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Abstract

The present work is devoted to the problem of differentiation of an Abelian function, defined by a family of plane algebraic curves, over the parameters of the family.

A precise formulation of the problem involves the language of Differential Geometry.

We give an effective solution, which is based on our theory of multivariate sigma-function. We obtain explicit expressions for the generators of the module of differentiations of a ring of Abelian functions. This result is equivalent, as we show, to an explicit construction of a Gauß-Manin connection and a Koszul connection in the appropriate vector bundles.

The families of curves, which we work with, are special deformations of the singularities $y^n - x^s$, where $\gcd(n, s) = 1$. Any algebraic curve has a bi-rationally equivalent model in such family. The choice of this type of families allows us to use methods and results of Singularity Theory.

In the course of exposition we outline the key classic results relevant to the problem.

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Theory of multivariate sigma-functions

V.M.Buchstaber and D.V.Leykin

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Applications

V.M.Buchstaber, D.V.Leykin, and M.V.Pavlov

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Work in progress.

Introduction

Theory of Abelian functions was a central topic of the 19th century mathematics. In mid-seventies of the last century a new wave arose of investigation in this field in response to the discovery that Abelian functions provide solutions of a number of challenging problems of modern Theoretical and Mathematical Physics.

In the cycle of our papers of 2000–05, we have developed a theory of multivariate sigma-function, an analogue of the classic Weierstrass sigma-function.

A sigma-function is defined on a cover of U , where U is the space of a bundle $p : U \to B$ defined by a family of plane algebraic curves of fixed genus. The base B of the bundle is the space of the family parameters and a fiber J_b over $b \in B$ is the Jacobi variety of the curve with the parameters b. A second logarithmic derivative of the sigma-function along the fiber is an Abelian function on J_b.

Thus, one can generate a ring F of fiber-wise Abelian functions on U. The problem to find derivations of the ring F along the base B is a reformulation of the classic problem of differentiation of Abelian functions over parameters. Its solution is relevant to a number of topical applications.

This work presents a solution of this problem recently found by the authors. Our method of solution essentially employs the results from Singularity Theory about vector fields tangent to the discriminant of a singularity $y^n - x^s$, $\gcd(n, s) = 1$.

An *Abelian function* is, in the classical sense, a meromorphic function on a complex Abelian torus

$$T^g = \mathbb{C}^g/\Gamma,$$

where $\Gamma \subset \mathbb{C}^g$ is a rank $2g$ lattice.

That is f is Abelian iff

$$f(u) = f(u + \omega), \quad \text{for all } u \in \mathbb{C}^g \text{ and } \omega \in \Gamma.$$

Abelian functions form a differential field.

Complex dimension g of the torus is called the genus of a field.

A plane algebraic curve defines a lattice Γ as the set of all periods of basis holomorphic differentials.

The resulting torus is called the Jacobian of a curve.

Suppose B is an open dense subset in $\mathbb{C}^d$.

We will consider a family of curves V, depending linearly on a parameter $b \in B$. We use V to define over B a space of Jacobians U.

The space U is naturally fibred,

$$p : U \longrightarrow B,$$

where the fiber over a point $b \in B$ is the Jacobian J_b of the curve with the parameter b.

Let $g = 1$ and $d = 2$,

$$V = \{(x, y, g_2, g_3) \in \mathbb{C}^2 \times B \mid y^2 = 4x^3 - g_2 x - g_3\}$$

is the family of Weierstrass elliptic curves, where

$$B = \{(g_2, g_3) \in \mathbb{C}^2 \mid g_2^3 - 27g_3^2 \neq 0\}.$$

This is the only case, when U and V are equivalent.

It is natural to consider on U the ring F of fiber-wise Abelian functions, i.e., the restriction of $f \in F$ to a fiber is an Abelian function.

Let $g = 1$ and $d = 2$.

F is generated by g_2, g_3 and elliptic Weierstrass functions

$$\wp(u, g_2, g_3) \quad \text{and} \quad \wp'(u, g_2, g_3).$$

The ring F attractes much interest since 1974, when S.P. Novikov discovered that relations in F are relevant to modern challenging problems of Mathematical Physics.

The main problem.

Find the generators of the F-module $\mathrm{Der}(F)$ of derivations of the ring F.

Let $g = 1$, $d = 2$. F.G.Frobenius (1849–1917) and L.Stickelberger found* the generators of $\mathrm{Der}(F)$,

$$L_0 = -u\partial_u + 4g_2\partial_{g_2} + 6g_3\partial_{g_3},$$
$$L_1 = \partial_u,$$
$$L_2 = -\zeta(u, g_2, g_3)\partial_u + 6g_3\partial_{g_2} + \frac{1}{3}g_2^2\partial_{g_3},$$

with the structure relations,

$$[L_0, L_k] = kL_k, \quad [L_1, L_2] = \wp(u, g_2, g_3)L_1.$$

B.A. Dubrovin† clarified the meaning of this result for reconstructing the differential geometry of the universal bundle of genus one Jacobians. He named the connection on this bundle the *FS-connection*.

