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A GSVD formulation of a domain decomposition method for planar eigenvalue problems

TIMO BETCKE

Institut Computational Mathematics, TU Braunschweig,

D-38023 Braunschweig, Germany.

In this article we present a modification of the domain decomposition method of Descloux and Tolley for planar eigenvalue problems. Instead of formulating a generalized eigenvalue problem our method is based on the generalized singular value decomposition. This approach is robust and at the same time highly accurate. Furthermore, we give an improved convergence analysis based on results from complex approximation theory. Several examples show the effectiveness of our method.

Keywords: eigenvalues, domain decomposition, generalized singular values, Method of Particular Solutions, conformal maps

1. Introduction

In 1983 Descloux and Tolley proposed a domain decomposition method for the planar eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in } \Omega \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain (see Descloux & Tolley (1983)). In each subdomain they used Fourier-Bessel functions that satisfy (1.1a) to approximate solutions of (1.1) and then set up a generalized eigenvalue problem, which modeled the compatibility conditions between the different subdomains. Based on power series estimates they proved exponential convergence of their method.

The use of particular solutions that satisfy (1.1a) but not necessarily (1.1b) was popularized in the paper by Fox *et al.* (1967). But their Method of Particular Solutions (MPS) is based on global basis functions that are supported on the whole domain rather than local approximations as used by Descloux and Tolley. Indeed, stability problems with the MPS on more complicated domains were one motivation for the work of Descloux and Tolley. The original MPS was recently revisited and improved by Betcke & Trefethen (2005).

Closely related ideas also appeared in the context of acoustic scattering, where instead of the eigenvalue problem (1.1) the solutions for a Helmholtz problem are sought (see for example Monk & Wang (1999)).

Unfortunately, the accuracy of the method of Descloux and Tolley is limited to $O(\sqrt{\epsilon_{mach}})$, where ϵ_{mach} is the machine accuracy. This was pointed out by Driscoll (1997). Using derivatives of eigenvalues he solved the accuracy problem and computed the first eigenvalues and eigenfunctions of the famous GWW isospectral drums (see Gordon *et al.* (1992)) to 12 digits of accuracy.

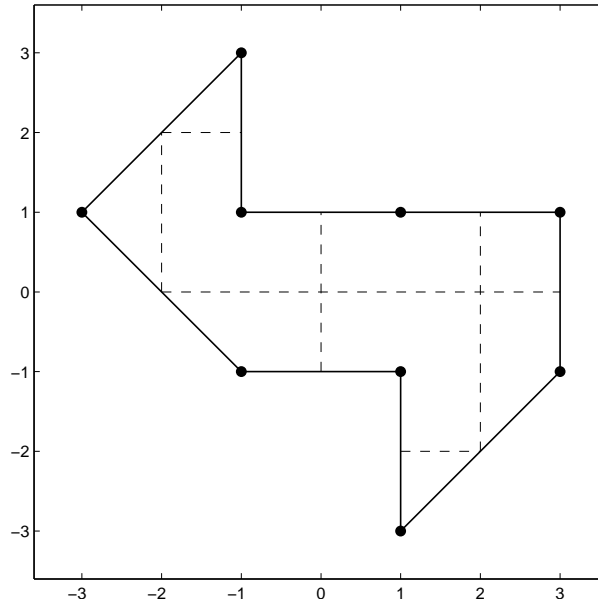


FIG. 1. A decomposition of the GWW-1 isospectral drum (see also Driscoll (1997), Figure 1.1). The black dots mark the corners z_j . Note the artificially introduced corner at $1 + 1i$.

In this article we present another solution to the limited accuracy problem of the method of Descloux and Tolley. It uses the generalized singular value decomposition (GSVD) instead of the generalized eigenvalue decomposition (GEVD). This approach is not only robust and highly accurate. In contrast to the methods of Descloux, Tolley and Driscoll it also avoids the explicit evaluation of boundary and domain integrals, which makes it easy to implement. Based on techniques from complex approximation theory we provide a new convergence analysis, which leads to sharper estimates than power series expansions.

2. The Method of Descloux and Tolley and Driscoll's modification

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, i.e. the boundary $\partial\Omega$ consists of piecewise straight arcs. For simplicity we assume that Ω is simply connected. But at the end of Section 6 we show how to apply the results of this paper to certain multiply connected domains. Let $\Omega_1, \dots, \Omega_p$ be simply connected subdomains of Ω with piecewise analytic boundary which form a partition of Ω , i.e. $\Omega_j \cap \Omega_l = \emptyset$ for $j \neq l$ and the closure of the union of the subdomains is $\overline{\Omega}$. Furthermore, for each subdomain Ω_j we assume that $\partial\Omega \cap \partial\Omega_j$ is a segment containing no corner of Ω or $\partial\Omega \cap \partial\Omega_j$ contains two segments of $\partial\Omega$ that intersect at a corner of $\partial\Omega$.

If $\partial\Omega_j$ contains no corner of $\partial\Omega$ we define a corner of angle π at a point on the segment $\partial\Omega \cap \partial\Omega_j$. Hence, we can assume that each subdomain Ω_j contains one corner of Ω with interior angle π/α_j at the position z_j . Such a decomposition of the GWW-1 isospectral drum is shown in Figure 1 (see also Driscoll (1997) Figure 1.1).

The boundary between two subdomains is denoted by $\Gamma_{kl} := \partial\Omega_k \cap \partial\Omega_l$. If Γ_{kl} consists only of a finite number of points we set $\Gamma_{kl} := \emptyset$.

In each subdomain Ω_j , $1 \leq j \leq p$ we define a local approximation space $\mathcal{A}_j(\lambda)$ of the form

$$\mathcal{A}_j(\lambda) := \left\{ \sum_{k=1}^{N_j} a_k^{(j)} J_{k\alpha_j}(\sqrt{\lambda}r) \sin k\alpha_j\theta; a_k^{(j)} \in \mathbb{R} \right\},$$

where $J_{k\alpha_j}$ is the Bessel function of the first kind of order $k\alpha_j$. The origin of the polar coordinates is z_j and the angular coordinate is chosen in such a way that every function in $\mathcal{A}_j(\lambda)$ is zero on the boundary arcs adjacent to the corner at z_j .

By $\mathcal{A}(\lambda) \subset \mathcal{C}^2(\bigcup_{j=1}^p \Omega_j)$ we denote the space of all functions u , which are in each subdomain Ω_j linear combinations of the Fourier-Bessel basis functions of $\mathcal{A}_j(\lambda)$. Hence, $u|_{\Omega_j} =: u^{(j)} \in \mathcal{A}_j(\lambda)$. Although the function u is not defined on the interfaces between the subdomains we can analytically continue each function $u^{(j)}$ across the internal interfaces to the neighboring subdomains.

A nonzero function $u \in \mathcal{A}(\lambda)$ can be continued to an eigenfunction of (1.1) on Ω if and only if for all $\Gamma_{kl} \neq \emptyset$ we have

$$u^{(k)} = u^{(l)} \quad \text{and} \quad \nabla u^{(k)} = \nabla u^{(l)}$$

on Γ_{kl} . If we define the quadratic functionals

$$\begin{aligned} \mathcal{T}(\lambda, u) &:= \sum_{k < l} \int_{\Gamma_{kl}} |u^{(k)} - u^{(l)}|^2 + |\nabla u^{(k)} - \nabla u^{(l)}|^2 ds, \\ \mathcal{M}(\lambda, u) &:= \sum_{k=1}^p \int_{\Omega_j} u(x, y)^2 dx dy, \end{aligned}$$

where $|\cdot|$ denotes the Euclidian norm, the continuity conditions are approximated by finding the local minima of

$$m(\lambda) = \min_{u \in \mathcal{A}(\lambda)} \frac{\mathcal{T}(\lambda, u)}{\mathcal{M}(\lambda, u)}. \tag{2.1}$$

The positions of the local minima are then approximations to eigenvalues of (1.1). This method is justified by the following result of Lemma 4.10 in Descloux & Tolley (1983).

LEMMA 2.1 There exists an eigenvalue λ_k of (1.1) such that

$$\frac{|\lambda - \lambda_k|}{\lambda_k} \leq C \sqrt{m(\lambda)},$$

where C is a constant that depends only on the domain decomposition $\Omega_1, \dots, \Omega_p$.

Similarly to Finite Element Methods we can rewrite (2.1) as a generalized eigenvalue problem of the form

$$T(\lambda)x(\lambda) = \mu(\lambda)M(\lambda)x(\lambda), \tag{2.2}$$

where $T(\lambda)$ is symmetric positive semi-definite, $M(\lambda)$ is symmetric positive definite and the vector $x(\lambda)$ contains all coefficients of the Fourier-Bessel expansions in the subdomains. The solution $m(\lambda)$ of (2.1) is now given as the smallest eigenvalue $\mu_1(\lambda)$ of (2.2).

The formulation of Descloux and Tolley has one drawback, which was analyzed by Driscoll (1997). Close to an eigenvalue λ_k of (1.1) the function $m(\lambda)$ behaves quadratically, which

leads to the effect that the minimum of $m(\lambda)$ and therefore also an eigenvalue λ_k can only be determined up to an accuracy of $O(\sqrt{\epsilon_{mach}})$. Driscoll solved this problem by computing the zeros of the derivative of $\mu(\lambda)$ instead of minimizing $\mu(\lambda)$. One obtains

$$\mu'(\lambda) = \frac{x(\lambda)(T'(\lambda) - \mu(\lambda)M'(\lambda))x(\lambda)}{x(\lambda)^T M(\lambda)x(\lambda)}.$$

Since the derivatives of Bessel functions are themselves simple linear combinations of Bessel functions the elements of $T'(\lambda)$ and $M'(\lambda)$ can be evaluated to high accuracy.

