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About a homoclinic pitchfork bifurcation in reversible systems with additional $\mathbb{Z}_2$-symmetry

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Abstract. This paper studies bifurcations from a homoclinic orbit to a degenerate fixed point. We consider reversible $\mathbb{Z}_2$-symmetric systems of ODEs and assume the existence of a symmetric homoclinic orbit to a fixed point which itself undergoes a pitchfork bifurcation. We are interested in bifurcations from the primary homoclinic orbit in an unfolding of the degenerate situation.

The studies are motivated by numerical investigations on a model-system of second order ODEs. There one finds a similar behaviour in the local and the global bifurcation. While locally two new fixed points are created numerical computations show that at the same time two homoclinic orbits to these fixed points bifurcate from the primary orbit. We call the global scenario a *reversible homoclinic pitchfork bifurcation*.

An analysis of this homoclinic bifurcation is performed in a general frame. Depending on the sign of a higher order coefficient in the normal form we distinguish two cases of the local pitchfork bifurcation: the eye case (which is the one encountered in the model-system) and the figure-eight case. Adopting Lin's method to the non-hyperbolic situation the bifurcation of one-homoclinic orbits to the local centre manifold of the fixed point is investigated. Rigorous existence results for homoclinic orbits to fixed points and periodic orbits are derived. The global bifurcation picture is found to depend crucially on the local bifurcation.

AMS classification scheme numbers: 37C29, 37C80, 37C25

1. Introduction

Homoclinic solutions of reversible ordinary differential equations (ODEs) have attracted a lot of attention. On one hand such solutions are of importance since they can describe solitary wave solutions of partial differential equations which are of interest in a variety of applications. On the other hand homoclinic solutions can strongly influence the dynamics of a system and are therefore of mathematical interest themselves. For instance, near such orbits bifurcations of periodic orbits can occur and under certain conditions complicated dynamics can arise. While the behaviour of homoclinic orbits to *hyperbolic* fixed points is a classic issue of the theory (see [5, 21] or the survey [1]) the case of homoclinic orbits to *non-hyperbolic* or degenerate fixed points has been considered only recently (see for instance [2, 3, 11]).

In this paper we will consider bifurcations from an orbit homoclinic to a degenerate fixed point with double zero eigenvalue which is supposed to bifurcate in a *reversible pitchfork bifurcation.* Thereby we will restrict to systems that are reversible with respect to *two* distinct linear involutions. We will study what generically happens to the homoclinic orbit in an unfolding. In particular, we will investigate the bifurcation of one-homoclinic orbits to the corresponding 'centre-manifolds', that is to the (locally invariant) manifolds in which the local bifurcation of the fixed point will be studied. Our analysis uses both analytical and geometrical tools. We employ Lin's method which has been proved to be a powerful tool for the investigation of bifurcations from and the dynamics near connecting orbits [16, 12, 21] and which was recently adapted to the case of orbits connecting non-hyperbolic fixed points in [11]. Geometrical methods are used to derive a complete description of the bifurcation scenario for one-homoclinic orbits.

Our studies are motivated by numerical investigations in [13] on a system of two second-order ODEs. Let us first describe these investigations shortly.

1.1. An introductory numerical example

We consider the following system of second order ODEs

$$\begin{aligned} \ddot{v} &= 2vw + 2\mu v \\ \ddot{w} &= -v^2 - w^2 + 2\mu w - \kappa, \end{aligned} \tag{1}$$

for $v, w : \mathbb{R} \to \mathbb{R}$, depending on parameters $\mu, \kappa \in \mathbb{R}$. This system has been obtained as an unfolding of a degenerate fixed point with fourfold eigenvalue zero in [22, 23]. There also analytical existence results for homoclinic and heteroclinic orbits have been derived. It is shown that a rich variety of bifurcations involving connecting orbits can be found in (1).

Here we will deal with a homoclinic bifurcation occurring for parameter values $\mu > 0$ and $\kappa = -3\mu^2$. We will consider (1) in $(v, \dot{v}, w, \dot{w})$-phase space. First observe that (1) is reversible with respect to the involutions

$$R_1 : (v, \dot{v}, w, \dot{w}) \to (v, -\dot{v}, w, -\dot{w})$$

and

$$R_2 : (v, \dot{v}, w, \dot{w}) \to (-v, \dot{v}, w, -\dot{w}).$$

Consequently, the map $S := R_1 R_2$ forms a $\mathbb{Z}_2$-symmetry for (1). Note that in addition (1) is a Hamiltonian system but this property will not be of concern here. (Readers not familiar with the notion of reversibility are referred to Section 2 where the basic terminology is introduced, see also [21] as a standard reference.)

For the following we will fix $\mu = 1$ and consider (1) as a system depending on a single parameter κ; this procedure is justified by a scaling property of (1), see [13]. Then for $\kappa = -3$ we find a fixed point $\xi_2 = (0, 0, 1 - \sqrt{1-\kappa}, 0) \in \mathrm{Fix}\,(R_1) \cap \mathrm{Fix}\,(R_2)$ which undergoes a reversible pitchfork bifurcation giving rise to two real saddles $\xi_3 = (-\sqrt{-3-\kappa}, 0, -1, 0)$, $\xi_4 = R_2\xi_3$ and turning from a real saddle into a saddle-centre itself if κ is decreased. (The fixed points are labelled as in the original papers [22, 23]

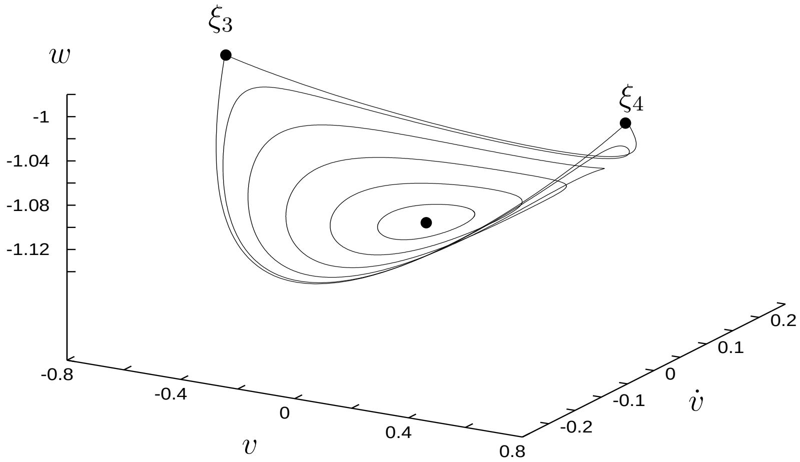
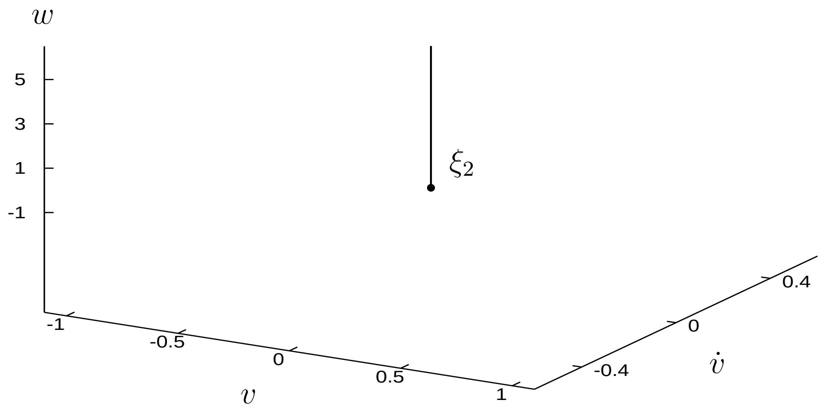


Figure 1. Phase portrait after the reversible pitchfork bifurcation of ξ_2. The figure shows plots of periodic orbits and the heteroclinic cycle connecting ξ_3 and ξ_4 for $\mu = 1$, $\kappa = -3.2$. (Projection in $(v, \dot{v}, w)$-space.)

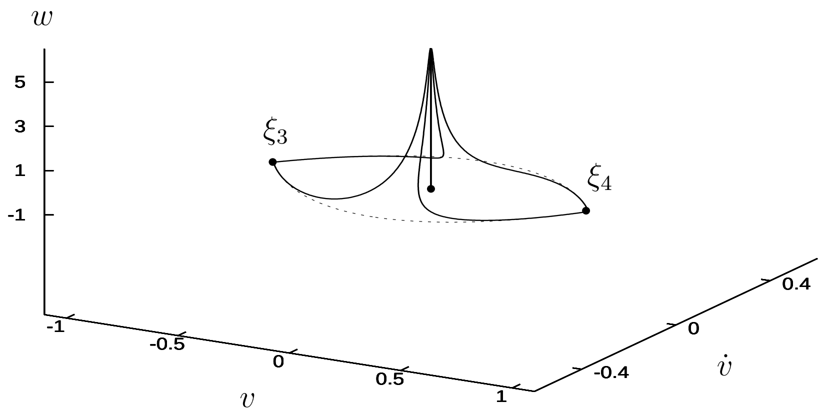
to which we also refer for the complete bifurcation diagram.) A numerical analysis of this local bifurcation for $\kappa < -3$ results in the phase portrait shown in Figure 1. In this figure we show the bifurcating periodic orbits that emerge in the pitchfork bifurcation of ξ_2. In addition, we can compute a small symmetric heteroclinic cycle connecting the fixed points ξ_3 and ξ_4. We remark, that since we cannot draw in $\mathbb{R}^4$ this plot shows a projection onto $(v, \dot{v}, w)$-space. It is not very hard to verify this bifurcation picture analytically, see [23].

Besides the information about the local behaviour near ξ_2 another result from [23] is of interest. Theorem 4.2 in that paper ensures the existence of a symmetric homoclinic orbit γ_{hom} to ξ_2 for all $\kappa < \mu^2$. In particular, γ_{hom} exists for $\mu = 1$, $\kappa = -3$ and the existence of this orbit is also unaffected by the local bifurcation of ξ_2. The reason for this is simple. The orbit is contained in the invariant subspace $\mathrm{Fix}\,(S)$ and within this space ξ_2 is hyperbolic for all $\kappa < \mu^2$. Therefore γ_{hom} cannot be destroyed. Nevertheless, the non-hyperbolicity of ξ_2 at $\kappa = -3$ makes it interesting to study bifurcations from γ_{hom}.

We have studied this situation numerically in [13] using the software package AUTO/HomCont, [6]. Since an analytical expression for γ_{hom} is known, [23], it is rather convenient to perform a bifurcation analysis of this orbit with AUTO/HomCont. For (1) we are mainly interested in the existence of orbits connecting fixed points of the system wherefore we restricted the computations to this point. We have used the continuation methods for symmetric connecting orbits in reversible systems that are implemented in AUTO.



a) $\mu = 1$, $\kappa = -2.8$



b) $\mu = 1$, $\kappa = -3.2$

Figure 2. Reversible homoclinic pitchfork bifurcation for (1). The figure shows how homoclinic solutions to $\xi_{3,4}$ bifurcate from the primary orbit γ_{hom} at $\mu = 1$, $\kappa = -3$. The orbit γ_{hom} is contained in the w-axis because of the chosen projection. Also incorporated is the heteroclinic cycle which is created in the reversible pitchfork bifurcation of ξ_2. Notice the different scales for the x, y-axes and the z-axis.