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*Crelles Journal, Bd. 92. S. 311–337. (1882)

†"Geometry of 2D topological field theories", Appendix C, Lect. Notes Math. 1620. (1994)

A differential-geometric approach to the problem.

One can realize U as the space of classes $[(u,b)]$ of pairs

$$(u,b) \in \mathbb{C}^g \times B \subset \mathbb{C}^{g+d}, \quad \text{where}$$
$$(u,b) \sim (u',b') \quad \text{iff} \quad b=b' \text{ and } u-u' \in \Gamma_b.$$

Here Γ_b is a rank $2g$ lattice in $\mathbb{C}^g$ defined by the curve with parameters b from the family of curves V.

Now, we have an action $\mu : U \times \mathbb{C}^g \longrightarrow U$

$$\mu([(u,b)],z) = [(u+z,b)],$$

with the following properties:

(1) The orbit space of μ is B.

(2) Fix $[(u,b)] \in U$. Then μ defines a map

$$\mu_{[(u,b)]} : \mathbb{C}^g \longrightarrow p^{-1}(b), \quad \mu_{[(u,b)]}(z) = [(u+z,b)],$$

which is a universal covering of the Jacobian $J = p^{-1}(b)$.

(3) Fix $z \in \mathbb{C}^g$. Then μ defines a map

$$\mu_z : U \longrightarrow U, \quad \mu_z([(u,b)]) = [(u+z,b)],$$

which induces an automorphism of F.

Since U is the space of the bundle $p : U \longrightarrow B$ with a fiber $J = T^g$, we have an exact sequence of bundles:

$$0 \longrightarrow \mathcal{T}_J U \longrightarrow \mathcal{T} U \longrightarrow \mathcal{T} B \longrightarrow 0$$

Due to the properties of μ we can fix the following basis sections in $\mathcal{T}_J U$,

$$(\partial_{u_1}, \dots, \partial_{u_g}),$$

which are derivations of F.

Note, that U has a zero section $s_0 = [(0, b)]$.

However, the trivial lift, with the help of s_0 and μ, of a vector field from $\mathcal{T} B$ <u>is not a derivation of F.</u>

The problem of derivations of F reduces to constructing a special basis of <u>horizontal sections</u> of $\mathcal{T} U$.

In other words, we have to construct a connection on U, which is 'smooth' with respect to the structure ring F.

Koszul connection.

Let $\pi : E \longrightarrow B$ be a complex vector bundle.

Notation:

$\mathfrak{X}(B)$ is the $C^\infty(B)$-module of vector fields on B;

ΓE is the space of smooth sections of $\pi : E \longrightarrow B$.

Then,*

A Koszul connection in a vector bundle $\pi : E \longrightarrow B$ is a map

$$\nabla \colon \mathfrak{X}(B) \times \Gamma E \longrightarrow \Gamma E, \quad (X, f) \mapsto \nabla_X(f),$$

which is bilinear and satisfies the two identities

$$\nabla_{\mu X}(f) = \mu \nabla_X(f), \quad \nabla_X(\mu f) = \mu \nabla_X(f) + X(\mu) f,$$

for all $X \in \mathfrak{X}(B)$, $f \in \Gamma E$, $\mu \in C^\infty(B)$.

The connection ∇ is flat if

$$\nabla_{[X,Y]}(f) = \nabla_X(\nabla_Y(f)) - \nabla_Y(\nabla_X(f)).$$

*Kirill C.H. Mackenzie. General theory of Lie groupoids and Lie algebroids. LMS Lect. Notes. Ser. 213, CUP (2005).

An accompanying problem.

Consider, associated with $p : U \longrightarrow B$, a complex vector bundle

$$\pi : E \longrightarrow B,$$

whose fiber F_b is the restriction of the ring F to the fiber $p^{-1}(b) = J_b \subset U$.

Since $\Gamma E = F$, we come to the problem

> Construct a flat Koszul connection on the vector bundle $\pi : E \longrightarrow B$ with a fiber F_b.

Basic facts about Abelian functions on a Jacobian of genus g.

(A_1) If $f \in F_b$, then $\partial_{u_i} f \in F_b$, $i = 1, \dots, g$.

(A_2) For any nonconstant $f_1, \dots, f_{g+1}$ from F_b there exists $P \in \mathbb{C}[z_1, \dots, z_{g+1}]$ such that

$$P(f_1, \dots, f_{g+1}) = 0, \quad \text{for all } u \in p^{-1}(b).$$

(A_3) If $f \in F_b$ is any nonconstant function, then any $h \in F_b$ is a rational function of $(f, \partial_{u_1} f, \dots, \partial_{u_g} f)$.

(A_4) There exists an entire function $\vartheta : \mathbb{C}^g \to \mathbb{C}$ such that

$$\partial_{u_i, u_j} \log \vartheta \in F_b, \quad i, j = 1, \dots, g.$$

A strategy of solution of the problem in general case.

One can understand the strategy of solution by following our treatment of the classic case.