In this article we provide a different solution to the problem of the limited accuracy. The idea is to directly minimize $m^{\frac{1}{2}}(\lambda)$ without forming $m(\lambda)$. This is achieved by going over from generalized eigenvalue to generalized singular value computations.

3. A quasi-matrix formulation

In order to formulate the domain decomposition as a generalized singular value problem we will make use of quasi-matrices.¹ These are matrices whose columns are not vectors but functions. An elegant description of such matrices was given in Stewart (1998). Battles & Trefethen (2005) (see also Battles (2006)) developed the `chebfun`-system, an extension of Matlab to continuous functions and operators, which can work with certain quasi-matrices.

Let f_1, \dots, f_n be functions defined in a domain Ω . Then we define the quasi-matrix A as

$$A := [f_1, \dots, f_n].$$

If $x \in \mathbb{R}^n$ the usual matrix-vector multiplication is defined for A as

$$Ax := \sum_{k=1}^n x_k f_k.$$

The result of this operation is a function. Similarly, the multiplication AX , where $X \in \mathbb{R}^{n \times m}$ is defined as for ordinary matrices. The result is a quasi-matrix. But the product of two quasi-matrices is not defined.

Let A and B be two quasi-matrices with column functions f_1, \dots, f_n and g_1, \dots, g_n defined on the domains Ω_A and Ω_B . If $\Omega_A \cap \Omega_B = \emptyset$ we define

$$\begin{bmatrix} A \\ B \end{bmatrix} := [h_1, \dots, h_n], \quad h_j(z) = \begin{cases} f_j(z) & z \in \Omega_A \\ g_j(z) & z \in \Omega_B \end{cases}, \quad 1 \leq j \leq n.$$

Hence, stacking up two quasi-matrices corresponds to extending the domain of definition of the column functions.

Although the multiplication of two quasi-matrices A and B is not defined the matrix of inner products of the column functions can be defined as

$$A^T B := (\langle f_i, g_j \rangle)_{ij}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the associated function space. Depending on the context we will use different definitions of $\langle \cdot, \cdot \rangle$.

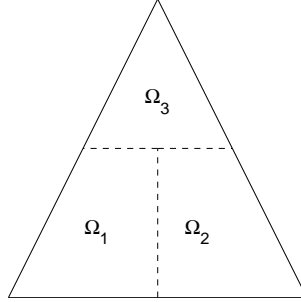


FIG. 2. The decomposition of a simple triangular domain.

Let us now formulate the domain-decomposition method of Descloux and Tolley with the help of quasi-matrices. We will do this for the simple triangle shown in Figure 2. Each sub-domain Ω_j , $1 \leq j \leq 3$ is associated with a quasi-matrix $A_j(\lambda)$ whose columns are the basis functions of $\mathcal{A}_j(\lambda)$. Take the internal boundary segment Γ_{12} . As norm on Γ_{12} we use the Sobolev-norm

$$\|u\|_{H^1(\Gamma_{12})}^2 = \int_{\Gamma_{12}} u^2(s) + |\nabla u(s)|^2 ds.$$

The error on Γ_{12} between the two local approximations $u^{(1)}$ and $u^{(2)}$ is then given as

$$\|u^{(1)} - u^{(2)}\|_{H^1(\Gamma_{12})} = \left\| \begin{bmatrix} A_1(\lambda) & -A_2(\lambda) \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \right\|_{H^1(\Gamma_{12})},$$

where $x^{(1)}$ and $x^{(2)}$ are the coefficient vectors of $u^{(1)}$ and $u^{(2)}$ in the Fourier-Bessel bases for $\mathcal{A}_1(\lambda)$ and $\mathcal{A}_2(\lambda)$.

By including the other two internal boundary segments this leads to the problem of minimizing

$$\left\| \begin{bmatrix} A_1(\lambda) & -A_2(\lambda) & 0 \\ A_1(\lambda) & 0 & -A_3(\lambda) \\ 0 & A_2(\lambda) & -A_3(\lambda) \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{bmatrix} \right\|_{H^1(\Gamma)} =: \|A_\Gamma(\lambda)x\|_{H^1(\Gamma)},$$

where $\|\cdot\|_{H^1(\Gamma)}$ is the H^1 -norm on $\Gamma := \Gamma_{12} \cup \Gamma_{13} \cup \Gamma_{23}$. The stacking up of quasi-matrices is well-defined in this case since the domain of the column functions in the first row block is Γ_{12} , in the second row block Γ_{13} and in the third row block Γ_{23} .²

From the choice of the norm and the definition of the quasi-matrix $A_\Gamma(\lambda)$ it follows that

$$\mathcal{T}(\lambda, u) = x^T A_\Gamma(\lambda)^T A_\Gamma(\lambda) x = \|A_\Gamma(\lambda)x\|_{H^1(\Gamma)}^2 \quad (3.1)$$

if $u \in \mathcal{A}(\lambda)$ is the function associated with the coefficient vector x . The matrix $A_\Gamma(\lambda)^T A_\Gamma(\lambda)$ is the matrix of H^1 -inner products between the column functions of $A_\Gamma(\lambda)$. It is an ordinary matrix and therefore the product $x^T A_\Gamma(\lambda)^T A_\Gamma(\lambda) x$ is well defined.

¹These matrices are also known as ‘‘column maps’’ (see De Boor (1991)) or ‘‘matrices with continuous columns’’ (see Trefethen & Bau (1997)).

²In a strict sense $\Gamma_{12} \cap \Gamma_{13} \cap \Gamma_{23} \neq \emptyset$ since they share one common point. Therefore, the stacking up of the matrices is not permitted. However, since function values at a single point do not influence the H^1 norm we can safely ignore this (for example by deleting the intersection point from the sets Γ_{12} , Γ_{13} and Γ_{23}).

Let us now give a quasi-matrix characterization of $\mathcal{M}(\lambda, u)$. By $\|u\|_{L^2(\Omega)}$ we denote the standard L^2 -norm on Ω for a function $u \in \mathcal{A}(\lambda)$. Then

$$\|u\|_{L^2(\Omega)} = \left\| \begin{bmatrix} A_1(\lambda) & 0 & 0 \\ 0 & A_2(\lambda) & 0 \\ 0 & 0 & A_3(\lambda) \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{bmatrix} \right\|_{L^2(\Omega)} =: \|A_\Omega(\lambda)x\|_{L^2(\Omega)}.$$

As in (3.1) it follows that

$$\mathcal{M}(\lambda, u) = x^T A_\Omega(\lambda)^T A_\Omega(\lambda)x = \|A_\Omega(\lambda)x\|_{L^2(\Omega)}^2.$$

We can now formulate the method of Descloux and Tolley as the minimization problem

$$\min_{\lambda} \min_{x \in \mathbb{R}^N \setminus \{0\}} \frac{\|A_\Gamma(\lambda)x\|_{H^1(\Gamma)}^2}{\|A_\Omega(\lambda)x\|_{L^2(\Omega)}^2} = \min_{\lambda} \min_{x \in \mathbb{R}^N \setminus \{0\}} \frac{x^T A_\Gamma(\lambda)^T A_\Gamma(\lambda)x}{x^T A_\Omega(\lambda)^T A_\Omega(\lambda)x}. \quad (3.2)$$

With

$$T(\lambda) = A_\Gamma(\lambda)^T A_\Gamma(\lambda) \quad \text{and} \quad M(\lambda) = A_\Omega(\lambda)^T A_\Omega(\lambda)$$

the formulation (3.2) leads to the generalized eigenvalue problem (2.2). But this involves the squaring of $\|A_\Gamma(\lambda)x\|_{H^1(\Gamma)}$ and $\|A_\Omega(\lambda)x\|_{L^2(\Omega)}$, which we want to avoid as this usually reduces the accuracy to which the minima of $t(\lambda)$ can be detected (see Driscoll (1997) or also Betcke (2006) for the explanation of this phenomenon in a closely related problem). It would be preferable to evaluate

$$t(\lambda) := \min_{x \in \mathbb{R}^N \setminus \{0\}} \frac{\|A_\Gamma(\lambda)x\|_{H^1(\Gamma)}}{\|A_\Omega(\lambda)x\|_{L^2(\Omega)}} \quad (3.3)$$

directly. The solution to this problem is given by the generalized singular value decomposition (GSVD).

4. A GSVD based method

The generalized singular value decomposition (GSVD) is a tool to find the stationary values of $\frac{\|Ax\|_2}{\|Bx\|_2}$, where $A \in \mathbb{R}^{n \times p}$, $B \in \mathbb{R}^{m \times p}$ and $\|\cdot\|_2$ is the usual Euclidian norm.

The concept of the GSVD was introduced by Van Loan (1976) and later generalized by Paige & Saunders (1981). We use a simplified version of the formulation in Paige & Saunders (1981) for the special case needed in this paper.

THEOREM 4.1 (GENERALIZED SINGULAR VALUE DECOMPOSITION) Let $A \in \mathbb{R}^{n \times p}$ and $B \in \mathbb{R}^{m \times p}$ be given with $n \geq p$. Define $Y = \begin{bmatrix} A \\ B \end{bmatrix}$ and assume that $\text{rank}(Y) = p$. There exist orthogonal matrices $U \in \mathbb{R}^{n \times n}$ and $W \in \mathbb{R}^{m \times m}$ and a nonsingular matrix $X \in \mathbb{R}^{p \times p}$ such that

$$A = UCX^{-1}, \quad B = WSX^{-1},$$

where $C \in \mathbb{R}^{n \times p}$ and $S \in \mathbb{R}^{m \times p}$ are diagonal matrices defined as $C = \text{diag}(c_1, \dots, c_p)$ and $S = \text{diag}(s_1, \dots, s_{\min\{m,p\}})$ with $0 \leq c_1 \leq \dots \leq c_p \leq 1$ and $1 \geq s_1 \geq \dots \geq s_{\min\{m,p\}} \geq 0$. Furthermore, it holds that $s_j^2 + c_j^2 = 1$ for $j = 1, \dots, \min\{m,p\}$ and $c_j = 1$ for $j = m+1, \dots, p$.