It turns out that when we move the parameter κ through $\kappa = -3$ then not only the fixed point ξ_2 bifurcates in a pitchfork bifurcation but also the orbit γ_{hom} follows a similar scenario. Indeed, for $\kappa < -3$ we find a new homoclinic orbit to the fixed point ξ_3, and - by reversibility - another one to ξ_4, both being copies of the original orbit γ_{hom}. We term this phenomenon a *reversible homoclinic pitchfork bifurcation.* An illustration of this bifurcation is given in Figure 2 where we show plots of the computed orbits. Note that in these plots the primary orbit γ_{hom} is completely contained in the w-axis due to the chosen projection. For an impression of the proportions we have also incorporated a plot of the heteroclinic cycle connecting ξ_3 and ξ_4.

The goal of this paper is to understand and describe the observed effects analytically. Instead of considering the specific system (1) we turn to a general class of $\mathbb{Z}_2$-symmetric, reversible ODEs possessing a fixed point which is assumed to bifurcate in a pitchfork bifurcation and which is connected to itself by a symmetric homoclinic orbit. Moreover, we do not restrict the analysis to four-dimensional systems but consider arbitrarily even-dimensional systems.

1.2. Contents and main results

The paper is organized as follows: In the next section we introduce the class of systems we will be dealing with. The assumptions about the fixed point and its local bifurcation are discussed in detail. We perform a centre manifold reduction and depending on the sign of a higher order term in the normal form of the reduced system we distinguish two cases: the eye case, which is the one encountered in the model-system (see also Figure 3), and the figure-eight case, see Figure 4.

The centre manifold reduction also allows to introduce suitable centre-stable and centre-unstable manifolds. In Sections 3.1 and 3.2 we employ Lin's method to study the intersection of these manifolds in some cross-section to the primary homoclinic orbit and compute one-homoclinic orbits to the centre manifolds. Note that as usual an orbit is called one-homoclinic if it is contained in some neighbourhood of the primary homoclinic orbit and if it intersects the cross-section exactly once. Using the symmetries of the system it is shown first that the primary orbit exists robustly. Afterwards we demonstrate that under suitable non-degeneracy assumptions for each parameter value there exist two families of one-homoclinic orbits to the centre manifolds. Their points of intersection with the cross-section lie on two smooth curves that intersect in a point corresponding to the primary homoclinic orbit (Theorem 3.8). We remark that the results so far do not depend on the type of the local bifurcation.

Finally, in Section 4 the asymptotic behaviour of the detected homoclinic solutions is discussed. For this we perform a *projection along stable fibres*. This method allows to directly read off the results from the bifurcation diagrams in Figure 6 and 7. We derive a complete description of bifurcating one-homoclinic orbits to fixed points and periodic orbits. The results depend considerably on the type of the local bifurcation. In the eye case we show that generically the local bifurcation is accompanied by the reversible homoclinic pitchfork bifurcation that was found numerically in the model-system. In addition we prove the emergence of a second heteroclinic cycle between the fixed points that are created in the reversible pitchfork bifurcation. By construction this cycle lies in a neighbourhood of the primary homoclinic orbit. It is therefore different from the (small) cycle shown in Figure 3 that emerges in the local bifurcation. The situation in the figure-eight case contrasts remarkably with this. Here we prove that generically no additional homoclinic orbits to fixed points bifurcate from the primary orbit. There are also differences in the bifurcation of homoclinic orbits to periodic orbits. In the eye case we find homoclinic orbits to each periodic orbit in the centre manifold. In the figure-eight case, however, only the orbits encircling the 'middle' fixed point are connected by homoclinic orbits, see Theorems 4.1 and 4.2 for precise statements.

2. Basic assumptions and conclusions

Throughout this paper we consider a family of ODEs

$$\dot{x} = f(x, \lambda), \quad (x, \lambda) \in \mathbb{R}^{2n+2} \times \mathbb{R} \tag{2}$$

with f smooth and λ as a real parameter. We will assume the system to be *reversible* and $\mathbb{Z}_2$-*symmetric*, more precisely we assume

(R1) There exist linear involutions $R_i : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$, $i = 1, 2$ with $R_1R_2 = R_2R_1$ such that

$$R_i f(x, \lambda) + f(R_i x, \lambda) = 0 \quad \forall (x, \lambda), \quad i = 1, 2.$$

Of course, we assume R_1, R_2 to be distinct (see (R2) below for a detailed statement). It is easy to see that (R1) implies a $\mathbb{Z}_2$-symmetry for (2), namely with $S := R_1R_2$ we have

$$Sf(x, \lambda) - f(Sx, \lambda) = 0 \quad \forall (x, \lambda). \tag{3}$$

In particular this equality shows that the space $\mathrm{Fix}\,(S) := \{x : Sx = x\}$ is an invariant subspace for (2). It is an immediate consequence from (R1) that within this space the involutions R_i agree and that (2) reduced to this space is also reversible with respect to the corresponding restriction R_S of the involutions R_i.

Remark. Using a more formal notion we can reformulate assumption (R1) by saying that (2) possesses the *reversing symmetry group* $G := \{I, R_1, R_2, S\}$; see [15] for the concept of reversing symmetry groups. We also note that since G is compact, we can introduce an inner product such that each R_i is self-adjoint. □

Let us discuss a further property associated to the reversing symmetry group of (2). In the following we will often use the fact that

$$R_i(\mathrm{Fix}\,(\pm R_j) \subset \mathrm{Fix}\,(\pm R_j), \quad i, j \in \{1, 2\}. \tag{4}$$

This is obvious for $i = j$. For $i \neq j$ let us choose $x \in \mathrm{Fix}\,(R_1)$. (The arguments for the other cases are the same.) Then

$$R_1(R_2x) = R_2R_1x = R_2x,$$

and therefore $R_2x \in \mathrm{Fix}\,(R_1)$, also. (Note that in (4) we even have equality since R_i is a bijective map.)

2.1. Assumptions involving the fixed point

Although our main interest lies in bifurcations of a global object, namely a homoclinic orbit, we will first describe the local bifurcation of the fixed point the homoclinic orbit is connected to. We will assume

(FP1) $f(0,0) = 0$, $\sigma(D_1 f(0,0)) = \{0\} \cup \{\pm\mu\} \cup \sigma^{ss} \cup \sigma^{uu}$, with 0 being a double, non-semisimple eigenvalue, $\mu \in \mathbb{R}^+$, and $|\Re(\tilde{\mu})| > \mu \; \forall \tilde{\mu} \in \sigma^{ss} \cup \sigma^{uu}$. (Here $\sigma^{ss(uu)}$ denotes the strong-stable (strong-unstable) spectrum of $D_1 f(0,0)$.)

An important property of (2) is that due to reversibility the R_i-image of any orbit $\widetilde{\Gamma}$ is again an orbit. As usual in the theory of reversible systems, we will call an orbit $\widetilde{\Gamma}$ *R_i-symmetric*, if $R_i\widetilde{\Gamma} = \widetilde{\Gamma}$. Thus, 0 is a R_i-symmetric fixed point (i=1,2) which, in particular, implies $R_i D_1 f(0,\lambda) + D_1 f(0,\lambda) R_i = 0$, i.e. $D_1 f(0,\lambda)$ is a R_i-reversible linear operator. Therefore, its spectrum is symmetric with respect to zero (in the complex plane) which yields that such fixed points generically occur in one-parameter families of reversible vector fields, see for instance [14].

In order to study the local bifurcation we first distinguish the involutions R_i. For this we denote the centre-subspace of $D_1 f(0,0)$ by $X^c_{\lambda=0}$ and observe that this space is invariant under R_i, $i = 1,2$. We want to study the situation when the involutions act differently (and non-trivially) on $X^c_{\lambda=0}$ and consequently assume that

(R2) $X^c_{\lambda=0} \not\subset \mathrm{Fix}\,(\pm R_i)$ for $i = 1,2$, and $X^c_{\lambda=0} \cap \mathrm{Fix}\,(S) = \{0\}$.

For the description of the local bifurcation of the fixed point 0 we can now apply centre-manifold theory and perform a reduction of the local problem to a family of 2-dimensional reversible vector fields. For this purpose, let us consider the extended system

$$\begin{aligned} \dot{x} &= f(x,\lambda) \\ \dot{\lambda} &= 0 \end{aligned} \tag{5}$$

which has a 3-dimensional local centre manifold $\mathfrak{W}^c_{loc}$ at the fixed point $(0,0)$. This manifold is foliated into 2-dimensional invariant slices $\{\lambda = \mathit{const.}\}$ which we will denote by $W^c_{loc,\lambda}$. Now, the centre-manifold-theorem [19, 8] shows that all small bounded solutions of (5) are contained in $\mathfrak{W}^c_{loc}$ which means that for (2) we can follow the evolution of small bifurcating solutions within the 2-dimensional slices $W^c_{loc,\lambda}$. Similar we define $W^{cs(cu)}_{loc,\lambda}$, $W^{s(u)}_{loc,\lambda}$ as slices of the local centre-(un)stable $\mathfrak{W}^{cs(cu)}_{loc}$ and (un)stable manifold $\mathfrak{W}^{s(u)}_{loc}$ of $(0,0)$ in (5). Dropping the *loc*-index we will later denote the globalized versions of these manifolds. By abuse of language we will, for instance, also term the slices W^{cs}_λ as a centre-stable manifold.

Note that centre-stable, centre-unstable and centre manifolds are not unique, in general. The next lemma shows that they can be chosen, such that the symmetries of (2) are preserved.

