Equalisers of frames in constructive set theory DRAFT

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1 Introduction

In a recent note Erik Palmgren has shown that the category of set-presented formal topologies has coequalisers in a sufficiently strong version of Martin-Löf's type theory.

Here we want to get a version of Palmgren's result in a sufficiently strong version of constructive set theory. We prefer to work with the category of set-presented class frames, a category that is equivalent to the opposite of the category of set-presented formal topologies. So we want to show that the category of set-presented class frames has equalisers.

Recall that a *class frame* is a partially ordered class having a top element $\top$, sups $\bigvee Y$ of arbitrary subsets Y and meets $a \wedge b$ of elements a, b such that the distributive law

$$a \wedge \bigvee Y = \bigvee \{a \wedge y \mid y \in Y\}$$

always holds. The (superlarge) category Frame of frames has maps that preserve the frame structure; i.e. the top, sups and meets.

A set-indexed family $\{\gamma(a)\}_{a\in S}$ of elements $\gamma(a)$ of a class frame A indexed by a set S is a *family of generators of* A if, for every $x \in A$, the class $S_x = \{a \in S \mid \gamma(a) \leq x\}$ is a set and $x = \bigvee S_x$. A function $C : S \to Pow(Pow(S))$ is a *set-presentation of* A *for the family of generators* if, for $a \in S$ and $U \in Pow(S)$,

$$\gamma(a) \leq \bigvee\{\gamma(b) \mid b \in U\} \iff (\exists V \in C(a))[V \subseteq U].$$

A *set-presented frame* is a frame having a set-indexed family of generators and a set-presentation for the family. The set-presented frames form a (superlarge) category sFrame, where a map in the category is just a frame map.

If A, A' are class frames and $F_1, F_2 : A \to A'$ are frame maps then we may form the subframe $\hat{A} = \{a \in A \mid F_1(a) = F_2(a)\}$ and easily show that it gives an equaliser of F_1, F_2 in the category of frames. Our task now is to show that if A, A' are set-presented then so is $\hat{A}$.

It will be convenient to use the set-presented formal topology associated with A. We first pre-order S by defining

$$a \leq b \iff \gamma(a) \leq \gamma(b)$$

for $a, b \in S$. To define the cover relation on S we let $\hat{\gamma}(U) = \bigvee\{\gamma(a) \mid a \in U\}$, for $U \in Pow(S)$ and define

$$a \triangleleft U \iff \gamma(a) \leq \hat{\gamma}(U).$$

The key to the construction of a set-presentation of $\hat{A}$ is the following result. The axiom $*$-**REA** will be explained later when we turn to the proof of the lemma. Let $eq(F_1, F_2)$ be the class $\{U \in Pow(S) \mid \hat{\gamma}(U) \in \hat{A}\}$.

Lemma: 1 (CZF + $*$-REA) *There is a* set-base $\hat{S}$ *for* $eq(F_1, F_2)$*; i.e. a subset* $\hat{S}$ *of* $eq(F_1, F_2)$ *such that if* $U \in eq(F_1, F_2)$ *then, for every* $b \in U$ *there is* $V \in \hat{S}$ *such that* $b \in V \subseteq U$.

As $\hat{S}$ is a set-base for $eq(F_1, F_2)$ the family $\{\hat{\gamma}(U)\}_{U\in\hat{S}}$ is easily seen to be a set-indexed family of generators of $\hat{A}$. We focus on the problem of getting a set-presentation $\hat{C} : \hat{S} \to Pow(Pow(S))$ for the family. For $U \in \hat{S}$ and $\mathcal{U} \in Pow(\hat{S})$ let

$$U \,\hat{\triangleleft}\, \mathcal{U} \iff \hat{\gamma}(U) \leq \bigvee\{\hat{\gamma}(V) \mid V \in \mathcal{U}\}.$$

We must find $\hat{C}$ such that for $U \in \hat{S}$ and $\mathcal{U} \in Pow(\hat{S})$

$$U \,\hat{\triangleleft}\, \mathcal{U} \iff (\exists \mathcal{V} \in \hat{C}(U))\ \mathcal{V} \subseteq \mathcal{U}.$$

