\, O_2(\langle E, C\rangle) = O_2(E)$. In [R2, Lemma 4.7] it is shown that B^o_D covers B/N and both A^o_C and A^o_D cover A/N and the same argument establishes the remainder of Lemma 5.5.

Recall that we have $I = \{1, 2, 3, 4, 5\}$ with the labelling



LEMMA 5.6. $\bigcap_{i, j \in I} O_2(P_{ij}) = O_2(\langle E, D\rangle) \cap H_1 \cap H$.

Proof. By Lemma 2.9(ix),

$$H_1 \cap H = \bigcap_{i, j \in I\setminus\{1\}} O_2(P_{ij}).$$

Thus

$$\begin{aligned}\bigcap_{i, j \in I} O_2(P_{ij}) &= O_2(\langle E, B\rangle) \cap O_2(\langle E, D\rangle) \cap O_2(\langle E, A\rangle) \\ &\quad \cap O_2(\langle E, C\rangle) \cap H_1 \cap H \\ &= (O_2(\langle E, D\rangle) \cap O_2(\langle E, A\rangle) \cap W) \\ &\quad \cap (O_2(\langle E, C\rangle) \cap M) \cap O_2(\langle E, B\rangle) \cap H_1 \cap H.\end{aligned}$$

From Lemma 3.1(i),

$$O_2(\langle E, D\rangle) \cap O_2(\langle E, A\rangle) \cap W = O_2(\langle E, D\rangle) \cap W$$

and

$$O_2(\langle E, C\rangle) \cap M = O_2(\langle E, B\rangle) \cap M.$$

Hence

$$\bigcap_{i,\, j \in I} O_2(P_{ij}) = O_2(\langle E, D\rangle) \cap O_2(\langle E, B\rangle) \cap H_1 \cap H.$$

Now $O_2(\langle E, D\rangle) \cap O_2(\langle E, B\rangle) = O_2(\langle E, D\rangle) \cap U$ whence

$$\bigcap_{i,\, j \in I} O_2(P_{ij}) = O_2(\langle E, D\rangle) \cap H_1 \cap H.$$

LEMMA 5.7. $T_1 \cap W \neq T_1 \cap O_2(\langle E, A\rangle)$.

Proof. We suppose $T_1 \cap W = T_1 \cap O_2(\langle E, A\rangle)$ holds and argue for a contradiction.

(5.7.1) $$N \cap O_2(\langle E, A\rangle) = N \cap W \trianglelefteq \langle A, B, D, E\rangle.$$

By Lemma 5.4(i) and (ii), $[N : N \cap O_2(\langle E, A\rangle)] = [N : N \cap W] = 2$. If $N \cap O_2(\langle E, A\rangle) \neq N \cap W$, then $N \cap O_2(\langle E, A\rangle) \cap W$ has index 2^2 in N. Note that B_D^o and B_E^o cover B/N and A_D^o and A_E^o cover A/N by Lemma 5.5. Applying Lemma 3.1(ii) to $\{D, A, B\}$ and $\{E, A, B\}$ gives

$$N \cap O_2(\langle E, A\rangle) \cap W \trianglelefteq \langle A, B\rangle = K.$$

Hence $O^2(K)$ centralizes $N/(N \cap O_2(\langle E, A\rangle) \cap W)$. Since, by Lemma 2.8(ii), N/T_1 is a non-central chief factor for K this forces $N = T_1(N \cap O_2(\langle E, A\rangle) \cap W)$. Thus

$$T_1 \cap N \cap O_2(\langle E, A\rangle) \cap W = T_1 \cap O_2(\langle E, A\rangle) \cap W$$

has index 2^2 in T_1 and so, by our supposition, $T_1 \cap W$ has index 2^2 in T_1. This is impossible since $[N : N \cap W] = 2$, and so we conclude that $N \cap O_2(\langle E, A\rangle) = N \cap W$. From Lemma 3.1(ii) we have $N \cap O_2(\langle E, A\rangle) \trianglelefteq \langle A, B, E\rangle$ and $N \cap W \trianglelefteq \langle A, B, D\rangle$ whence (5.7.1) holds.

(5.7.2)
(i) $$\bigcap_{i,\, j \in I} O_2(P_{ij}) = H_1 \cap H$$
(ii) $$[S : H_1 \cap H] = 2^8.$$

From (5.7.1) we deduce that $N \cap W \leqslant O_2(\langle D, E \rangle)$. Since, by Lemma 2.9(ix),

$$H_1 \cap H = \bigcap_{i, j \in I \setminus \{1\}} O_2(P_{ij}) \leqslant N \cap W,$$

this yields $H_1 \cap H \leqslant O_2(\langle D, E \rangle)$. Then (i) follows from Lemma 5.6.

(ii) By Lemma 2.9(xiii), $[S : H_0 \cap H_1] = 2^7$. Now $[H_0 : H] = 2$ by Lemma 2.8 whence $H \cap H_1 = H \cap H_0 \cap H_1$ has index at most 2 in $H_0 \cap H_1$. We prove that $H \cap H_1 \neq H_0 \cap H_1$. Supposing $H \cap H_1 = H_0 \cap H_1$ we seek a contradiction. Lemma 2.9(ix) then gives

$$M \cap N \cap P \cap U \cap V \cap W = H \cap H_1 = H_0 \cap H_1$$

whence $H_0 \cap H_1 \leqslant N \cap V \cap W$. Since $[S : N \cap V \cap W] = 2^7$ by Lemma 2.9(xiv), we infer that $H_0 \cap H_1 = N \cap V \cap W$. Since $N \cap W \trianglelefteq D$ by Lemma 2.9(vii), we have $N \cap V \cap W \trianglelefteq D$ and so $H_0 \cap H_1 \trianglelefteq D$. Hence, using Lemma 2.9(x) we obtain

$$H_1 = (H_0 \cap H_1)\, H_2 \trianglelefteq D.$$

But then Lemma 2.9(viii) implies $H_1 = H_2$, whereas $H_2 < H_1$. Therefore we must have $H \cap H_1 \neq H_0 \cap H_1$ and hence $[S : H \cap H_1] = 2^8$.

(5.7.3) $$\left[S : \bigcap_{i, j \in I \setminus \{5\}} O_2(P_{ij}) \right] = 2^7.$$

Working mod Z we see that

$$\bigcap_{i, j \in I \setminus \{5\}} O_2(P_{ij}) = \begin{pmatrix} 1 & 0 & 0 & * & * \\ & 1 & 0 & 0 & * \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & \bigcirc & & & 1 \end{pmatrix}.$$

So Z has index 2^3 in $\bigcap_{i, j \in I \setminus \{5\}} O_2(P_{ij})$ and hence Lemma 5.1(ii) gives (5.7.3).

(5.7.4) $$[Z : Z \cap H_1 \cap H] \leqslant 2.$$

Clearly we have

$$Z \leqslant \bigcap_{i, j \in I \setminus \{5\}} O_2(P_{ij})$$

and, using (5.7.2)(i),

$$H_1 \cap H \leqslant \bigcap_{i,\, j \in I \setminus \{5\}} O_2(P_{ij}).$$

Combining (5.7.2)(ii) and (5.7.3) gives

$$\left[\bigcap_{i,\, j \in I \setminus \{5\}} O_2(P_{ij}) : H_1 \cap H \right] = 2$$

from which (5.7.4) follows

From Lemma 2.8 we have $M \geqslant H_0 \geqslant R$ with $[M : H_0] = 2^2$. Using Lemma 5.2 this implies that $[Z : Z \cap H_0] = 2^2$. Consequently, as $H_0 \geqslant H_1 \cap H$, we see that $[Z : Z \cap H_1 \cap H] \geqslant 2^2$, contrary to (5.7.4). This contradiction completes the proof of Lemma 5.7.

Our next result, which utilizes Lemma 5.7, marks the beginning of our investigation of subgroups below T_3.

LEMMA 5.8. (i) *$T_3 \cap O_2(\langle E, A \rangle) = T_3 \cap O_2(\langle E, B \rangle)$ is a normal subgroup of K which has index 2 in T_3.*

(ii) *$ZT_3 = core_{\langle B, C, D \rangle} S$ and $[T_3 : T_3 \cap Z] = 2^4$.*

Proof. (i) By Lemma 3.1(ii), $N \cap O_2(\langle E, A \rangle) = N \cap O_2(\langle E, B \rangle)$ is a normal subgroup of K and so since $T_3 \trianglelefteq K$ and $T_3 \leqslant N$, $T_3 \cap O_2(\langle E, A \rangle) = T_3 \cap O_2(\langle E, B \rangle)$ is a normal subgroup of K. Since $[N : N \cap O_2(\langle E, B \rangle)] = 2$ by Lemma 5.4(i), Lemma 5.7 implies that $[T_1 \cap W : T_1 \cap W \cap O_2(\langle E, B \rangle)] = 2$. From Lemma 2.9(i), $T_1 \cap W/T_2$ is a chief factor for K and thus $[T_2 : T_2 \cap O_2(\langle E, B \rangle)] = 2$ which in turn, as T_2/T_3 is a chief factor for K forces $T_3 \cap T_2 \cap O_2(\langle E, B \rangle) = T_3 \cap O_2(\langle E, B \rangle)$ to have index 2 in T_3.

(ii) Since $Z \leqslant O_2(\langle E, B \rangle)$, using (i) we see that

$$Z \cap T_3 \leqslant O_2(\langle E, B \rangle) \cap T_3 < T_3.$$

Consequently $Z < ZT_3 \trianglelefteq \langle B, C, D \rangle$. From Lemma 5.1(i) and properties of $L_5(2)$ we have that $core_{\langle B, C, D \rangle} S/Z$ is a chief factor of $\langle B, C, D \rangle$ of order 2^4. Therefore $ZT_3 = core_{\langle B, C, D \rangle} S$ and hence $[T_3 : T_3 \cap Z] = 2^4$.

LEMMA 5.9. (i) *$[T_3 \cap O_2(\langle E, B \rangle) : T_3 \cap O_2(\langle E, B \rangle) \cap O_2(\langle E, D \rangle)] = 2$*

(ii) *$T_3 \cap O_2(\langle E, B \rangle) \cap O_2(\langle E, D \rangle) \trianglelefteq \langle K, D \rangle$.*

(In particular, $T_3/T_3 \cap O_2(\langle E, B \rangle)$ and $T_3 \cap O_2(\langle E, B \rangle)/T_3 \cap O_2(\langle E, B \rangle) \cap O_2(\langle E, D \rangle)$ are both central chief factors for K.)

Proof. Using the description of $\langle B, C, D, E\rangle/Z$ given earlier in this section, we have (mod Z)

$$O_2(\langle E, D\rangle) = \begin{pmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ \bigcirc & & & & 1 \end{pmatrix} \qquad O_2(\langle E, B\rangle) = \begin{pmatrix} 1 & 0 & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & 0 \\ \bigcirc & & & & 1 \end{pmatrix}$$

and

$$\text{core}_{\langle B, C, D\rangle} S = \begin{pmatrix} 1 & 0 & 0 & 0 & * \\ & 1 & 0 & 0 & * \\ & & 1 & 0 & * \\ & & & 1 & * \\ \bigcirc & & & & 1 \end{pmatrix}.$$

Clearly

$$(\text{core}_{\langle B, C, D\rangle} S) \cap O_2(\langle E, D\rangle) \cap O_2(\langle E, B\rangle) = \begin{pmatrix} 1 & 0 & 0 & 0 & * \\ & 1 & 0 & 0 & * \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ \bigcirc & & & & 1 \end{pmatrix}.$$

Hence $(core_{\langle B, C, D\rangle} S) \cap O_2(\langle E, D\rangle) \cap O_2(\langle E, B\rangle)$ has index 2^2 in $core_{\langle B, C, D\rangle} S$. Noting that

$$Z \leqslant (core_{\langle B, C, D\rangle} S) \cap O_2(\langle E, D\rangle) \cap O_2(\langle E, B\rangle),$$

Lemma 5.8(ii) yields that

$$\begin{aligned} &T_3 \cap (core_{\langle B, C, D\rangle} S) \cap O_2(\langle E, D\rangle) \cap O_2(\langle E, B\rangle) \\ &\quad = T_3 \cap O_2(\langle E, D\rangle) \cap O_2(\langle E, B\rangle) \end{aligned}$$

has index 2^2 in T_3. Since $T_3 \cap O_2(\langle E, B\rangle)$ has index 2 in T_3 by Lemma 5.8(i), (i) now follows.