Lemma 2.1. *The manifolds $W^{cs}_{loc,\lambda}$, $W^{cu}_{loc,\lambda}$ can be chosen such that $W^{cs}_{loc,\lambda} = R_i W^{cu}_{loc,\lambda}$.*

Proof. We consider (5) which is easily seen to possess the reversing symmetry group $\mathfrak{G} := \{\mathfrak{I}, \mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{S}\}$ where

$$\mathfrak{R}_i := \begin{pmatrix} R_i & 0 \\ 0 & 1 \end{pmatrix}$$

and $\mathfrak{S} = \mathfrak{R}_1\mathfrak{R}_2$. The key argument of the proof is that we can choose $\mathfrak{W}_{loc}^{cs(cu)}$ such that $\mathfrak{R}_i\mathfrak{W}_{loc}^{cs} = \mathfrak{W}_{loc}^{cu}$. In fact, this is a standard result for systems that are reversible with respect to *one* involution, [10] or for systems symmetric with respect to a compact symmetry group, [4]. The corresponding proofs can easily be generalized to the case of a compact reversing symmetry group. And then the symmetry properties of the corresponding slices $W_{loc,\lambda}^{cs(cu)}$ are apparent. □

For later purposes we will further simplify equation (2) around 0. Let $\mathfrak{X}^{cs(cu)}$ denote the centre-stable (centre-unstable) subspace of the linearisation at (0,0) in (5). We can then assume that

$$\mathfrak{W}_{loc}^{cs(cu)} \subset \mathfrak{X}^{cs(cu)}.$$

In fact, there exists a transformation $\mathfrak{T}$ which pushes the manifolds into their respective subspaces and moreover $\mathfrak{T}$ can be chosen such that the symmetries of (5) are preserved, since $\mathfrak{G}$ is a compact reversing symmetry group; see for instance [11] for a computation of $\mathfrak{T}$. Now setting $\mathfrak{W}_{loc}^{c} := \mathfrak{W}_{loc}^{cs} \cap \mathfrak{W}_{loc}^{cu}$ we obtain a 'flat' centre manifold for (5) and by Lemma 2.1 we have $\mathfrak{R}_i\mathfrak{W}_{loc}^{c} = \mathfrak{W}_{loc}^{c}$ for $i = 1, 2$. Finally, for the invariant slices we conclude that

$$W_{loc,\lambda}^{cs(cu)} \subset X_{\lambda}^{cs(cu)}, \tag{6}$$

where $X_{\lambda}^{cs(cu)}$ denote the corresponding slices of the linear spaces $\mathfrak{X}^{cs(cu)}$, and moreover that $R_i W_{loc,\lambda}^{c} = W_{loc,\lambda}^{c}$ for $i = 1, 2$.

Therefore the vector field in $W_{loc,\lambda}^{c}$ is also reversible with respect to two distinct involutions. Let us denote the corresponding system by

$$\dot{y} = g(y, \lambda). \tag{7}$$

Assumption (FP1) yields

$$g(0,0) = 0, \quad D_1 g(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and in suitably chosen coordinates we find the following normal form for the corresponding involutions R_i

$$R_1 : y := (y_1, y_2) \mapsto (y_1, -y_2), \quad R_2 : (y_1, y_2) \mapsto (-y_1, y_2).$$

(Note that by slight abuse of notation we will use the same symbols for objects in $W_{loc,\lambda}^{c}$ and in the full phase space.)

The bifurcation of the fixed point $x = 0$ can be studied using a normal form of the vector field. More precisely, due to results by Dumortier [7] bifurcations of fixed points in planar systems can be studied in polynomial systems, that is fixed points of finite codimension in planar systems are *finitely determined.* For polynomial systems one can then perform a transformation into some normal form which can be chosen such that

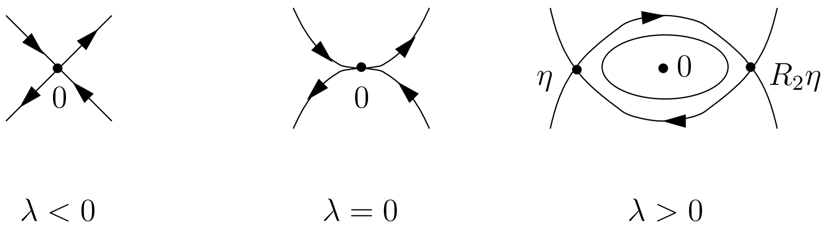


Figure 3. Phase portraits for the reversible pitchfork bifurcation I: The eye case with normal form (9).

the symmetries of the system are preserved. In our case this normal form is given in [14, 15] by

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \sum_{k\ odd} a_k(\lambda) y_1^k. \end{aligned} \tag{8}$$

Note that the normal form (8) for the reversible system (7) is also Hamiltonian.

By (FP1) we have $a_1(0) = 0$ in (8) and we will consider a generic bifurcation by demanding

(FP2) $a_3(0) \neq 0$.

Depending on the sign of $a_3(0)$ we are then led to two unfoldings of the corresponding singular systems. If $a_3(0) > 0$ we obtain

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -\lambda y_1 + y_1^3. \end{aligned} \tag{9}$$

Note that we have neglected terms of order larger than three in (9). These terms do not influence the qualitative local behaviour of the systems since (9) describes a versal unfolding of the singularity, see again [14].

The (Hamiltonian) system (9) is easily analysed and gives the corresponding phase portraits in Figure 3. Here we find the situation that was encountered for the illustrating example (1). For $\lambda > 0$ there exist two additional fixed points η, $R_2\eta$ which are saddles while 0 has turned from a real saddle into a centre. Moreover, we find a (small) symmetric heteroclinic cycle which connects $\eta, R_2\eta$. Because of the shape of this cycle we will refer to this bifurcation scenario as the *eye case*.

If $a_3(0) < 0$ a versal unfolding is given by

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \lambda y_1 - y_1^3. \end{aligned} \tag{10}$$

The phase portrait can be found in Figure 4. For $\lambda > 0$ the additional fixed points η, $R_2\eta$ are centres in this case and the fixed point at 0 has become a saddle which is connected to itself by two homoclinic orbits. Therefore we will name this scenario the *figure-eight case*. (Note that the different sign in front of the λ-term was only chosen to find a similar behaviour for the bifurcating fixed points compared to (9).)

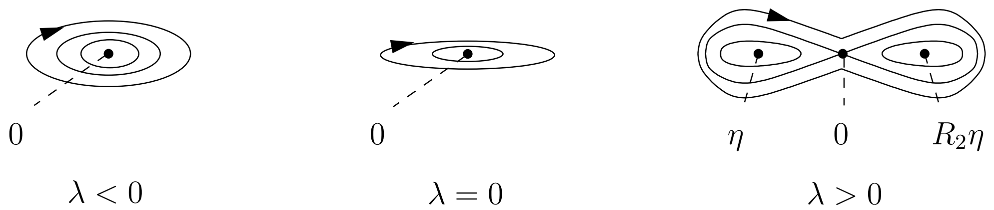


Figure 4. Phase portraits for the reversible pitchfork bifurcation II: The figure-eight case with normal form (10).

Let us take another look at the symmetries of (2). It is important to note that the local bifurcation scenarios (9) and (10) imply the existence of a hyperbolic fixed point $0 \in \mathrm{Fix}\,(R_1) \cap \mathrm{Fix}\,(R_2)$ for $\lambda < 0$ and $\lambda > 0$, respectively. Therefore we have a splitting

$$\mathbb{R}^{2n+2} = \mathrm{Fix}\,(R_i) \oplus \mathrm{Fix}\,(-R_i), \quad i = 1, 2$$

with $\dim(\mathrm{Fix}\,(R_i)) = \dim(\mathrm{Fix}\,(-R_i)) = n + 1$, see [21]. Similar conclusions can be drawn for the system reduced to $\mathrm{Fix}\,(S)$ wherefore we conclude that this space must be even-dimensional.

Remark. We see that the parameter λ was chosen to control the local bifurcation of the fixed point 0. Since, however, we are interested in the bifurcation of a homoclinic orbit one would expect at a first sight that this requires additional parameters. But in the next section we will assume sufficient transversality conditions for the homoclinic orbit which ensure our problem to be of codimension one. □

2.2. Assumptions about the homoclinic orbit

Let us now describe the homoclinic orbit which will be of concern in the following.

(H1) For $\lambda = 0$ equation (2) possesses a solution $\gamma(\cdot)$ homoclinic to the origin, i.e. $\lim_{t\to\pm\infty} \gamma(t) = 0$. Moreover, the corresponding orbit $\Gamma := \{\gamma(t) : t \in \mathbb{R}\}$ is symmetric with respect to both R_1 and R_2.

It is well known, see [21], that a homoclinic orbit is R_i-symmetric if and only if it intersects the fixed space $\mathrm{Fix}\,(R_i) := \{x : R_i x = x\}$ at a unique intersection point, say $\gamma(0)$. Therefore, if we choose $\gamma(0) \in \mathrm{Fix}\,(R_1)$, then $R_2\gamma(0) \in \mathrm{Fix}\,(R_1)$ by (4) and thus we have $\gamma(0) \in \mathrm{Fix}\,(R_2)$. This implies $\gamma(0) \in \mathrm{Fix}\,(S)$ and the invariance of $\mathrm{Fix}\,(S)$ shows

$$\Gamma \subset \mathrm{Fix}\,(S).$$

From (R2) we can immediately conclude that the orbit Γ is a global object which lies in the intersection of $W^s_{\lambda=0}$ and $W^u_{\lambda=0}$. Thus, it does not approach the origin at the lowest possible speed and generically such a situation would result in a reversible orbit-flip bifurcation, see [17]. Here, however, this behaviour is forced by the symmetries of Γ

and the forthcoming analysis will prove that no orbit-flip bifurcation occurs. Finally, $\Gamma \subset \mathrm{Fix}\,(S)$ trivially shows that $\mathrm{Fix}\,(S) \neq \{0\}$ and therefore $\dim(\mathrm{Fix}\,(S)) \geq 2$.

Let us impose a transversality condition upon the homoclinic orbit Γ. In order to consider a generic situation we will assume that at $\lambda = 0$ the stable manifold of 0 intersects the centre-unstable manifold as cleanly as possible. More precisely, for $\lambda = 0$ we denote the tangent space of the (un)-stable manifold of 0 at the point $\gamma(0)$ by $T_{\gamma(0)}W^{s(u)}_{\lambda=0}$ and demand

(H2) $\dim(T_{\gamma(0)}W^{s}_{\lambda=0} \cap T_{\gamma(0)}W^{cu}_{\lambda=0}) = 1.$

We note that this assumption is automatically fulfilled in $\mathbb{R}^4$, since in this case $\dim W^s_{\lambda=0} = 1$.

Assumption (H2) has important consequences for the dynamics near Γ. Let us first consider the situation in the invariant subspace $\mathrm{Fix}\,(S)$. Here the fixed point at the origin is hyperbolic and moreover, the trace of the centre-unstable manifold $W^{cu}_{\lambda=0}$ is the unstable manifold W^u_S of this fixed point. This immediately follows from $\mathrm{Fix}\,(S) \cap W^c_{loc,\lambda} = \{0\}$, because of (R2). Thus, under (H2) the intersection of W^u_S and W^s_S along Γ only contains the vector field direction, i.e. the orbit is non-degenerate within $\mathrm{Fix}\,(S)$. As a consequence, we find that the intersection of W^u_S and $\mathrm{Fix}\,(R_S)$ is transverse, see Lemma 4 in [21]. (Recall that R_S is the restriction of the R_i to $\mathrm{Fix}\,(S)$.) This point will be of importance in the proof of Lemma 2.2 below.

