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MODEL-UPDATING FOR SYMMETRIC QUADRATIC EIGENVALUE PROBLEMS

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Abstract

This paper concerns quadratic matrix functions of the form $L(\lambda) = M\lambda^2 + D\lambda + K$ where M, D, K are real and symmetric $n \times n$ matrices with $M > 0$. Given complete spectral information on $L(\lambda)$, it is shown how new systems of the same type can be generated with updated eigenvalues and/or eigenvectors. A general purpose algorithm is formulated and illustrated with problems having no real eigenvalues, or a mixture of real and non-real eiegnvalues, or only real eigenvalues. The methods also apply for matrix polynomials of higher degree.

Key Words: Vibrating systems, updating, pole placement.
AMS subject classifications: Primary 74A15; Secondary 15A29.
Abbreviated title: Model Updating.

1 Introduction

This paper concerns quadratic matrix functions of the form

$$L(\lambda) = M\lambda^2 + D\lambda + K, \tag{1}$$

where M, D, K are real and symmetric $n \times n$ matrices. It will be assumed throughout that M is positive definite (written $M > 0$). Positivity properties

for D and K occur in many problem areas, but are not necessary for the algorithms developed here. Matrix functions of this kind appear frequently in problems of classical mechanics where M, D, K are known as the mass, damping, and stiffness matrices, respectively.

The set all of eigenvalues of $L(\lambda)$ (zeros of the determinant, $\det L(\lambda)$) form the *spectrum*, $\sigma(L)$, of $L(\lambda)$, and are of great physical interest. There are $2n$ eigenvalues (counting algebraic multiplicities) and their location in the complex plane (necessarily symmetric about the real line) determines vital physical properties of any underlying system.

The model-updating problem considered here is briefly as follows: Given complete spectral information on $L(\lambda)$, suppose that the locations of some (or all) eigenvalues are seen to be unfavourable. What changes in M, D, K will produce a favourable re-location of the spectrum? Frequently, "unfavourable" eigenvalue distributions concern either clustered eigenvalues, or eigenvalues close to the imaginary axis, and adjustments are to be made to moderate such properties. However, there is no hypothesis in this work requiring that updates be "small" in any sense. When posed in the context of the "transfer function" $L(\lambda)^{-1}$, the problem could also be described as that of "pole placement".

Problems of this kind have been considered by several authors (see [1], [2], [4], [7], et al.). In particular, interesting solutions are proposed in the first of these papers and, as in this work, solutions are sought by Carvalho et al. in [1] which *maintain the symmetry of the coefficient matrices after disturbance*. The techniques proposed here are different, and more general in the sense that the condition $K > 0$ is not imposed. We take advantage of the detailed study of inverse problems begun in [5] and recently studied more closely in [9] and [8], for example.

In the theory developed in these references, the notion of a *self-adjoint* triple of spectral data plays a vital role. To the authors' knowledge this is the first time that an algorithmic approach has been taken to the determination of these self-adjoint triples. They are, of course, a vital part of the updating strategy proposed. We also remark that the theory and techniques developed here apply immediately to polynomial functions $L(\lambda)$ of higher degree - under much the same conditions.

There is high computational expense in this methodology resulting from the calculation of complete spectral data for the unperturbed problem, but we take it that, in practice, this information is likely to be already known. Indeed, much of this information is necessary before sensible "updates" can

be formulated.

The method developed here is illustrated on small artificial problems, but it is clear that accessible software can be utilized. There is no explicit restriction on the size of $L(\lambda)$, but stability problems are to be expected as the size increases, as this will generally imply clustering of eigenvalues.

2 Spectral data

The complexity of our problem is significantly increased if $L(\lambda)$ has **both** real eigenvalues and eigenvalues in complex-conjugate pairs. We will formulate the spectral data assuming that both are present, but we make the simplifying (generic) hypothesis that all eigenvalues are distinct. Extension of the process to admit multiple semisimple eigenvalues is not expected to be difficult, but the inclusion of *defective* eigenvalues would be more formidable.