2 MV-set-bases

The following notion will be useful. If $\{X_i\}_{i\in I}$ is a set-indexed family of sets let $\mathsf{MV}_{i\in I}X_i$ be the class of all families $\{R_i\}_{i\in I}$ such that R_i is an inhabited subset of X_i for each $i \in I$. It will be convenient to identify such a family $\{R_i\}_{i\in I}$ with the set $R = \{(i, x) \mid i \in I \; \& \; x \in R_i\}$

A subset Y of $\mathsf{MV}_{i\in I}X_i$ is an MV*-set-base for* $\{X_i\}_{i\in I}$ if, for every $R' \in \mathsf{MV}_{i\in I}X_i$ there is $R \in Y$ such that $R \subseteq R'$; i.e. $R_i \subseteq R'_i$ for all $i \in I$.

Proposition: 2 *If Y is an* MV*-set-base for $\{X_i\}_{i\in I}$ then, for any T,*

$$(\forall i \in I)(\exists x \in X_i)[(i, x) \in T] \iff (\exists R \in Y)(\forall (i, x) \in R)[(i, x) \in T].$$

Using Subset Collection and Strong Collection we can prove the following result.

Lemma: 3 *Let $\{X_i\}_{i\in I}$ be a set-indexed family of sets, let A be a set and let $J : A \to Pow(I)$. Then there is a function Q defined on A such that $Q(a)$ is an* MV*-set-base for $\{X_i\}_{i\in J(a)}$ for each $a \in A$.*

3 The set-presentation of $\hat{A}$

Observe that

$$\begin{array}{rcl} U \mathrel{\hat{\lhd}} \mathcal{U} & \iff & \bigvee\{\gamma(a) \mid a \in U\} \leq \bigvee\{\gamma(b) \mid b \in \cup\mathcal{U}\} \\ & \iff & (\forall a \in U)\; \gamma(a) \leq \bigvee\{\gamma(b) \mid b \in \cup\mathcal{U}\} \\ & \iff & (\forall a \in U)(\exists V \in C(a))\; V \subseteq \cup\mathcal{U} \end{array}$$

By Lemma 3 we get a function Q_1 defined on $\hat{S}$ such that $Q_1(U)$ is an MV-set-base for $\{C(a)\}_{a\in U}$ for all $U \in \hat{S}$. By Proposition 2 it follows that

$$U \mathrel{\hat{\lhd}} \mathcal{U} \iff (\exists R \in Q_1(U))(\forall (a, V) \in R)[V \subseteq \cup\mathcal{U}]$$

Using the lemma again we get a function Q_2 defined on $\overline{Q_1} = \bigcup_{U\in\hat{S}} Q_1(U)$ such that $Q_2(R)$ is an MV-set-base for $\{\hat{S}_b\}_{(x,b)\in\overline{R}}$ for all $R \in \overline{Q_1}$, where $\hat{S}_b = \{U \in \hat{S} \mid b \in U\}$ and $\overline{R} = \{((a, V), b) \mid (a, V) \in R \; \& \; b \in V\}$. It follows that, if $R \in \overline{Q_1}$ then

$$\begin{array}{rcl} (\forall (a, V) \in R)[V \subseteq \cup\mathcal{U}] & \iff & (\forall (x, b) \in \overline{R})(\exists U \in \hat{S}_b)[U \in \mathcal{U}] \\ & \iff & (\exists R_1 \in Q_2(R))(\forall (y, U) \in R_1)[U \in \mathcal{U}] \\ & \iff & (\exists R_1 \in Q_2(R))[\mathcal{V}_{R_1} \subseteq \mathcal{U}] \end{array}$$

where $\mathcal{V}_{R_1} = \{U \in \hat{S} \mid \exists y\; (y, U) \in R_1\}$.

For each $U \in \hat{S}$ let $\overline{Q_2}(U) = \bigcup_{R \in Q_1(U)} Q_2(R)$. Then

$$\begin{array}{rcl} U \,\hat{\lhd}\, \mathcal{U} & \iff & (\exists R_1 \in \overline{Q_2}(U)[\mathcal{V}_{R_1} \subseteq \mathcal{U}] \\ & \iff & (\exists \mathcal{V} \in \hat{C}(U))[\mathcal{V} \subseteq \mathcal{U}] \end{array}$$

where $\hat{C}(U) = \{\mathcal{V}_{R_1} \mid R_1 \in \overline{Q_2}(U)\}$.
■

4 Proof of Lemma 1

We first formulate the axiom $*$-**REA**.