If $m < p$ we define $s_{m+1} = \dots = s_p = 0$. Then $s_j^2 + c_j^2 = 1$ for all $j = 1, \dots, p$. The values $\sigma_j = c_j/s_j$ are called the generalized singular values of the pencil $\{A, B\}$. If $s_j = 0$ then $\sigma_j = \infty$. The j th column x_j of X is the right generalized singular vector associated with σ_j .

The generalized singular value pairs (c_j, s_j) satisfy $s_j^2 A^T A x_j = c_j^2 B^T B x_j$ and therefore $A^T A x_j = \sigma_j^2 B^T B x_j$ if σ_j is a finite generalized singular value of the pencil $\{A, B\}$. This shows that if B is the identity matrix I , the generalized singular values of $\{A, I\}$ are just the singular values of A .

The finite generalized singular values can also be described by a minmax characterization that can be derived from similar minmax characterizations of singular values;

$$\sigma_j = \min_{\substack{H \subset \mathbb{R}^p \\ \dim(H)=j}} \max_{\substack{x \in H \\ Bx \neq 0}} \frac{\|Ax\|_2}{\|Bx\|_2}. \tag{4.1}$$

A short proof is for example contained in Betcke (2005). How can we use the GSVD for the domain decomposition method from the previous section? The idea is to approximate the pencil $\{A_\Gamma(\lambda), A_\Omega(\lambda)\}$ by a pencil of ordinary matrices. Hence, we want to discretize $A_\Gamma(\lambda)$ and $A_\Omega(\lambda)$ using ordinary matrices $\tilde{A}_{H^1(\Gamma)}(\lambda)$ and $\tilde{A}_{L^2(\Omega)}(\lambda)$ such that

$$t(\lambda) = \min_{x \in \mathbb{R}^N \setminus \{0\}} \frac{\|A_\Gamma(\lambda)x\|_{H^1(\Gamma)}}{\|A_\Omega(\lambda)x\|_{L^2(\Omega)}} \approx \min_{x \in \mathbb{R}^N \setminus \{0\}} \frac{\|\tilde{A}_{H^1(\Gamma)}(\lambda)x\|_2}{\|\tilde{A}_{L^2(\Omega)}(\lambda)x\|_2}. \tag{4.2}$$

Let us first discretize $A_\Omega(\lambda)$. In each subdomain Ω_k we choose m_k interior discretization points $w_j^{(k)}$, $1 \leq j \leq m_k$. The quasi-matrix $A_k(\lambda)$ is now discretized by evaluating the columns of $A_k(\lambda)$ at the points $w_j^{(k)}$. We therefore have

$$A_k(\lambda) = [\Phi_1(z), \dots, \Phi_{N_k}(z)] \rightarrow \begin{bmatrix} \Phi_1^{(k)}(w_1^{(k)}) & \dots & \Phi_{N_k}^{(k)}(w_1^{(k)}) \\ \vdots & \vdots & \vdots \\ \Phi_1^{(k)}(w_{m_k}^{(k)}) & \dots & \Phi_{N_k}^{(k)}(w_{m_k}^{(k)}) \end{bmatrix} =: \tilde{A}_k(\lambda),$$

where the functions $\Phi_1^{(k)}, \dots, \Phi_{N_k}^{(k)}$ are the basis functions of $\mathcal{A}_k(\lambda)$. Hence, $A_\Omega(\lambda)$ is discretized as

$$A_\Omega(\lambda) \rightarrow \begin{bmatrix} \tilde{A}_1(\lambda) & 0 & 0 \\ 0 & \tilde{A}_2(\lambda) & 0 \\ 0 & 0 & \tilde{A}_3(\lambda) \end{bmatrix} =: \tilde{A}_{L^2(\Omega)}(\lambda).$$

When discretizing $A_\Gamma(\lambda)$ we must be slightly more careful because of the H^1 -inner product, which is used on the internal boundary segments. Let $A_{\Gamma,x}(\lambda)$ and $A_{\Gamma,y}(\lambda)$ be the quasi-matrices containing as columns the partial derivatives of the column functions of $A_\Gamma(\lambda)$ in the x - and y -direction. Then

$$\|A_\Gamma(\lambda)x\|_{H^1(\Gamma)}^2 = \|A_\Gamma(\lambda)x\|_{L^2(\Gamma)}^2 + \|A_{\Gamma,x}(\lambda)x\|_{L^2(\Gamma)}^2 + \|A_{\Gamma,y}(\lambda)x\|_{L^2(\Gamma)}^2. \tag{4.3}$$

By choosing discretization points on the internal boundary lines and evaluating the basis functions on these points we can proceed as in the case of $A_\Omega(\lambda)$ and obtain the discretized matrices

$\tilde{A}_{H^1(\Gamma)}(\lambda)$, $\tilde{A}_{L^2(\Gamma),x}(\lambda)$ and $\tilde{A}_{L^2(\Gamma),y}(\lambda)$. It follows that

$$A_\Gamma(\lambda) \rightarrow \begin{bmatrix} \tilde{A}_{L^2(\Gamma)}(\lambda) \\ \tilde{A}_{L^2(\Gamma),x}(\lambda) \\ \tilde{A}_{L^2(\Gamma),y}(\lambda) \end{bmatrix} =: \tilde{A}_{H^1(\Gamma)}(\lambda).$$

Our modified GSVD based domain decomposition method now has the form

$$\min_{\lambda} \min_{x \in \mathbb{R}^N \setminus \{0\}} \frac{\|\tilde{A}_{H^1(\Gamma)}(\lambda)x\|_2}{\|\tilde{A}_{L^2(\Omega)}(\lambda)x\|_2} = \min_{\lambda} \sigma_1(\lambda), \quad (4.4)$$

where $\sigma_1(\lambda)$ is the smallest generalized singular value of $\{\tilde{A}_{H^1(\Gamma)}(\lambda), \tilde{A}_{L^2(\Omega)}(\lambda)\}$. In Matlab the GSVD of two matrices A and B can be easily computed with the command `gsvd(A,B)`.

The discretization of the column functions can also be interpreted in terms of quadrature rules. If all discretization points have the equal weight w in the quadrature rule the matrix $w\tilde{A}_{H^1(\Gamma)}^T \tilde{A}_{H^1(\Gamma)}$ is an approximation to the matrix of H^1 -inner products $A_\Gamma^T A_\Gamma$. But in our experiments it turned out that it is not necessary to choose the points such that the error of certain quadrature rules becomes small. A healthy number of equally distributed points on the interfaces and some randomly chosen interior points always worked well enough to determine the eigenvalues.

Let us try this on the GWW-1 isospectral drum from Figure 1. On each internal boundary segment we use 40 equally distributed discretization points. The interior of each subdomain is discretized using 10 randomly chosen points. In each subdomain Ω_k we use $4n/\alpha_k$ basis functions for some $n \in \mathbb{N}$ (see also Driscoll (1997)).

How to choose the number of basis functions is discussed in Section 5. Figure 3 shows the convergence for the first eigenvalue on this domain. The eigenvalue approximation is obtained in each step by minimizing $\sigma_1(\lambda)$ with the Matlab function `fminsearch`. More efficient minimization methods that utilize the V-shaped form of the curve of $\sigma_1(\lambda)$ close to an eigenvalue are possible. But we will not go into this here. To compute the relative error we used the eigenvalue approximation $\lambda_1 \approx 2.5379439997986$ obtained for $N = 18$. The dashed curve shows the value $\sigma_1(\lambda)$ for each N evaluated at the corresponding eigenvalue approximation. The rate of convergence seems very similar to the rate observed by Driscoll (compare with Figure 3.2 in Driscoll (1997)). The advantage of our method is that we avoid a squared formulation, making the problem better conditioned. Furthermore, we need neither the explicit evaluation of integrals nor do we have to compute derivatives of eigenvalues, which makes the GSVD approach easier to implement than the methods of Descloux, Tolley and Driscoll.