So suppose that there are $2r$ real eigenvalues ($0 \leq r \leq n$). When $r < n$ the non-real eigenvalues in the upper half of the complex plane are determined by a complex diagonal matrix $J_c = U_1 + iW$ of size $(n-r) \times (n-r)$ with $W > 0$. The complex conjugate eigenvalues make up the diagonal entries of $\bar{J}_c$. Then there are $2r$ real eigenvalues which are distributed between the diagonal entries of two $r \times r$ real diagonal matrices U_2 and U_3. The way in which these two matrices are formed will be discussed in what follows.

A complex (canonical) diagonal $2n \times 2n$ matrix including all the eigenvalues is now

$$J = \begin{bmatrix} J_c & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & \bar{J}_c \end{bmatrix} = \begin{bmatrix} U_1 + iW & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & U_1 - iW \end{bmatrix}. \tag{2}$$

A right eigenvector (say $x_j \neq 0$) can be associated with each diagonal entry of J (each eigenvalue), and these form the columns of an associated $n \times 2n$ matrix of eigenvectors, say X. Here, with our hypotheses on the spectrum, we may define an $n \times 2n$ matrix of eigenvectors of $L(\lambda)$ in the form

$$X = \begin{bmatrix} X_c & X_{R1} & X_{R2} & \overline{X_c} \end{bmatrix}, \tag{3}$$

where X_c is an $n \times (n-r)$ matrix of (generally) non-real eigenvectors corresponding to the eigenvalues of J_c, matrices X_{R1} and X_{R2} are $n \times r$ real

matrices of eigenvectors corresponding to the real eigenvalues in U_2 and U_3, respectively. Note that the structure of X is consistent with that of J in (2).

The two matrices (X, J) form a *Jordan pair* of matrices for $L(\lambda)$ and necessarily satisfy the condition

$$\det \begin{bmatrix} X \\ XJ \end{bmatrix} \neq 0. \tag{4}$$

Canonical structures of the function $L(\lambda)$ require the definition of a second matrix. With X, J formulated as above, this matrix takes the form

$$P = \begin{bmatrix} 0 & 0 & 0 & I_{n-r} \\ 0 & I_r & 0 & 0 \\ 0 & 0 & -I_r & 0 \\ I_{n-r} & 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

and we observe that $P^* = P$ and $(PJ)^* = PJ$. This imposes a constraint on the distribution of the $r$eal eigenvalues between U_2 and U_3. The eigenvalues in U_2, U_3 have *positive type* and *negative type*, respectively. If it known *a priori* that there are no real eigenvalues, then corresponding blocks of J, X, and P simply do not appear. A bi-product of our algorithms is the determination of the types of the real eigenvalues - if any. (See [5], or Chapter 12 of [6] for the theory, and [8] for an expository discussion.)

It is shown in [8] that if M, D, K are real and symmetric and $M > 0$, then X can be defined so that the two conditions

$$XPX^* = 0, \qquad X(JP)X^* = M^{-1} > 0 \tag{6}$$

hold. (Then (X, J, PX^*) is known as a *self-adjoint* triple.)

The *moments* of the system are then the hermitian matrices

$$\Gamma_j = X(J^j P)X^*, \tag{7}$$

for all integers j for which J^j is defined. Furthermore, when conditions (6) hold the coefficents of $L(\lambda)$ can be determined recursively in terms of the moments (and hence X, J, P):

$$M = \Gamma_1^{-1}, \quad D = -M\Gamma_2 M, \quad K = -M\Gamma_3 M + D\Gamma_1 D. \tag{8}$$

3 Model updating

We assume that in model updating the number of real and non-real eigenvalues is invariant. Then the facts above suggest the following

Strategy for model updating

- Given a system with real symmetric coefficients M, D, K and $M > 0$, compute the matrices X, P, J above.
- Make the updates in X and J to produce $\hat{X}$, $\hat{J}$ in such a way that:
  (a) the canonical matrix P will not be disturbed, and
  (b) conditions (6) are maintained (i.e. $\hat{X}P\hat{X}^* = 0$, and $\hat{X}(\hat{J}P)\hat{X}^* > 0$).
- Compute the moments defined by $\hat{X}$, P, $\hat{J}$, and hence new coefficients $\hat{M}$, $\hat{D}$, $\hat{K}$ (as in (7) and (8)).