We employ the following properties of Weierstrass sigma-function $\sigma : \mathbb{C}^3 \to \mathbb{C}$.

(a) $\sigma(u, g_2, g_3)$ is entire in $(u, g_2, g_3) \in \mathbb{C}^3$.

(b) $\partial_u^2 \log(\sigma(u, g_2, g_3)) = -\wp(u, g_2, g_3) \in F$,

whenever

$$(g_2, g_3) \in B = \{(g_2, g_3) \in \mathbb{C}^2 \mid g_2^3 - 27g_3^2 \neq 0\},$$

which is sufficient to generate the whole ring F.

(c) $\sigma(u, g_2, g_3)$ is a solution of the system

$$Q_0(\sigma) = 0, \quad Q_0 = 4g_2\partial_{g_2} + 6g_3\partial_{g_3} - u\partial_u + 1,$$

$$Q_2(\sigma) = 0, \quad Q_2 = 6g_3\partial_{g_2} + \frac{1}{3}g_2^2\partial_{g_3} - \frac{1}{2}\partial_u^2 - \frac{1}{24}g_2u^2,$$

where the operators depend polynomially on $b \in B$.

K.Weierstrass (1815–97) discovered Q_0 and Q_2 in 1894.

Observe, that due to (b) the equations (c) convert into derivations of F.

Let $\boxed{\ell_2 = 6g_3\partial_{g_2} + \frac{1}{3}g_2^2\partial_{g_3}.}$

Since

$$\ell_2(g_2^3 - 27g_3^2) = 0,$$

ℓ_2 is a field on B.

Now, $Q_2 = \ell_2 - \frac{1}{2}\partial_u^2 - \frac{1}{24}g_2u^2$.

Divide $Q_2(\sigma) = 0$ by σ and rearrange the terms using

$$\zeta = \partial_u \log\sigma \quad \text{and} \quad \wp = -\partial_u^2 \log\sigma.$$

We obtain

$$\ell_2(\log(\sigma)) - \frac{1}{2}\zeta^2 + \frac{1}{2}\wp - \frac{1}{24}g_2u^2 = 0.$$

Apply ∂_u, and as $[\partial_u, \ell_2] = 0$,

$$\ell_2(\zeta) + \zeta\wp + \frac{1}{2}\wp' - \frac{1}{12}g_2u = 0,$$

apply ∂_u again,

$$-\ell_2(\wp) + \zeta\wp' - \wp^2 + \frac{1}{2}\wp'' - \frac{1}{12}g_2 = 0,$$

and finally:

$$(\ell_2 - \zeta\partial_u)\wp = \frac{1}{2}\wp' - \wp^2 - \frac{1}{12}g_2 \in F.$$

We have recovered the operator $\boxed{L_2 = \ell_2 - \zeta\partial_u}$.

In their original work of 1882, Frobenius and Stickelberger used a completely different techniques.

By our construction, the operator L_2 is a special horizontal section of $\mathcal{T}U$, which respects the structure ring F.

Our strategy leads to the solution of general case, as soon as one presents an entire function on $\mathbb{C}^{g+d}$, whose properties generalize the above properties (b) and (c) of Weierstrass σ.

What classic Abelian functions theory had in store.

A function with (b) is at hand. One takes any Riemann θ-function. But, since it depends on the choice of a basis in Γ, it is impossible to find a θ-function with (c).

Klein's project (c. 1886)
Modify θ to obtain an entire function, which
(1.) depends on a lattice Γ in the whole;
(2.) is a covariant of Möbius transforms of a curve.

F.Klein (1849-1925) gave a review* of the outcome, which is the hyperelliptic sigma-function. It was proven, that Kleinian σ has both (a) and (b).

Still, the claim (2.) restricts Klein's theory to hyperelliptic curves and, even in this case, creates artificial complications in the operators (c).

Gesammelte Mathematische Abhandlungen, vol. 3, S. 317-322, (1923) H.F.Baker (1866–1956) abandoned (2.) and demonstrated for $g = 2$ that a theory of sigma-function can be constructed without any reference to θ. Baker's book is a realization of the following.

Weierstrass principle
One has to work with a canonical model of a curve.

Elliptic sigma-function owes its advantages to Weierstrass' cubic equation,

$$y^2 = 4x^3 - g_2 x - g_3.$$

Weierstrass† proposed, for a pair (n, s), $\gcd(n, s) = 1$, the class of models

$$V = \{(x, y; \lambda) \in \mathbb{C}^{2+d} \mid y^n = x^s + \sum_{i,j \geq 0}^{q(i,j)>0} \lambda_{q(i,j)} x^i y^j\},$$

where $q(i, j) = (n - j)(s - i) - ij$ and $d = ns - g$.

Curves in Weierstrass' (n, s)-class are of genus not greater than $g = (n - 1)(s - 1)/2$.

For hyperelliptic curves $(n, s) = (2, 2g + 1)$.

*Multiply periodic functions. Part I. (1907)

†Abel'schen Funktionen. Ges. Werke, vol. 4. (1904)

A contribution from Singularity Theory.