Let $d \in D$ be such that

$$d = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & \bigcirc & \\ 0 & 0 & 1 & & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\text{mod } Z).$$

Matrix calculation reveals that

$$O_2(\langle E, B\rangle) \cap O_2(\langle E, B\rangle)^d = O_2(\langle E, D\rangle) \cap O_2(\langle E, B\rangle) \qquad (\text{mod } Z).$$

Since $Z^d = Z$ we conclude that

$$(5.9.1) \quad O_2(\langle E, B\rangle) \cap O_2(\langle E, B\rangle)^d = O_2(\langle E, D\rangle) \cap O_2(\langle E, B\rangle).$$

Using Lemma 3.4(i), since $T_3 \cap O_2(\langle E, B\rangle) \trianglelefteq K$, yields that

$$(T_3 \cap O_2(\langle E, B\rangle)) \cap (T_3 \cap O_2(\langle E, B\rangle))^d$$

is normalized by $\langle B_D^o, A_D^o\rangle$. Because $T_3 \trianglelefteq D$, we see that

$$\begin{aligned}(T_3 \cap O_2(\langle E, B\rangle)) &\cap (T_3 \cap O_2(\langle E, B\rangle))^d \\ &= T_3 \cap O_2(\langle E, B\rangle) \cap O_2(\langle E, B\rangle)^d \\ &= T_3 \cap O_2(\langle E, D\rangle) \cap O_2(\langle E, B\rangle),\end{aligned}$$

using (5.9.1). Thus, as $d^2 \in S$, $T_3 \cap O_2(\langle E, D\rangle) \cap O_2(\langle E, B\rangle)$ is normalized by $\langle A_D^o, B_D^o, S, d\rangle$, which, since $\langle A_D^o, B_D^o\rangle$ covers K/N by Lemma 5.5(i), (ii) equals $\langle K, D\rangle$. This proves part (ii) and completes the proof of the lemma.

6. THE SUBGROUPS T_4, T_5, T_6, AND T_7

This section sees us descending rapidly down the subgroup lattice of S. The Replication Lemma plays a prominent role in these investigations as do certain facts about T_1/T_3. Our first two lemmas prepare the ground for Theorem 6.3, in which T_4 and T_5 are constructed and certain of their properties are established.

LEMMA 6.1. *There exists $e_1, e_2 \in E \backslash S$ such that*

(i) $e_1^2, e_2^2 \in Z$, $e_1 e_2^{-1} \notin S$; *and*

(ii) $T_3 \cap T_3^{e_1} \leqslant Z$ *and* $T_3 \cap T_3^{e_2} \leqslant Z$.

Proof. Recall that $H_2 = core_{\langle B, C, D\rangle} S$. By Lemma 5.8(ii), $ZT_3 = H_2$ and so

$$Z(T_3 \cap T_3^e) \leqslant ZT_3 \cap (ZT_3)^e = H_2 \cap H_2^e.$$

where $e \in E$.

Working mod Z and making the same identification of S, B, C, D, E as at the beginning of Section 5, we have

$$E = \begin{pmatrix} 1 & & & & \\ & 1 & & \bigcirc & \\ & & 1 & & \\ & & & * & * \\ & \bigcirc & & * & * \end{pmatrix} \qquad \text{and} \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & * \\ & 1 & 0 & 0 & * \\ & & 1 & 0 & * \\ & & & 1 & * \\ \bigcirc & & & & \end{pmatrix}.$$

Taking

$$\bar{e}_1 = \begin{pmatrix} 1 & & & & \\ & 1 & & \bigcirc & \\ & & 1 & & \\ & & & 1 & 0 \\ & \bigcirc & & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \bar{e}_2 = \begin{pmatrix} 1 & & & & \\ & 1 & & \bigcirc & \\ & & 1 & & \\ & & & 0 & 1 \\ & \bigcirc & & 1 & 0 \end{pmatrix}$$

we calculate that $(H_2/Z) \cap (H_2/Z)^{\bar{e}_1} = (H_2/Z) \cap (H_2/Z)^{\bar{e}_2} = 1$. Now choose $e_1, e_2 \in E$ such that $e_i Z = \bar{e}_i$ $(i = 1, 2)$. Then we have $T_3 \cap T_3^{e_i} \leqslant Z$ $(i = 1, 2)$ and, since $\bar{e}_1 \bar{e}_2^{-1} \notin S/Z$, part (i) also holds.

LEMMA 6.2. *T_1/T_3 contains no proper non-trivial subgroups which are normalized by $\langle A_E^o, B_E^o, C_E^o \rangle$.*

Proof. Let $T_3 < X \leqslant T_1$ be such that X is normalized by $\langle A_E^o, B_E^o, C_E^o \rangle$. We first establish

(6.2.1) Let Y be a subgroup of T_1 which is normalized by B_E^o. If $Y(T_1 \cap W) = T_1$, then $Y(T_1 \cap H_2) = T_1$.

By Lemma 5.5, B_E^o covers B/M. From Lemma 2.9(ii), $T_1/T_1 \cap H_2$ is a chief factor for $\langle B, C \rangle$ of order 2^3, whence B normalizes $Y(T_1 \cap H_2)/T_1 \cap H_2$. Since $T_1 \cap W/T_1 \cap H_2$ is the only proper non-trivial B-invariant subgroup of $T_1/T_1 \cap H_2$, the hypothesis $Y(T_1 \cap W) = T_1$ forces $Y(T_1 \cap H_2) = T_1$, so proving (6.2.1).

(6.2.2) $T_1 > T_1 \cap W > T_2 > T_3$ is a K-chief series for T_1/T_3 with $|T_1/T_1 \cap W| = 2$, $|T_1 \cap W/T_2| = 2^6$, and $|T_2/T_3| = 2^4$.

This is just Lemma 2.9(i).

(6.2.3) $X \trianglelefteq K$.

Since $[N : N \cap O_2(\langle E, B\rangle)] = 2$, Lemma 5.8(i) implies that $(N \cap O_2(\langle E, B\rangle))\, T_3 = N$. Therefore, as $O_2(\langle E, B\rangle) \leqslant B^o_E$ and $\langle A^o_E, B^o_E\rangle$ covers K/N by Lemma 5.5, $N_K(X)$ covers K/T_3. This yields (6.2.3).

(6.2.4) $$T_2 \leqslant X.$$

Supposing $T_2 \nleqslant X$ we deduce a contradiction and put $\bar{X} = X/T_3$. In view of (6.2.2) and (6.2.3) this forces $X \cap T_2 = T_3$. Since K normalizes X, (6.2.2) also yields that $|\bar{X}| = 2$, 2^6 or 2^7. Now $|\bar{X}| = 2$ cannot occur since $T_1/T_1 \cap W$ being the only central K-chief factor in the K composition series in (6.2.2) means $T_1 = X(T_1 \cap W)$ whence $|\bar{X}| \geqslant 2^3$ by (6.2.1). So we have $|\bar{X}| = 2^6$ or 2^7 with, as $[O^2(K), T_1] = T_1 \cap W$, $T_1 \cap W \leqslant XT_2$. From Lemma 2.9(iii) there exists subgroups Y_1 and Y_2 such that

$$T_1 \cap W > Y_1 > T_2 > Y_2 > T_3$$

with $Y_i \trianglelefteq L$ $(i = 1, 2)$. So $Y_1 \leqslant XT_2$ and therefore $Y_1 = (X \cap Y_1)\, T_2$. Also, $X \cap Y_1 \trianglelefteq L$ by (6.2.3). However, since $X \cap T_2 = T_3$, we have that $X \cap Y_1/Y_2$ is a non-trivial proper L-invariant subgroup of Y_1/Y_2, contradicting Lemma 2.9(iii). This is the desired contradiction and so we have proved (6.2.4).

If $X = T_2$, then we obtain, using Lemma 2.9(v),

$$T_2 \trianglelefteq \langle A, B, D, C^o_E\rangle = \langle A, B, C, D\rangle,$$

contrary to our hypothesis that $[S : core_{\langle A, B, C, D\rangle}\, S] = 2^{21}$. So $X > T_2$. Suppose $|X/T_2| = 2$. Then (6.2.2) and (6.2.3) again give $X(T_1 \cap W) = T_1$ and thus $X(T_1 \cap H_2) = T_1$. Since $T_1 \cap H_2 > T_2$ we obtain $|X/T_2| \geqslant 2^3$ and therefore $|X/T_2| \neq 2$. Using (6.2.2) and (6.2.3) yet again we see that $X \geqslant T_1 \cap W$. If $X = T_1 \cap W$, then $T_1 \cap W \trianglelefteq \langle B, C^o_E\rangle = \langle B, C\rangle$ which is impossible since $T_1 > T_1 \cap W > T_1 \cap H_2$ and $T_1/T_1 \cap H_2$ is a $\langle B, C\rangle$-chief factor. Therefore the only possibility left is $X = T_1$, and we have proved Lemma 6.2.

Let $e_1, e_2 \in E \backslash S$ be as described in Lemma 6.1. We define $T_5 = T_3 \cap T_3^{e_1}$ and $T_4 = T_2^{e_1^{-1}} \cap T_3 \cap O_2(E)$.

THEOREM 6.3. (i) $T_5 \trianglelefteq \langle A, B, C, E\rangle$.

(ii) *For any* $e \in E \backslash S$, $T_5 = T_3 \cap T_3^e$.

(iii) $T_5 < T_4 < T_3 \cap O_2(\langle E, B\rangle) \cap O_2(\langle E, D\rangle) < T_3 \cap O_2(\langle E, B\rangle) < T_3$ *is a chief series for* K *within* T_3/T_5 *where the chief factors have, respectively, orders* 2^4, 2^6, 2, *and* 2.

(iv) $T_3 \cap Z \geqslant T_4$.

(v) T_4/T_5 *is the unique non-trivial* $\langle A^o_E, B^o_E \rangle$*-invariant subgroup contained in* $T_3 \cap Z/T_5$ *and also the unique non-trivial* $\langle A^o_D, B^o_D \rangle$*-invariant subgroup contained in* $T_3 \cap Z/T_5$.

(vi) $T_3(T_3 \cap O_2(E))^{e_1} = T_1 = T_3(T_3 \cap O_2(E))^{e_2}$.

(vii) $T_3 \cap (T_3 \cap O_2(E))^{e_1} = T_5$.

(viii) $T_2/T_3 \underset{K}{\cong} T_4/T_5$. *In particular,* T_4/T_5 *has two A-chief factors each of order* 2^2.

Proof. We will apply the Replication Lemma with $X = T_3$, $P_k = E$, $P^o_{\mathscr{J}} = \langle A^o_E, B^o_E, C^o_E \rangle$, and $g = e_1$. Put $X_E = O_2(E) \cap T_3$.

Combining Lemmas 5.8(ii) and 6.1(ii) gives that $[T_3 : T_3 \cap T_3^{e_1}] \geqslant 2^4$ and so Lemma 3.4(ii) predicts that

$$T_3 X_E^{e_1} > T_3. \tag{6.3.1}$$

Also, from Lemma 3.4(i), we have

$$\langle A^o_E, B^o_E, C^o_E \rangle \text{ normalizes } T_3 X_E^{e_1}. \tag{6.3.2}$$

We next show that

$$T_3 X_E^{e_1} \leqslant T_1. \tag{6.3.3}$$

By Lemma 5.5, $\langle A^o_E, B^o_E \rangle$ covers K/N and so Lemma 3.4(iii) forces $T_3 X_E^{e_1} \leqslant N$. Likewise, $\langle B^o_E, C^o_E \rangle$ covering J/M by Lemma 5.5 yields $T_3 X_E^{e_1} \leqslant M$. So

$$N > N \cap M \geqslant T_3 X_E^{e_1}.$$

By Lemma 2.8(ii), N/T_1 is a chief factor for K and so, using (6.3.2) and Lemma 5.5, we observe that

$$T_1(T_3 X_E^{e_1}) \trianglelefteq \langle N, A^o_E, B^o_E \rangle = K. \tag{6.3.4}$$

Consequently $T_3 X_E^{e_1} \leqslant T_1$, establishing (6.3.3).

Together, Lemma 6.2, (6.3.1), and (6.3.3) show that

$$T_3 X_E^{e_1} = T_1. \tag{6.3.5}$$

Because $K^o_E = \langle A^o_E, B^o_E \rangle$ covers K/N, it follows that the K-chief series

$$T_3 < T_3 < T_1 \cap W < T_1$$

between T_1 and T_3 (as given by Lemma 2.9(i)) is also a K_E^o-chief series. Noting that $e_1^2 \in Z \leqslant O_2(E)$ we may use Lemma 3.4(v) to deduce from (6.3.5) that

$$X_E \cap X_E^{e_1} = T_3^{e_1^{-1}} \cap X_E < T_2^{e_1^{-1}} \cap X_E < (T_1 \cap W)^{e_1} \cap X_E < X_E \tag{6.3.6}$$

is a K_E^o-chief series between X_E and $X_E \cap X_E^{e_1}$. (The orders of the respective quotients are 2^4, 2^6, and 2.)