To conclude this section let us slightly generalize the current method. In the formulations of Descloux, Tolley and Driscoll it was always assumed that all local basis functions are zero on $\partial\Omega$ making it unnecessary to match the zero boundary conditions (1.1b). But we can easily extend the method to also work for basis functions which are not automatically zero on the boundary. Let the quasi-matrix $A_{\partial\Omega}$ be defined as

$$A_{\partial\Omega}(\lambda) := \begin{bmatrix} A_1(\lambda) & & & \\ & \ddots & & \\ & & & A_p(\lambda) \end{bmatrix},$$

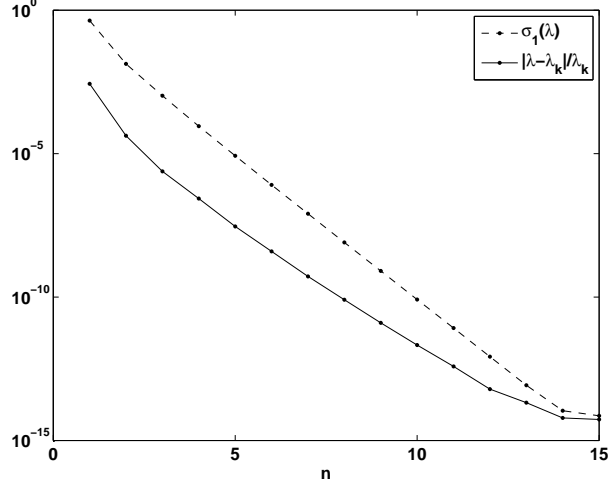


FIG. 3. Convergence of the first eigenvalue (solid line) and the corresponding values $\sigma_1(\lambda)$ (dashed line) for a growing number of basis functions on the GWW-1 isospectral drum (compare also with Driscoll (1997) Figure 3.2).

where the columns of the quasi-matrix $A_j(\lambda)$ are the basis functions in $\mathcal{A}_j(\lambda)$, $1 \leq j \leq p$. The error on $\partial\Omega$ is measured using the L^2 -norm

$$\|u\|_{L^2(\partial\Omega)}^2 := \int_{\partial\Omega} u^2(s) ds.$$

In quasi-matrix notation this is

$$\|u\|_{L^2(\partial\Omega)} = \|A_{\partial\Omega}(\lambda)x\|_{L^2(\partial\Omega)}.$$

The value $t(\lambda)$ from (4.2) becomes

$$\tilde{t}(\lambda) = \min_{x \in \mathbb{R}^N \setminus \{0\}} \frac{\left(\|A_\Gamma(\lambda)x\|_{H^1(\Gamma)}^2 + \|A_{\partial\Omega}(\lambda)x\|_{L^2(\partial\Omega)}^2 \right)^{1/2}}{\|A_\Omega(\lambda)x\|_{L^2(\Omega)}},$$

which corresponds to using the modified functional

$$\tilde{\mathcal{T}}(\lambda, u) := \mathcal{T}(\lambda, u) + \sum_{k=1}^p \int_{\partial\Omega \cap \partial\Omega_k} u^2(s) ds,$$

instead of $\mathcal{T}(\lambda, u)$.

By choosing a set of boundary collocation points we can discretize $A_{\partial\Omega}$ in the usual way by evaluating the column functions of the matrices $A_j(\lambda)$ at the collocation points belonging to $\partial\Omega \cap \partial\Omega_j$. We obtain the matrix $\tilde{A}_{L^2(\partial\Omega)}(\lambda)$ and instead of $\tilde{A}_{H^1(\Gamma)}(\lambda)$ we use the matrix

$$\tilde{A}_{L^2(\partial\Omega), H^1(\Gamma)}(\lambda) := [\tilde{A}_{L^2(\partial\Omega)}(\lambda)^T \quad \tilde{A}_{H^1(\Gamma)}(\lambda)^T]^T.$$

If we have only one subdomain Ω_1 with $\Omega_1 = \Omega$ there are no internal boundary lines and the quasi-matrix $A_\Gamma(\lambda)$ is empty. Then we only need to minimize the error $\|A_{\partial\Omega}(\lambda)x\|_{L^2(\partial\Omega)}/\|A_\Omega(\lambda)x\|_{L^2(\Omega)}$, which after discretization is the GSVD formulation of the Method of Particular Solutions (see Betcke (2006)).

Similarly to Lemma 2.1 the eigenvalue error can be bounded by \tilde{t} as

$$\frac{|\lambda - \lambda_k|}{\lambda_k} \leq C\tilde{t}(\lambda). \quad (4.5)$$

The proof is just a slight modification of the Lemmas 4.6, 4.9 and 4.10 in Descloux & Tolley (1983).³ This result can also be interpreted as a generalization of classical error bounds for the MPS (see Moler & Payne (1968), Kuttler & Sigillito (1978), Still (1988)). Since in an interval $[a, b]$ containing an eigenvalue λ_k of (1.1) we have

$$\min_{\lambda \in [a, b]} \tilde{t}(\lambda) \leq \tilde{t}(\lambda_k)$$

it follows from (4.5) that the rate of convergence of eigenvalue approximations can be estimated by the rate with which $\tilde{t}(\lambda_k) \rightarrow 0$ for a growing number of basis functions. This is investigated in the following Section.

5. Convergence analysis

Consider the example domain in Figure 4. The four corners of the domain are $z_1 = 0$, $z_2 = 0.3 + 1/\tan(\frac{3\pi}{8})$, $z_3 = 0.3 + 1/\tan(\frac{3\pi}{8}) + 1i$, $z_4 = 1/\tan(\frac{3\pi}{8}) + 1i$. The corresponding interior angles π/α_k are defined by $\alpha_1 = \frac{8}{3}$, $\alpha_2 = 2$, $\alpha_3 = 2$, $\alpha_4 = \frac{8}{5}$. By reflection one can show that eigenfunctions on this domain can have singularities only at z_1 and z_4 .

The dashed line in Figure 5 shows the convergence of the generalized singular value $\sigma_1(\lambda)$ for a growing number N of basis functions in each subdomain, where the subdomains are defined as in Figure 4. After $N = 40$ we achieve an accuracy of about 10^{-5} with an overall number of 160 basis functions. Let us now try the same with a domain decomposition into the two subdomains $G_1 = \Omega_1 \cup \Omega_2$ and $G_2 = \Omega_3 \cup \Omega_4$. The corresponding convergence curve is shown as a solid line in Figure 5. With $N = 40$ basis functions in each subdomain the value $\sigma_1(\lambda)$ is close to machine precision, a drastic improvement to the other decomposition although we are only using half the number of basis functions.

This example shows that a finer decomposition does not automatically lead to an improved accuracy with this method. The opposite can be the case. In this section we will analyze the convergence and provide theorems that will allow us to compute asymptotic convergence rates that give a very good match with the observed rates.

We denote the sup-norm of a function f in a set S by

$$\|f\|_{\infty, S} := \sup_{z \in S} |f(z)|.$$

³By using the inequality $\int_\Omega h^2(x)dx \leq C \int_{\partial\Omega} h^2(s)ds$ (see Kuttler (1972)) for a function h that is harmonic in Ω , where C is a domain dependent constant, Lemma 4.6 can be proved with the modified functional. Lemma 4.9 and from that the eigenvalue bound in Lemma 4.10 for $\tilde{t}(\lambda)$ follow directly.

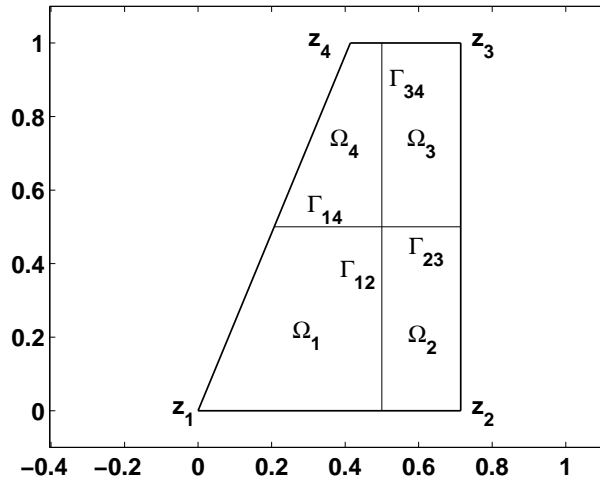


FIG. 4. The domain decomposition of a quadrilateral.

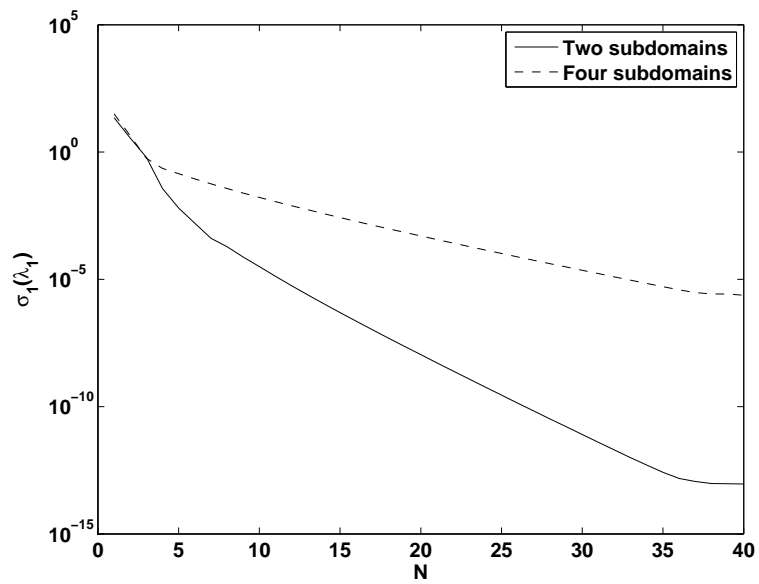


FIG. 5. Comparison of the convergence of $\sigma(\lambda_1)$ for a growing number N of basis functions in each subdomain in the cases of division into two subdomains and four subdomains.

Let (λ_k, u_k) be an eigenpair of (1.1) on Ω . $\tilde{\mathcal{T}}(\lambda, u)$ can be estimated as

$$\begin{aligned}
\tilde{\mathcal{T}}(\lambda_k, u) &= \sum_{j < l} \int_{\Gamma_{jl}} |u^{(j)}(s) - u^{(l)}(s)|^2 + |\nabla u^{(j)}(s) - \nabla u^{(l)}(s)|^2 ds \\
&+ \sum_{j=1}^p \int_{\partial\Omega \cap \partial\Omega_j} |u^{(j)}(s)|^2 ds \\
&\leq C_1 \sum_{j < l} \|\nabla u^{(j)} - \nabla u_k\|_{\infty, \Gamma_{jl}}^2 + \|\nabla u^{(l)} - \nabla u_k\|_{\infty, \Gamma_{jl}}^2 \\
&+ C_2 \sum_{j=1}^p \|u_k - u^{(j)}\|_{\infty, \Omega_j}^2,
\end{aligned} \tag{5.1}$$

where C_1, C_2 are constants which depend on the subdomains Ω_j .