Definition: 4 *A regular set $\mathcal{A}$ is defined to be a $*$-*regular set *if the following conditions hold.*

1. *It is* union-closed*; i.e. if $X \in \mathcal{A}$ then $\cup X \in \mathcal{A}$.*
2. *It has the* Relation Reflection Property*; i.e. if $a_0 \in X \subseteq \mathcal{A}$ and $R \subseteq X \times X$ such that*

   $$(\forall a \in X)(\exists b \in X)[(a,b) \in R]$$

   then there is $X_0 \in \mathcal{A}$ such that $a_0 \in X_0 \subseteq X$ and

   $$(\forall a \in X_0)(\exists b \in X_0)[(a,b) \in R].$$

Note that, assuming **DC**, any regular set that is union-closed has the Relation Reflection Property and so is $*$-regular.

Definition: 5 (The axiom $*$-REA) *Every set is a subset of a $*$-regular set.*

Note that $*$-**REA** is a theorem of **CZF** + **DC** + u**REA**, where u**REA** is the axiom that every set is a subset of a union-closed regular set.

To prove the lemma we let $\mathcal{A}$ be a large enough $*$-regular set. We will explain what large enough means as we go on. For any set X let $Pow_{\mathcal{A}}(X) = \mathcal{A} \cap Pow(X)$. As $\mathcal{A}$ is a set so is each $Pow_{\mathcal{A}}(X)$. Let $\hat{S} = \{U \in \mathcal{A} \mid U \in eq(F_1, F_2)\}$. As $\mathcal{A}$ is a set we can use Replacement and Restricted Separation to see that $\hat{S}$ is a set. We show that $\hat{S}$ is a set-base for $eq(F_1, F_2)$. For $i = 1, 2$ and $U \in Pow(S)$ let

$$G_i U = F_i(\hat{\gamma}(U)).$$

Note that

$$U \in eq(F_1, F_2) \iff G_1 U = G_2 U.$$

Let $G_1 U = G_2 U$. We must show that if $b \in U$ then there is $V \in \mathcal{A}$ such that $b \in V \subseteq U$ and $G_1 V = G_2 V$. So let $G_1 U = G_2 U$ and let $b \in U$. Let $X = Pow_{\mathcal{A}}(U)$. We will assume that the set $\{\{b\} \mid b \in S\}$ is a subset of $\mathcal{A}$ so that $\{b\} \in X$. We will use the following lemma.

Lemma: 6

1. $(\forall V_1 \in X)(\exists V_2 \in X)\ [G_1 V_1 \leq' G_2 V_2]$.
2. $(\forall V_2 \in X)(\exists V_1 \in X)\ [G_2 V_2 \leq' G_1 V_1]$.

Proof: It suffices to prove 1 as we get 2 by interchanging the roles of G_1 and G_2. For $V \in Pow(S)$ and $i = 1, 2$ let $T_i V = \bigcup_{a \in V} S'_{G_i\{a\}}$. Note that, by assuming that the set $\{S'_{G_i\{a\}} \mid i \in \{1,2\}\ \&\ a \in S\}$ is a subset of $\mathcal{A}$, we get that $T_i V \in \mathcal{A}$ for all $V \in X$. Observe that

$$\begin{array}{ll} G_i V & = F_i(\hat{\gamma}(V)) \\ & = \bigvee'\{G_i\{a\} \mid a \in V\} \\ & = \bigvee'\{\bigvee'\{\gamma'(a') \mid a' \in S'_{G_i\{a\}}\}\} \\ & = \bigvee'\{\gamma'(a') \mid a' \in T_i V\} \end{array}$$

So, for $V_1, V_2 \in Pow(S)$,

$$\begin{array}{lll} G_1 V_1 \leq' G_2 V_2 & \iff & (\forall a' \in T_1 V_1)[\gamma'(a') \leq' \bigvee'\{\gamma'(b') \mid b' \in T_2 V_2\}] \\ & \iff & (\forall a' \in T_1 V_1)(\exists V' \in C'(a')[V' \subseteq T_2 V_2] \end{array}$$