Suppose $X_E = T_3$ holds. Then $T_3/T_3 \cap T_3^{e_1}$ has exactly one central chief K_E^o-factor by (6.3.6). However, combining Lemmas 6.1(ii) and 5.9 we obtain

$$\begin{aligned} X_E \cap X_E^{e_1} \leqslant T_3 \cap T_3^{e_1} \leqslant Z \cap T_3 < T_3 \cap O_2(\langle E, B\rangle) \cap O_2(\langle E, D\rangle) \\ < T_3 \cap O_2(\langle E, B\rangle) < T_3, \end{aligned}$$

where $T_3/T_3 \cap O_2(\langle E, B\rangle)$ and $T_3/T_3 \cap O_2(\langle E, B\rangle) \cap O_2(\langle E, D\rangle)$ are central chief factors for K_E^o. Thus $X_E \neq T_3$ and so, since $[T_3 : T_3 \cap O_2(\langle E, B\rangle)] = 2$ and $T_3 \cap O_2(\langle E, B\rangle) \leqslant T_3 \cap O_2(E) = X_E$, we deduce that

$$T_3 \cap O_2(\langle E, B\rangle) = X_E. \tag{6.3.7}$$

Now we show that

$$T_5 = X_E \cap X_E^{e_1}. \tag{6.3.8}$$

Suppose (6.3.8) is false. Then, by Lemmas 3.4(vi) and 6.1(ii), $T_3 \cap T_3^{e_1}/X_E \cap X_E^{e_1}$ is a subgroup of order 2 normalized by K_E^o and contained in $Z \cap T_3/X_E \cap X_E^{e_1}$.

Refining

$$\begin{aligned} X_E \cap X_E^{e_1} < T_3 \cap T_3^{e_1} < T_3 \cap O_2(\langle E, B\rangle) \cap O_2(\langle E, D\rangle) \\ < T_3 \cap O_2(\langle E, B\rangle) = X_E \end{aligned}$$

to a K_E^o-chief series then produces a K_E^o-chief series for $X_E/X_E \cap X_E^{e_1}$ which has at least two central chief factors. This is incompatible with the information in (6.3.6). Thus we conclude that $T_5 = X_E \cap X_E^{e_1}$.

$$T_4 \leqslant T_3 \cap Z. \tag{6.3.9}$$

Since $O^2(K_E^o)$ acts trivially upon $T_3/T_3 \cap O_2(\langle E, B\rangle) \cap O_2(\langle E, D\rangle)$ and T_4/T_5 is a non-central chief factor for K_E^o by (6.3.6), clearly

$$T_4 \leqslant T_3 \cap O_2(\langle E, B\rangle) \cap O_2(\langle E, D\rangle).$$

Suppose $T_4 \not\leqslant T_3 \cap Z$ holds. Using Lemma 5.8(ii) we see that

$$T_3/T_3 \cap Z \underset{K}{\cong} (core_{\langle B, C, D\rangle} S)/Z$$

and so, as $(core_{\langle B, C, D\rangle} S)/Z$ is a 4-dimensional irreducible $GF(2)\,L_4(2)$-module, we infer that

$$(T_3 \cap O_2(\langle E, B\rangle) \cap O_2(\langle E, D\rangle))/T_3 \cap Z$$

is a 2-dimensional irreducible module for B. Now B_E^o covers $B/O_2(B)$ (because it covers B/N) and T_4 is normalized by B_E^o which means that

$$T_4(T_3 \cap Z) \trianglelefteq B.$$

So

$$T_3 \cap O_2(\langle E, B\rangle) \cap O_2(\langle E, D\rangle) = T_4(T_3 \cap Z).$$

In particular, T_4/T_5 contains a B_E^o-invariant subgroup of index 2^2. Appealing to the Replication Lemma (Lemma 3.4(v)) with $P^o_{\mathcal{J}} = B_E^o$ yields that T_2/T_3 also possesses a B_E^o-invariant subgroup of index 2^2. Because B_E^o covers B/N (by Lemma 5.5(i)) and T_2/T_3 is a K-chief factor, this means T_2/T_3 contains an B-invariant subgroup of index 2^2, which contradicts Lemma 2.9(iv). Thus we conclude that $T_4 \leqslant T_3 \cap Z$ must hold, so proving (6.3.9).

(6.3.10) T_4/T_5 is the unique non-trivial subgroup of $T_3 \cap Z/T_5$ normalized by K_E^o.

Let $T_5 < Y \leqslant T_3 \cap Z$ be such that Y/T_5 is a non-trivial subgroup of $T_3 \cap Z/T_5$ which is normalized by K_E^o. Since, by (6.3.9),

$$T_4 Y \leqslant T_3 \cap Z \neq T_3 \cap O_2(\langle E, B\rangle) \cap O_2(\langle D, B\rangle)$$

and K_E^o acts irreducibly on $T_3 \cap O_2(\langle E, B\rangle) \cap O_2(\langle D, B\rangle)/T_4$ by (6.3.6), $Y \leqslant T_4$. Then K_E^o acting irreducibly on T_4/T_5 forces $Y = T_4$, so giving (6.3.10).

We now consider $T_3 \cap T_3^{e_2}$ where e_2 is as given in Lemma 6.1. By Lemma 6.1(ii), $T_3 \cap T_3^{e_2} \leqslant T_3 \cap Z$ and by Lemma 3.4(i), $T_3 \cap T_3^{e_2}$ is normalized by $\langle A_E^o, B_E^o, C_E^o\rangle$. The arguments used to prove (6.3.5) and (6.3.8) will also yield

$$(6.3.11)\quad \begin{aligned} &\text{(i)}\quad T_3(T_3\cap O_2(E))^{e_2}=T_1.\\ &\text{(ii)}\quad T_3\cap T_3^{e_2}=(T_3\cap O_2(E))\cap(T_3\cap O_2(E))^{e_2}\\ &\qquad\qquad (=T_3\cap(T_3\cap O_2(E))^{e_2}).\end{aligned}$$

In particular, from (6.3.11) we infer that $|T_3\cap T_3^{e_2}|=|T_5|$. We now show that $T_3\cap T_3^{e_2}=T_5$. Suppose $T_3\cap T_3^{e_2}\neq T_5$ and argue for a contradiction. Then $(T_3\cap T_3^{e_2})\,T_5/T_5$ is a non-trivial subgroup of $T_3\cap Z/T_5$ which is normalized by $\langle A_E^o, B_E^o, C_E^o\rangle$. Now (6.3.10) dictates that $T_4=(T_3\cap T_3^{e_2})\,T_5$. In particular, T_4 is normalized by C_E^o. Application of part (v) of the Replication Lemma yields that T_2 is also normalized by C_E^o. Therefore, calling on Lemma 2.9(v), we have

$$T_2\trianglelefteq\langle A, B, D, C_E^o\rangle=\langle A, B, C, D\rangle,$$

against the hypothesis that $|S/core_{\langle A, B, C, D\rangle}\,S|=2^{21}$. With this contradiction we have proved that $T_3\cap T_3^{e_2}=T_5$ $(=T_3\cap T_3^{e_1})$. Because e_1^2, $e_2^2\in S$ and $T_3\trianglelefteq S$, it follows that e_1 and e_2 normalize T_5. From Lemma 6.1(i) we have that $\langle e_1, e_2\rangle$ covers $E/O_2(E)$ whence, as $T_5\trianglelefteq S\cap S^{e_1}=O_2(E)$, we obtain $T_5\trianglelefteq E$. So now, by Lemma 3.4(i),

$$T_5\trianglelefteq\langle A_E^o, B_E^o, C_E^o, E\rangle=\langle A, B, C, E\rangle,$$

which proves part (i) of the theorem.

For $e\in E\backslash S$, since $e_1e_2^{-1}\notin S$ by Lemma 6.1(i), we have $e\in Se_1$ or $e\in Se_2$ and thus, as $T_3\trianglelefteq S$, $T_3^e=T_3^{e_1}$ or $T_3^e=T_3^{e_2}$. So $T_3\cap T_3^e=T_3\cap T_3^{e_1}$ or $T_3\cap T_3^{e_2}$, so proving (ii).

Turning to part (iii) we have

$$T_5\leqslant T_4\leqslant T_3\cap Z\leqslant T_3\cap O_2(\langle E, B\rangle)\cap O_2(\langle D, B\rangle),$$

using (6.3.9). Since $O^2(K_E^o)$ acts trivially on $T_3/T_3\cap O_2(\langle E, B\rangle)\cap O_2(\langle D, B\rangle)$ by Lemma 5.9, (6.3.6) and (6.3.7) imply that

$$(T_1\cap W)^{e_1}\cap X_E\leqslant T_3\cap O_2(\langle E, B\rangle)\cap O_2(\langle D, B\rangle).$$

Hence, by orders, $(T_1\cap W)^{e_1}\cap X_E=T_3\cap O_2(\langle E, B\rangle)\cap O_2(\langle D, B\rangle)$. Looking at $T_4=T_2^{e_1^{-1}}\cap X_E$, we show that $T_4\trianglelefteq K$. We already have that T_4 is normalized by K_E^o. Recall that $[K:K_E^o]=2$. By (6.3.7), X_E/T_5 admits K. Let $g\in K\backslash K_E^o$. Then $T_5\leqslant T_4T_4^g\leqslant X_E$ and K normalizes $T_4T_4^g$ (note that $T_4\trianglelefteq X_E$). Consulting (6.3.6) we see that $T_4T_4^g=T_4$ is the only possibility and so $T_4\trianglelefteq K$. Thus we have verified that

$$T_5<T_4<T_3\cap O_2(\langle E, B\rangle)\cap O_2(\langle D, B\rangle)<T_3\cap O_2(\langle E, B\rangle)<T_3$$

is a chief series for K within T_3/T_5. By (6.3.6) and (6.3.7) the sizes of the chief factors are as stated in (iii).

Part (iv) was established in (6.3.9). By Lemma 5.5, $K_D^o = \langle A_D^o, B_D^o \rangle$ covers K/N and hence K_D^o acts irreducibly upon $T_3 \cap O_2(\langle E, B \rangle) \cap O_2(\langle D, B \rangle)/T_4$ and T_4/T_5. Arguing as in (6.3.10) yields that T_4/T_5 is the unique non-trivial K_D^o-invariant subgroup contained in $T_3 \cap Z/T_5$ and so (v) holds. For part (vi) see (6.3.5) and (6.3.11) and part (vii) follows from Lemma 3.5(iv) and (6.3.8).

We next establish part (viii). By Lemma 2.9(iv) and properties of 4-dimensional $\hat{S}_6$-modules a chief series for A between T_2 and T_3 will be of the form $T_2 > Y > T_3$ where $|T_2/Y| = 2^2 = |Y/T_3|$. Since A_E^o covers $A/O_2(A)$ this is also a chief series for A_E^o between T_2 and T_3. Applying Lemma 3.4(v) with $P_{\mathcal{J}}^o = A_E^o$ yields that

$$T_4 > Y^{e_1^{-1}} \cap T_3 \cap O_2(E) > T_5$$

is a A_E^o-chief series between T_4 and T_5. Hence x_A acts fixed-point-freely on T_4/T_5. By properties of 4-dimensional $\hat{S}_6$-modules we then deduce that $T_2/T_3 \underset{K}{\cong} T_4/T_5$ and so (viii) is proven.

The proof of Theorem 6.3 is complete.

LEMMA 6.4. $T_5 \ntrianglelefteq G$.

Proof. Set $K^o = \langle A_D^o, B_D^o \rangle$. We begin by establishing

$$(6.4.1) \qquad [x_D, K^o] \leqslant N.$$

Recall that $[K : K^o] = 2$ (see Lemma 3.1). Hence $A \cap K^o = A_D^o$ and $B \cap K^o = B_D^o$. Put $\bar{K}^o = K^o/O_2(K^o)$ and use the usual bar notation. Now by Lemma 5.5(i), (ii), K^o covers K/N whence, as $O_2(K^o) = K^o \cap N$, $\bar{K}^o \cong \hat{S}_6$. From $K^o \trianglelefteq K$ we have $S_1 = S \cap K^o \in Syl_2\, K^o$. Thus $\bar{S}_1 \cong D_8 \times Z_2$. Also, it is straightforward to show that $\{A_D^o, B_D^o\}$ is a minimal parabolic system for K^o with $A_D^o \cap B_D^o = S_1$ and $A_D^o/O_2(A_D^o) \cong S_3 \cong B_D^o/O_2(B_D^o)$. Observe that

$$S_1 = S \cap K^o = S \cap A_D^o \geqslant W.$$

Since $[x_D, A_D^o] \leqslant W \leqslant S_1$, we have that x_D normalizes S_1 and A_D^o. By Lemma 3.1(ii), x_D normalizes $N \cap W = O_2(K^o)$. Therefore x_D acts upon $\bar{S}_1$, and, because $D_8 \times Z_2$ has only two cyclic subgroups of order 4, it follows that x_D centralizes $\bar{S}_1$. From $[x_D, A_D^o] \leqslant W$ we further note that

$[x_D, \overline{A_D^o}] \leqslant \bar{W}$ and therefore 3 divides $|C_{\overline{A_D^o}}(x_D)|$. Hence, as $[\overline{A_D^o} : \bar{S}_1] = 3$, x_D centralizes $\overline{A_D^o}$. A similar argument shows that x_D centralizes $\bar{B}_D^o$. Therefore, since $\langle \overline{A_D^o}, B_D^o \rangle = \overline{K^o}$, x_D centralizes $\bar{K}^o$. So

$$[x_D, K^o] \leqslant O_2(K^o) = N \cap W \leqslant N.$$

This proves (6.4.1).