Hence, we need to estimate the error of approximating the eigenfunction u_k and its derivative by Fourier-Bessel functions and their derivatives. Descloux and Tolley assumed that the maximum distance of the points in the closure of each subdomain Ω_j to the expansion point z_j is smaller than the distance of z_j to the next nearest singularity of u_k in order to show that on each subdomain any eigenfunction u_k can be expanded into an absolutely convergent series of the form

$$u_k(r, \theta) = \sum_{k=1}^{\infty} a_k J_{\alpha_j k}(\sqrt{\lambda_k} r) \sin \alpha_j k \theta. \tag{5.2}$$

The rate of convergence of the first N terms of (5.2) to u_k for growing N can be estimated in the same way as a power series. Using this Descloux and Tolley were able to give bounds on the exponential rate of convergence.

However, power series estimates only give optimal convergence rates on circles. On other domains the optimum rate of convergence that can be achieved with polynomial approximation is usually much better than that of power series expansions. In the following sections we show how the error in the domain decomposition method can be bounded by these optimal rates of polynomial approximation of certain functions in the complex plane.

This new analysis also allows to relax the restriction of Descloux and Tolley on the domain decomposition. For our analysis we have the weaker assumption that the continuation towards ∞ of the two arcs adjacent to the corner z_j do not intersect with the subdomain Ω_j . A domain decomposition that violates this restriction is shown in Figure 6. The continuation of the arcs adjacent to z_2 intersect Ω_2 in the left half of the domain. Since every linear combination of Fourier-Bessel sine functions around z_2 is zero on these arcs it is not possible to accurately approximate an eigenfunction on Ω_2 that is not zero on the intersection of these arcs with Ω_2 . This restriction will become important in Lemma 5.4.

5.1 Some results from Vekua's theory

Our results are based on Vekua's theory of elliptic partial differential equations with analytic coefficient functions (see Vekua (1948), Schryer (1972), Eisenstat (1974), Still (1989), Still (1992), Melenk (1999)). The presentation here follows closely the one given in Eisenstat (1974).

Before we proceed let us introduce some further notation. For $x, y \in \mathbb{R}$ let $z = x + iy \in \mathbb{C}$. The complex conjugate of z is denoted by $\bar{z} = x - iy$. For a set $\Omega \subset \mathbb{C}$ the closure of Ω is

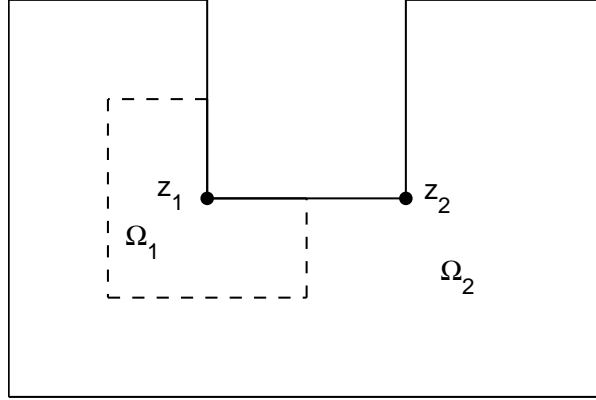


FIG. 6. This domain decomposition is not valid for our convergence analysis since the continuation of the arcs adjacent to Ω_2 towards ∞ intersect Ω_2 .

denoted by $\overline{\Omega}$. The reflection of Ω at the real axis is defined as $\Omega^* := \{z \in \mathbb{C} \mid \bar{z} \in \Omega\}$. Let ϕ be an analytic function in Ω . Then we denote by $\bar{\phi}$ the function defined as $\bar{\phi}(z) := \phi(\bar{z})$ for $z \in \Omega^*$. For two sets $\Omega_1, \Omega_2 \subset \mathbb{C}$ we also define

$$\Omega_1 \times \Omega_2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \in \Omega_1, z_2 \in \Omega_2\}.$$

Let $u(x, y)$ be a solution of (1.1a) in a simply connected domain Ω and define

$$U(z, z^*) := u\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right). \tag{5.3}$$

It follows that $U(z, \bar{z}) = u(x, y)$. Since u is real analytic in Ω there exist small neighborhoods $U_1 \subset \Omega$ of z and $U_2 \subset \Omega^*$ of \bar{z} such that U can be analytically continued into $U_1 \times U_2 \subset \mathbb{C}^2$ as a holomorphic function of the two complex variables z and z^* . Vekua showed that this continuation does not only exist in the small but that $\Omega \subset U_1$ and $\Omega^* \subset U_2$ for solutions of elliptic PDEs with analytic coefficients. Furthermore, he showed that there exists a 1 – 1 map between the solutions of elliptic PDEs with analytic coefficients and holomorphic functions.

Fix $z_0 \in \Omega$ and let

$$I[\phi; z_0](z, z^*) := \frac{1}{2} \left\{ G(z, \bar{z}_0, z, z^*)\phi(z) + \int_{z_0}^z \phi(t)H(t, \bar{z}_0, z, z^*)dt + G(z_0, z^*, z, z^*)\bar{\phi}(z^*) + \int_{\bar{z}_0}^{z^*} \bar{\phi}(t^*)H^*(z_0, t^*, z, z^*)dt^* \right\}, \tag{5.4}$$

where ϕ is holomorphic in Ω . In the special case of the elliptic PDE $\Delta u + \lambda u = 0$ the functions

G , H and H^* are defined as⁴

$$\begin{aligned} G(t, t^*, z, z^*) &:= J_0(\sqrt{\lambda}\sqrt{(z-t)(z^*-t^*)}) \\ H(t, t^*, z, z^*) &:= -\frac{\partial}{\partial t}G(t, t^*, z, z^*) \\ H^*(t, t^*, z, z^*) &:= -\frac{\partial}{\partial t^*}G(t, t^*, z, z^*). \end{aligned}$$

For $z^* = \bar{z}$ this simplifies to

$$\begin{aligned} I[\phi; z_0](z, \bar{z}) &= \operatorname{Re} \left\{ G(z, \bar{z}_0, z, \bar{z})\phi(z) + \int_{z_0}^z \phi(t)H(t, \bar{z}_0, z, \bar{z})dt \right\} \\ &:= \operatorname{Re}\{V[\phi; z_0]\}(z). \end{aligned} \quad (5.5)$$

Vekua showed the following theorem.

THEOREM 5.1 (VEKUA (1948)) Let Ω be simply connected and fix $z_0 \in \Omega$. Then there exists a unique function ϕ holomorphic in Ω with $\phi(z_0)$ real such that

$$\begin{aligned} u(x, y) &= \operatorname{Re}\{V[\phi; z_0]\}(z), \quad z = x + iy \in \Omega \\ U(z, z^*) &= I[\phi; z_0](z, z^*), \quad (z, z^*) \in \Omega \times \Omega^*. \end{aligned}$$

Moreover,

$$\phi(z) = 2U(z, \bar{z}_0) - U(z_0, \bar{z}_0)G(z_0, \bar{z}_0, z, \bar{z}_0). \quad (5.6)$$

If Ω is bounded the Vekua operator $\operatorname{Re}\{V[\phi; z_0]\}$ is bounded by

$$\|\operatorname{Re}\{V[\phi; z_0]\}\|_{\infty, \Omega} \leq \|G\|_{\infty} \|\phi\|_{\infty, \Omega} + \int_{z_0}^z \|H\|_{\infty} \|\phi\|_{\infty, \Omega} d|t| \leq K_V \|\phi\|_{\infty, \Omega}$$

(see Eisenstat (1974)). Similarly, we can bound the function $U(z, z^*) = I[\phi, z_0]$ in $\Omega \times \Omega^*$ as

$$\begin{aligned} \|U\|_{\infty, \Omega \times \Omega^*} &\leq \frac{1}{2} \left\{ \|G\|_{\infty} \|\phi\|_{\infty, \Omega} + \int_{z_0}^z \|H\|_{\infty} \|\phi\|_{\infty, \Omega} d|t| \right. \\ &\quad \left. + \|G\|_{\infty} \|\bar{\phi}\|_{\infty, \Omega^*} + \int_{\bar{z}_0}^{\bar{z}^*} \|H^*\|_{\infty} \|\bar{\phi}\|_{\infty, \Omega^*} d|t^*| \right\} \\ &\leq K_I \|\phi\|_{\infty, \Omega} \end{aligned} \quad (5.7)$$

since $\|\phi\|_{\infty, \Omega} = \|\bar{\phi}\|_{\infty, \Omega^*}$. The constants K_V and K_I only depend on the domain Ω and the value λ . $\|G\|_{\infty}$, $\|H\|_{\infty}$ and $\|H^*\|_{\infty}$ are the suprema of these functions over the domain $\Omega \times \Omega^* \times \Omega \times \Omega^* \subset \mathbb{C}^4$.

We can therefore bound the error of approximating an eigenfunction u_k by the error of approximating an associated holomorphic function ϕ_k and use complex approximation theory to establish the rate of convergence. This idea was used in Eisenstat (1974) to give algebraic convergence estimates for the approximation of solutions of elliptic PDEs by particular solutions.

⁴ G is the complex Riemann function of the elliptic PDE $\Delta u + \lambda u = 0$ (see Henrici (1957) for a beautiful survey of complex Riemann functions and Vekua's theory).

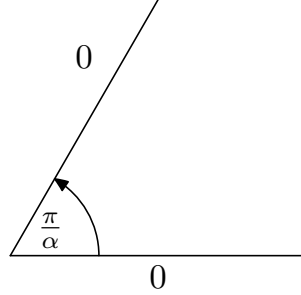


FIG. 7. A wedge with interior angle π/α . Any Fourier-Bessel function of the form $J_{\alpha k}(\sqrt{\lambda}r) \sin \alpha k \theta$ with $\lambda > 0$ and $k = 1, 2, \dots$ satisfies the eigenvalue equation and the zero boundary conditions on this domain.