Clearly, one can choose to perturb the eigenvalues alone, or the eigenvectors alone, or both together.

A difficulty in this program may arise in ensuring that, for the perturbed system, the condition $XPX^* = 0$ is maintained after updating. However, *this problem does not arise if (as is often the case) the eigenvector matrix, X, is not to be changed ($\hat{X} = X$), and adjustments are made to the eigenvalues only.* Also, as it will be shown below, this problem can be resolved in another way if it is known that there are no real eigenvalues.

Notice also that, by continuity, the positivity condition $X(JP)X^* > 0$ will be preserved provided the perturbations are small enough. However, even if the positivity condition is not satisfied but $X(JP)X^*$ is nonsingular, then an updated system can be generated. In this case $\hat{M}$ will not be positive definite.

The major task is now to compute the triple X, P, J for the *unperturbed* system (and hence a *self-adjoint triple*, X, J, PX^*).

4 Computing a selfadjoint triple

Given our hypothesis that all eigenvalues are simple, the determination of a self-adjoint triple reduces to finding an appropriate "normalisation" of eigenvectors. This normalisation can be formulated in terms of the first order *real,*

symmetric pencil forming a *linearization* of $L(\lambda)$, namely $A\lambda - B$, where

$$A := \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \tag{9}$$

and observe that A^{-1} exists.

A characteristic property of a self-adjoint triple is the fact that X, J can be used to simultaneously reduce A and B to canonical form as follows (when all eigenvalues are distinct [1]):

$$\begin{bmatrix} X \\ XJ \end{bmatrix}^* A \begin{bmatrix} X \\ XJ \end{bmatrix} = P, \qquad \begin{bmatrix} X \\ XJ \end{bmatrix}^* B \begin{bmatrix} X \\ XJ \end{bmatrix} = PJ. \tag{10}$$

(These conditions are equivalent to those of equations (6)-(8)). After a little manipulation, it follows that, also,

$$A^{-1}B \begin{bmatrix} X \\ XJ \end{bmatrix} = J \begin{bmatrix} X \\ XJ \end{bmatrix}. \tag{11}$$

On the other hand, suppose we are given *any* set of $2n$ eigenvectors, one associated with each eigenvalue and arranged as in (3) to form an $n \times 2n$ matrix V_0. Then the (generally complex) matrix $V := \begin{bmatrix} V_0 \\ V_0 J \end{bmatrix}$, with diagonal J as in (2), is necessarily nonsingular ($(V,\ J)$ is a Jordan pair). Indeed, we will proceed on the assumption that V, J are generated by applying some general purpous eigenvalue/eigenvector algorithm to the pencil $\lambda A - B$. Notice that we can identify the computed eigenvalues and associated eigenvectors immediately (some real and some in conjugate pairs), but this does not give immediate access to the sign characteristic. Consequently, we cannot write down matrix P immediately.

We proceed with the retrieval of the sign characteristic and then to the required normalization of the eigenvectors.

Proposition 1 *If all eigenvalues are distinct, the matrices $V^T AV$ and $V^T BV$ are (generally complex) diagonal.*

(Note the curious entry of transposition rather than transposition with conjugation; cf. the Appendix of [9].)

[1] cf. Theorem 6.1 of [10], for example.