In the next section we will determine one-homoclinic orbits to $W^c_{loc,\lambda}$ by investigating the intersection of W^{cs}_λ and W^{cu}_λ. Related studies in [11] reveal that the results crucially depend on the relative position of the tangent spaces of the centre-(un)stable manifold and the fixed spaces of the involutions. Due to the symmetries of our problem we can determine this position (see Lemmas 2.2 and 3.7). As a first result in this context we obtain

Lemma 2.2. *Under the assumptions above the intersection of $W^{cs}_{\lambda=0}$ and $W^{cu}_{\lambda=0}$ is non-transverse with*

$$\dim\left(T_{\gamma(0)}W^{cs}_{\lambda=0} \cap T_{\gamma(0)}W^{cu}_{\lambda=0}\right) = 3.$$

Proof. First observe that it suffices to prove the non-transversality of the intersection of the manifolds. The second assertion is then an immediate consequence of (H2) and of the dimension of $W^{cs(cu)}_{\lambda=0}$.

So, seeking a contradiction let us assume that the intersection of $W^{cs(cu)}$ is transverse and let us introduce a space Y^c by setting

$$\mathrm{span}\{f(\gamma(0),0)\} \oplus Y^c := T_{\gamma(0)}W^{cs}_{\lambda=0} \cap T_{\gamma(0)}W^{cu}_{\lambda=0},$$

(all appearing decompositions are assumed to be orthogonal with respect to the R_i-invariant inner product). Since $\dim W^{cs(cu)}_{\lambda=0} = n+2$ we have $\dim Y^c = 1$. Moreover, again by counting dimensions we see that

$$\dim\left(T_{\gamma(0)}W^{cs}_{\lambda=0} \cap \mathrm{Fix}\,(R_i)\right) = 1,$$

for $i = 1, 2$. By reversibility components of $T_{\gamma(0)}W^{cs}_{\lambda}$ which are contained in $\mathrm{Fix}\,(R_i)$ also belong to W^{cu}_{λ}, and therefore $Y^c \subset \mathrm{Fix}\,(R_1) \cap \mathrm{Fix}\,(R_2)$, i.e. we have $Y^c \subset \mathrm{Fix}\,(S)$. We will show that this is impossible because of (H2). For this introduce $\widetilde{Y}^{cs}$ by letting

$$T_{\gamma(0)}W^{cs}_{\lambda=0} = \mathrm{span}\{f(\gamma(0), 0)\} \oplus Y^c \oplus \widetilde{Y}^{cs}.$$

An important observation is that

$$R_1\widetilde{Y}^{cs} = R_2\widetilde{Y}^{cs}. \tag{11}$$

In fact, since the spaces $T_{\gamma(0)}W^{cs}_{\lambda=0}$ and $\mathrm{span}\{f(\gamma(0), 0)\} \oplus Y^c$ are invariant under S this also applies to $\widetilde{Y}^{cs}$, i.e. we have $S\widetilde{Y}^{cs} = \widetilde{Y}^{cs}$. In particular, $R_1R_2\widetilde{Y}^{cs} = R_1R_1\widetilde{Y}^{cs}$, and this yields (11). So we can similarly decompose

$$T_{\gamma(0)}W^{cu}_{\lambda=0} = \mathrm{span}\{f(\gamma(0), 0)\} \oplus Y^c \oplus \widetilde{Y}^{cu},$$

with $\widetilde{Y}^{cu} := R_i\widetilde{Y}^{cs}$.

We will represent $W^{cs(cu)}_{\lambda=0}$ as graphs of functions

$$h^{cs(cu)} : \mathrm{span}\{f(\gamma(0), 0)\} \oplus Y^c \oplus \widetilde{Y}^{cs(cu)} \to \widetilde{Y}^{cu(cs)},$$

with $Dh^{cs(cu)}(0) = 0$. Choosing $(y_c, h^{cu}(y_c)) \in W^{cu}$ with $y_c \in Y^c \subset \mathrm{Fix}\,(S)$ we have

$$R_1(y_c, h^{cu}(y_c)) \in W^{cs}, \quad R_2(y_c, h^{cu}(y_c)) \in W^{cs}$$

and thus $R_1h^{cu}(y_c) = R_2h^{cu}(y_c)$, wherefore $(y_c, h^{cu}(y_c)) \in \mathrm{Fix}\,(S)$.

Thus, we can again consider the reduced system within $\mathrm{Fix}\,(S)$. We recall that this system is R_S-reversible, and as above we denote the unstable manifold of the (hyperbolic) fixed point 0 by W^u_S. Then $W^u_S = W^{cu}_{\lambda=0} \cap \mathrm{Fix}\,(S)$, and a consequence of our considerations is that W^u_S intersects $\mathrm{Fix}\,(R_S)$ non-transversally. Indeed, letting $\dim \mathrm{Fix}\,(S) = 2k$ (recall that because of reversibility $\mathrm{Fix}\,(S)$ is even-dimensional) we have

$$\dim T_{\gamma(0)}W^u_S = \dim \mathrm{Fix}\, R_S = k,$$

and since $y_c \in \mathrm{Fix}\,(R_1) \cap \mathrm{Fix}\,(R_2)$ the above considerations yield

$$\dim \left(T_{\gamma(0)}W^u_S \cap \mathrm{Fix}\, R_S\right) \geq 1.$$

But as it has already been discussed before, assumption (H2) implies a transverse intersection of W^u_S and $\mathrm{Fix}\,(R_S)$. So we derive the desired contradiction. □

3. Detection of one-homoclinic orbits to the centre manifold

The goal of our analysis is the description of bifurcating homoclinic orbits to the centre-manifolds $W^c_{loc,\lambda}$ introduced in the last section. For this we employ Lin's method, [16, 21]. The original version of this method allows to study the dynamics near orbits connecting hyperbolic fixed points. Since we deal here with a homoclinic orbit to a degenerate fixed point we use a generalization from [11]. Although our procedure runs completely along the same lines as in [11] we will include some technical details for the sake of self-containment of the present paper. We remark again that we concentrate on the bifurcation of one-homoclinic orbits. This corresponds to the 'first step' of the original version of Lin's method. Therefore, the spirit of our analysis will also be similar to [20].

We first remind the reader that we have chosen $\gamma(0) \subset \mathrm{Fix}\,(R_1) \cap \mathrm{Fix}\,(R_2)$. At this point we will introduce a cross-section Σ to Γ by decomposing

$$\mathbb{R}^{2n+2} = \mathrm{span}\{f(\gamma(0), 0)\} \oplus Y^s \oplus Y^u \oplus Z$$

with $\mathrm{span}\{f(\gamma(0), 0)\} \oplus Y^{s(u)} = T_{\gamma(0)} W^{s(u)}$ and setting

$$\Sigma := \gamma(0) + \{Y^s \oplus Y^u \oplus Z\}.$$

Again, we stress the fact that this decomposition is assumed to be orthogonal with respect to an R_i-invariant inner product. Note that Z is complementary to the sum of the tangent spaces of the stable and unstable manifolds of 0 and therefore (H2) implies $\dim Z = 3$. Using Lin's method we will eventually detect one-homoclinic orbits by solving a bifurcation equation in Z.

Let us discuss how the symmetries of (2) are reflected in this decomposition. Recall that $\dim(\mathrm{Fix}\,(R_i)) = \dim(\mathrm{Fix}\,(-R_i)) = n + 1$. Now, using $R_i W^s = W^u$, and therefore $R_i Y^s = Y^u$ for $i = 1, 2$ we obtain the equivalent of Lemma 2.4 and Lemma 2.5 in [11].

Lemma 3.1. *The space $Y^s \oplus Y^u$ contains $(n-1)$-dimensional subspaces of both $Fix\,(R_i)$ and $Fix\,(-R_i)$ for each $i = 1, 2$. The space Z contains a two-dimensional subspace Y_i of $Fix\,(R_i)$ and a one-dimensional subspace of $Fix\,(-R_i)$ for each $i = 1, 2$.*

Even more so, we have $Y_1 \neq Y_2$, because assuming $Y_1 = Y_2$ we can conclude $Z \subset \mathrm{Fix}\,(S)$. However, within the invariant subspace $\mathrm{Fix}\,(S)$ assumption (H2) implies that Γ is a non-degenerate orbit homoclinic to the hyperbolic fixed point 0. Since Z is complementary to the sum of the tangent spaces of the corresponding stable and unstable manifold we must have $\dim Z = 1$ in contradiction to the above.

An immediate consequence of this observation is the next lemma

Lemma 3.2. *For Z there exists a decomposition into one-dimensional subspaces X_i*

$$Z = X_1 \oplus X_2 \oplus X_3, \tag{12}$$

where $X_1 \subset Fix\,(R_1) \cap Fix\,(-R_2)$, $X_2 \subset Fix\,(-R_1) \cap Fix\,(R_2)$, $X_3 \subset Fix\,(R_1) \cap Fix\,(R_2)$.

Proof. We first consider the situation in $\Sigma \cap \mathrm{Fix}\,(S)$. We recall again, that here the one-dimensional space $Z \cap \mathrm{Fix}\,(S)$ is complementary to the sum of the tangent spaces of the stable and unstable manifolds of 0 and since Γ is non-degenerate in $\mathrm{Fix}\,(S)$ we know from [21] that

$$(Z \cap \mathrm{Fix}\,(S)) \subset (\mathrm{Fix}\,(R_1) \cap \mathrm{Fix}\,(R_2)),$$

which shows the existence of X_3.

Now let $X_1 := Z \cap \mathrm{Fix}\,(-R_2)$. Then by Lemma 3.1 it holds $\dim X_1 = 1$. Moreover, by (4) we have $R_1 X_1 \subset X_1$ and thus, either $X_1 \subset \mathrm{Fix}\,(R_1)$ or $X_1 \subset \mathrm{Fix}\,(-R_1)$; and the second possibility is excluded because it would imply $X_1 \subset \mathrm{Fix}\,(S)$ in contradiction to $\dim Z \cap \mathrm{Fix}\,(S) = 1$ and $X_1 \cap X_3 = \{0\}$. In a similar manner we obtain the assertion for X_2. $\square$

In a first step of Lin's method we will look for one-homoclinic orbits to the origin. In order to cope with non-hyperbolicity we will restrict to orbits approaching 0 with a certain exponential rate.