A similar technique was also used in Monk & Wang (1999) to obtain algebraic estimates for the use of particular solutions in the context of acoustic scattering.

An important role in Vekua's theory is played by the solutions of $-\Delta u = \lambda u$ associated with the functions z^k and iz^k given by the following Lemma (see also Example 3.14 in Melenk (1999) or Lemma 1 in Still (1992)), which follows immediately by straightforward calculation.

LEMMA 5.1 Let Ω be a wedge with interior angle π/α (see Figure 7). Then

$$\begin{aligned} -\frac{2^{\alpha k} \Gamma(\alpha k + 1)}{\sqrt{\lambda}^{\alpha k}} J_{\alpha k}(\sqrt{\lambda}r) \sin \alpha k \theta &= \operatorname{Re}\{V[iz^{\alpha k}, 0]\}(z), \\ \frac{2^{\alpha k} \Gamma(\alpha k + 1)}{\sqrt{\lambda}^{\alpha k}} J_{\alpha k}(\sqrt{\lambda}r) \cos \alpha k \theta &= \operatorname{Re}\{V[z^{\alpha k}, 0]\}(z). \end{aligned}$$

Melenk calls these special solutions generalized harmonic polynomials since in the case of the Laplace equation $-\Delta u = 0$ the functions $V[z^k, 0]$ and $V[iz^k, 0]$ are just the standard harmonic polynomials $r^k \cos k\theta$ and $-r^k \sin k\theta$. From Lemma 5.1 it follows that approximating in a subdomain Ω_j with a Fourier-Bessel series corresponds to polynomial approximation in the complex plane.

5.2 Bounding the rate of convergence

Let Ω_j be a subdomain of Ω and z_j the corner at which we expand with Fourier-Bessel functions. We assume that for each subdomain Ω_j $1 \leq j \leq p$ an eigenfunction u_k of (1.1) has at most one singularity in Ω_j , which lies at the corner z_j . This is always satisfied for the original domain decomposition of Descloux and Tolley. In our modified domain decomposition, where we allow each subdomain Ω_j to contain more than one corner of Ω this has to be explicitly ensured in order for the exponential convergence results developed in this section to hold.

Without restriction we choose $z_j = 0$ and the orientation of the wedge at the corner such that the right arc is part of the x -axis (see for example the corner at z_1 in Figure 4). From Lemma 5.1 we obtain

$$\left\| u_k - \sum_{l=1}^N a_l J_{\alpha_j l}(\sqrt{\lambda_k} r) \sin \alpha_j l \theta \right\|_{\infty, \Omega_j} \leq K_V \left\| \phi_k^{(j)} - \sum_{l=1}^N \tilde{a}_l iz^{\alpha_j l} \right\|_{\infty, \Omega_j}, \quad (5.8)$$

where $\phi_k^{(j)}$ is the holomorphic function associated with $u_k|_{\Omega_j}$ around z_j , i.e. $u_k = \operatorname{Re}\{V[\phi_k, z_j]\}$ in Ω_j , and $\tilde{a}_l = -\frac{\sqrt{\lambda_k}^{\alpha_j l}}{2^{\alpha_j l} \Gamma(\alpha_j l + 1)} a_l$. The following Lemma is identical to the results of §6, Example 3 in Still (1989). For completeness we give the proof here.

LEMMA 5.2 There exists $R > 0$ such that for $z \in \overline{\Omega_j}$ and $|z| < R$ the function $\phi_k^{(j)}$ has the absolutely convergent expansion

$$\phi_k^{(j)}(z) = \sum_{l=1}^{\infty} \tilde{b}_l i z^{\alpha_j l}, \quad \tilde{b}_l \in \mathbb{R}. \quad (5.9)$$

Proof. By expanding $u_k(r, \theta)$ into a Fourier series for a fixed $r > 0$ it is possible to show that there exists $R > 0$ such that $u_k(r, \theta)$ has the absolutely convergent expansion

$$u_k(r, \theta) = \sum_{l=1}^{\infty} b_l J_{\alpha_j l}(\sqrt{\lambda_k} r) \sin \alpha_j l \theta \quad (5.10)$$

for all $r < R$ and that,

$$\sum_{l=1}^{\infty} |b_l| |J_{\alpha_j l}(\sqrt{\lambda_k} r)| < \infty \quad (5.11)$$

(see Descloux & Tolley (1983) or Still (1989)). Bessel functions can be estimated as

$$\frac{|z|^\nu}{2^\nu \Gamma(\nu + 1)} (1 - \epsilon) \leq |J_\nu(z)| \leq \frac{|z|^\nu}{2^\nu \Gamma(\nu + 1)}, \quad |z| \leq R$$

for every $\epsilon > 0$ and $\nu > \nu_0(R, \epsilon)$ sufficiently large. This follows directly from a power series expansion (see Still (1989)). Let $\tilde{b}_l := -\frac{\sqrt{\lambda_k}^{\alpha_j l}}{2^{\alpha_j l} \Gamma(\alpha_j l + 1)} b_l$. Then for l large enough the terms in (5.9) can be bounded by

$$|\tilde{b}_l| \cdot |z|^{\alpha_j l} \leq \frac{1}{1 - \epsilon} |J_{\alpha_j l}(\sqrt{\lambda_k} r)| \cdot |b_l|.$$

Together with (5.11) the absolute convergence of (5.9) follows. \square

We need the absolute convergence close to a corner for the proof of the following Lemma.

LEMMA 5.3 Let $w = z^{\alpha_j}$. Define $\tilde{\phi}_k^{(j)}(w) := \phi_k^{(j)}(w^{\frac{1}{\alpha_j}})$ and $\overline{\Omega_j^{\alpha_j}} := \{z^{\alpha_j} | z \in \overline{\Omega_j}\}$. Choose for z^{α_j} and $w^{\frac{1}{\alpha_j}}$ a common branch cut outside $\overline{\Omega_j}$ and $\overline{\Omega_j^{\alpha_j}}$. Then $\tilde{\phi}_k^{(j)}$ can be analytically continued across $\overline{\Omega_j^{\alpha_j}}$.

Proof. Due to the domain decomposition the function $u_k|_{\Omega_j}$ can be analytically continued across $\overline{\Omega_j}$ except possibly close to zero. It follows that the function $\phi_k^{(j)}$ and therefore $\tilde{\phi}_k^{(j)}$ can also be analytically continued there. From Lemma 5.2 it follows that $\tilde{\phi}_k^{(j)}$ has the absolutely convergent series representation

$$\tilde{\phi}_k^{(j)}(w) = \sum_{l=1}^{\infty} \tilde{b}_l i w^l \quad (5.12)$$

close to zero. Hence, $\tilde{\phi}_k^{(j)}$ can also be analytically continued in a neighborhood of zero. \square

Lemma 5.3 states that the singularity of $\phi_k^{(j)}$ at $z_j = 0$ can be canceled out by the transformation $w = z^{\alpha_j}$. Furthermore, from (5.8) it follows that approximating u_k in Ω_j with Fourier-Bessel sine functions corresponds to polynomial approximation in the domain $\Omega_j^{\alpha_j}$ with polynomials that have purely imaginary coefficients. By one further transformation this can be turned into a standard polynomial approximation problem.

LEMMA 5.4 Let the set D be defined as the closure of the union of $\overline{\Omega_j^{\alpha_j}}$ with its reflection at the real axis. Define the continuation f of $\tilde{\phi}_k^{(j)}$ into the whole of D as

$$f(w) := \begin{cases} \tilde{\phi}_k^{(j)}(w); & w \in \overline{\Omega_j^{\alpha_j}} \\ -\tilde{\phi}_k^{(j)}(\bar{w}); & \bar{w} \in \overline{\Omega_j^{\alpha_j}} \end{cases}$$

Then f is analytic in D and the best approximating polynomial p_N of maximal degree N for f in D in the sup-norm has purely imaginary coefficients. Furthermore, it is identical to the best approximating polynomial for $\tilde{\phi}_k^{(j)}$ on $\overline{\Omega_j^{\alpha_j}}$ in the sup-norm from the set of polynomials of maximal degree N with purely imaginary coefficients.

Proof. From (5.12) it follows that close to zero $\text{Re}\{\tilde{\phi}_k^{(j)}(w)\} = 0$ for real w . Furthermore, $\tilde{\phi}_k^{(j)}$ can be analytically continued across the whole of $\overline{\Omega_j^{\alpha_j}}$. Hence, by analytic continuation along the real line we have $\text{Re}\{\tilde{\phi}_k^{(j)}(w)\} = 0$ on $\mathbb{R} \cap \overline{\Omega_j^{\alpha_j}}$. It follows that f defines an analytic continuation of $\tilde{\phi}_k^{(j)}$ into the whole of D .

Since p_N is the best approximating polynomial for f in D it follows that \bar{p}_N is the best approximating polynomial for \bar{f} . But since $\bar{f}(w) = -f(w)$ for $w \in D$ we have $\bar{p}_N = -p_N$. Let

$$p_N(w) = \sum_{j=0}^N c_j w^j.$$

It follows that

$$0 = \bar{p}_N(w) + p_N(w) = \sum_{j=0}^N (c_j + \bar{c}_j) w^j = \sum_{j=0}^N 2\text{Re}\{c_j\} w^j$$

and therefore $\text{Re}\{c_j\} = 0$ for $j = 1, \dots, N$. Hence, p_N has purely imaginary coefficients. Since p_N and f are symmetric around the real axis and $f = \tilde{\phi}_k^{(j)}$ in $\overline{\Omega_j^{\alpha_j}}$ it follows that p_N is also the best approximating polynomial for $\tilde{\phi}_k^{(j)}$ from the space of polynomials of maximum degree N with purely imaginary coefficients. \square

In Lemma 5.4 the restriction on the domain decomposition that the continuation of the arcs adjacent to z_j does not intersect Ω_j becomes important since the analytic continuation f to the whole of D is only well defined if the subdomain $\Omega_j^{\alpha_j}$ is restricted to the upper half plane.