Proof: We have $(A^{-1}B)V = VJ$, whence $AVJ = BV$ and $(V^TAV)J = (V^TBV)$. Since V^TAV and V^TBV are symmetric we also have $J(V^TAV) = V^TBV$. Thus, $(V^TAV)J = J(V^TAV)$ and the result follows easily from this. □

Let $S^2 = V^TAV$, a diagonal matrix. It can be assumed that the entries of J and the columns of V_0 are arranged as in (2) and (3). Notice that for a *real* eigenvalue λ_j the corresponding diagonal entry of S^2 is

$$\left[\begin{array}{cc} v_j^T & \lambda_j v_j^T \end{array}\right] \left[\begin{array}{cc} D & M \\ M & 0 \end{array}\right] \left[\begin{array}{c} v_j \\ \lambda_j v_j \end{array}\right] = v_j^T(2\lambda_j M + D)v_j, \tag{12}$$

and the sign associated with λ_j (in the sign characteristic) is just the sign of $v_j^T(2\lambda_j M + D)v_j$). (See Theorem 12.4.2 of [6], for example). Thus the signs of the real diagonal entries of V^TAV determine the signs ϵ_j $(= \pm 1)$ associated with the real eigenvalues. This admits the construction of the canonical matrix P and the corresponding order of the real eigenvalues in (2). Also, we may write

$$S^2 = \mathrm{diag}\left[\begin{array}{cccc} S_1^2, & S_2^2, & -S_3^2, & (\bar{S}_1)^2 \end{array}\right],$$

where $S_2^2 > 0,\ S_3^2 > 0$ and S_1 has non-real diagonal entries.

Now we go on to the definition of X. Let S be a diagonal square root of S^2 of the form

$$\mathrm{diag}\left[\begin{array}{cccc} S_1 & S_2 & iS_3 & \bar{S}_1 \end{array}\right],$$

and let V_0 be partitioned accordingly: $V_0 = \left[\begin{array}{cccc} V_c & V_{R1} & V_{R2} & \bar{V}_c \end{array}\right]$ where V_{R1} and V_{R2} are real.

Proposition 2 *If $X := V_0S^{-1}$ then (X, J, PX^*) is a self-adjoint Jordan triple.*

Proof: Observe first that, if $Q := |P|$ (the permutation matrix whose entries are the absolute values of those of P), then

$$V_0^* = QV_0T \quad \text{and} \quad J^*Q = QJ.$$

Thus,

$$\begin{aligned} V^*AV &= \left[\begin{array}{cc} V_0^* & J^*V_0^* \end{array}\right] A \left[\begin{array}{c} V_0 \\ V_0J \end{array}\right] = \left[\begin{array}{cc} QV_0^T & J^*QV_0^T] \end{array}\right] A \left[\begin{array}{c} V_0 \\ V_0J \end{array}\right] \\ &= \left[\begin{array}{cc} QV_0^T & QJV_0^T] \end{array}\right] A \left[\begin{array}{c} V_0 \\ V_0J \end{array}\right] \\ &= Q(V^TAV) = QS^2. \end{aligned}$$

Now $\begin{bmatrix} X \\ XJ \end{bmatrix} = \begin{bmatrix} V_0 S^{-1} \\ V_0 S^{-1} J \end{bmatrix} = \begin{bmatrix} V_0 \\ V_0 J \end{bmatrix} S^{-1} = V S^{-1}$, and so

$$\begin{bmatrix} X \\ XJ \end{bmatrix}^* A \begin{bmatrix} X \\ XJ \end{bmatrix} = S^{-*}(V^*AV)S^{-1} = S^{-*}Q(V^TAV)S^{-1}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \bar{S}_1^{-1} \\ 0 & S_2^{-1} & 0 & 0 \\ 0 & 0 & iS_3^{-1} & 0 \\ S_1^{-1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_1 & 0 & 0 & 0 \\ 0 & S_2 & 0 & 0 \\ 0 & 0 & iS_3 & 0 \\ 0 & 0 & 0 & \bar{S}_1 \end{bmatrix} = P.$$

Thus the first of equations (10) holds, and it can be shown in a similar way that the second is also true. □

The construction of a self-adjoint Jordan triple is now complete. The computational steps required to implement the analysis are summarised in the Appendix to this paper.

As some technicalities disappear when there are no real eigenvalues, and because this is a case of practical interest, it will be considered first in the next section. The more general case of mixed, real and non-real eigenvalues will be discussed in Section 6. A simple example of a (hyperbolic) system with all real eigenvalues appears in Section 7.