Singularity Theory studies a function

$$f(x,y,\lambda) = y^n - x^s - \sum_{j=0}^{n-2} \sum_{i=0}^{s-2} \lambda_{q(i,j)} x^i y^j,$$

as *miniversal unfolding* of *Pham singularity* $y^n - x^s$.

Miniversal unfolding has $2g$ parameters λ.

The number $m = \#\{\lambda_k \mid k < 0\}$ is the *modality* of f.

One relates to f the discriminant variety $\Sigma \subset \mathbb{C}^{2g}$,

$$(\lambda \in \Sigma) \;\Leftrightarrow\; (\exists\,(x,y) \in \mathbb{C}^2 \colon f_x = f_y = 0 \text{ at } (x,y,\lambda)).$$

We use a construction*, which is based on a theorem due to V.M. Zakalyukin†, of holomorphic vector fields tangent to Σ.

The fields define a holomorphic function $\Delta(\lambda) \in \mathbb{C}(\lambda)$, such that

$$\Sigma = \{\lambda \in \mathbb{C}^{2g} \mid \Delta(\lambda) = 0\},$$

a vector field ℓ is a tangent to Σ iff

$$\ell(\Delta(\lambda)) = \phi(\lambda)\Delta(\lambda),$$

where

$$\phi(\lambda) \in \mathbb{C}[[\lambda]].$$

In the space of holomorphic fields tangent to Σ, there exists a unique basis $\mathcal{L} = (\ell_1, \ldots, \ell_{2g})^t$ such that

$$\mathcal{L} = T(\lambda)\partial_\lambda, \quad T(\lambda) = T(\lambda)^t, \quad \Delta(\lambda) = \det T(\lambda),$$

where $T(\lambda)$ is the matrix of Arnold's convolution.

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*Funct. Anal. Appl. 36 (2002), no. 4, 267–280.

†Funct. Anal. Appl. 10 (1976), no. 2, 139–140.

The family of (n, s)-curves.

Genus of a curve in Weierstrass' (n, s)-class is $\leq (n-1)(s-1)/2$.

Genus of a miniversal unfolding, if $b \notin \Sigma$, is $\geq (n-1)(s-1)/2$.

We impose the condition

$$\lambda_{q(i,j)} = 0, \quad \text{when } q(i,j) < 0,$$

on miniversal unfolding, or, equivalently,

$$\lambda_{q(s-1,j)} = \lambda_{q(i,n-1)} = 0$$

on Weierstrass' model, and obtain a family of curves of constant genus $g = (n-1)(s-1)/2$ over $B = \mathbb{C}^{2g-m} \backslash \Sigma$.

Our (n, s)-models are the intersection of the classes of miniversal unfoldings and Weierstrass' models.

The structure of a $\mathbb{C}[\lambda]$-module in the space of sections of $\mathcal{T}B$.

In what follows we use an obvious renumbering of λ_k.

Under the condition $\lambda_k = 0$ for $k < 0$,

(1) the holomorphic symmetric matrix $T(\lambda)$ becomes a matrix over $\mathbb{C}[\lambda_1, \dots, \lambda_{2g-m}]$;

(2) $\Delta(\lambda) \in \mathbb{C}[\lambda_1, \dots, \lambda_{2g-m}]$ and

$$B = \{\lambda \in \mathbb{C}^{2g-m} \mid \Delta(\lambda) \neq 0\};$$

(3) the holomorphic frame $\mathcal{L}$ becomes the $2g$-dimensional basis of $\mathbb{C}[\lambda_1, \dots, \lambda_{2g-m}]$-module of global sections of $(2g-m)$-dimensional bundle $\mathcal{T}B$.

Fix the notation for the frame

$$\mathcal{L} = (\ell_1, \dots, \ell_{2g})^t = T(\lambda)(\underbrace{0, \dots, 0}_{m}, \partial_{\lambda_1}, \dots, \partial_{\lambda_{2g-m}})^t$$

and its structure functions

$$[\ell_i, \ell_j] = \sum_{h=1}^{2g} c_{ij}^h(\lambda)\ell_h, \qquad c_{ij}^h(\lambda) \in \mathbb{C}[\lambda].$$

Gauß-Manin connection on the bundle of (n, s)-curves punctured at (∞).

The equation $f(x, y, \lambda) = 0$ in $\mathbb{C}^{2+2g-m}$ defines the family V of (n, s)-curves over $B = \mathbb{C}^{2g-m} \backslash \Sigma$.

Consider the bundle $\overset{\circ}{p} : \overset{\circ}{V} \longrightarrow B$ whose fiber is the curve

$$\overset{\circ}{V}_b = \{(x, y) \in \mathbb{C}^2 \mid f(x, y, b) = 0\}$$

with a puncture at infinity.

Let $H^1(\overset{\circ}{V}_b, \mathbb{C})$ be the linear $2g$-dimensional vector space of holomorphic 1-forms on $\overset{\circ}{V}_b$.

Consider associated with $\overset{\circ}{p} : \overset{\circ}{V} \longrightarrow B$ locally trivial vector bundle $\varpi : \Omega^1 \longrightarrow B$ whose fiber is $H^1(\overset{\circ}{V}_b, \mathbb{C})$.