3.1. Homoclinic orbits to the origin

Following Lin's method for hyperbolic fixed points we would look for solutions $\gamma^{\pm}$ of (2) defined on $\mathbb{R}^{\pm}$ which start in Σ with a difference lying in a certain space and which approach 0 for $t \to \pm\infty$. Since 0 is a non-hyperbolic fixed point in our case we will restrict to exponentially decaying solutions. (A geometric explanation for this is given in the remark on page 16.) So, let us choose α such that $0 < \alpha < \mu$ (see (FP1) for the definition of μ) and look for solutions $\gamma^{\pm}$ that fulfill

($\mathbf{P}_\gamma$)
- (i) The orbits of $\gamma^{\pm}$ are near Γ
- (ii) $\gamma^+(0),\, \gamma^-(0) \in \Sigma$
- (iii) $\sup\, \{e^{\pm\alpha t}||\gamma^{\pm}(t)|| : t \in \mathbb{R}^{\pm}\} < \infty$
- (iv) $\gamma^+(0) - \gamma^-(0) \in Z$

Such solutions will be detected as perturbations of Γ for which we introduce functions $v^{\pm}$ defined on $\mathbb{R}^{\pm}$ by

$$\gamma^{\pm}(t) = \gamma(t) + v^{\pm}(t), \quad t \in \mathbb{R}^{\pm}.$$

We will formulate an equivalent problem to (P_γ) for $v^{\pm}$. First, the functions have to solve the equation

$$\dot{v} = D_1 f(\gamma(t), 0)v + h(t, v, \lambda) \tag{13}$$

where $h(t, v, \lambda) = f(\gamma(t) + v, \lambda) - f(\gamma(t), 0) - D_1 f(\gamma(t), 0)v$. In order to satisfy the exponential rate for $\gamma^{\pm}$ we introduce spaces

$$V_\alpha^+ := \{v \in C^0([0,\infty), \mathbb{R}^{2n+2}) : \sup_{t\geq 0} e^{\alpha t}||v(t)|| < \infty\}$$
$$V_\alpha^- := \{v \in C^0((-\infty,0], \mathbb{R}^{2n+2}) : \sup_{t\leq 0} e^{-\alpha t}||v(t)|| < \infty\}.$$

The adopted version of (P_γ) then reads

($\mathbf{P}_v$)
- (i) $||v^{\pm}(t)||$ is small for all $t \in \mathbb{R}^{\pm}$
- (ii) $v^+(0),\, v^-(0) \in Y^s \oplus Y^u \oplus Z$
- (iii) $v^+ \in V_\alpha^+$, $v^- \in V_\alpha^-$
- (iv) $v^+(0) - v^-(0) \in Z$

In order to find solutions of (13) that fulfill (P_v) we use the fact that the variational equation along the homoclinic orbit Γ

$$\dot{v} = D_1 f(\gamma(t), 0)v \tag{14}$$

possesses exponential trichotomies on $\mathbb{R}^{\pm}$, see [11]. This means, there exist projections $P_u^{\pm}(t)$, $P_s^{\pm}(t)$, $P_c^{\pm}(t)$ such that $id = P_u^{\pm}(t) + P_s^{\pm}(t) + P_c^{\pm}(t)$ $\forall t \in \mathbb{R}^{\pm}$ and

$$\Phi(t,s)P_i^{\pm}(s) = P_i^{\pm}(t)\Phi(t,s), \quad i = u, s, c,$$

where $\Phi(\cdot,\cdot)$ denotes the transition matrix of (14). Moreover, for $t \geq s \geq 0$ and for all α_c with $\mu > \alpha > \alpha_c > 0$ we have

$$\begin{array}{ll} ||\Phi(t,s)P_s^+(s)|| \leq Ke^{-\alpha(t-s)}, & ||\Phi(s,t)P_u^+(t)|| \leq Ke^{-\alpha(t-s)}, \\ ||\Phi(t,s)P_c^+(s)|| \leq Ke^{\alpha_c(t-s)}, & ||\Phi(s,t)P_c^+(t)|| \leq Ke^{\alpha_c(t-s)}. \end{array}$$

Using reversibility one can define $P_i^-(t)$ such that similar relations hold on $\mathbb{R}^-$. Of importance for us is that

$$\text{im } P_s^+(t) = T_{\gamma(t)}W^s_{\lambda=0}, \quad \text{im } P_u^-(t) = T_{\gamma(t)}W^u_{\lambda=0}, \tag{15}$$

and that we can choose

$$\ker P_s^+(0) = Z \oplus Y^u, \quad \ker P_u^-(0) = Z \oplus Y^s.$$

These results are proved in [9], see also [11].

Solutions of (13) satisfy the following fixed point problem

$$\begin{aligned} v^+(t) = \Phi(t,0)\eta^+ \;&+\; \int_0^t \Phi(t,s)P_s^+(s)h(s,v^+,\lambda)ds \\ &\qquad - \int_t^\infty \Phi(t,s)(id - P_s^+(s))h(s,v^+,\lambda)ds \\ v^-(t) = \Phi(t,0)\eta^- \;&-\; \int_t^0 \Phi(t,s)P_u^-(s)h(s,v^-,\lambda)ds \\ &\qquad + \int_{-\infty}^t \Phi(t,s)(id - P_u^-(s))h(s,v^-,\lambda)ds, \end{aligned} \tag{16}$$

where $\eta^+ \in T_{\gamma(0)}W^s_{\lambda=0}$, $\eta^- \in T_{\gamma(0)}W^u_{\lambda=0}$. Expanding f in the definition of h we obtain the estimate

$$||h(t,v,\lambda)|| \leq c_1||v||^2 + c_2||\lambda||(||\gamma(t)|| + ||v||).$$

Thus, $v^{\pm}, \in V_\alpha^{\pm}$ implies $h(\cdot, v^{\pm}(\cdot), \lambda) \in V_\alpha^{\pm}$. Combining this with the exponential trichotomy we see that the right-hand side in (16) is a map

$$T_{\gamma(0)}W^{s(u)}(0) \times \mathbb{R}^2 \times V_\alpha^{\pm} \to V_\alpha^{\pm}.$$

Therefore the exponentially bounded solution of (13) are exactly the solutions of (16), considered in $V_\alpha^{\pm}$. By the Implicit Function Theorem this problem can be solved around $(\eta^{\pm}, v^{\pm}, \lambda) = (0,0,0)$ for $v^{\pm} = v^{\pm}(\eta^{\pm}, \lambda)$. Now, regarding the requirements on $v^{\pm}(\eta^{\pm},\lambda)(0)$ in (P_v) we decompose

$$\begin{aligned} v^+(\eta^+,\lambda)(0) &= \eta^+ + y_u(\eta^+,\lambda) + z^+(\eta^+,\lambda) \\ v^-(\eta^-,\lambda)(0) &= \eta^- + y_s(\eta^-,\lambda) + z^-(\eta^-,\lambda), \end{aligned}$$

with $y_{s(u)} \in Y^{s(u)}$, $z^{\pm} \in Z$. By (P_v) (iv) and (15) we must have

$$\eta^+ = y_s(\eta^-,\lambda), \quad \eta^- = y_u(\eta^+,\lambda),$$

which again can be solved for $\eta^{\pm} = \eta^{\pm}(\lambda)$. We thus obtain in complete analogy to Lemma 2.7 in [11]

Lemma 3.3. *For each λ sufficiently small there exists a unique pair of solutions $(\gamma^+(\lambda), \gamma^-(\lambda))$ of problem (P_γ).*

Remark. We see that the solution of (P_γ) is not affected by the change of dimension of the stable (unstable) manifold of the fixed point 0 for $\lambda \neq 0$. This (rather surprising) fact can be explained by the assumed exponential bound for the solutions, because for parameter values where 0 is hyperbolic we look for solutions of (P_γ) that are contained in the *strong stable (unstable) manifold* of 0. Therefore the change in the dimension of the whole stable (unstable) manifolds is not important. □

We can find homoclinic orbits approaching the origin with some minimal exponential rate by solving the bifurcation equation

$$\xi^\infty(\lambda) := \gamma^+(\lambda)(0) - \gamma^-(\lambda)(0) = 0.$$

The uniqueness of the pair $(\gamma^+(\lambda), \gamma^-(\lambda))$ then immediately implies

$$R_i\gamma^+(\lambda)(0) = \gamma^-(\lambda)(0), \quad i = 1, 2.$$

Therefore $\xi^\infty(\lambda) \in \mathrm{Fix}\,(-R_i)$ for $i = 1, 2$ and since in Z

$$(\mathrm{Fix}\,(-R_1) \cap \mathrm{Fix}\,(-R_2)) = \{0\}$$

we conclude

Theorem 3.4. *For all λ sufficiently small there exists a homoclinic orbit $\Gamma(\lambda)$ to 0 which is symmetric with respect to both involutions.*

Remark.

a) The geometric reason for this result is very clear. We have already established the fact that within $\mathrm{Fix}\,(S)$ the orbit Γ is non-degenerate. An application of Lemma 4 from [21] as in the proof of Lemma 2.2 then shows that the unstable manifold W^u_S of 0 intersects $\mathrm{Fix}\,(R_S)$ transversally. So there is no chance for destroying this homoclinic connection.

b) There may exist additional homoclinic orbits to 0 which approach the fixed point with a smaller exponential rate and in fact such a smaller rate would be generic. Only the symmetries of (2) prevent Γ from switching to the lowest exponential rate available (reversible orbit-flip bifurcation). We will deal with the existence of such orbits in the next part. ◇

3.2. Homoclinic orbits to $W^c_{loc,\lambda}$

In the second step of Lin's method we will determine one-homoclinic orbits to $W^c_{loc,\lambda}$. This time the loss of hyperbolicity near the fixed point is overcome by looking for solutions of (2) on a finite time-interval. Choosing a 'time' T we seek solutions $x^+ : [0, T] \to \mathbb{R}^{2n+2}$ and $x^- : [-T, 0] \to \mathbb{R}^{2n+2}$ that satisfy

($\mathbf{P}_x$) (i) The orbits of $x^{\pm}$ are near Γ
(ii) $x^+(0),\, x^-(0) \in \Sigma$
(iii) $x^+(0) \in W^{cs}_{\lambda}, x^-(0) \in W^{cu}_{\lambda}$
(iv) $x^+(0) - x^-(0) \in Z$.

Provided T is chosen large enough we can formulate an equivalent demand to (P_x) (iii) by requiring

($\widetilde{\mathbf{P}}_x$) (iii) $x^+(T) \in W^{cs}_{loc,\lambda}, x^-(T) \in W^{cu}_{loc,\lambda}$

This time $x^{\pm}$ will be described as perturbations of $\gamma^{\pm}(\lambda)$

$$x^{\pm}(t) = \gamma^{\pm}(\lambda)(t) + v^{\pm}(t);$$

again giving a corresponding problem for $v^+ : [0,T] \to \mathbb{R}^{2n+2}$, $v^-[-T,0] \to \mathbb{R}^{2n+2}$. So we will determine solutions of

$$\dot{v} = D_1 f(\gamma^{\pm}(\lambda)(t), \lambda)v + h(t,v,\lambda) \tag{17}$$

with

$$h(t,v,\lambda) = f(\gamma^{\pm}(\lambda)(t) + v, \lambda) - f(\gamma^{\pm}(\lambda)(t), \lambda) - D_1 f(\gamma^{\pm}(\lambda)(t), \lambda)v$$

and we will require the solutions of (17) to satisfy

($\mathbf{P}^c_v$) (i) $||v^{\pm}(t)||$ is small for $t \in [-T,0]$, $t \in [0,T]$, respectively
(ii) $v^+(0),\, v^-(0) \in Y^s \oplus Y^u \oplus Z$
(iii) $v^+(T) \in W^{cs}_{loc,\lambda}, v^-(-T) \in W^{cu}_{loc,\lambda}$
(iv) $v^+(0) - v^-(0) \in Z$

Note that (P^c_v) (iii) uses the linear structure in $W^{cs(cu)}_{loc,\lambda}$ which is guaranteed in (6) for T sufficiently large.