Now let $V_1 \in X$. Then, as $V_1 \subseteq U$, $G_1 V_1 \leq' G_2 U$ so that

$$(\forall a' \in T_1 V_1)(\exists V' \in C'(a'))[V' \subseteq T_2 U].$$

We assume that the set $\bigcup_{a' \in S'} C'(a')$ is a subset of $\mathcal{A}$. So, for $V' \in \bigcup_{a' \in S'} C'(a')$, $V' \in \mathcal{A}$ so that, using the regularity of $\mathcal{A}$, as $U \subseteq S \subseteq \mathcal{A}$,

$$\begin{array}{lll} V' \subseteq T_2 U & \iff & (\forall b' \in V')(\exists b \in \mathcal{A})[b \in U\ \&\ b' \in T_2\{b\}] \\ & \iff & (\exists W \in \mathcal{A})[W \subseteq U\ \&\ V' \subseteq T_2 W] \end{array}$$

So

$$(\forall a' \in T_1 V_1)(\exists V' \in C'(a'))(\exists W \in \mathcal{A})[W \subseteq U\ \&\ V' \subseteq T_2 W]$$

so that

$$(\forall a' \in T_1 V_1)(\exists W \in \mathcal{A})[W \subseteq U\ \&\ (\exists V' \in C'(a')[V' \subseteq T_2 W].$$

As $T_1V_1 \in \mathcal{A}$ and $\mathcal{A}$ is regular there is a set $\mathcal{W} \in \mathcal{A}$ such that $\mathcal{W} \subseteq Pow(U)$ and

$$(\forall a' \in T_1V_1)(\exists W \in \mathcal{W})(\exists V' \in C'(a')[V' \subseteq T_2W].$$

Now let $V_2 = \cup\mathcal{W}$. Then $V_2 \in \mathcal{A}$ and

$$(\forall a' \in T_1V_1)(\exists V' \in C'(a'))[V' \subseteq T_2V_2].$$

It follows that $G_1V_1 \leq' G_2V_2$.
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Corollary: 7 $(\forall V \in X)(\exists V' \in X)[\ G_1V \leq' G_2V' \ \ \&\ \ G_2V \leq' G_1V'\ ]$

Proof: Given $V \in X$, by the lemma there are $V_1, V_2 \in X$ such that

$$[\ G_1V \leq' G_2V_2 \ \ \&\ \ G_2V \leq' G_1V_1\ ]$$

Let $V' = V_1 \cup V_2$. Then, using the assumption that $2 = \{\emptyset, \{\emptyset\}\} \in \mathcal{A}$, we get that $\{V_1, V_2\} \in \mathcal{A}$ so that $V' = \cup\{V_1, V_2\} \in \mathcal{A}$ and hence $V' \in X$. Also $G_iV_i \leq' G_iV'$ for $i = 1, 2$ so that

$$[\ G_1V \leq' G_2V' \ \ \&\ \ G_2V \leq' G_1V'\ ].$$

■

As $\{b\} \in X$ and $X \subseteq \mathcal{A}$ we may apply the Relation Reflection Property to the corollary to get a set $X_0 \in \mathcal{A}$ such that $\{b\} \in X_0 \subset Pow(U)$ and

$$(*) \qquad (\forall V \in X_0)(\exists V' \in X_0)[\ G_1V \leq' G_2V' \ \ \&\ \ G_2V \leq' G_1V'\ ].$$

Let $V = \cup X_0$. Then $V \in X$ and $b \in V \subseteq U$. Also

$$\hat{\gamma}(V) = \bigvee\{\gamma(a) \mid a \in V\} = \bigvee\{\hat{\gamma}(V_0) \mid V_0 \in X_0\}$$

so that, for $i = 1, 2$, $G_iV = \bigvee'\{G_iV_0 \mid V_0 \in X_0\}$. Using $(*)$ we can now easily show that $G_1V = G_2V$ and we are done.
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We now summarise the assupmtions we have made about the $*$-regular set $\mathcal{A}$. We have assumed that $\mathcal{A}$ is a superset of the set

$$\{\{b\} \mid b \in S\} \cup \{S'_{G_i\{a\}} \mid i \in \{1, 2\} \ \&\ a \in S\} \cup \bigcup_{a' \in S'} C'(a') \cup \{2\}.$$