5 Elliptic systems

Consider the case in which all eigenvalues appear in non-real conjugate pairs (both before and after updating). Such systems are said to be *elliptic*. The first observation is that equations (2) and (3) take the more simple form

$$J = J_c \oplus \overline{J_c}, \qquad X = \begin{bmatrix} X_c & \overline{X_c} \end{bmatrix}.$$

The computational steps required are summarised in the Appendix to this paper, and will simply be illustrated here by working through a simple example.

Example 1: (This example is taken from [3].) The oscillations of a mass-spring system lead to analysis of the 3×3 function $L(\lambda) = M\lambda^2 + D\lambda + K$ where

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 6 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 10 \end{bmatrix}.$$

Apply the "matlab" instruction of Step 2 of the algorithm in the Appendix (or equivalent), and complete Step 3 to find V and J_c. It is found that (with truncated numbers) the eigenvalues are the diagonal elements of J with

$$J_c = \begin{bmatrix} -0.0826 + i(1.4502) & 0 & 0 \\ 0 & -0.7530 + i(0.8576) & 0 \\ 0 & 0 & -0.5144 + i(1.2469) \end{bmatrix}.$$

We observe that the problem is, indeed, elliptic.

Complete Steps 4, 5, and 6 of the algorithm resulting in $P = \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix}$ and evaluation of the matrix Z. The normalised eigenvectors are the columns of the 3×6 matrix X making up the first three rows of Z, and X_c is the first 3×3 partition of X:

$$X_c = \begin{bmatrix} 0.3775 - 0.4667i & 0.2440 - 0.0283i & 0.1447 - 0.0656i \\ -0.1482 - 0.0455i & 0.4103 - 0.3670i & 0.0186 - 0.2322i \\ -0.0290 - 0.0066i & -0.0343 + 0.1257i & 0.2083 - 0.1720i \end{bmatrix}.$$

This completes the preparatory steps. (It can now be checked that $\Gamma_0 = 0$ and $\Gamma_1 = M^{-1}$, for example.)

Suppose now that the first eigenvalue, $-0.0826 + i(1.4502)$, is thought to be too close to the imaginary axis. We generate an updated system with this eigenvalue replaced by $-\mu + i(1.4502)$ where $\mu > 0.0826$ and with the eigenvectors unchanged. So the new matrix $\hat{J}$ is formed from

$$\hat{J}_c = \begin{bmatrix} -\mu + i(1.4502) & 0 & 0 \\ 0 & -0.7530 + i(0.8576) & 0 \\ 0 & 0 & -0.5144 + i(1.2469) \end{bmatrix},$$

and X_c, X are unchanged. It can then be verified that $X(P\hat{J})X^* > 0$. We give the results below rounded to two decimal places for convenience.

When the critical eigenvalue is $-0.1826 + i(1.4502)$ the coefficient matrices become:

$$M = \begin{bmatrix} 0.99 & -0.03 & -0.01 \\ & 2.02 & 0.01 \\ & & 5.00 \end{bmatrix}, \quad D = \begin{bmatrix} 0.20 & -0.10 & -0.03 \\ & 3.02 & -1.00 \\ & & 6.00 \end{bmatrix}, \quad K = \begin{bmatrix} 2.03 & -0.98 & 0.03 \\ & 2.96 & -0.02 \\ & & 9.99 \end{bmatrix}.$$

When the critical eigenvalue is $-0.2826 + i(1.4502)$ the coefficient matrices become:

$$M = \begin{bmatrix} 0.97 & -0.06 & -0.03 \\ & 2.04 & 0.02 \\ & & 5.01 \end{bmatrix}, \quad D = \begin{bmatrix} 0.41 & -0.21 & -0.07 \\ & 3.04 & -1.00 \\ & & 5.99 \end{bmatrix}, \quad K = \begin{bmatrix} 2.07 & -0.97 & 0.05 \\ & 2.92 & -0.04 \\ & & 9.99 \end{bmatrix}.$$