The search for solutions of this problem again relies on exponential trichotomies of the equations

$$\dot{v} = D_1 f(\gamma^{\pm}(\lambda)(t), \lambda)v.$$

Similar to Section 3.1 we get solutions

$$v^{\pm} = v^{\pm}(\eta^{\pm}, \lambda), \tag{18}$$

where this time

$$\eta^+ \in (Y^s \oplus Y^u \oplus Z) \cap T_{\gamma^+(\lambda)(0)} W^{cs}_{\lambda}, \quad \eta^- \in (Y^s \oplus Y^u \oplus Z) \cap T_{\gamma^-(\lambda)(0)} W^{cu}_{\lambda}$$

according to (P_v)(iii) - again we refer to [11] for a detailed exposition.

In order to manage (P_v)(ii), (iv) we involve the space Y^c introduced as in the proof of Lemma 2.2 by

$$\mathrm{span}\{f(\gamma(0), 0)\} \oplus Y^c = T_{\gamma(0)} W^{cs}_{\lambda=0} \cap T_{\gamma(0)} W^{cu}_{\lambda=0}$$

and use the refined decomposition

$$\mathbb{R}^{2n+2} = \mathrm{span}\{f(\gamma(0),0)\} \oplus Y^s \oplus Y^u \oplus \widehat{Z} \oplus Y^c,$$

with $\widehat{Z} \oplus Y^c = Z$. Because of Lemma 2.2 we have $\dim Y^c = 2$, and we find that $R_i\widehat{Z} = \widehat{Z}$. We obtain a representation of $\Sigma \cap W_\lambda^{cs(cu)}$ as the graph of some function

$$h^{cs(cu)}(\cdot,\lambda) : Y^c \oplus Y^{s(u)} \to Y^{u(s)} \oplus \widehat{Z},$$

which yields

$$\begin{aligned}\eta^+ &= y_c^+ + y_s + D_1 h^{cs}(0,\lambda)(y_c^+, y_s)\\ \eta^- &= y_c^- + y_u + D_1 h^{cu}(0,\lambda)(y_c^-, y_u)\end{aligned}$$

with some $y_c^\pm \in Y^c$, $y_{s(u)} \in Y^{s(u)}$. Substituting η^+ with this relation in (18) we obtain

$$v^+ = v^+(y_c^+, y_s, \lambda), \quad v^- = v^-(y_c^-, y_u, \lambda).$$

In a similar way as in the preceding subsection we decompose $v^\pm$ at $t=0$ to find

$$\begin{array}{rcl} v^+(y_c^+, y_s, \lambda)(0) &=& y_c^+ + y_s + y_u^+(y_c^+, y_s, \lambda) + z^+(y_c^+, y_s, \lambda)\\ v^-(y_c^-, y_u, \lambda)(0) &=& y_c^- + y_u + y_s^-(y_c^-, y_u, \lambda) + z^-(y_c^-, y_u, \lambda). \end{array} \tag{19}$$

Again, (P_v)(iv) implies

$$y_s = y_s^-(y_c^-, y_u, \lambda), \quad y_u = y_u^+(y_c^+, y_s, \lambda), \tag{20}$$

and this system of equations can be solved for $y_s = y_s(y_c^+, y_c^-, \lambda)$, $y_u = y_u(y_c^+, y_c^-, \lambda)$. Putting things together we obtain

Lemma 3.5. *For sufficiently small λ and for sufficiently small $y_c^+, y_c^- \in Y^c$ there exists a unique pair $(x^+(y_c^+, y_c^-, \lambda), x^-(y_c^+, y_c^-, \lambda))$ of solutions of problem (P_x).*

For the detection of one-homoclinic orbits to $W_{loc,\lambda}^c$ it remains to solve

$$\xi(y_c^+, y_c^-, \lambda) := x^+(y_c^+, y_c^-, \lambda)(0) - x^-(y_c^+, y_c^-, \lambda)(0) = 0, \tag{21}$$

which can be written

$$\begin{array}{rcl} \xi(y_c^+, y_c^-, \lambda) = (y_c^+ - y_c^-) &+& (z^+(y_c^+, y_s(y_c^+, y_c^-, \lambda), \lambda)\\ &-& z^-(y_c^-, y_u(y_c^+, y_c^-, \lambda), \lambda))\,. \end{array} \tag{22}$$

Here we have used the representation (19) and the fact that because of Theorem 3.4 we have

$$\gamma^+(\lambda)(0) - \gamma^-(\lambda)(0) = 0, \quad \forall \lambda.$$

In order to solve (21) we must have $y_c^+ = y_c^- =: y_c$ since these are the Y^c-components of ξ. Introducing $\tilde{\xi}(y_c, \lambda) := \xi(y_c, y_c, \lambda)$ it therefore suffices to consider the bifurcation equation

$$\tilde{\xi}(y_c, \lambda) = 0. \tag{23}$$

We will view $\tilde{\xi}$ as a map $\tilde{\xi} : Y^c \times \mathbb{R} \to \widehat{Z}$.

Our solution of (23) will to a large extend invoke the symmetries of (2). So we have to consider their consequences for the equation. Let us explore this point before we go on with the solution of (23).

We return to the presentation (18): $v^+ = v^+(\eta^+, \lambda)$, $v^- = v^-(\eta^-, \lambda)$. Due to the reversibility of the fixed point equation similar to (16) it holds that $R_i v^\pm(\eta^\pm, \lambda)(t) = v^\mp(\eta^\mp, \lambda)(-t)$ (as usual $i = 1, 2$). An immediate consequence for (20) is

$$\begin{aligned} R_i y_u^+(y_c^+, y_s, \lambda) &= y_s^-(R_i y_c^+, R_i y_s, \lambda), \\ R_i z^+(y_c^+, y_s, \lambda) &= z^-(R_i y_c^+, R_i y_s, \lambda). \end{aligned} \tag{24}$$

For the solutions of (20) we thus obtain

$$R_i y_u(y_c^+, y_c^-, \lambda) = y_s(R_i y_c^-, R_i y_c^+, \lambda). \tag{25}$$

These properties will be used to detect symmetries in (23) in Section 3.2.1.

The last result of this part shows that we have a one-to-one correspondence between solutions (y_c, λ) of (23) with $y_c \in \mathrm{Fix}\,(R_i)$ and R_i-symmetric one-homoclinic orbits near the primary one Γ.

Lemma 3.6. *Suppose that the pair (y_c, λ) solves the bifurcation equation (23) and let $x(y_c, \lambda)(\cdot)$ denote the corresponding solution of (2) with orbit $\Xi(y_c, \lambda)$. Then $\Xi(y_c, \lambda)$ is R_i-symmetric if and only if $y_c \in Fix\,(R_i)$.*

Proof. For the proof we note first that $x(y_c, \lambda)(0) \in \mathrm{Fix}\,(R_i)$ is equivalent to

$$\begin{aligned} &R_i \gamma^+(\lambda)(0) + R_i v^+(y_c, y_s(y_c, y_c, \lambda), \lambda)(0) \\ &\qquad\qquad = \gamma^-(\lambda)(0) + v^-(y_c, y_u(y_c, y_c, \lambda), \lambda)(0). \end{aligned}$$

Now suppose $y_c \in \mathrm{Fix}\,(R_i)$. Since $R_i \gamma^+(\lambda)(0) = \gamma^-(\lambda)(0)$ by Theorem 3.4 we only have to consider the $v^\pm$-part. Here the above symmetries provide

$$\begin{aligned} R_i v^+(y_c, y_s(y_c, y_c, \lambda), \lambda)(0) &= v^-(R_i y_c, R_i y_s(y_c, y_c, \lambda), \lambda)(0) \\ &= v^-(R_i y_c, y_u(R_i y_c, R_i y_c, \lambda), \lambda)(0), \end{aligned}$$

and from $y_c \in \mathrm{Fix}\,(R_i)$ and the equivalence above we obtain $x(y_c, \lambda)(0) \in \mathrm{Fix}\,(R_i)$ and therefore the symmetry of the orbit.

On the other hand $v^+(y_c, y_s(y_c, y_c, \lambda), \lambda)(0) = v^-(y_c, y_u(y_c, y_c, \lambda), \lambda)(0)$ and therefore the only Y^c-component in $x(y_c, \lambda)(0)$ is y_c because of (19). Since the symmetry of $\Xi(y_c, \lambda)$ is equivalent to $x(y_c, \lambda)(0) \in \mathrm{Fix}\,(R_i)$ this requires $y_c \in \mathrm{Fix}\,(R_i)$. □

3.2.1. Geometry in Σ Before we solve the bifurcation equation (23) we will return to a discussion of geometric properties of the primary homoclinic orbit Γ. In Lemma 2.2 we have already shown that Γ results from a non-transverse intersection of $W^{cs}_{\lambda=0}$ and $W^{cu}_{\lambda=0}$. Now we investigate the relative position of these manifolds with respect to $\mathrm{Fix}\,(R_i)$. We will follow the convention in [11] and will call Γ *elementary* provided that $W^{cs}_{\lambda=0}$ intersects $\mathrm{Fix}\,(R_i)$ transversally for some $i = 1, 2$. In the other case when this intersection is non-transverse the orbit will be called *non-elementary.* So this is nothing but a suitably adopted version of the terms introduced in [21] for symmetric homoclinic orbits to hyperbolic fixed points. (Note however, that in general elementary homoclinic orbits in this sense will not survive perturbations in contrast to the hyperbolic case.)

We will show that due to the symmetries of the system Γ must be elementary.

Lemma 3.7. *Consider (2) and assume (R1), (R2), (FP1), (FP2), (H1), and (H2). Then it holds $W^{cs}_{\lambda=0} \pitchfork Fix(R_i)$ at $\gamma(0)$ where $i = 1, 2$.*

Proof. The proof is by contradiction, so let us assume that for instance $W^{cs}_{\lambda=0}$ intersects $\mathrm{Fix}\,(R_1)$ non-transversally, which implies

$$\dim\left(T_{\gamma(0)}W^{cs}_{\lambda=0} \cap \mathrm{Fix}\,(R_1)\right) \geq 2.$$

Because of (H2) we therefore have $Y^c = T_{\gamma(0)}W^{cs}_{\lambda=0} \cap \mathrm{Fix}\,(R_1)$ and $\widehat{Z} \subset \mathrm{Fix}\,(-R_1) \cap \mathrm{Fix}\,(R_2)$. Applying the decomposition (12) we thus have $Y^c = X_1 \oplus X_3$ and $\widehat{Z} = X_2$. The idea of the proof is to show that for each $y_c \in X_3$ we have $\tilde{\xi}(y_c, 0) = 0$. Since $(y_c, 0) \in \mathrm{Fix}\,(S)$ this would amount to a family of homoclinic orbits to 0 in $\mathrm{Fix}\,(S)$ and as in the proof of Lemma 2.2 we derive a contradiction to the non-degeneracy hypothesis (H2).