Since $T_3 \cap Z \trianglelefteq D$, $T_4 T_4^d \leqslant T_3 \cap Z$ for any $d \in D$. So N normalizes $T_4 T_4^d$ since $T_3 \cap O_2(\langle E, B \rangle) \cap O_2(\langle E, D \rangle)/T_4$ is an $\langle A, B \rangle$-chief factor by Theorem 6.3(iii). By Lemma 3.4(i), $\langle A_D^o, B_D^o \rangle$ normalizes $T_4 T_4^d$ and hence $T_4 T_4^d \trianglelefteq \langle A, B \rangle$, using Lemma 5.5(i), (ii). Therefore $T_4 = T_4 T_4^d$ by Theorem 6.3(iii). So $T_4 = T_4^d$ for all $d \in D$ and hence, by Theorem 6.3(iii), $T_4 \trianglelefteq \langle A, B, D \rangle$.

Now, supposing $T_5 \trianglelefteq G$ we obtain contradictory information about $C_{T_4/T_5}(x_B)$. Our next statement makes such deductions possible.

(6.4.2)

$$x_D \text{ centralizes } T_3 \cap Z/T_4.$$

Set $X = T_3 \cap O_2(\langle E, B \rangle) \cap O_2(\langle E, D \rangle)$. By Lemma 5.9(ii), $X \trianglelefteq \langle A, B, D \rangle$ and so X/T_4 admits $\langle A, B, D \rangle$. Because $T_3/T_3 \cap Z \underset{B}{\cong} (core_{\langle B, C, D \rangle} S)/Z$ by Lemma 5.8(ii), we see using Lemma 5.9 that B acts irreducibly upon $X/Z \cap T_3$. Since $\langle B, D \rangle$ acts on $X/Z \cap T_3$, x_D centralizes $X/Z \cap T_3$ by Lemma 2.3. So $[X, x_D] \leqslant Z \cap T_3$. Put $\bar{X} = X/T_4$. Then $[\bar{X}, x_D] \neq \bar{X}$. Since $\bar{X}$ is a chief factor for $\langle A, B \rangle$, N centralizes $\bar{X}$. Consequently (6.4.1) yields that K^o normalizes $[\bar{X}, x_D]$. Therefore, using Lemma 5.5(i), (ii), we obtain that $\langle A, B \rangle = \langle N, K^o \rangle$ normalizes $[\bar{X}, x_D]$. Hence $[\bar{X}, x_D] = 1$, from which (6.4.2) follows.

Since B acts irreducibly upon $T_3 \cap O_2(\langle E, B \rangle) \cap O_2(\langle E, D \rangle)/Z \cap T_3$ and, by Theorem 6.3(iii), $T_3 \cap O_2(\langle E, B \rangle) \cap O_2(\langle E, D \rangle)/T_4$ is an irreducible $GF(2)$ $\hat{S}_6$-module, Lemma 2.4(v) yields that $|C_{Z \cap T_3/T_4}(x_B)| = 2^2$. From Theorem 6.3(viii) and Lemma 2.4(iii), $|C_{T_4/T_5}(x_B)| = 2^2$ whence $|C_{Z \cap T_3/T_5}(x_B)| = 2^4$. Note that $Z \cap T_3/T_5$ admits $\langle B, C, D \rangle$. Now $\langle B, C \rangle/M \cong L_3(2) \cong \langle D, C \rangle/V$ implies that $|C_{Z \cap T_3/T_5}(x_D)| = 2^4$ by Lemma 2.5. In view of (6.4.2) and $[Z \cap T_3 : T_4] = 2^4$, we then infer that x_D acts fixed-point-freely on T_4/T_5. Therefore T_4/T_5 must have two $\langle B, D \rangle$ chief factors both of order 2^2 by Lemma 2.3. But then, by Lemma 2.3, x_B must centralize T_4/T_5, whereas $|C_{T_4/T_5}(x_B)| = 2^2$. From this contradictory state of affairs we conclude that $T_5 \not\trianglelefteq G$, so completing the proof of Lemma 6.4.

Since $\langle D, E \rangle/O_2(\langle D, E \rangle) \cong L_3(2)$, by Lemma 3.2(i) we may choose $d_3 \in D \backslash S$ and $e_3 \in E \backslash S$ for which d_3^2, e_3^2, $(d_3 e_3)^3 \in O_2(\langle D, E \rangle)$. Define $T_6 = T_5 \cap T_5^{d_3}$.

LEMMA 6.5. (i) $T_6 \trianglelefteq \langle A, B, D, E\rangle$.

(ii) $[T_5 : T_6] = 2^4$ *and* $T_6 = T_5 \cap T_5^d$ *for all* $d \in D\backslash S$.

(iii) $T_4 \trianglelefteq \langle A, B, D\rangle$.

(iv) T_4/T_5 *and* T_5/T_6 *are isomorphic* $\langle A, B\rangle$*-modules.*

In particular T_5/T_6 *contains a unique proper non-trivial* A*-invariant subgroup (which has order* 2^2*).*

Proof. By Theorem 6.3(i) and (ii), $T_5 = T_3 \cap T_3^{e_3} \trianglelefteq S$ and, since $T_3 \trianglelefteq D$ (as $T_3 = core_{\langle A, B, C, D\rangle} S$), using Lemma 3.2(ii) gives

$$T_6 = T_5 \cap T_5^{d_3} \trianglelefteq \langle D, E\rangle.$$

Because $T_5 \trianglelefteq \langle A, B\rangle$ by Theorem 6.3(i) and $\langle D, B\rangle/U \cong S_3 \times S_3 \cong \langle D, A\rangle/W$, Lemma 3.4(i) implies that

$$T_5 \cap T_5^{d_3} \trianglelefteq \langle A_D^o, B_D^o\rangle.$$

Consequently

$$T_6 \trianglelefteq \langle A_D^o, B_D^o, D, E\rangle = \langle A, B, D, E\rangle,$$

proving (i).

Since $T_5 \ntrianglelefteq G$ by Lemma 6.4, Theorem 6.3(i) implies $T_5 \neq T_5^{d_3}$. Applying the Replication Lemma with $X = T_5$, $P_k = D$, and $P_{\mathscr{J}} = \langle A, B\rangle$ gives (note that $T_5 \leqslant T_2 \leqslant O_2(D)$)

$$T_5 T_5^{d_3} \text{ is normalized by } \langle A_D^o, B_D^o\rangle.$$

Recalling that $T_5 \leqslant T_3 \cap Z$ and $T_3 \cap Z \trianglelefteq D$ (see, for example, the subgroup lattice in the Appendix) we have $T_5 T_5^{d_3} \leqslant T_3 \cap Z$. Appealing to Theorem 6.3(v) (as $T_5 < T_5 T_5^{d_3}$) yields that $T_5 T_5^{d_3} = T_4$. Thus $[T_5 : T_6] = 2^4$.

Let $d \in D\backslash S$. Since $T_6 \trianglelefteq D$, clearly $T_6 \leqslant T_5 \cap T_5^d$. Now $T_5 T_5^d$ is normalized by $\langle A_D^o, B_D^o\rangle$ and $T_5 T_5^d \leqslant T_3 \cap Z$. Therefore, as $T_5 < T_5 T_5^d$ (because $T_5 \ntrianglelefteq G$ by Lemma 6.4), Theorem 6.3(v) forces $T_5 T_5^d = T_4$ whence $[T_5 : T_5 \cap T_5^d] = 2^4$. This implies $T_5 \cap T_5^d = T_6$, as required.

From $T_5 T_5^{d_3} = T_4$ together with $T_5 \trianglelefteq S$ and $d_3^2 \in S$ we conclude that d_3 normalizes T_4. So

$$T_4 \trianglelefteq \langle A, B, d_3\rangle = \langle A, B, D\rangle,$$

by Theorem 6.3(iii), proving (iii).

Using the Replication Lemma with $P_K = D$ and $P_{\hat{\mathscr{J}}}^o = A_D^o$ yields, because of Theorem 6.3(viii) and Lemma 5.5(ii), that A_D^o has two non-central chief factors within T_5/T_6. Therefore $K = \langle A, B \rangle$ acts non-trivially and hence T_5/T_6 is a chief factor for K with A possessing two non-central chief factors. Consequently T_4/T_5 and T_5/T_6 are isomorphic K-modules. This proves part (iv) and completes the proof of the lemma.

LEMMA 6.6. *$T_6 \ntrianglelefteq G$.*

Proof. Suppose the result is false. Then T_5/T_6 admits $\langle A, B, C \rangle$ by Theorem 6.3(i). Combining Lemmas 6.5(ii), (iv) and Lemma 2.4(iii) yields that $C_{T_5/T_6}(x_B)$ has order 2^2. Hence $C_{T_5/T_6}(x_C)$ also has order 2^2 by Lemma 2.5. Therefore by Lemma 2.3, $\langle C, A \rangle$ cannot act irreducibly upon T_5/T_6. Because x_A acts fixed-point-freely on T_5/T_6, by Lemma 2.3, T_5/T_6 must have two $\langle C, A \rangle$ chief factors both of order 2^2 in a chief series. But then Lemma 2.3 forces x_C to centralize T_5/T_6, a contradiction. With this contradiction we have established the lemma.

Let $c_4 \in C \backslash S$ and $d_4 \in D \backslash S$ be chosen so that c_4^2, d_4^2, $(c_4 d_4)^3 \in O_2(\langle C, D \rangle)$. Now define T_7 to be $T_6 \cap T_6^{c_4}$.

LEMMA 6.7. (i) *$[T_6 : T_7] = 2^2$ or 2^4.*

(ii) *$T_7, \trianglelefteq \langle A, C, D, E \rangle$.*

(iii) *x_A acts fixed-point-freely on T_6/T_7.*

Proof. Since $T_6 \trianglelefteq \langle A, B, D, E \rangle$ and $T_6 \ntrianglelefteq G$ by Lemma 6.6, $T_6 T_6^{C_4} > T_6$. Because $T_6 \trianglelefteq A$ and $T_5 \trianglelefteq C$ we infer that $T_6 T_6^{C_4} \leqslant T_5$ and that $T_6 T_6^{C_4}$ is normalized by A_C^o. So, since A_C^o covers A/N by Lemma 5.5(iii) and T_5/T_6 is a chief factor for K, $T_6 T_6^{c_4} \trianglelefteq A$. Appealing to Theorem 6.3(viii) gives that $[T_6 T_6^{c_4} : T_6] = 2^2$ or 2^4, hence we obtain (i).

From Lemma 6.5(ii) we have that $T_6 = T_5 \cap T_5^{d_4}$. Now $T_5 \trianglelefteq C$ by Theorem 6.3(i) and $T_6 \trianglelefteq S$ by Lemma 6.5(i) imply that $T_7 \trianglelefteq \langle C, D \rangle$, using Lemma 3.2(ii). So, as $T_6 \trianglelefteq \langle A, E \rangle$, we have

$$T_7 \trianglelefteq \langle C, D, A_C^o, E_C^o \rangle = \langle A, C, D, E \rangle,$$

which establishes (ii).

Finally,

$$T_6 T_6^{c_4}/T_6 \underset{A_C^0}{\cong} T_6/T_7,$$

Theorem 6.3(viii), and Lemma 6.5(iv) imply that the A_C^o chief factors of T_6/T_7 have order 2^2. Thus x_A acts fixed-point-freely on T_6/T_7.

This proves the lemma.

7. THE INDEX OF T_7 IN T_6

The whole of this section is concerned with sharpening the conclusion of Lemma 6.7(i), the end product being

THEOREM 7.1. $[T_6 : T_7] = 2^2$.

One approach that might be used to eliminate the $[T_6 : T_7] = 2^4$ possibility is to examine the way groups such as $\langle A, C, E\rangle$ and $\langle A, D\rangle$ act (respectively) on T_5/T_7 and T_4/T_7. This particular strategy (in my hands) has met with little success. Another strategy is to gather more information from higher up the subgroup lattice of S about subgroups "close" to T_7. One starting point for this might be to see what may be gleaned from certain subgroups of small index in Z (other likely candidates such as T_3, T_5, and T_6 appear to have already been bled dry). Since $[Z : T_7] = 2^{31}$ (assuming $[T_6 : T_7] = 2^4$) this is a daunting prospect!