A connection in Ω^1 is a Gauß-Manin connection on $\overset{\circ}{V}$.

Since (∞) belongs to all curves from V, we can construct a global section of Ω^1 by taking the classical basis of Abelian differentials of first and second kind.

Let $D(x, y, \lambda)$ be the vector

$$D(x, y, \lambda) = \big(D_1(x, y, \lambda), \ldots, D_{2g}(x, y, \lambda)\big)$$

of canonical basis 1-forms from $H^1(\overset{\circ}{V}_b, \mathbb{C})$.

Its matrix of periods Ω satisfies the Legendre relation*

$$\Omega^t J \Omega = 2\pi\imath J, \qquad \text{where} \quad J = \begin{pmatrix} 0_g & 1_g \\ -1_g & 0_g \end{pmatrix}.$$

Classic theory of Abelian differentials asserts that such basis exists and provides a means to construct it†.

*The particular case of Riemann-Hodge relations.

†H.F.Baker, Abelian Functions, CUP, 1997

The essential part of $D(x, y, \lambda)$, i.e., the part that provides the Legendre relation, is defined by the classic formula

$$D(x_1, y_1)^t J D(x_2, y_2) = \{\Phi_{1,2} - \Phi_{2,1}\} \mathrm{d}x_1 \mathrm{d}x_2,$$

$$\text{where} \quad \Phi_{1,2} = \frac{1}{f_y(x_1, y_1, \lambda)} \frac{\mathrm{d}}{\mathrm{d}x_2}\Big(\frac{f(x_1, y_2, \lambda)}{(x_1 - x_2)(y_1 - y_2)}\Big).$$

The calculation is carried out 'on the curve', i.e. with the assumption $f(x_i, y_i, \lambda) = 0$, $i = 1, 2$.

The Christoffel coefficient of the Gauß-Manin connection

$$\Gamma_j = (\Gamma_{j,i}^k), \quad i, j, k = 1, \ldots, 2g,$$

associated to the field ℓ_j is uniquely defined by the relation

The holomorphic vector-valued 1-form

$$\ell_j(D(x, y, \lambda)) + \Gamma_j D(x, y, \lambda)$$

is exact 'on the curve'.

The properties of our sigma-function.*

(a) $\sigma(u,\lambda)$ is an entire function of $u \in \mathbb{C}^g$ and $\lambda \in \mathbb{C}^d$.

(b) $\partial_{u_i,u_j} \log(\sigma(u,\lambda)) = -\wp_{ij}(u,\lambda) \in F$
whenever $\lambda \in B$, $i,j = 1,\dots,g$.

(c) $\sigma(u,\lambda)$ is a solution of the system

$$Q_j \sigma(u,\lambda) = 0, \quad j = 1,\dots,2g.$$

The operators have the form $Q_j = \ell_j - \frac{1}{2}H_j - \delta_j(\lambda)$, with $\ell_j \in \mathcal{L}$ and

$$H_j = \alpha_j^{kl}(\lambda)\partial_{u_k}\partial_{u_l} + 2\beta_{jk}^{l}(\lambda)u_k\partial_{u_l} + \gamma_{jkl}(\lambda)u_k u_l,$$
$$\delta_j(\lambda) = \frac{1}{8}\ell_j(\log\Delta(\lambda)) + \frac{1}{2}\beta_{jk}^{k}(\lambda),$$

where the summation from 1 to g extends over the repeated indices.

The coefficients $\alpha_j^{kl}(\lambda) = \alpha_j^{lk}(\lambda)$, $\beta_{jk}^{l}(\lambda)$ and $\gamma_{jkl}(\lambda) = \gamma_{jlk}(\lambda)$ are polynomials of λ.

*Funct. Anal. Appl. 38 (2004), no. 2, 88–101.

The annihilators Q_j of the sigma-function and a quantum oscillator.

Write the system of equations

$$Q_j\sigma(u,\lambda) = 0, \qquad j = 1,\dots,2g.$$

in the form of the Schrödinger equations

$$\ell_j(\sigma) = \left\{\frac{1}{2}H_j + \delta_j(\lambda)\right\}\sigma,$$

of a multidimensional quantum harmonic oscillator with multiple 'times'.

The formalism of quantum oscillator:

H_j is a set of 'quadratic Hamiltonians',

ℓ_j are derivatives over 'times',

δ_j is 'the energy of an oscillator mode'.

The realization of sigma-function in the form of an average of the 'ground state wave-function' (a multi-dimensional Gaussian function) over a lattice* suggests a natural interpretation of sigma-function as the 'wave-function of the coherent state' of the oscillator.

*see MIMS EPrint: 2005.50.

The Gauß-Manin connection and the annihilators Q_j of the sigma-function.