We shall show first that

$$\tilde{\xi}(y_c, 0) = -\tilde{\xi}(R_2 y_c, 0). \tag{26}$$

The simple proof of this assertion uses the representation (22). We find that

$$\xi(y_c, y_c, 0) = z^+(y_c, y_s(y_c, y_c, 0), 0) - z^-(y_c, y_u(y_c, y_c, 0), 0)$$

and because of (24) we have

$$\begin{aligned}\xi(R_2 y_c, R_2 y_c, 0) &= z^+(R_2 y_c, R_2 y_u(y_c, y_c, 0), 0) - z^-(R_2 y_c, R_2 y_s(y_c, y_c, 0), 0)\\ &= z^-(y_c, y_u(y_c, y_c, 0), 0) - z^+(y_c, y_s(y_c, y_c, 0), 0),\end{aligned}$$

since $z^{\pm} \in \mathrm{Fix}\,(R_2)$. This proves (26).

From the symmetry (26) we immediately deduce that for $y_c \in \mathrm{Fix}\,(R_2)$ we have $\tilde{\xi}(y_c, 0) = 0$, i.e. we deduce that $\tilde{\xi}_{|X_3 \times \{0\}} \equiv 0$. Then the application of Lemma 3.6 gives the R_1- and R_2-symmetry of the corresponding orbits, i.e. for the corresponding solutions $x(y_c, 0)(\cdot)$ we have $x(y_c, 0)(0) \in \mathrm{Fix}\,(R_1) \cap \mathrm{Fix}\,(R_2) \subset \mathrm{Fix}\,(S)$. By invariance of $\mathrm{Fix}\,(S)$ it holds $x(y_c, 0)(t) \in \mathrm{Fix}\,(S)\ \forall t \in \mathbb{R}$.

Hence, the above assumption implies the existence of a 1-parameter family of one-homoclinic solutions in $\mathrm{Fix}\,(S)$ connecting $W^c_{loc,\lambda=0}$ - and thus the fixed point 0 - to itself. Assumption (H2), however, implies the non-degeneracy of Γ in $\mathrm{Fix}\,(S)$ which gives a contradiction. $\square$

We recapitulate the result, namely that $T_{\gamma(0)}W^{cs}_{\lambda=0} \pitchfork \mathrm{Fix}\,(R_i)$, for $i = 1, 2$. By transversality this relation persists for λ small and we conclude that

$$\dim\left(T_{\gamma(0)}W^{cs}_{\lambda} \cap \mathrm{Fix}\,(R_i)\right) = 1 \text{ for } i = 1, 2. \tag{27}$$

In view of the decomposition (12) this results in $Y_c = X_1 \oplus X_2$ and $\widehat{Z} = X_3$. To see this choose $y \in Y_c \setminus \mathrm{Fix}\,(-R_1)$. By Lemma 3.1 such y exists. Then we have for $Y := \mathrm{span}(y + R_1 y) \subset \mathrm{Fix}\,(R_1)$ that $R_2 Y \subset Y$, and therefore we conclude that either $Y = X_1$ or $Y = X_3$. The latter possibility can be ruled out since we would find that in this case either $Y_c = X_1 \oplus X_3$ or $Y_c = X_2 \oplus X_3$ in contradiction to (27). We refer to Figure 5 for an impression of the geometric relations in Z.

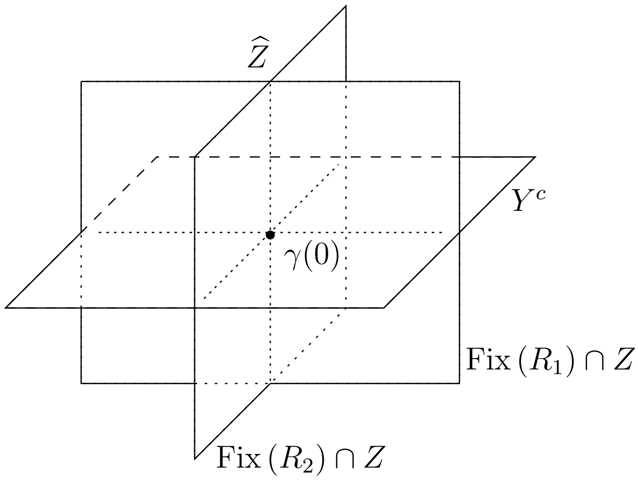


Figure 5. Position of the fixed spaces of the involutions in Z

For the solution of (23) we identify $\tilde{\xi} : X_1 \oplus X_2 \times \mathbb{R} \to \widehat{Z}$ with a map $\tilde{\xi}_\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In a manner similar to the proof of Lemma 3.7 the symmetries of (2) yield in the present situation

$$\tilde{\xi}_\lambda(y_1, y_2) = -\tilde{\xi}_\lambda(-y_1, y_2), \quad \tilde{\xi}_\lambda(y_1, y_2) = -\tilde{\xi}_\lambda(y_1, -y_2);$$

note again that this is valid for all λ. In particular,

$$\tilde{\xi}_\lambda(0, \cdot) \equiv 0 \text{ and } \tilde{\xi}_\lambda(\cdot, 0) \equiv 0 \tag{28}$$

for all λ. Thus, we can write

$$\tilde{\xi}_\lambda(y_1, y_2) = y_1 y_2 \cdot r_\lambda(y_1, y_2),$$

and in order to describe the solution set of (23) completely, we impose the following non-degeneracy condition

(ND) $r_{\lambda=0}(0, 0) \neq 0$.

This condition is equivalent to assuming $D^2\tilde{\xi}_{\lambda=0}(0, 0)$ to be non-singular and it ensures that the zero level set of $\tilde{\xi}_\lambda$ is given in (28). Again applying Lemma 3.6 we find for each λ sufficiently small a curve of intersection points of W^{cs}_λ and W^{cu}_λ in Fix (R_1) and one in Fix (R_2). Let us summarize this in the next

Theorem 3.8. *Under the assumptions (R1), (R2), (FP1), (FP2), (H1), (H2), and (ND) we find curves $\mathcal{C}^\Sigma_1$ and $\mathcal{C}^\Sigma_2$ in Σ such that for each point of $\mathcal{C}^\Sigma_i$ the orbit through this point is a R_i-symmetric one-homoclinic orbit to $W^c_{loc,\lambda}$. The curves $\mathcal{C}^\Sigma_1, \mathcal{C}^\Sigma_2$ intersect in a unique point which corresponds to the homoclinic orbit to* 0 *provided by Theorem 3.4. There exist no other one-homoclinic orbits than those described above.*

4. The bifurcation scenario

In the last section we have seen that for each λ sufficiently small the intersection of W^{cs}_λ and W^{cu}_λ in Σ consists of two curves $\mathcal{C}^\Sigma_{1,2}$. To derive a complete description of the homoclinic bifurcation it remains to study the asymptotic behaviour of the corresponding homoclinic orbits. For this we use the same technique as in [11] and project the solution set of (23) along the stable fibres of $W^c_{loc,\lambda}$ onto this manifold. For simplicity we will restrict to 4-dimensional problems in this section, i.e. we set $n = 1$ in (2) since this allows a very convenient geometric arguing.

Let us introduce the method. Restricting to $\mathbb{R}^4$ we have $\dim W^{cs}_\lambda = \dim W^{cu}_\lambda = 3$ and we can think of W^{cs}_λ as being foliated into 1-dimensional fibres, i.e. from each $x \in W^c_{loc,\lambda}$ there originates a 1-dimensional manifold $M_{x,\lambda} \subset W^{cs}_\lambda$, that is $W^{cs}_\lambda = \cup_{x \in W^c_{loc,\lambda}} M_{x,\lambda}$. In particular, $M_{0,\lambda}$ is nothing but the homoclinic orbit $\Gamma(\lambda)$. We infer that for λ sufficiently small there exists a neighbourhood U of 0 (independent of λ) such that for each $x \in U$ the fibre $M_{x,\lambda}$ intersects Σ transversally which shows that the projection along the fibre is injective. Moreover, this projection is smooth in both x and λ, see [18].

Finally, the fibres enjoy a certain invariance property that is of fundamental importance for the following discussion. Let $\phi_t(x)$ denote the solution of (2) with $\phi_0(x) = x$, then we have

$$\phi_t(M_{x,\lambda}) \subset M_{\phi_t(x),\lambda} \tag{29}$$

as long as $\phi_\tau(x) \in W^c_{loc,\lambda}$ $\forall \tau \in [0, t]$. In particular each point in a fibre of a basis-point which is part of a stable manifold of an orbit or fixed point will be transported to this point under the flow.

For the discussion of one-homoclinic orbits to $W^c_{loc,\lambda}$ it suffices to consider the projection of $\mathcal{C}^\Sigma_{1,2}$ at $\lambda = 0$ since we can infer the images for λ sufficiently small from this by continuity. We see that the existence of one-homoclinic orbits does mainly depend on the dynamics in $W^c_{loc,\lambda}$ and not on the behaviour in Σ. This, for instance, is contrary to the situation that is analysed in [11].

Coming back to the problem, the above results immediately show that the images $\mathcal{C}_{1,2}$ of $\mathcal{C}^\Sigma_{1,2}$ under the projection are curves which intersect only in 0. We impose a final transversality condition concerning the projection.

(P) $\quad \mathcal{C}_i \pitchfork \mathrm{Fix}\,(R_1)$ in $W^c_{loc,\lambda=0}$ for $i = 1, 2$.

Geometrically, we demand that $\mathcal{C}^\Sigma_{1,2}$ are projected on curves that are *not* tangent to the fixed space of the involution R_1. A further discussion of this assumption will be postponed to Section 5 after the bifurcation results have been formulated. For the moment (P) allows a complete classification of one-homoclinic orbits to $W^c_{loc,\lambda}$. We will treat each type of the local bifurcation separately.

4.1. The eye case - the reversible homoclinic pitchfork bifurcation

Let us redraw Figure 3 including the curves $\mathcal{C}_{1,2}$ of points $x \in W^c_{loc,\lambda}$ whose fibres intersect Σ in points on one-homoclinic orbits to $W^c_{loc,\lambda}$. We then obtain Figure 6.