These results suggest that the update can be achieved in large measure by adjustments to the damping matrix, D, only. □

In this example there has been no perturbation of the eigenvectors; i.e. the columns of X, or X_c. As noted above, if X is perturbed there seems to be a difficulty with maintaining the condition $X^*PX = 0$. However, Prells has shown in [12] (see also [9]) how this can be controlled. Thus, the normalisations required in this theory imply that (for an elliptic system) the real and imaginary parts of X_c are strongly connected. Thus, if we write the real and imaginary parts, $X_c = X_R + iX_I$ then, *necessarily*, there is a real orthogonal matrix Θ such that

$$X_c = X_R(I - i\Theta). \tag{13}$$

Indeed, when X is properly normalised, X_R is nonsingular and $X_R^{-1}X_I$ is an orthogonal matrix. Furthermore, the positive definite properties of the coefficients M, D, K depend on Θ, and not on X_R (see Theorem 9 of [9]).

However, for present purposes, the important observation is that, if X_c has the form (15), then the condition $X^*PX = 0$ is automatically satisfied. Furthermore, if updates are made in X_R only then, by not changing Θ, positivity properties of M, D, K will be preserved.

6 Mixed problems

Consider application of the algorithm of the Appendix to a "mixed" problem, i.e. with both real and non-real eigenvalues. Steps 1 and 2 of the algorithm in the Appendix can be completed as before.

For Step 3: Suppose that there are $2r$ real eigenvalues with $0 < r < n$. As mentioned above, it is known that there must be r eigenvalues of positive type and r of negative type, and this classification is required in constructing the canonical matrices J of (2) and P of equation (5). These signs are determined using equation (13). We illustrate with a simple example.

Example 2: Consider the problem with

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad K = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix},$$

and note that all three matrices are positive definite.

At Step 2 of the algorithm it is found that the problem is, indeed, mixed with (truncated) eigenvalues

$$-1.0656 \pm i(2.1742), \quad -0.4175, \quad -2.4513.$$

Using the criterion (13) it is found that, of the real eigenvalues, -0.4175 has positive type and -2.4513 has negative type. This admits definition of the permutation Q of Step 4, and hence the suitably organised matrices V, J, and P. In fact:

$$J = \begin{bmatrix} -1.0656 + i(2.1742) & 0 & 0 & 0 \\ 0 & -0.4175 & 0 & 0 \\ 0 & 0 & -2.4513 & 0 \\ 0 & 0 & 0 & -1.0656 - i(2.1742) \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

After completion of Steps 4, 5, and 6, the normalised eigenvectors are obtained from matrix X_E:

$$X = \begin{bmatrix} -0.3021 + i(0.3333) & 0.2967 & -i(0.2201) & -0.3021 - i(0.3333) \\ -0.1567 + i(0.0774) & -0.6438 & i(0.6721) & -0.1567 - i(0.0774) \end{bmatrix}.$$

Now we are in position to make updates. Consider the substantial shift of spectrum to

$$-4 \pm 4i, \quad -1, \quad -4,$$

and leave the eigenvectors invariant. Thus, J is to be modified and P and X are to be unchanged; it can be verified that, with the new J, the condition $X(PJ)X^* > 0$ still holds (ensuring positive definiteness of the new mass matrix).

Finally, apply formulae (7) and (8) to obtain the the coefficients of the updated system:

$$\hat{M} = \begin{bmatrix} 0.5334 & -0.118 \\ & 0.6959 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} 3.8308 & 0.8350 \\ & 4.0086 \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} 14.8328 & 6.1635 \\ & 5.7632 \end{bmatrix}.$$

7 A hyperbolic example

This illustration is based on Example 8 of [8]. There, an inverse problem is solved with specified eigenvalues and the coefficient matrices M, D, K are generated. The eigenvalues are -1, -2, -3, -4 all of positive type and -5, -6, -7, -8 all of negative type. Here we reverse the process and, starting with the (truncated) coefficients found in that example, we reconstruct the spectrum. Indeed, a selfadjoint triple is to be found (made up of matrices J, P, X) so that updates can readily be made.