An operator Q_j is defined if we know the (polynomial in λ) matrices

$$\alpha_j = (\alpha_j^{kl}), \quad \beta_j = (\beta_{jk}^l), \quad \text{and} \quad \gamma_j = (\gamma_{jkl}).$$

Let

$$A_j(\lambda) = J \begin{pmatrix} \alpha_j & (\beta_j)^t \\ \beta_j & \gamma_j \end{pmatrix}.$$

Theorem. *The set of matrices $\{\alpha_j, \beta_j, \gamma_j\}$ defines the operators Q_j, $j = 1, \ldots, 2g$, such that*

(1) $$[Q_i, Q_j] = \sum_{h=1}^{2g} c_{ij}^h(\lambda) Q_h$$

(2) $$Q_j(\sigma(u, \lambda)) = 0$$

if and only if $\boxed{A_j = \Gamma_j.}$

Observe, that our operators Q_j and vector fields ℓ_j obey the same commutation relations.

A lifting process

Define a map t that takes the operators Q_j to vector fields by the formula

$$\boxed{t(Q_j) = \ell_i - (\alpha_j^{kl}\zeta_k(u,\lambda) + \beta_{jk}^l u_k)\partial_{u_l},}$$

where $\zeta_i(u,\lambda) = \partial_{u_i} \log \sigma(u,\lambda)$.

Lemma. *Let $f \in F$ and $Q(\sigma) = 0$, then $t(Q)(f) \in F$.*

Define a lift of the basis fields $\mathcal{L}$ to horizontal sections of $\mathcal{T}U$ by the formula

$$\boxed{p^*(\ell_i) = t(Q_i).}$$

Summary: take a field ℓ on B, construct the heat operator $Q = \ell - \frac{1}{2}H - \delta$ associated to ℓ, and then apply t to Q obtain the lift $p^*(\ell) = t(Q)$.

Solution of the main problem: a basis of $\mathrm{Der}(F)$.

Theorem. *The lifting process gives the following basis of the F-module $\mathrm{Der}(F)$:*

$$\mathcal{F} = (\partial_{u_1}, \dots, \partial_{u_g}, p^*(\ell_1), \dots, p^*(\ell_{2g})).$$

The coordinate frame $\mathcal{F}$ is subject to the relations

$$[\partial_{u_q}, \partial_{u_r}] = 0,$$

$$[p^*(\ell_i), \partial_{u_q}] = -(\alpha_i^{kl}\wp_{lq}(u,\lambda) - \beta_{iq}^k)\partial_{u_k},$$

$$[p^*(\ell_i), p^*(\ell_j)] = p^*([\ell_i, \ell_j]) + \\ + \frac{1}{2}(\alpha_i^{kl}\alpha_j^{qr} - \alpha_j^{kl}\alpha_i^{qr})\wp_{klq}(u,\lambda)\partial_{u_r},$$

where $i, j = 1, \dots, 2g$, $k, l, q, r = 1, \dots, g$.

The frame $\mathcal{F}$ has zero curvature and nontrivial torsion. Introduce the second order linear operators X_i, $i = 1, \dots, 2g$,

$$X_i(\,\cdot\,) = \frac{1}{2}\alpha_i^{kl}[[\,\cdot\,, \partial_{u_k}], \partial_{u_l}]; \qquad [X_i, X_j] = 0.$$

Now, we have the torsion formula

$$\boxed{[p^*(\ell_i), p^*(\ell_j)] - p^*([\ell_i, \ell_j]) = X_i(p^*(\ell_j)) - X_j(p^*(\ell_i)).}$$

Action of $p^*(\ell_i)$ and ∂_{u_k} on F.

$$\partial_{u_k}(\wp_{qr}) = \wp_{kqr},$$

$$\partial_{u_k}(\lambda_j) = 0,$$

$$p^*(\ell_i)(\wp_{qr}) = \frac{1}{2}\alpha_i^{kl}\Big(\wp_{klqr} - 2\wp_{kq}\wp_{lr}\Big) + \\ + \beta_{iq}^{k}\wp_{kr} + \beta_{ir}^{k}\wp_{kq} - \gamma_{iqr},$$

$$p^*(\ell_i)(\lambda_j) = T_i^j(\lambda).$$

The singular set of $p^*(\ell_i)$.

$$p^*(\ell_i)\Delta(\lambda) = \ell_i\Delta(\lambda) = \phi(\lambda)\Delta(\lambda),$$

$$p^*(\ell_i)\sigma(u,\lambda)^k = \psi(u,\lambda)\sigma(u,\lambda)^{k-2},$$

where $\phi(\lambda) \in \mathbb{C}[\lambda]$ and $\psi(u,\lambda) \in \mathbb{C}[\lambda][[u]]$

The coefficients of vector fields $p^*(\ell_i)$ become singular at the points where $\sigma(u,\lambda)$ vanishes.

A solution of the accompanying problem.

Consider the operators $Q_i^{(\sigma)}$, $i = 1, \ldots, 2g$,

$$Q_i^{(\sigma)}(f) = \sigma(u,\lambda)^{-1} Q_i(\sigma(u,\lambda) f(u,\lambda)),$$

where $f(u,\lambda)$ is a differentiable function.