However, starting lower down S and building a subgroup chain from T_2 we are able to establish Theorem 7.1.

We present the proof of Theorem 7.1 in a series of lemmas. Let $e_1 \in E\backslash S$ be as given in Lemma 6.1 and put $R_2 = T_2 \cap T_2^{e_1}$.

In Fig. 6, we give a subgroup lattice which exhibits the main subgroups involved in the proof of Theorem 7.1.

LEMMA 7.2. *The following statements hold*

(i) $R_2 \trianglelefteq \langle A, B, D, E\rangle$ *and* $R_2 = T_2 \cap T_2^e$ *for all* $e \in E\backslash S$;

(ii) $R_2 \cap T_3 = T_4$; *and*

(iii) $[T_2 : R_2] = 2^8$ *and* $[R_2 : T_4] = 2^4$.

Proof. By the lattice in Lemma 2.9, $H_2 \geqslant T_2$. So, since $T_2 \geqslant T_3$, Lemma 5.8(ii) forces $H_2 = ZT_3 = ZT_2$. Now $[H_2 : Z] = 2^4$ (because $[S : H_2] = 2^6$ and $[S : Z] = 2^{10}$) whence we deduce that

(7.2.1) $$[T_2 : T_2 \cap Z] = 2^4.$$

Clearly, from $ZT_3 = ZT_2$, we have $ZT_3/Z = ZT_2/Z$. Therefore, appealing to Lemma 6.1(ii) we infer that $T_2 \cap T_2^{e_1} \leqslant Z$. Thus

(7.2.2) $$R_2 \leqslant T_2 \cap Z,$$

(7.2.3) $$T_5 \leqslant T_3 \cap (T_2 \cap O_2(E))^{e_1} \leqslant R_2.$$

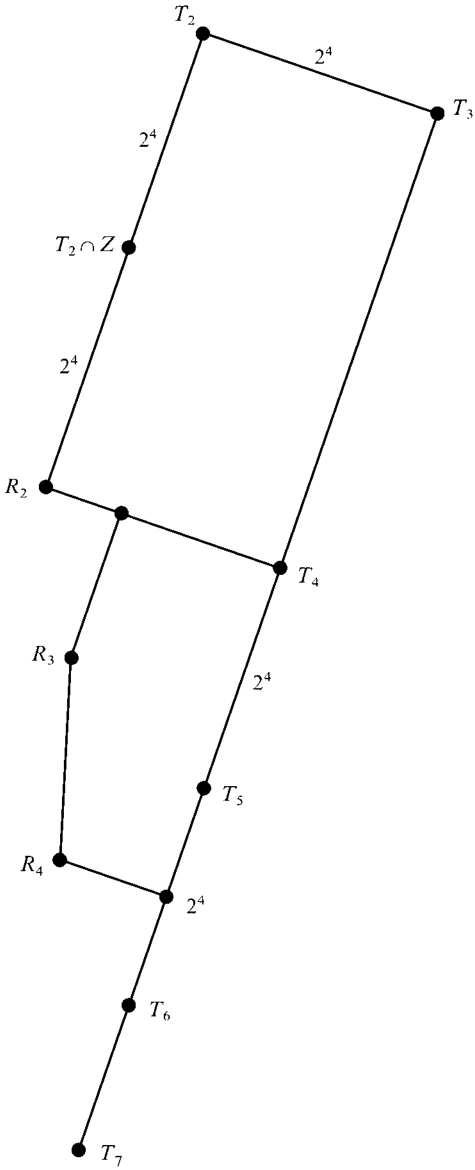


FIGURE 6

By definition $R_2 = T_2 \cap T_2^{e_1}$, and so

$$R_2 \geqslant T_2 \cap (T_2 \cap O_2(E))^{e_1} \geqslant T_3 \cap (T_3 \cap O_2(E))^{e_1} = T_5,$$

using Theorem 6.3(vii).

From Theorem 6.3(iii) we have that $[T_3 : T_3 \cap O_2(E)] = 2$. Since $T_2 \geqslant T_3$, this gives $[T_2 : T_2 \cap O_2(E)] = 2$. We next prove that

$$(T_2 \cap O_2(E))^{e_1} T_2 = T_1. \tag{7.2.4}$$

Using Theorem 6.3(vi) we clearly have

$$(T_2 \cap O_2(E))^{e_1} T_2 \geqslant (T_3 \cap O_2(E))^{e_1} T_3 = T_1.$$

Now Lemmas 5.5(i), (ii), and 3.4(iii) (with $P_k = E$ and $P_{\mathcal{J}} = K$) imply that

$$N \geqslant (T_2 \cap O_2(E))^{e_1} T_2.$$

Assume (7.2.4) is false and argue for a contradiction. Since $(T_2 \cap O_2(E))^{e_1} T_2$ is normalized by $\langle A_E^o, B_E^o \rangle$ by Lemma 3.4(i) and N/T_1 is a chief factor for K by Lemma 2.8(ii), we conclude that

$$N = (T_2 \cap O_2(E))^{e_1} T_2.$$

Because

$$[N : (T_2 \cap O_2(E))^{e_1}] = [N : T_2 \cap O_2(E)] = 2[N : T_2] = 2 \cdot 2^{13} = 2^{14},$$

we observe that

$$[T_2 : T_2 \cap (T_2 \cap O_2(E))^{e_1}] = 2^{14}. \tag{7.2.5}$$

If $T_2 \cap (T_2 \cap O_2(E))^{e_1} \leqslant T_3$, then $[T_2 : T_3] = 2^4$ implies that $[T_3 : T_2 \cap (T_2 \cap O_2(E))^{e_1}] = 2^{10}$. Now by Lemma 3.4(iv), $T_2 \cap (T_2 \cap O_2(E))^{e_1}$ is normalized by e_1 and consequently $[T_3 : T_3 \cap T_3^{e_1}] \leqslant 2^{10}$. Hence $[T_3 : T_5] \leqslant 2^{10}$ which contradicts $[T_3 : T_5] = 2^{12}$ (see, for example, the subgroup lattice in the Appendix). Therefore

$$T_2 \cap (T_2 \cap O_2(E))^{e_1} \nleqslant T_3.$$

From Lemma 2.9(i) we have that T_2/T_3 is a chief factor for K whence, since $\langle A_E^o, B_E^o \rangle$ normalizes $T_2 \cap (T_2 \cap O_2(E))^{e_1}$, we obtain

$$T_2 = T_3(T_2 \cap (T_2 \cap O_2(E))^{e_1}).$$

Using (7.2.5) this implies that

$$T_3 \cap (T_2 \cap (T_2 \cap O_2(E))^{e_1}) = T_2 \cap (T_2 \cap O_2(E))^{e_1}$$

has index 2^{14} in T_3. But then (7.2.3) forces $[T_3 : T_5] \geqslant 2^{14}$, which is contrary to $[T_3 : T_5] = 2^{12}$. This is the desired contradiction which completes the proof of (7.2.4).

Since, using the lattice in Lemma 2.9,

$$[T_1 : (T_2 \cap O_2(E))^{e_1}] = [T_1 : T_2]\, 2 = 2^7 \cdot 2 = 2^8,$$

(7.2.4) implies that $T_2 \cap (T_2 \cap O_2(E))^{e_1}$ has index 2^8 in T_2. If $T_2 \cap (T_2 \cap O_2(E))^{e_1} \leqslant T_3$ were to hold, then, as $T_2 \cap (T_2 \cap O_2(E))^{e_1}$ is normalized by e_1, we obtain $[T_3 : T_3 \cap T_3^{e_1}] \leqslant 2^4$. So $[T_3 : T_5] \leqslant 2^4$ whereas $[T_3 : T_5] = 2^{12}$. Thus

$$T_2 \cap (T_2 \cap O_2(E))^{e_1} \nleqslant T_3$$

and then T_2/T_3 being a chief factor for K yields, as above, that

$$T_2 = (T_2 \cap (T_2 \cap O_2(E))^{e_1})\, T_3. \tag{7.2.6}$$

Consequently $T_3 \cap (T_2 \cap O_2(E))^{e_1}$ has index 2^8 in T_3. Therefore $[T_3 \cap (T_2 \cap O_2(E))^{e_1} : T_5] = 2^4$ (as $[T_3 : T_5] = 2^{12}$). Note by (7.2.2) that

$$T_3 \cap (T_2 \cap O_2(E))^{e_1} \leqslant T_3 \cap R_2 \leqslant T_3 \cap T_2 \cap Z = T_3 \cap Z.$$

So, since $T_3 \cap (T_2 \cap O_2(E))^{e_1}$ is normalized by $\langle A_E^o, B_E^o \rangle$, Theorem 6.3(v) implies

$$T_3 \cap (T_2 \cap O_2(E))^{e_1} = T_4. \tag{7.2.7}$$

Suppose $T_2 \cap (T_2 \cap O_2(E))^{e_1} \neq R_2$. Then $R_2/(T_2 \cap (T_2 \cap O_2(E))^{e_1})$ has order 2 and is normalized by $\langle A_E^o, B_E^o \rangle$. Hence, by (7.2.6) and (7.2.7), $R_2 \cap T_3/T_4$ also has order 2 and is normalized by $\langle A_E^o, B_E^o \rangle$. Since $R_2 \cap T_3 \leqslant T_3 \cap Z$ by (7.2.2), we have contradicted Theorem 6.3(v). Therefore

$$T_2 \cap (T_2 \cap O_2(E))^{e_1} = R_2$$

and consequently

$$R_2 \cap T_3 = T_2 \cap (T_2 \cap O_2(E))^{e_1} \cap T_3 = T_4$$

by (7.2.7). Part (iii) also follows at this point.

We now establish part (i). Let e_2 be as given in Lemma 6.1. Arguing as for (7.2.2) we deduce that

$$T_2 \cap T_2^{e_2} \leqslant T_2 \cap Z,$$

and, by Lemma 3.4(i), $\langle A_E^o, B_E^o \rangle$ normalizes $T_2 \cap T_2^{e_2}$. Also the arguments for R_2 will yield that $[T_2 : T_2 \cap T_2^{e_2}] = 2^8$. So, if $R_2 \neq T_2 \cap T_2^{e_2}$, then

$$R_2 < R_2(T_2 \cap T_2^{e_2}) \leqslant T_2 \cap Z.$$

Since $[T_2 : R_2] = 2^8$, (7.2.1) implies that

$$[R_2(T_2 \cap T_2^{e_2}) : R_2] \leqslant 2^4.$$

By the Replication Lemma (Lemma 3.4(v) with $g = e_1$) and (7.2.4), T_1/T_2 contains a non-trivial subgroup of order less than or equal to 2^4 which is normalized by $\langle A_E^o, B_E^o \rangle$. However, by Lemma 2.9(i), a $\langle A_E^o, B_E^o \rangle$-chief series of T_1/T_2 has two chief factors one of dimension 1 and one of dimension 6. From this contradictory situation we conclude that $R_2 = T_2 \cap T_2^{e_2}$. Hence, referring to Lemma 6.1(i), we have

$$R_2 \trianglelefteq \langle O_2(E), e_1, e_2, \rangle = E$$

and

$$R_2 = T_2 \cap T_2^e \qquad \text{for all} \quad e \in E \backslash S.$$

So

$$R_2 \trianglelefteq \langle A_E^o, B_E^o, E \rangle = \langle A, B, E \rangle.$$

Now, for a suitable choice of $d \in D \backslash S$, we are able to apply Lemma 3.2(ii) with $Y = R_2$ (since $T_2 \trianglelefteq D$ by Lemma 2.9(v)) to obtain

$$R_2 \cap R_2^d \trianglelefteq \langle D, E \rangle.$$

Since $T_4 \trianglelefteq D$ by Lemma 6.5(iii) and $T_4 \leqslant R_2$ by part (ii), evidently $R_2 \cap R_2^d \geqslant T_4$. Because

$$T_4 \cap T_4^{e_1} \leqslant T_3 \cap T_3^{e_1} = T_5,$$

we have $[T_4 : T_4 \cap T_4^{e_1}] \geqslant 2^4$ by Theorem 6.3(iii). Consequently, as $R_2 \cap R_2^d \trianglelefteq E$, $[R_2 \cap R_2^d : T_4] \geqslant 2^4$ and this, since $[R_2 : T_4] = 2^4$, forces $R_2 = R_2^d$. Thus

$$R_2 \trianglelefteq \langle A, B, E, d \rangle = \langle A, B, D, E \rangle,$$

and the proof of Lemma 7.2 is complete.

LEMMA 7.3. $R_2 \ntrianglelefteq G$.