We quote from [8]:

$$M = \begin{bmatrix} 0.1886 & 0.0269 & -0.0168 & -0.0051 \\ & 0.2896 & 0.0690 & 0.0707 \\ & & 0.2694 & 0.0808 \\ & & & 0.42342 \end{bmatrix}, \; D = \begin{bmatrix} 1.3771 & 0.0808 & -0.0673 & -0.0253 \\ & 2.1582 & 0.3451 & 0.4242 \\ & & 2.6162 & 0.5657 \\ & & & 4.3939 \end{bmatrix},$$

$$K = \begin{bmatrix} 1.1886 & 0.0539 & -0.0505 & -0.0202 \\ & 3.1582 & 0.4141 & 0.5657 \\ & & 5.4242 & 0.9697 \\ & & & 10.7879 \end{bmatrix},$$

After Steps 2 and 3 of the algorithm it is found that the (truncated) eigenvalues are:

$$-1.0001, \; -2.0002, \; -3.0002, \; -3.9872 \quad \text{of positive type,}$$

$$-5.0030, \; -6.0033, \; -7.0258, \; -7.9963 \quad \text{of negative type.}$$

The variation from integer values is to be expected given the truncated entries of M, D, and K. At Step 6 we obtain

$$X = \begin{bmatrix} 1.0001 & -0.0001 & -0.0001 & -0.0005 & -0.2012i & -0.8032i & 0.3959i & 0.3972i \\ 0 & 1.0002 & 0.0003 & -0.0025 & -0.8025i & -0.1932i & -0.4015i & -0.3972i \\ 0 & 0.0001 & -1.0001 & -0.0022 & -0.3999i & 0.4040i & -0.1871i & 0.8013i \\ 0 & 0 & -0.0001 & 0.9909 & -0.3880i & 0.3894i & 0.7983i & -0.2061i \end{bmatrix}.$$

The eigenvectors are not to be updated and the new spectrum is taken to be

$$-1 \quad -2 \quad -3 \quad -4 \quad \text{of positive type,}$$

$$-10 \quad -11 \quad -12 \quad -13 \quad \text{of negative type.}$$

The updated coefficients are found to be :

$$M = \begin{bmatrix} 0.0965 & 0.0065 & -0.0045 & -0.0009 \\ & 0.1152 & 0.0114 & 0.0084 \\ & & 0.1112 & 0.0105 \\ & & & 0.1358 \end{bmatrix}, \; D = \begin{bmatrix} 1.1928 & 0.195 & -0.0182 & -0.0041 \\ & 1.4607 & 0.0570 & 0.0526 \\ & & 1.6672 & 0.0752 \\ & & & 2.1046 \end{bmatrix},$$

$$K = \begin{bmatrix} 1.0963 & 0.0130 & -0.0137 & -0.0032 \\ & 2.4604 & 0.0684 & 0.0714 \\ & & 4.0004 & 0.1310 \\ & & & 6.2459 \end{bmatrix},$$

□

8 Confining perturbations to D and K.

Our analysis has assumed that all three coefficient matrices M, D, and K are accessible for the implementation of updating. Given our hypothesis that $M > 0$, it is possible to confine the perturbations to the damping and stiffness matrices, but at the expense of perturbing eigenvectors. This may be particularly useful if the precise nature of the eigenvectors is unimportant. The main advantage of such a strategy is likely to be that updates in D and K can be achieved by state and velocity feedback mechanisms (at least in theory).