Observe, that the map $(\,\cdot\,)^{(\sigma)} : Q_i \mapsto Q_i^{(\sigma)}$ preserves the bracket,

$$[Q_i^{(\sigma)}, Q_j^{(\sigma)}] = ([Q_i, Q_j])^{(\sigma)} = c_{ij}^h Q_h^{(\sigma)},$$

and that $Q_i^{(\sigma)}(1) = 0$.

Since $Q_i(\sigma) = 0$, we have

$$\boxed{Q_i^{(\sigma)}(f) = p^*(\ell_i)(f) - X_i(f),}$$

thus, if $f \in F$, then $Q_i^{(\sigma)}(f) \in F$.

Theorem. *The operators $Q_i^{(\sigma)}$, $i = 1, \ldots, 2g$, define a flat Koszul connection in the complex vector fiber bundle $\pi : E \longrightarrow B$ by the formula*

$$\boxed{\nabla_{\ell_j}(f) = Q_j^{(\sigma)}(f),}$$

where $f \in F = \Gamma E$.

Torsion of the Koszul connection.

Let $Y, Y' \in \mathrm{Der}(F)$ and denote by p the projection $p : \mathrm{Der}(F) \to \mathcal{T}B$.

The torsion $T_\nabla : \mathrm{Der}(F) \times \mathrm{Der}(F) \to \mathrm{Der}(F)$ is defined by

$$T_\nabla(Y, Y') = \nabla_{p(Y)}(Y') - \nabla_{p(Y')}(Y) - [Y, Y'].$$

Explicitly,

$$\begin{aligned}
T_\nabla(\partial_{u_q}, \partial_{u_r}) &= 0, \\
T_\nabla(p^*(\ell_i), \partial_{u_q}) &= -[p^*(\ell_i), \partial_{u_q}] \\
&= (\alpha_i^{kl} \wp_{lq}(u, \lambda) - \beta_{iq}^k)\partial_{u_k}, \\
T_\nabla(p^*(\ell_i), p^*(\ell_j)) &= Q_i(p^*(\ell_j)) - Q_j(p^*(\ell_i)) - \\
&\quad - [p^*(\ell_i), p^*(\ell_j)] \\
&= -(X_i(p^*(\ell_j)) - X_j(p^*(\ell_i)))
\end{aligned}$$

and by our calculation

$$= p^*([\ell_i, \ell_j]) - [p^*(\ell_i), p^*(\ell_j)].$$

g=2. The basis $\{\ell_i\}$.

The symmetric matrix T, which transforms the standard fields $\partial_{\lambda_4}, \partial_{\lambda_6}, \partial_{\lambda_8}, \partial_{\lambda_{10}}$ to the basis fields $\ell_0, \ell_2, \ell_4, \ell_6$,

$$T = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ * & \frac{40\lambda_8 - 12\lambda_4^2}{5} & \frac{50\lambda_{10} - 8\lambda_4\lambda_6}{5} & -\frac{4\lambda_4\lambda_8}{5} \\ * & * & \frac{20\lambda_4\lambda_8 - 12\lambda_6^2}{5} & \frac{30\lambda_4\lambda_{10} - 6\lambda_6\lambda_8}{5} \\ * & * & * & \frac{4\lambda_6\lambda_{10} - 8\lambda_8^2}{5} \end{pmatrix}$$

$$[\ell_0, \ell_k] = k\ell_k, \quad k = 2, 4, 6;$$
$$[\ell_2, \ell_4] = 2\ell_6 - \frac{8}{5}\lambda_4\ell_2 + \frac{8}{5}\lambda_6\ell_0;$$
$$[\ell_2, \ell_6] = -\frac{4}{5}\lambda_4\ell_4 + \frac{4}{5}\lambda_8\ell_0;$$
$$[\ell_4, \ell_6] = 2\lambda_4\ell_6 - \frac{6}{5}\lambda_6\ell_4 + \frac{6}{5}\lambda_8\ell_2 - 2\lambda_{10}\ell_0;$$

$g=2$. The operators $\{H_i\}$.

$$H_0 = \underline{u_1\partial_{u_1} + 3u_3\partial_{u_3}} - 3$$

$$10H_2 = \underline{5\partial_{u_1}^2 + 10u_1\partial_{u_3} - 8\lambda_4 u_3\partial_{u_1}} - 3\lambda_4 u_1^2 + (15\lambda_8 - 4\lambda_4^2)u_3^2$$

$$5H_4 = \underline{5\partial_{u_1}\partial_{u_3} + 5\lambda_4 u_3\partial_{u_3} - 6\lambda_6 u_3\partial_{u_1}} - 5\lambda_4 - \lambda_6 u_1^2 + 5\lambda_8 u_1 u_3 + 3(5\lambda_{10} - \lambda_4\lambda_6)u_3^2$$

$$10H_6 = \underline{5\partial_{u_3}^2 - 6\lambda_8 u_3\partial_{u_1}} - 5\lambda_6 - \lambda_8 u_1^2 + 20\lambda_{10} u_1 u_3 - 3\lambda_4\lambda_8 u_3^2$$

g=2. The operators $\{t(Q_i)\}$.