We can now discuss which type of orbits we find. Let us start with $\lambda \leq 0$. Note first that (P) forbids intersection points of $\mathcal{C}_{1,2}$ with the stable and unstable manifolds

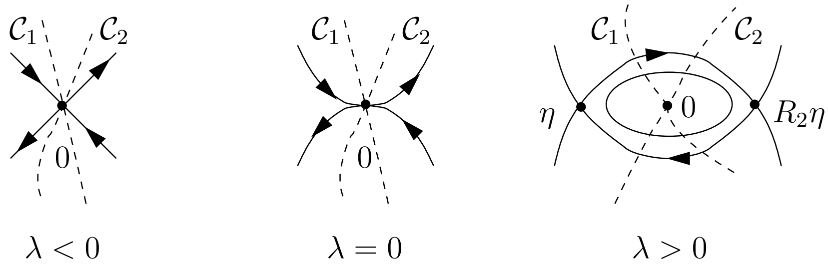


Figure 6. Dynamics in $W^c_{loc,\lambda}$ together with the curves $\mathcal{C}_{1,2}$ (dashed) needed for the detection of one-homoclinic orbits I: The eye case

of 0 in $W^c_{loc,\lambda}$ others than the origin, since these manifolds become tangent to Fix (R_1) as $\lambda \to 0$. Therefore the curves $\mathcal{C}_{1,2}$ only contain points whose orbits leave $W^c_{loc,\lambda}$ (apart from 0). And hence, there only exists the one-homoclinic orbit to 0 whose existence has been established in Theorem 3.4.

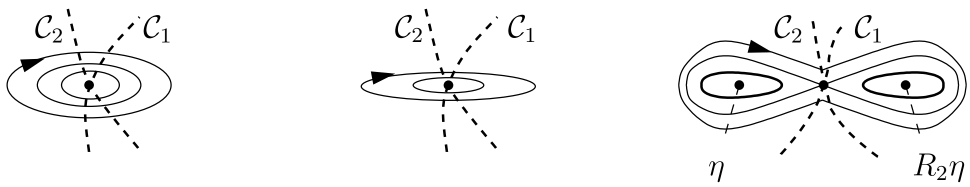
For $\lambda > 0$ we find a large variety of one-homoclinic orbits. Again intersection points of $\mathcal{C}_i$ with bounded solutions in $W^c_{loc,\lambda}$ are of interest. First we observe from Figure 6 that each periodic orbit Γ_p in $W^c_{loc,\lambda}$ is intersected by each $\mathcal{C}_i$ two times. Using the invariance property (29) of the fibre we see that every such intersection point corresponds to a symmetric one-homoclinic orbit to the periodic orbit. First, (29) shows that the corresponding solution that starts in Σ approaches Γ_p as $t \to \infty$. Moreover, considering a point $\Gamma_p \cap \mathcal{C}_i$ we know from the construction that this solution starts in Fix (R_i) which immediately shows that for $t \to -\infty$ it approaches $R_i\Gamma_p = \Gamma_p$.

Finally, we discuss the intersection points of $\mathcal{C}_i$ with the heteroclinic cycle. To these points there correspond solutions in W^{cs}_λ that approach the fixed points $\eta, R_2\eta$ as $t \to \infty$ by the invariance property (29). Moreover, the intersections of the cycle with $\mathcal{C}_1$ give rise to R_1-symmetric solutions which therefore approach $R_1\eta = \eta, R_2\eta$ as $t \to -\infty$, i.e. these solutions are homoclinic to the fixed points. Similarly, the intersection points of the cycle with $\mathcal{C}_2$ show the existence of a (large) heteroclinic cycle near the primary homoclinic orbit Γ. We hence obtain the next theorem.

Theorem 4.1 (One-homoclinic orbits in the eye case)**.** *Consider (2) under the assumptions (R1), (R2), (FP1), (FP2), (H1), (H2), (ND), and (P) and assume moreover that the normal form for the local bifurcation of the fixed point $x = 0$ is given by (9). Then for each λ sufficiently small there exists a homoclinic orbit to the origin which is symmetric with respect to both R_1 and R_2.*

In addition for $\lambda > 0$ every periodic orbit in $W^c_{loc,\lambda}$ is connected to itself by two pairs of R_1-symmetric and R_2-symmetric homoclinic orbits, respectively. Moreover, to η and $R_2\eta$ there exists one R_1-symmetric homoclinic orbit and these fixed points are connected by a symmetric heteroclinic cycle.

Remark. We claim that for the example (1) this theorem also proves the occurrence of the reversible homoclinic pitchfork bifurcation as introduced in Section 1.1. In fact, since



$\lambda < 0$ $\lambda = 0$ $\lambda > 0$

Figure 7. Dynamics in $W^c_{loc,\lambda}$ together with the curves $\mathcal{C}_{1,2}$ (dashed) needed for the detection of one-homoclinic orbits II: The figure-eight case.

this is a 4-dimensional problem the assumptions of Section 2 can easily be checked or they are even automatically fulfilled. Moreover, it is important that assumptions (ND) and (P) were only imposed to exclude the existence of additional solutions homoclinic to $W^c_{loc,\lambda}$. Thus, they are not important for the existence of homoclinics solutions to the fixed points $\xi_{3,4}$ in this example.

We finally note that also the existence of the large heteroclinic cycle could be numerically verified for the example (1), see [13]. □

4.2. The figure-eight case

The procedure for the second type of the local pitchfork bifurcation is completely analogous to the one above. We start again by plotting the centre manifolds $W^c_{loc,\lambda}$ together with the curves $\mathcal{C}_i$ in Figure 7. (Recall that the behaviour in Σ was completely independent from the bifurcation of 0.)

We can analyse the intersection points of $\mathcal{C}_i$ with orbits in $W^c_{loc,\lambda}$ in a similar fashion. So for $\lambda \leq 0$ we see that each curve $\mathcal{C}_i$ intersects each periodic orbit surrounding the centre 0 two times. Hence, there exist two R_i-symmetric homoclinic orbits to each periodic orbit in $W^c_{loc,\lambda}$.

The analysis for $\lambda > 0$ requires a closer look at the figure-eight in $W^c_{loc,\lambda}$. It is important that for $\lambda \to 0$ the stable and unstable manifold of the saddle 0 ($\lambda > 0$) become tangent to $\mathrm{Fix}\,(R_1)$. This is an immediate result of an analysis of the corresponding (Hamiltonian) normal form system (9). So for λ sufficiently small assumption (P) prevents intersections of the curves $\mathcal{C}_i$ with the region bounded by the figure-eight. Therefore we only find intersections of $\mathcal{C}_i$ with the periodic orbits surrounding all three fixed points.

Theorem 4.2 (One-homoclinic orbits in the figure-eight case)**.** *Consider (2) under the assumptions (R1), (R2), (FP1), (FP2), (H1), (H2), (ND) and (P) and assume moreover that the normal form for the local bifurcation of the fixed point $x = 0$ is given by (9). Then for each λ sufficiently small exists a homoclinic orbit to the origin which is symmetric with respect to both R_1 and R_2. No other one-homoclinic orbits to fixed points exist.*

In addition for $\lambda \leq 0$ every periodic orbit in $W^c_{loc,\lambda}$ is connected to itself by two pairs of R_1-symmetric and R_2-symmetric homoclinic orbits, respectively. For $\lambda > 0$ we find such two pairs of homoclinic orbits for the periodic orbits encircling all three fixed points.

5. Discussion

In this paper we have studied bifurcations from an orbit homoclinic to a non-hyperbolic fixed point that is supposed to undergo a reversible pitchfork bifurcation. Using Lin's method we have described the bifurcation of one-homoclinic orbits by considering the intersection of centre-stable and centre-unstable manifolds $W^{cs(cu)}_\lambda$ (as introduced in Section 2.1) in some cross-section Σ. It is interesting to note that this intersection is essentially determined by the symmetries of (2). In particular, the intersection is independent of the concrete type of the local bifurcation we assume for the fixed point. Nevertheless, the local bifurcation strongly influences the global bifurcation. This influence has been made clear by a projection from Σ onto $W^c_{loc,\lambda}$ along stable fibres. Here we have imposed a transversality condition (P) which shall now be considered more closely. Let us for simplicity restrict the discussion to the figure-eight case.

Suppose that (P) is violated such that for instance $\mathcal{C}_1$ is tangent to $\mathrm{Fix}\,(R_1)$ at $\lambda = 0$. Then we would find that $\mathcal{C}_1$ intersects the figure-eight in $W^c_{loc,\lambda}$ for λ sufficiently small. By the same arguments as in Section 4 the points of intersection correspond to R_1-symmetric homoclinic orbits to the origin. So in this situation we find another version of a homoclinic pitchfork bifurcation. Here the two additional homoclinic orbits are asymptotic to the same fixed point 0. One could treat this scenario as a codimension-two bifurcation by letting $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2$ where λ_1 is the original parameter in (2) and controls the local bifurcation of 0, whereas λ_2 controls how the curves $\mathcal{C}^\Sigma_{1,2}$ are projected onto $W^c_{loc,\lambda}$. Thus, λ_2 has a very precise geometric meaning since it describes the relations of the stable fibres of $W^c_{loc,\lambda}$. It would, however, also be useful to associate an analytical meaning to this parameter. We will not explore this point further here, but we remark that the qualitative results can already be inferred from Figure 7 by just thinking of $\mathcal{C}_{1,2}$ being rotated in $W^c_{loc,\lambda}$. (In a similar manner problems of higher codimension are created when assumption (ND) is violated. Since this assumption concerns the relation in Σ one has to adopt (P) appropriately in this case.)

Another interesting project for further studies is a description of recurrent dynamics near Γ. Let us turn to the eye case and provide some motivation. By a well known result of Devaney [5], see also [21], symmetric non-degenerate homoclinic orbits in reversible systems are accompanied by a family of symmetric periodic orbits with period tending to infinity. As it has been observed before, assumption (H2) implies the non-degeneracy of Γ in $\mathrm{Fix}\,(S)$. It therefore also ensures the existence of a family of periodic orbits in $\mathrm{Fix}\,(S)$ near Γ. In the eye case we can also expect that the two homoclinic orbits to η, $R_2\eta$ are non-degenerate because of their robust existence for $\lambda > 0$. This gives the possibility of two further families of periodic orbits. Now, one could look for connecting orbits between these families of periodic orbits which may result in the existence of very complicated dynamics near Γ.

Of particular relevance is the investigation of bifurcating n-homoclinic orbits to the centre manifold. These orbits are also contained in the intersection of W_λ^{cs} and W_λ^{cu} such that in principal our approach can be used to detect them. There are, however, additional difficulties since these orbits have to pass the centre manifold. Thus, the analysis requires a precise knowledge of the behaviour in a neighbourhood of this manifold. No analytical results exist at the moment.

We finally remark that the methods used in this paper can also be adapted to different bifurcation scenarios involving homoclinic orbits to non-hyperbolic fixed points. Examples of such problems have been found in models from nonlinear optics or in the theory of water waves. A discussion of several cases can be found in a joint paper with A. R. Champneys, [24].

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