Proof. By Lemma 7.2(ii), (iii), $T_2 = R_2 T_3$ (see the lattice in Fig. 6). If $R_2 \trianglelefteq G$ were to hold, then we obtain $T_2 \trianglelefteq \langle A, B, C, D \rangle$. Hence

$$T_3 < T_2 \leqslant core_{\langle A, B, C, D \rangle} S = T_3,$$

a contradiction. Therefore $R_2 \ntrianglelefteq G$.

Choose $c_5 \in C_E^o \backslash S$ and $d_5 \in D \backslash S$ such that c_5^2, d_5^2, $(c_5 d_5)^3 \in V$. Since $\langle C, D \rangle / V \cong L_3(2)$ and C_E^o covers C/V by Lemma 5.5(iv), this is possible by Lemma 3.2(i). Now put

$$R_3 = R_2 \cap R_2^{c_5}$$

and

$$R_4 = R_3 \cap R_3^{d_5}.$$

LEMMA 7.4. *We have that* $R_4 \cap T_5 \trianglelefteq C$ *and that* $T_6 \leqslant R_4 \cap T_5 < T_5$.

Proof. Let $c \in C \backslash S$. If $R_2^c = R_2$, then by Lemma 7.2(i), $R_2 \trianglelefteq G$, whereas $R_2 \ntrianglelefteq G$ by Lemma 7.3. In particular $R_3 < R_2$.

(7.4.1) $$R_3 \ngeqslant T_4.$$

Suppose $R_3 \geqslant T_4$ were to hold. Because $R_2 \trianglelefteq E$ by Lemma 7.2(i), R_3 is normalized by E_C^o by Lemma 3.4(i). But $R_3 < R_2$ and Lemma 7.2(iii) means $[R_3 : T_4] < 2^4$ whence, for $e \in E_C^o \backslash S$, $[T_4 : T_4 \cap T_4^e] < 2^4$, and then $[T_4 : T_5] < 2^4$ by Theorem 6.3(ii). This contradicts Theorem 6.3(iii). Therefore (7.4.1) must hold.

(7.4.2) $$R_4 \trianglelefteq \langle D, C \rangle.$$

First we show that $R_3 \trianglelefteq S$. By Lemma 7.2(i), $R_2 \trianglelefteq S$ and so $R_3 = R_2 \cap R_2^{c_5}$ is clearly normalized by $O_2(C)$. Choose $e \in E_C^o$. Then, using Lemma 7.2(i),

$$R_3 = (T_2 \cap T_2^e) \cap (T_2 \cap T_2^e)^{c_5} = T_2 \cap T_2^e \cap T_2^{c_5} \cap T_2^{ec_5}$$

since $[e, c_5] \in O_2(\langle C, E \rangle)$ and $T_2 \trianglelefteq S$. Thus

$$R_3 = (T_2 \cap T_2^{c_5}) \cap (T_2 \cap T_2^{c_5})^e.$$

From Lemma 2.9(vi), $T_2 \cap T_2^{c_5} \trianglelefteq S$ and hence $R_3 \trianglelefteq O_2(E)$. Therefore $R_3 \trianglelefteq \langle O_2(C), O_2(E) \rangle = S$. Recalling that $R_2 \trianglelefteq D$, Lemma 3.2(ii) yields (7.4.2)

From Theorem 6.3(i), $T_5 \trianglelefteq C$. So, since $R_2 \geqslant T_4 \geqslant T_5$, we obtain

$$R_3 = R_2 \cap R_2^{c_5} \geqslant T_5.$$

Now $T_6 \trianglelefteq D$ by Lemma 6.5(i) which then implies that

$$R_4 = R_3 \cap R_3^{d_5} \geqslant T_6.$$

We claim that $R_4 \ngeqslant T_5$. Suppose $R_4 \geqslant T_5$ where to hold; then

$$T_5 \leqslant R_4 \cap T_4 \leqslant T_4$$

with $R_4 \cap T_4 \trianglelefteq D$ by (7.4.2) and Lemma 6.5(iii). A consequence of Lemma 6.5(ii) is that there is no D-invariant subgroup properly between T_5 and T_4. Thus either $T_5 = R_4 \cap T_4$ or $R_4 \cap T_4 = T_4$. Using Theorem 6.3(i) the former possibility yields $T_5 \trianglelefteq G$, contrary to Lemma 6.4. The latter gives $T_4 \leqslant R_4 \leqslant R_3$ which contradicts (7.4.1), so verifying our claim. Therefore we have shown that $T_6 \leqslant R_4 \cap T_5 < T_5$. Combining (7.4.2) and Theorem 6.3(i) we see that $R_4 \cap T_5 \trianglelefteq C$.

The proof of the lemma is complete.

Proof of Theorem 7.1. Lemma 7.4. implies $T_6 T_6^{C_4} \leqslant R_4 \cap T_5 < T_5$ and so $[T_6 : T_6 \cap T_6^{c_4}] = [T_6 T_6^{c_4} : T_6] < 2^4$. Hence $[T_6 : T_7] = 2^2$ by Lemma 6.7(i), so establishing Theorem 7.1.

8. ANOTHER SUBGROUP OF S

After our sojourn in Section 7 we now continue our voyage of discovery down the subgroup lattice of S. The next subgroup we meet is T_8 and it is this group that is the subject of this section.

The ultimate state of play with respect to the groups R_5, R_6, R_7, R_8, and R_9 that we introduce shortly is displayed in a subgroup lattice in Fig. 7.

Let $b_6 \in B \backslash S$ be chosen so that there exists $c_6 \in C_A^o \backslash S$ such that b_6^2, c_6^2, $(b_6 c_6)^3 \in M$, and define $T_8 = T_7 \cap T_7^{b_6}$. By Lemmas 3.2(i) and 5.5(vi) such b_6 and c_6 exist.

LEMMA 8.1. (i) $T_8 \trianglelefteq \langle B, C, D, E \rangle$

(ii) $[T_7 : T_8] \leqslant 2^2$.

Proof. Note that $T_6 T_6^{c_6}$ is normalized by A_C^o and so, as in Lemma 6.7, we obtain $[T_6 : T_6 \cap T_6^{c_8}] = 2^2$, whence $T_6 \cap T_6^{c_8} = T_7$. Because $T_6 \trianglelefteq B$ and $T_7 \trianglelefteq S$ by Lemmas 6.5(i) and 6.7(ii), Lemma 3.2(ii) gives $T_8 \trianglelefteq \langle B, C \rangle$. As $T_7 \trianglelefteq \langle D, E \rangle$, T_8 is normalized by $\langle D_B^o, E_B^o \rangle$ by Lemma 3.4(i)

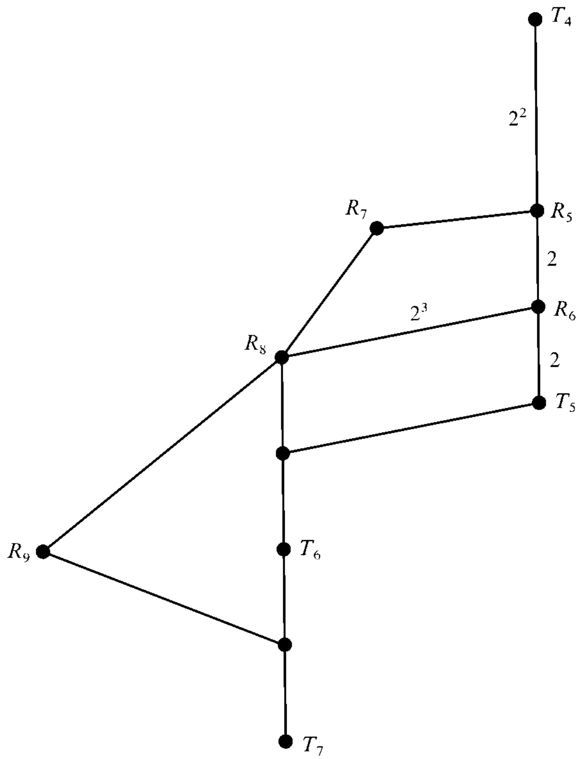


FIGURE 7

and we have (i). Because $[T_6 : T_7] = 2^2$ and $T_6 \trianglelefteq B$ by Lemma 6.5(i) and Theorem 7.1, (ii) also follows.

In similar spirit to Section 7, we now build certain subgroups of T_4 so as to pinpoint the index of T_8 in T_7.

Let $c_7 \in C_E^o \backslash S$ and $d_7 \in D_B^o \backslash S$ be chosen so that c_7^2, d_7^2, $(d_7 c_7)^3 \in V$. Since C_E^o covers C/V and D_B^o covers D/V by Lemma 5.5(iii), (iv) such a choice is possible. Put $R_5 = T_4 \cap T_4^{c_7}$. Observe that, since $T_5 \trianglelefteq C$, $R_5 \geqslant T_5$. Also we choose $b_7 \in B_D^o \backslash S$ such that $b_7^2 \in S$.

Further we define

$$R_6 = R_5 \cap R_5^{b_7}$$

$$R_7 = R_5 \cap R_5^{d_7}$$

$$R_8 = R_6 \cap R_6^{d_7}.$$

LEMMA 8.2. (i) $R_7 \trianglelefteq \langle C, D\rangle$

(ii) $R_8 = R_7 \cap R_7^{b_7} \trianglelefteq \langle B, D\rangle$

(iii) $[T_5 : T_5 \cap R_8] = 2^3$

(iv) $R_8 \geqslant T_6$.

Proof. (i) By Lemma 2.9(vi), $T_2 \cap T_2^{c_7} \trianglelefteq \langle A, C\rangle$. Under the correspondence in the Replication Lemma, $T_2 \cap T_2^{c_7}$ corresponds to $(T_3^{e_1} \cap T_2 \cap T_2^{c_7})^{e_1^{-1}}$ (using the fact that $(T_3 \cap O_2(E))^{e_1} \cap T_2 \leqslant O_2(E)$) which equals $T_3 \cap T_3^{c_7} \cap T_2^{e_1^{-1}} \cap T_2^{c_7 e_1^{-1}}$ (since $T_3 \trianglelefteq C$).

Since E_C^o covers $E/O_2(\langle E, D\rangle)$ by Lemma 5.5(vii), E_C^o clearly covers $E/O_2(E)$. Hence, without loss of generality, we may suppose e_1 (as introduced in Lemma 6.1) belongs to E_C^o.

Now $c_7 \in C_E^o$ and $e_1 \in E_C^o$ yield that

$$T_3 \cap T_3^{c_7} \cap T_2^{e_1^{-1}} \cap T_2^{c_7 e_1^{-1}} = (T_3 \cap T_2^{e_1^{-1}}) \cap (T_3 \cap T_2^{e_1^{-1}})^{c_7}$$

$$= T_4 \cap T_4^{c_7} = R_5.$$

Appealing to the Replication Lemma we see that $[T_4 : R_5] = 2^2$ and R_5 is normalized by $\langle A_E^o, C_E^o\rangle$. Since T_4/T_5 is a chief factor for K,

$$R_5 \trianglelefteq \langle A_E^o, C_E^o, N\rangle = \langle A, C\rangle$$

as A_E^o covers A/N. In particular $R_5 \trianglelefteq S$ and now an application of Lemma 3.2(ii) (since $T_4 \trianglelefteq D$ by Lemma 6.5(iii)) yields $R_7 = R_5 \cap R_5^{d_7} \trianglelefteq \langle C, D\rangle$.

(ii) First we observe that, for any $b \in B_E^o \backslash S$, $(T_2 \cap T_2^{c_7}) \cap (T_2 \cap T_2^{c_7})^b$ corresponds to $R_5 \cap R_5^b$ in the application of the Replication Lemma. Consequently $R_5 \cap R_5^b$ is normalized by B_E^o. Also we note that $[R_5 : R_5 \cap R_5^b] = 2$ and $R_5 \cap R_5^b \geqslant T_5$. Since N acts trivially upon T_4/T_5 and B_E^o covers B/N by Lemma 5.5(i), we have $R_5 \cap R_5^b \trianglelefteq B$. Recalling that $R_5 \trianglelefteq A$ and $[R_5 : R_5 \cap R_5^b] = 2$, K acting irreducibly upon T_4/T_5 implies $R_6 = R_5 \cap R_5^b$. In particular, $R_6 \trianglelefteq S$ and $[R_6 : T_5] = 2$.

Consider

$$R_7 \cap R_7^{b_7} = (R_5 \cap R_5^{d_7}) \cap (R_5 \cap R_5^{d_7})^{b_7}$$

$$= (R_5 \cap R_5^{b_7}) \cap (R_5 \cap R_5^{b_7})^{d_7}$$

$$= R_6 \cap R_6^{d_7}$$

$$= R_8,$$

since $[b_7, d_7] \in U$ (by choice of b_7 and d_7) and $R_5 \trianglelefteq S$. Now $R_7 \trianglelefteq S$ by (i), $R_6 \trianglelefteq S$, and $b_7^2, d_7^2 \in S$ imply

$$R_8 \trianglelefteq \langle O_2(B), O_2(D), b_7, d_7 \rangle = \langle B, D \rangle.$$

This proves part (ii).