Suppose that the updating process of $L(\lambda)$ has been completed and results in a system $\hat{L}(\lambda) = \hat{M}\lambda^2 + \hat{D}\lambda + \hat{K}$. Since $M > 0$ and $\hat{M} > 0$ both matrices have unique positive definite square-roots, $M^{\frac{1}{2}}$ and $\hat{M}^{\frac{1}{2}}$. It is clear that the system

$$M^{\frac{1}{2}}\hat{M}^{-\frac{1}{2}}\hat{L}(\lambda)\hat{M}^{-\frac{1}{2}}M^{\frac{1}{2}}$$

has the spectrum of $\hat{L}$ and the leading coefficient, M, of the original system, L. The cost of this maneouvre is to replace each eigenvector, x_j of $\hat{L}(\lambda)$ by the transformed vector $M^{-\frac{1}{2}}\hat{M}^{\frac{1}{2}}x_j$. This simplifies, of course, if the original system happened to be *monic*, i.e. had $M = I$. Let us illustrate by re-examining Example 2.

Example 4: Note that for the undisturbed problem of Example 2, $M = I$. After applying the strategy immediately above it is found that, for the updated system,

$$\hat{M} = \begin{bmatrix} 0.5334 & -0.0118 \\ & 0.6959 \end{bmatrix},$$

and then

$$M = I_2, \quad D = \begin{bmatrix} 7.2126 & 1.4955 \\ & 5.7874 \end{bmatrix}, \quad K = \begin{bmatrix} 28.0250 & 10.4572 \\ & 8.4693 \end{bmatrix},$$

and the eigenvectors are premultiplied by

$$\hat{M}^{\frac{1}{2}} = \begin{bmatrix} 0.7303 & -0.0075 \\ & 0.8342 \end{bmatrix}.$$

□

9 Conclusions

It has been shown how spectral updating of $L(\lambda)$ (or "pole placement" of the transfer function $L(\lambda)^{-1}$) can be achieved when complete spectral data for $L(\lambda)$ is suitably formulated. In the course of the analysis, it is shown that the (formerly) theoretical spectral analysis of real symmetric matrix polynomials can be used to advantage in a computational setting. The reader is reminded that, although second degree polynomials are the objects of study of this paper, the methods apply immediately to real symmetric matrix polynomials of any degree with positive definite leading coefficient.

Attention has been confined to real symmetric systems, but it is clear that, if symmetry is not an issue, then the techniques used here can be applied more widely - to nonself-adjoint problems. (In particular, the notion of positive and negative real eigenvalue types does not arise.) Updates of selected parts of the spectrum and/or eigenvectors can be made once a *Jordan triple* of eigenvalues and eigenvectors for the undisturbed system has been determined (see [5], [11], for example). This may be a topic for further investigation.

10 APPENDIX: Computing a self-adjoint triple for problems with distinct eigenvalues

This discussion applies quite generally to the real symmetric eigenvalue problem $(\lambda A - B)\xi = 0$ with $\det A \neq 0$ provided that no eigenvalues are defective.

1. Use the matrices M, D, K to form matrices A and B of (9).

2. Apply a standard algorithm to compute eigenvector and eigenvalue matrices V and J of $A^{-1}B$.
   (For example, in "Matlab" call $[V, J] = \text{eig}(\text{inv}(\text{A})*\text{B})$.) (Ensure that real eigenvalues have real eigenvectors.)

3. If there are no real eigenvalues go directly to the next step.
   Otherwise, compute $v_j^* A v_j$ for each real eigenvalue to determine their types (see (13)).

4. Apply a permutation matrix Q (so that $Q^T J Q \rightarrow J$ and $V Q^T \rightarrow V$) to obtain the eigenvalue ordering of equation (2) and record the corresponding matrix P of (5).

5. Compute a (generally complex) diagonal matrix $S := (V^T AV)^{-\frac{1}{2}}$.
   (In "Matlab", $S = \text{sqrt}(\text{inv}(V.' * A * V)$.)

6. Compute $X_E = VS$.
   (In "Matlab", $XE = V * S$.)
   The first n rows of X_E form matrix X.
   (In "Matlab", $X = XE(1:n,:)$.)
   THE COLUMNS OF X ARE THE NORMALISED EIGENVECTORS.

7. Confirm by checking that $XPX^* = 0,\ (XJPX^*)^{-1} = M$.
   (In "Matlab", $X * P * X' = 0,\ \text{inv}(X * J * P * X') = M$.)

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