We are now able to give the first convergence result.

THEOREM 5.2 Let (λ_k, u_k) be an eigenpair of (1.1). In each subdomain Ω_j there exists a Fourier-Bessel expansion of the form

$$u^{(j)}(r, \theta) = \sum_{l=1}^{N_j} a_l J_{\alpha_j l}(\sqrt{\lambda_k} r) \sin \alpha_j l \theta$$

and a number $\rho_j > 1$ such that

$$\|u_k - u^{(j)}\|_{\infty, \Omega_j} = O(R^{-N_j})$$

for every $1 < R < \rho_j$ as $N_j \rightarrow \infty$.

Proof. Let the function f be defined as in Lemma 5.4. From Lemma 5.3 we know that $\tilde{\phi}_k^{(j)}$ is analytic on $\overline{\Omega_j^{\alpha_j}}$. Hence, f is analytic on the closed set D . Let p_{N_j} be the best approximating polynomial of f on D from the set of polynomials of maximal degree N_j . From the maximum convergence theorem of complex approximation theory it follows that there exists $\rho_j > 1$ such that

$$\|f - p_{N_j}\|_{\infty, D} = O(R^{-N_j})$$

for every $1 < R < \rho_j$ but for no $R > \rho_j$ (Walsh, 1960, p. 79). Since p_{N_j} has purely imaginary coefficients there exist coefficients $\tilde{a}_l \in \mathbb{R}$ such that

$$\begin{aligned} \left\| u_k - \sum_{l=1}^{N_j} a_l J_{\alpha_j l}(\sqrt{\lambda_k} r) \sin \alpha_j l \theta \right\|_{\infty, \Omega_j} &\leq K_V \left\| \phi_k^{(j)} - \sum_{l=1}^{N_j} \tilde{a}_l i z^{\alpha_j l} \right\|_{\infty, \Omega_j} \\ &= K_V \left\| \tilde{\phi}_k^{(j)} - \sum_{l=1}^{N_j} \tilde{a}_l i w^l \right\|_{\infty, \Omega_j^{\alpha_j}} \\ &= K_V \left\| f - \sum_{l=1}^{N_j} \tilde{a}_l i w^l \right\|_{\infty, D} \\ &= O(R^{-N_j}) \end{aligned}$$

for $1 < R < \rho_j$ and $a_l := -\frac{2^{\alpha_j l} \Gamma(\alpha_j l + 1)}{\sqrt{\lambda^{\alpha_j l}}} \tilde{a}_l$. \square

A similar result was also shown in Lemma 4.1 of Descloux & Tolley (1983) using different techniques. But the approach presented here allows to establish tighter bounds ρ_j on the exponential convergence. This is discussed in Section 6.

It is now left to bound $\|\nabla u_k - \nabla u^{(j)}\|_{\infty, \Gamma_{jl}}$ on the interface Γ_{jl} between Ω_j and a neighboring subdomain Ω_l .

LEMMA 5.5 Let Γ_{jl} be a nonempty interface between Ω_j and a neighboring subdomain Ω_l . Let $\phi_k^{(j)}$ be the holomorphic function associated with $u_k|_{\Omega_j}$ around z_j and $\phi^{(j)}$ be the holomorphic function associated with $u^{(j)}$ around z_j . From

$$\|\phi_k^{(j)} - \phi^{(j)}\|_{\infty, \Omega_j} = O(R^{-N_j})$$

for every $1 < R < \rho_j$ it follows that

$$\|\nabla u_k - \nabla u^{(j)}\|_{\infty, \Gamma_{jl}} = O(R^{-N_j})$$

for every $1 < R < \rho_j$.

Proof. Fix $1 < \tilde{R} < \rho_j$. Assume that

$$\|\phi_k^{(j)} - \phi^{(j)}\|_{\infty, \Omega_j} = O(\tilde{R}^{-N_j}).$$

Therefore, also $\|\phi_k^{(j)} - \phi^{(j)}\|_{\infty, \Gamma_{jl}} = O(\tilde{R}^{-N_j})$. From the overconvergence of polynomial approximation in the complex plane (Walsh, 1960, §4.6-4.7) it follows that for arbitrary $\delta > 0$ there exists a neighborhood W of Γ_{jl} such that

$$\|\phi_k^{(j)} - \phi^{(j)}\|_{\infty, W} = O((\tilde{R} - \delta)^{-N_j}).$$

Together with the boundedness of the Vekua operator we obtain

$$\|U_k - U^{(j)}\|_{\infty, W \times W^*} \leq K_I \|\phi_k - \phi^{(j)}\|_{\infty, W} = O((\tilde{R} - \delta)^{-N_j}),$$

where $U_k(z, z^*)$ and $U^{(j)}(z, z^*)$ are the holomorphic extensions of $u_k(x, y)$ and $u^{(j)}(x, y)$ into $W \times W^* \subset \mathbb{C}^2$ as defined in (5.3). To simplify the notation we define $\hat{U} := U_k - U^{(j)}$.

Choose $\epsilon > 0$ such that $K_\epsilon(z) := \{\xi \in \mathbb{C} : |\xi - z| = \epsilon\} \subset W$ for all $z \in \Gamma_{jl}$. Using Cauchy's integral theorem we obtain

$$\frac{\partial}{\partial z} \hat{U}(z, z^*) = \frac{1}{2\pi i} \int_{K_\epsilon(z)} \frac{\hat{U}(\xi, z^*)}{(\xi - z)^2} d\xi$$

for $z \in \Gamma_{jl}$. It follows that

$$\left\| \frac{\partial}{\partial z} \hat{U} \right\|_{\infty, \Gamma_{jl} \times \Gamma_{jl}^*} \leq \frac{1}{\epsilon} \|\hat{U}\|_{W \times W^*} = O((\tilde{R} - \delta)^{-N_j}). \quad (5.13)$$

Similarly, we conclude that

$$\left\| \frac{\partial}{\partial z^*} \hat{U} \right\|_{\Gamma_{jl} \times \Gamma_{jl}^*} \leq \frac{1}{\epsilon} \|\hat{U}\|_{W \times W^*} = O((\tilde{R} - \delta)^{-N_j}). \quad (5.14)$$

Since

$$\begin{aligned} \frac{\partial}{\partial x} u(x, y) &= \frac{\partial}{\partial z} U(z, \bar{z}) + \frac{\partial}{\partial z^*} U(z, \bar{z}), \\ \frac{\partial}{\partial y} u(x, y) &= i \left[\frac{\partial}{\partial z} U(z, \bar{z}) - \frac{\partial}{\partial z^*} U(z, \bar{z}) \right] \end{aligned}$$

the proof follows from (5.13), (5.14) and the fact that for all $1 < R < \rho_j$ there exists a value $1 < \tilde{R} < \rho_j$ and $\delta > 0$ such that $R := \tilde{R} - \delta$. \square

As immediate consequence of Theorem 5.2 and Lemma 5.5 the exponential convergence of the method of Descloux and Tolley follows.

THEOREM 5.3 Let (λ_k, u_k) be an eigenpair of (1.1) with $\|u_k\|_\Omega = 1$. Then there exist numbers $\rho_j > 1$ such that

$$\min_{u \in \mathcal{A}(\lambda_k)} \frac{\tilde{\mathcal{T}}(\lambda_k, u)}{\mathcal{M}(\lambda_k, u)} = \sum_{j=1}^p O(R_j^{-2N_j})$$

for all $1 < R_j < \rho_j$, $1 \leq j \leq p$.

Proof. From (5.1), Theorem 5.2 and Lemma 5.5 it follows immediately that there exist local Fourier-Bessel expansions $u^{(j)}$ and numbers $\rho_j > 1$ such that

$$\tilde{T}(\lambda_k, u) = \sum_{j=1}^p O(R_j^{-2N_j}) \quad (5.15)$$

for all $1 < R_j < \rho_j$, $1 \leq j \leq p$ and u defined by $u|_{\Omega_j} = u^{(j)}$.

We can estimate $\mathcal{M}(\lambda_k, u)$ as

$$\begin{aligned} \mathcal{M}(\lambda_k, u) &= \sum_{j=1}^p \|u^{(j)}\|_{L^2(\Omega_j)}^2 \\ &\geq \sum_{j=1}^p \left[\|u_k\|_{L^2(\Omega_j)} - \|u^{(j)} - u_k\|_{L^2(\Omega_j)} \right]^2 \\ &\geq \sum_{j=1}^p \left[\|u_k\|_{L^2(\Omega_j)} - C_j \|u^{(j)} - u_k\|_{\infty, \Omega_j} \right]^2 \\ &= \sum_{j=1}^p \left[\|u_k\|_{L^2(\Omega_j)} - O(R_j^{-N_j}) \right]^2 \rightarrow \sum_{j=1}^p \|u_k\|_{L^2(\Omega_j)}^2 = 1 \end{aligned} \quad (5.16)$$

for $N_j \rightarrow \infty$, $1 \leq j \leq p$. The constants C_j only depend on Ω_j . Let $0 < \epsilon < 1$ it follows that there exists N_0 such that $\mathcal{M}(\lambda, u) \geq 1 - \epsilon$ for $N_j > N_0$, $1 \leq j \leq p$. Together with (5.15) we find

$$\min_{v \in \mathcal{A}(\lambda_k)} \frac{\tilde{T}(\lambda_k, v)}{\mathcal{M}(\lambda_k, v)} \leq \frac{1}{1 - \epsilon} \tilde{T}(\lambda_k, u) = \sum_{j=1}^p O(R_j^{-2N_j})$$

for $N_j \rightarrow \infty$. □

Since for the quasi-matrix formulation we have

$$\tilde{t}(\lambda_k) = \min_{x \in \mathbb{R}^N \setminus \{0\}} \frac{\left(\|A_\Gamma(\lambda)x\|_{H^1(\Gamma)}^2 + \|A_{\partial\Omega}(\lambda)x\|_{L^2(\partial\Omega)}^2 \right)^{1/2}}{\|A_\Omega(\lambda)\|_{L^2(\hat{\Omega})}} = \min_{u \in \mathcal{A}(\lambda_k)} \sqrt{\frac{\tilde{T}(\lambda_k, u)}{\mathcal{M}(\lambda_k, u)}}$$

we immediately obtain

COROLLARY 5.1 Let $N_j := k_j N$ for a number $k_j \in \mathbb{N}$ and define $\rho := \min_{1 \leq j \leq p} \rho_j^{k_j}$, where the ρ_j are defined as in Theorem 5.3. Then

$$\tilde{t}(\lambda_k) = O(R^{-N})$$

for every $1 < R < \rho$.