(iii) Using Lemma 6.5(ii) and (iii) we see that

$$T_4 = T_5 T_5^{d_7} \leqslant R_6 R_6^{d_7} \leqslant T_4.$$

Therefore $[R_6 : R_8] = [R_6 : R_6 \cap R_6^{d_7}] = [T_4 : R_6] = 2^3$.

If $R_8 \leqslant T_5$, then $R_8 \trianglelefteq D$ implies for $d \in D \backslash S$ that $R_8 \leqslant T_5 \cap T_5^d = T_6$, using Lemma 6.5(ii). But then $[T_5 : T_6] \leqslant 2^3$, contradicting Lemma 6.5(ii). So $R_8 \nleqslant T_5$ and then $[R_6 : T_5] = 2$ and $[R_6 : R_8] = 2^3$ yield (iii).

(iv) From $R_6 \geqslant T_5$ we have $R_8 = R_6 \cap R_6^{d_7} \geqslant T_5 \cap T_5^{d_7} = T_6$, as required.

LEMMA 8.3. $T_7 \ntrianglelefteq G$.

Proof. Suppose $T_7 \trianglelefteq G$ were to hold. So T_6/T_7 admits $\langle A, B \rangle$. Hence, since $[T_6 : T_7] = 2^2$ by Theorem 7.1, $O^2(\langle A, B \rangle)$ centralizes T_6/T_7. Thus x_A does not act fixed-point-freely on T_6/T_7, contrary to Lemma 6.7(iii)

Put $R_9 = R_8 \cap R_8^{c_7}$.

LEMMA 8.4. (i) $[T_6 : R_9 \cap T_6] = [R_9 \cap T_6 : T_7] = 2$ *and* $R_9 \cap T_6 \trianglelefteq B$.

(ii) $[T_7 : T_8] = 2$.

Proof. By Lemma 8.2, $R_8 \trianglelefteq S$ and $R_8 = R_7 \cap R_7^{b_7}$ with $R_7 \trianglelefteq C$ and therefore $R_9 \trianglelefteq \langle B, C \rangle$ by Lemma 3.2(ii). Now $R_8 \geqslant T_6$ by Lemma 8.2 and so $T_7 \trianglelefteq C$ yields $R_9 \geqslant T_7$.

If $R_9 \cap T_6 = T_7$, then as $R_9 \trianglelefteq B$ and $T_6 \trianglelefteq B$, we have $T_7 \trianglelefteq B$. But then $T_7 \trianglelefteq G$ by Lemma 6.7(ii) contrary to Lemma 8.3. Thus $R_9 \cap T_6 > T_7$.

Suppose $R_9 \geqslant T_6$. Because $T_5 \trianglelefteq C$ we clearly have $R_9 \cap T_5 \trianglelefteq C$. Hence $R_9 \cap T_5 > T_6$ because $T_6 \trianglelefteq \langle A, B, D, E \rangle$ and $T_6 \ntrianglelefteq G$. Since $[T_6 : T_7] = 2^2$ by Theorem 7.1 we must have $[R_9 \cap T_5 : T_6] \geqslant 2^2$. Therefore $R_9 \cap T_5$ has index less than or equal to 2^2 in T_5 and hence $[T_5 : R_8 \cap T_5] \leqslant 2^2$. But this cannot happen by Lemma 8.2(iii). So $R_9 \ngeqslant T_6$. Thus

$$T_7 < R_9 \cap T_6 < T_6$$

with $R_9 \cap T_6 \trianglelefteq B$. So $[R_9 \cap T_6 : T_7] = 2$ and this together with Lemma 8.3 proves that $T_8 = T_7 \cap T_7^{b_6}$ has index 2 in T_7. Since $[T_6 : T_7] = 2^2$ we have also established (i), and so the lemma is proven.

9. NORMAL SUBGROUPS OF G

In this section we reap the fruits of our earlier labours in the form of discovering subgroups of S which are normal in G. Let T_9 denote the core of T_8 in A. The subgroup T_{13} will be defined during the course of the proof of Theorem 9.1.

THEOREM 9.1. *Exactly one of the following holds.*

(a) $T_9 \trianglelefteq G$ *and* $[T_8 : T_9] = 2$.

(b) $T_{13} \trianglelefteq G$ *and* $[S : T_{13}] = 2^{46}$. *Further we have the following chain of subgroups,*

$$T_8 > T_9 > T_{10} > T_{11} > T_{12} > T_{13},$$

where

(i) $[T_8 : T_9] = 2^2$, $[T_9 : T_{10}] = [T_{10} : T_{11}] = [T_{11} : T_{12}] = [T_{12} : T_{13}] = 2$;

(ii) $T_9 \trianglelefteq \langle A, C, D, E \rangle$, $T_{10} \trianglelefteq \langle A, B, D, E \rangle$, $T_{11} \trianglelefteq \langle A, B, C, E \rangle$, $T_{12} \trianglelefteq \langle A, B, C, D \rangle$; *and*

(iii) T_6/T_{10} *is an* $\langle A, B \rangle$*-chief factor of order* 2^6.

Proof. We begin by noting that

$$T_8 \ntrianglelefteq A \qquad (\text{and so } T_9 < T_8). \tag{9.1.1}$$

Suppose $T_8 \trianglelefteq A$ were to hold. Then T_6/T_8 admits $\langle A, B \rangle$ by Lemmas 6.5(i) and 8.1(i). From Theorem 7.1 and Lemma 8.4 we have that $|T_6/T_8| = 2^3$. Hence $O^2(\langle A, B \rangle)$ centralizes T_6/T_8 whereas x_A acts fixed-point-freely on T_6/T_7 by Lemma 6.7(iii). Thus we conclude that $T_8 \ntrianglelefteq A$.

We next show that with a suitable choice of X_i and P_i Hypothesis 4.1 holds. If we take $A = P_k$, $B = P_j$, and $C = P_i$ we clearly obtain the required subdiagram in Hypothesis 4.1. For the X_i we take

$$X_1 = T_6$$

$$X_2 = R_9 \cap T_6$$

$$X_3 = T_7$$

and

$$X_4 = T_8.$$

Consulting Lemma 8.4 we see that $[X_i : X_{i+1}] = 2$ for $i = 1, 2, 3$. From Lemma 8.4(i), Lemma 6.7(ii), and Lemma 8.1(i) we have that $R_9 \cap T_6 \trianglelefteq B$, $T_7 \trianglelefteq A$, and $T_8 \trianglelefteq \langle B, C \rangle$. Whilst x_A acting fixed-point-freely on T_6/T_7 means $R_9 \cap T_6 \ntrianglelefteq A$ and Lemma 8.3 dictates that $T_7 \ntrianglelefteq B$. Thus, since $|S/T_1| \neq 2^9$ by the hypothesis of Theorem 1.3, Hypothesis 4.2 is satisfied. So, using Theorem 4.2 and (9.1.1) we obtain that one of the following holds:

(9.1.2)
(a) $[T_8 : T_9] = 2$ and $T_9 \trianglelefteq \langle A, B \rangle$; and
(b) $[T_8 : T_9] = 2^2$ and for any $b \in B \backslash S$, $T_9 \cap T_9^b \trianglelefteq \langle A, B \rangle$ with $[T_9 : T_9 \cap T_9^b] = 2$.

Before discussing the two possibilities in (9.1.2) we observe that

(9.1.3) $$T_9 \trianglelefteq \langle A, C, D, E \rangle.$$

Since $T_8 \trianglelefteq \langle C, D, E \rangle$ by Lemma 8.1(i), Lemma 3.4(i) implies that $\langle C_A^o, D_A^o, E_A^o \rangle$ normalizes $T_8 \cap T_8^a$ for any $a \in A$. Since T_9 is the intersection of subgroups like $T_8 \cap T_8^a$ $(a \in A)$, $\langle C_A^o, D_A^o, E_A^o \rangle$ must normalize T_9. So, since $T_9 = core_A\, T_8$, we obtain (9.1.3).

If (9.1.2)(a) holds, then (9.1.3) implies

$$T_9 \trianglelefteq \langle A, B, C, D, E \rangle = G$$

and we obtain alternative (a) of the theorem. So, for the remainder of the proof of Theorem 9.1 we assume (9.1.2)(b) holds and show that (b) must result. Note that this means we have $T_9 \ntrianglelefteq \langle A, B \rangle$. Letting $b_8 \in B \backslash S$ and $c_8 \in C \backslash S$ be such that b_8^2, c_8^2, $(b_8 c_8)^3 \in M$ we put

$$T_{10} = T_9 \cap T_9^{b_8}$$

and

$$T_{11} = T_{10} \cap T_{10}^{c_8}.$$

Combining Lemma 3.4(i), (9.1.2)(b), and (9.1.3) we obtain

(9.1.4) $$T_{10} \trianglelefteq \langle A, B, D, E \rangle \text{ and } [T_9 : T_{10}] = 2,$$

(9.1.5) $$T_{10} \ntrianglelefteq G \text{ (and so } T_{10} > T_{11}).$$

Suppose (9.1.5) is false. Then $T_8 > T_9 > T_{10}$ is a chain of $\langle C, D \rangle$-invariant subgroups by Lemma 8.1(i) and (9.1.3). By (9.1.2)(b) and (9.1.4),

$[T_8 : T_9] = 2^2$ and $[T_9 : T_{10}] = 2$ and consequently $O^2(\langle C, D\rangle)$ centralizes T_8/T_{10}. In particular x_C centralizes T_8/T_{10}. Thus, as T_8/T_{10} also admits $\langle B, C\rangle$, we deduce that $O^2(\langle B, C\rangle)$ also centralizes T_8/T_{10}. This and (9.1.3) yield $T_9 \trianglelefteq \langle A, B\rangle$, contrary to $T_9 \ntrianglelefteq \langle A, B\rangle$. Thus (9.1.5) is proven.

(9.1.6) $T_{11} \trianglelefteq \langle A, B, C, E\rangle$ and $[T_{10} : T_{11}] = 2.$

Since $T_9 \trianglelefteq C$ and $T_{10} \trianglelefteq S$, Lemma 3.2(ii) readily yields $T_{11} \trianglelefteq \langle B, C\rangle$. From $T_{10} \trianglelefteq \langle A, E\rangle$, Lemma 3.4(i) implies that T_{11} is normalized by $\langle A^o_C, E^o_C\rangle$ whence $T_{11} \trianglelefteq \langle A, B, C, E\rangle$. Because $T_9 \trianglelefteq C$ and $[T_9 : T_{10}] = 2$ by (9.1.3) and (9.1.4), $[T_{10} : T_{11}] \leqslant 2$. Hence, by (9.1.5), $[T_{10} : T_{11}] = 2$, as required.

(9.1.7) $T_{11} \ntrianglelefteq G.$

If $T_{11} \trianglelefteq G$ were to hold, then, by (9.1.3), T_9/T_{11} would admit $\langle C, D\rangle$. Since $\langle C, D\rangle/V \cong L_3(2)$ and $|T_9/T_{11}| = 2^2$ by (9.1.4) and (9.1.6), $O^2(\langle C, D\rangle)$ must centralize T_9/T_{11}. But then (9.1.4) yields $T_{10} \trianglelefteq G$, contradicting (9.1.5). Therefore $T_{11} \ntrianglelefteq G$.

Now let $c_9 \in C\backslash S$ and $d_9 \in D\backslash S$ be such that $c_9^2, d_9^2, (c_9 d_9)^3 \in V$. By (9.1.5), $T_{11} = T_{10} \cap T_{10}^{c_9}$.

Put $T_{12} = T_{11} \cap T_{11}^{d_9}$.

(9.1.8) $T_{12} \trianglelefteq \langle A, B, C, D\rangle$ and $[T_{11} : T_{12}] = 2.$

Since $T_{10} \trianglelefteq D$ by (9.1.4) and $T_{11} \trianglelefteq S$ by (9.1.6), Lemma 3.2(ii) gives $T_{12} \trianglelefteq \langle C, D\rangle$. Employing Lemma 3.4(i) also yields that T_{12} is normalized by $\langle A^o_D, B^o_D\rangle$ and so $T_{12} \trianglelefteq \langle A, B, C, D\rangle$. By (9.1.4) and (9.1.6), $T_{10} \trianglelefteq D$ and $[T_{10} : T_{11}] = 2$. Hence $[T_{11} : T_{12}] = 2$ by (9.1.7), so proving (9.1.8).