$$t(Q_0) = \ell_0 - u_1\partial_{u_1} - 3u_3\partial_{u_3},$$
$$t(Q_2) = \ell_2 - \zeta_1\partial_{u_1} - u_1\partial_{u_3} - \frac{4}{5}\lambda_4 u_3\partial_{u_1}$$
$$t(Q_4) = \ell_4 - \zeta_3\partial_{u_1} - \zeta_1\partial_{u_3} + \lambda_4 u_3\partial_{u_3} - \frac{6}{5}\lambda_6 u_3\partial_{u_1}$$
$$t(Q_6) = \ell_6 - \zeta_3\partial_{u_3} - \frac{3}{5}\lambda_8 u_3\partial_{u_1}.$$

g=2. The frame $\mathcal{F}$ structure.

Notation: $L_i = p^*(\ell_i) = t(Q_i)$.

$$[L_2, L_4] = p^*[\ell_2, \ell_4] + \frac{1}{2}(\wp_{1,1,3}\partial_{u_1} - \wp_{1,1,1}\partial_{u_3}),$$
$$[L_2, L_6] = p^*[\ell_2, \ell_6] + \frac{1}{2}(\wp_{1,3,3}\partial_{u_1} - \wp_{1,1,3}\partial_{u_3}),$$
$$[L_4, L_6] = p^*[\ell_4, \ell_6] + \frac{1}{2}(\wp_{3,3,3}\partial_{u_1} - \wp_{1,3,3}\partial_{u_3}).$$

g=2. Linearization.

Let $\lambda_i = \mu_i \xi$, and pass to the limit as $\xi \to 0$.

$$\sigma(u,0) = u_3 - \frac{1}{3}u_1^3.$$

Change the variables: $u_1 = \xi_1 + \xi_2$, $u_3 = (\xi_1^3 + \xi_2^3)/3$, then $\sigma \longrightarrow -\xi_1\xi_2(\xi_1+\xi_2)$.

$\lim\limits_{\xi\to 0} L_i = M_i$, where

$$\begin{aligned} M_0 &= 4\mu_4\partial_{\mu_4} + 6\mu_6\partial_{\mu_6} + 8\mu_8\partial_{\mu_8} + 10\mu_{10}\partial_{\mu_{10}} \\ &\quad - \xi_1\partial_{\xi_1} - \xi_2\partial_{\xi_2}, \end{aligned}$$

$$\begin{aligned} M_2 &= 6\mu_6\partial_{\mu_4} + 8\mu_8\partial_{\mu_6} + 10\mu_{10}\partial_{\mu_8} \\ &\quad - \xi_1^{-1}\partial_{\xi_1} - \xi_2^{-1}\partial_{\xi_2}, \end{aligned}$$

$$\begin{aligned} M_4 &= 8\mu_8\partial_{\mu_4} + 10\mu_{10}\partial_{\mu_6} + \\ &\qquad + \frac{1}{2}\varphi\Big(\frac{\xi_2^2\partial_{\xi_1} - \xi_1^2\partial_{\xi_2}}{\xi_1 - \xi_2} + (\xi_1+\xi_2)^2\frac{\partial_{\xi_1} - \partial_{\xi_2}}{\xi_1 - \xi_2}\Big), \end{aligned}$$

$$M_6 = 10\mu_{10}\partial_{\mu_4} + \varphi\frac{\partial_{\xi_1} - \partial_{\xi_2}}{\xi_1 - \xi_2},$$

and $\quad \varphi = \dfrac{1}{\xi_1\xi_2(\xi_1+\xi_2)^2}.$

A polynomial Lie algebra structure

$$[L_i, d_q] = -(\alpha_i^{kl} x_{lq} - \beta_{iq}^k) d_k,$$

$$[L_i, L_j] = c_{ij}^h L_h + \frac{1}{2}(\alpha_i^{kl}\alpha_j^{qr} - \alpha_j^{kl}\alpha_i^{qr}) x_{klq} d_r,$$

$$[L_i, x_{qr}] = \frac{1}{2}\alpha_i^{kl}\Big(x_{klqr} - 2x_{kq}x_{lr}\Big) + \\ + \beta_{iq}^k x_{kr} + \beta_{ir}^k x_{kq} - \gamma_{iqr},$$

$$[L_i, \lambda_a] = T_i^a, \quad [d_k, x_{lq\ldots}] = x_{klq\ldots},$$

$$[d_q, d_r] = [d_r, \lambda_a] = [\lambda_a, \lambda_e] = \\ = [\lambda_e, x_{lq\ldots}] = [x_{lq\ldots}, x_{kr\ldots}] = 0,$$

where $T_i^j, \alpha_j^{kl}, \beta_{jk}^l, \gamma_{jkl}, c_{ij}^h \in \mathbb{C}[\lambda]$,

$$a, e = 1, \ldots, 2g - m,$$
$$h, i, j = 1, \ldots, 2g,$$
$$k, l, q, r = 1, \ldots, g.$$