(9.1.9) $T_{12} \ntrianglelefteq G.$

Suppose $T_{12} \trianglelefteq G$. Then T_{10}/T_{12} admits $\langle D, E\rangle$ by (9.1.4). Since $|T_{10}/T_{12}| = 2^2$ by (9.1.6) and (9.1.8) and $\langle D, E\rangle/O_2(\langle D, E\rangle) \cong L_3(2)$, $O^2(\langle D, E\rangle)$ must centralize T_{10}/T_{12} whence $T_{11} \trianglelefteq G$ by (9.1.6). This contradicts (9.1.7) and so we must have $T_{12} \ntrianglelefteq G$.

Next let $d_{10} \in D\backslash S$ and $e_{10} \in E\backslash S$ be such that $d_{10}^2, e_{10}^2, (d_{10} e_{10})^3 \in O_2(\langle D, E\rangle)$. From (9.1.7), $T_{12} = T_{11} \cap T_{11}^{d_{10}}$. Putting $T_{13} = T_{12} \cap T_{12}^{e_{10}}$ we finally establish

(9.1.10) $T_{13} \trianglelefteq G$ and $[T_{12} : T_{13}] = 2.$

Using Lemma 3.2(ii) yet again, as $T_{11} \trianglelefteq E$ and $T_{12} \trianglelefteq S$ by (9.1.6) and (9.1.8), gives $T_{13} \trianglelefteq \langle D, E \rangle$. Combining $T_{12} \trianglelefteq \langle A, B, C \rangle$ (by (9.1.8)) and Lemma 3.4(i) we then obtain

$$T_{13} \trianglelefteq \langle A_E^o, B_E^o, C_E^o, D, E \rangle = G.$$

Taken together, $T_{11} \trianglelefteq E$, $[T_{11} : T_{12}] = 2$ and (9.1.9) yield that $[T_{12} : T_{13}] = 2$.

Combining (9.1.2)(b), (9.1.4), (9.1.6), (9.1.8), and (9.1.10) yields that $[T_8 : T_9] = 2^2$ and $[T_9 : T_{10}] = [T_{10} : T_{11}] = [T_{11} : T_{12}] = [T_{12} : T_{13}] = 2$. So we have (b)(i). From (9.1.3), (9.1.4), (9.1.6), and (9.1.8) we obtain (b)(ii).

We now establish (b)(iii). By Lemma 6.5(i) and (9.1.4), T_6/T_{10} admits $\langle A, B \rangle$. Theorem 7.1, Lemma 8.4(ii), and (b)(i) imply that $|T_6/T_{10}| = 2^6$. From (b)(i), (ii), and Lemma 8.4(ii), $T_6 > T_7 > T_9 > T_{10}$ is a chain of A-invariant subgroups with $[T_7 : T_9] = 2^3$ and $[T_9 : T_{10}] = 2$. Since $T_8 \ntrianglelefteq A$ by (9.1.1), $|C_{T_7/T_9}(x_A)| = 2$. By Lemma 6.7(iii), x_A acts fixed-point-freely on T_6/T_7 and so we deduce that $|C_{T_6/T_{10}}(x_A)| = 2^2$. From Lemma 8.1(i) and (b)(i), $T_8 \trianglelefteq B$ and $[T_6 : T_8] = [T_8 : T_{10}] = 2^3$. Since $T_7 \ntrianglelefteq B$ by Lemmas 6.7(ii) and 8.3 and $T_9 \ntrianglelefteq B$, we infer that $|C_{T_6/T_{10}}(x_B)| = 2^2$. Now Lemma 2.4(i) and Lemma 2.4(iii) together with $|C_{T_6/T_{10}}(x_B)| = 2^2 = |C_{T_6/T_{10}}(x_A)|$ forces $\langle A, B \rangle$ to act irreducibly upon T_6/T_{10}. Thus T_6/T_{10} is an $\langle A, B \rangle$-chief factor of order 2^6.

Consulting Theorem 6.3(iii) and Lemma 6.5(ii) we see that $[T_3 : T_5] = 2^{12}$ and $[T_5 : T_6] = 2^4$, and from (b)(iii) we have $[T_6 : T_{10}] = 2^6$. Thus, using (b)(ii), we infer that $[T_3 : T_{13}] = 2^{25}$. Recalling that $[S : T_3] = 2^{21}$ this implies that $[S : T_{13}] = 2^{46}$. Since we have already proved that $T_{13} \trianglelefteq G$ in (9.1.10), we have shown that (b) holds and this completes the proof of Theorem 9.1.

10. CONCLUSION OF THE PROOF

We are at last in a position to identify S_o.

THEOREM 10.1. *$S_o = T_{13}$ with $[S : S_o] = 2^{46}$.*

Proof. First we prove that

(10.1.1) $$T_9 \ntrianglelefteq G.$$

We assume that $T_9 \trianglelefteq G$ and seek to derive a contradiction. Hence, by Theorem 6.3(i), T_5/T_9 admits $\langle A, B, C\rangle$. Also we have

$$(10.1.2)\qquad \begin{array}{l} T_5 > T_6 > T_9 \text{ is an } \langle A, B\rangle \text{ chief series between} \\ T_5 \text{ and } T_9 \text{ with } |T_5/T_6| = 2^4 = |T_6/T_9|. \end{array}$$

Lemma 6.5(ii) gives $[T_5 : T_6] = 2^4$ while a combination of Theorems 7.1 and 9.1 yields $[T_6 : T_9] = 2^4$. From Lemma 6.5(iv) and Theorem 6.3(iii) we obtain that T_5/T_6 is an $\langle A, B\rangle$ chief factor. Since x_A acts fixed-point-freely upon T_6/T_7 by Lemma 6.7(iii), T_6/T_9 must also be an $\langle A, B\rangle$-chief factor. So (10.1.2) holds.

Let $T_9 < X \leqslant T_5$ be such that $\langle A, B, C\rangle$ acts irreducibly upon X/T_9. Put $C^* = C_{\langle A, B, C\rangle}(X/T_9)$. Since T_1 is a normal 2-subgroup of $\langle A, B, C\rangle$ we see that $T_1 \leqslant C^*$. Let $\langle \xi\rangle \in Syl_3\, O_{2,3}(\langle A, B\rangle)$. In view of (10.1.2) and Lemma 2.4(ii), ξ must centralize both of T_5/T_6 and T_6/T_9. Thus $\xi \in C^*$. Because N/T_1 is an $\langle A, B\rangle$ chief of order 2^6 by Lemma 2.8(ii), $N = [N, \xi]\, T_1$ by Lemma 2.4(iv). Hence, from $\xi \in C^*$ and $C^* \trianglelefteq \langle A, B, C\rangle$, we deduce that $[N, \xi] \trianglelefteq C^*$. Consequently $N \leqslant C^*$. Consulting the subgroup lattice in Lemma 2.8 gives

$$T < N \cap R \leqslant C^* \cap R \leqslant R.$$

Clearly $C^* \cap R \trianglelefteq \langle B, C\rangle$ and so $C^* \cap R = R$ by Lemma 2.8(i). Therefore $N_o = RN \leqslant C^*$ and so the normal closure of N_o in $\langle A, B\rangle$, $\langle N_o^{\langle A, B\rangle}\rangle$, must also be contained in C^*. Since $N_o/N = Z(B/N)$, the normal subgroup structure of $\hat{S}_6$ yields $\langle N_o^{\langle A, B\rangle}\rangle = \langle A, B\rangle$. Thus $\langle A, B\rangle \leqslant C^*$.

However,

$$X/T_9 \underset{\langle A, B\rangle}{\cong} T_5/T_9,\ T_5/T_6, \text{ or } T_6/T_9$$

by (10.1.2) and so $\langle A, B\rangle$ cannot possibly centralize X/T_9. This is the desired contradiction that establishes (10.1.1).

Combining (10.1.1) with Theorem 9.1 yields that $T_{13} \trianglelefteq G$ with $[S : T_{13}] = 2^{46}$. Clearly then $T_{13} \leqslant S_o$. Also we have $S_o \leqslant core_{\langle A, B, C, D\rangle}\, S = T_3$. Hence $S_o = S_o^{e_1} \leqslant T_3 \cap T_3^{e_1} = T_5$. Since for $i \in \{5, 6, 7, 8, 9, 10, 11, 12\}$ we have $T_{i+1} = T_i \cap T_i^{x_i}$ for some $x_i \in G$, this forces $S_o \leqslant T_{13}$. Thus $S_o = T_{13}$, and the theorem is proved.

Assembling earlier results we now complete the

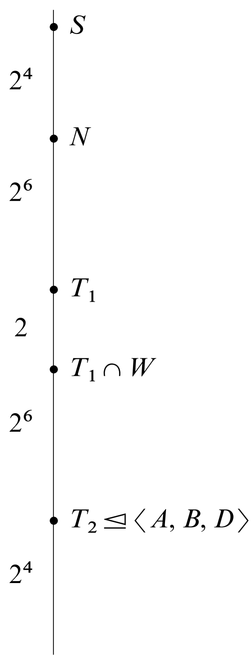
Proof of Theorem 1.3. Since we have shown in Theorem 10.1 that $[S : S_o] = 2^{46}$, it only remains to examine $\langle A, B\rangle$ chief series of S/T_{13}. Taking $X_o = N$, $X_1 = T_1$, $X_2 = T_1 \cap W$, $X_3 = T_2$, and $X_4 = T_3$, we see from

Lemmas 2.8(ii) and 2.9(i) that these form a part of an $\langle A, B \rangle$ chief series for $O_2(\langle A, B \rangle)/S_o$ of the desired kind. Consulting Theorem 6.3 we see that setting $X_5 = T_3 \cap O_2(\langle E, B \rangle)$, $X_6 = T_3 \cap O_2(\langle E, B \rangle) \cap O_2(\langle E, D \rangle)$, $X_7 = T_4$, and $X_8 = T_5$ gives us a further portion of an $\langle A, B \rangle$ chief series for $O_2(\langle A, B \rangle)/S_o$ where $|X_4/X_5| = |X_5/X_6| = 2$, $|X_6/X_7| = 2^6$, and $|X_7/X_8| = 2^4$. With $X_9 = T_6$ we have that X_8/X_9 is an $\langle A, B \rangle$-chief factor of order 2^4. Finally, taking $X_{10} = T_{10}$, $X_{11} = T_{11}$, $X_{12} = T_{12}$, and $X_{13} = T_{13}$ and appealing to Theorems 9.1 and 10.1 we obtain an $\langle A, B \rangle$-chief series for $O_2(\langle A, B \rangle)/S_o$ of the kind stated in the theorem.

The proof of Theorem 1.3 is now complete.

APPENDIX

Here we give a subgroup lattice incorporating some of the important subgroups encountered in the proof of Theorem 1.3. The groups N, T_1, $T_1 \cap W$, T_2, and T_3 are introduced in Section 2. The first appearance of T_3, $T_3 \cap O_2(\langle B, E \rangle)$, $T_3 \cap O_2(\langle B, E \rangle) \cap O_2(\langle E, D \rangle)$, $T_3 \cap Z$, T_4, T_5, T_6, T_7 occurs in Section 5 with the subgroups between T_3 and T_5 being the centerpiece of Theorem 6.3. In Section 8, T_8 is defined and the remaining subgroups below T_8 are described in Section 9.



- $T_3 \trianglelefteq \langle A, B, C, D \rangle$

2

- $T_3 \cap O_2(\langle B, E \rangle) \trianglelefteq \langle A, B \rangle$

2

- $T_3 \cap O_2(\langle B, E \rangle) \cap O_2(\langle E, D \rangle) \trianglelefteq \langle A, B, D \rangle$

2^2

- $T_3 \cap Z \trianglelefteq \langle B, C, D \rangle$

2^4

- $T_4 \trianglelefteq \langle A, B \rangle$

2^4

- $T_5 \trianglelefteq \langle A, B, C, E \rangle$

2^4

- $T_6 \trianglelefteq \langle A, B, D, E \rangle$

2^2

- $T_7 \trianglelefteq \langle A, C, D, E \rangle$

2

- $T_8 \trianglelefteq \langle B, C, D, E \rangle$

2^2

- $T_9 = \mathrm{core}_A\, T_8 \trianglelefteq \langle A, C, D, E \rangle$

2

- $T_{10} \trianglelefteq \langle A, B, D, E \rangle$

2

$T_{11} \trianglelefteq \langle A, B, C, E \rangle$

2

- $T_{12} \trianglelefteq \langle A, B, C, D \rangle$

2

- $T_{13} \trianglelefteq G$

ACKNOWLEDGMENTS

This paper was written in 1987 but, partly due to the author's delinquency, has remained unpublished until now. In the intervening years there has been considerable progress in the study of minimal parabolic systems and other related group geometries, as may be seen by consulting [I]. Of particular relevance to this work are the results of Heiss [H] and Ivanov and Shpectorov [IS] classifying flag transitive geometries with diagram ○——○ ═̃═ ○, which, if used here, would shorten some of our proofs (though this classification relies crucially upon certain coset enumerations performed by computer). Finally, I thank Corinna Wiedorn for her careful reading of this paper.

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