The numbers ρ_j are the optimal convergence rates on the subdomains Ω_j . Consider the example that $\rho_1 \approx \rho_2^2$ and assume that we use the same number N of Fourier-Bessel terms on both subdomains. Then the error on Ω_1 asymptotically behaves like $O(\rho_1^{-2N})$, while the error on Ω_2 behaves like $O(\rho_2^{-N})$. Since the overall convergence is determined by the subdomain with the slowest rate of convergence we are wasting expansion terms on Ω_1 . A better strategy

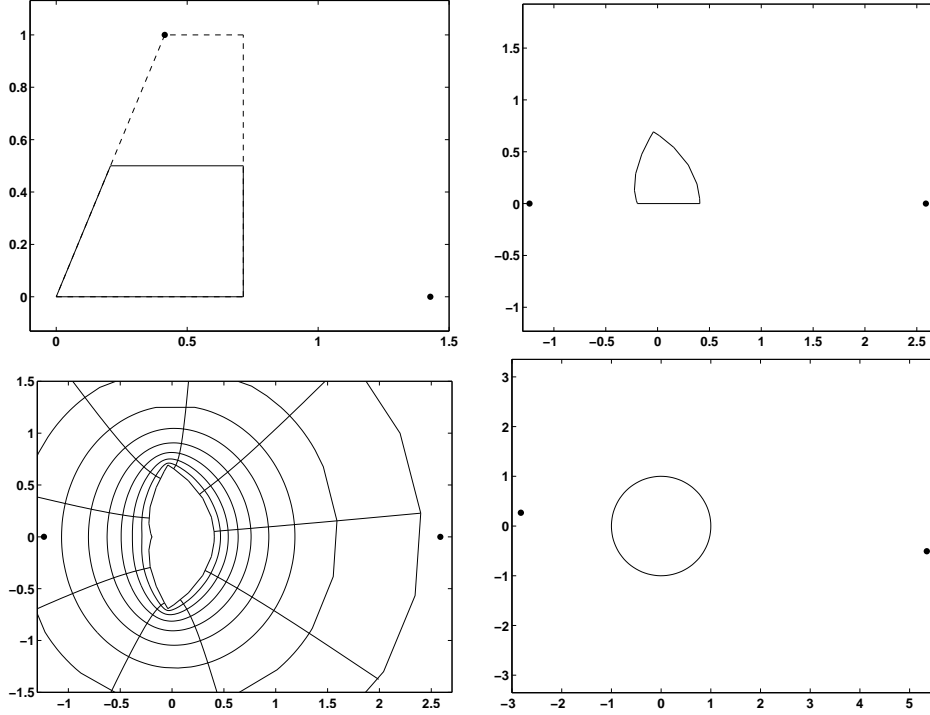


FIG. 8. The four steps to compute the convergence rate of approximating an eigenfunction u_k on the subdomain $G_1 = \Omega_1 \cup \Omega_2$ from Figure 4.

is to use N expansion terms on Ω_1 and $2N$ expansion terms on Ω_2 . A similar strategy was proposed by Descloux and Tolley to equalize the convergence rates on the different subdomains. They also proposed the rule of thumb to choose the number of Fourier-Bessel terms in each subdomain Ω_j proportional to the interior angle π/α_j of the corner at the expansion point z_j . If the numbers ρ_j are known then from our analysis it follows that an optimal number of expansion terms in each subdomain is $k_j N$, where $k_j \in \mathbb{N}$ is determined such that

$$\rho_1^{k_1} \approx \rho_2^{k_2} \approx \dots \approx \rho_N^{k_N}.$$

6. Computing the optimal rate of convergence

In this section we demonstrate how to compute the optimal convergence rates ρ_j . Let us return to the domain from Figure 4 with the decomposition $G_1 := \Omega_1 \cup \Omega_2$ and $G_2 := \Omega_3 \cup \Omega_4$. We first compute the value ρ_1 on G_1 . The only singularity of u_k in $\overline{G_1}$ is at the point $z_1 = 0$. This is also a singularity of the holomorphic function $\phi_k^{(1)}$ associated with u_k such that $u_k = \text{Re}\{V[\phi_k^{(1)}; z_1]\}$ in G_1 . Hence, from Lemma 5.3 it follows that after the transformation $w = z^{\frac{8}{3}}$ the function $\tilde{\phi}_k^{(1)}(w) := \phi_k^{(1)}(w^{\frac{3}{8}})$ is analytic on the whole of $\overline{G_1^{8/3}}$. Denote by D the closed set consisting of the union of $G_1^{\frac{8}{3}}$ with its reflection at the real line. From Lemma 5.4 it follows that the rate of convergence is determined by polynomial approximation on D .

The maximum convergence rate ρ_j of polynomial approximation on D can be determined by a conformal map Φ of the exterior of D to the exterior of the unit disc. For a point $w \in \mathbb{C} \setminus D$ denote by $|\Phi(w)|$ the conformal distance of w to D . Then $\rho_j = |\Phi(w_s)|$, where w_s is the singularity of the analytic continuation of $\tilde{\phi}_k^{(1)}$ with the smallest conformal distance to D .⁵ Hence, we need to determine the singularity w_s and the conformal map $\Phi(w)$.

The situation is shown in Figure 8. The upper left plot shows the original domain Ω from Figure 4 and the subdomain G_1 . The eigenfunction u_k has singularities at z_1 and z_4 . It follows that the unique analytic continuation of $\phi_k^{(1)}$ to the whole of Ω has singularities at these two points. Furthermore, by reflecting u_k at the right boundary line these singularities are also reflected. The image z'_1 of z_1 is $z'_1 = 2/\tan(\frac{3\pi}{8}) + 0.6$. By further reflection one can obtain other singularities, but these are too far away from G_1 to have a chance of influencing the value ρ_1 on G_1 . In the upper right picture of Figure 8 the domain G_1 and the singularities are shown after the map $w = z^{8/3}$. The singular corner at 0 has been straightened out by this map. In the lower left picture the domain $G_1^{8/3}$ is reflected at the real axis. The picture also shows some level curves of the conformal map $\Phi(w)$ of the exterior of this domain to the exterior of the unit disk, which were computed by Driscoll's "Schwarz-Christoffel toolbox".⁶ The lower right picture shows the positions of the two singularities after the map to the exterior of the unit disk. The map of $z_4^{8/3}$ has a closer distance to the unit disk than the map of $z_1^{8/3}$. It follows that $\rho_1 = |\Phi(z_4^{8/3})| \approx 2.82$. Similarly, on G_2 we obtain the value $\rho_2 \approx 1.86$. Since the slowest convergence rate dominates the overall convergence of the domain decomposition method it follows that

$$\tilde{t}(\lambda) = \min_{u \in \mathcal{A}(\lambda_k)} \sqrt{\frac{\tilde{T}(\lambda_k, u)}{\mathcal{M}(\lambda_k, u)}} = O(1.86^{-N}).$$

In Figure 9 we compare the observed convergence rates on this domain with the estimate 1.86^{-N} . At first the observed convergence seems to be faster than the estimated value. But then the slope of the observed rate slowly approaches the estimated value leading to a good match between them. One always has to keep in mind that the estimated rates are asymptotics for $N \rightarrow \infty$. The transient convergence behavior might differ from this.

Let us compare the estimated rate of 1.86^{-N} with the rates obtained by power series estimates. The radius r_1 of G_1 is

$$r_1 = \sup_{z \in G_1} |z - z_1| \approx 0.87,$$

while the closest singularity is z_4 with $|z_4 - z_1| \approx 1.08$. A power series estimate leads to the rate of convergence $\left(\frac{|z_4 - z_1|}{r_1}\right)^{8/3} \approx 1.78$. This is much slower than predicted by our value $\rho_1 \approx 2.82$. On G_2 the difference is even more striking. Here, we would obtain with a power series estimate an exponential rate of 1.05. Our computed value is $\rho_2 \approx 1.86$. Hence, by power series estimates

⁵Proofs of these results can be found in the books by Gaier (1987) and Walsh (1960). A beautiful short introduction is also given in Embree & Trefethen (1999).

⁶Available at <http://www.math.udel.edu/~driscoll/software/SC/index.html>. Trefethen found a way to compute the map $\Phi(w)$ of the exterior of D to the exterior of the unit disc with the Schwarz-Christoffel toolbox without having to transform the domain G_1 to $G_1^{8/3}$ and reflecting it at the real line first. We will not go into the details here.

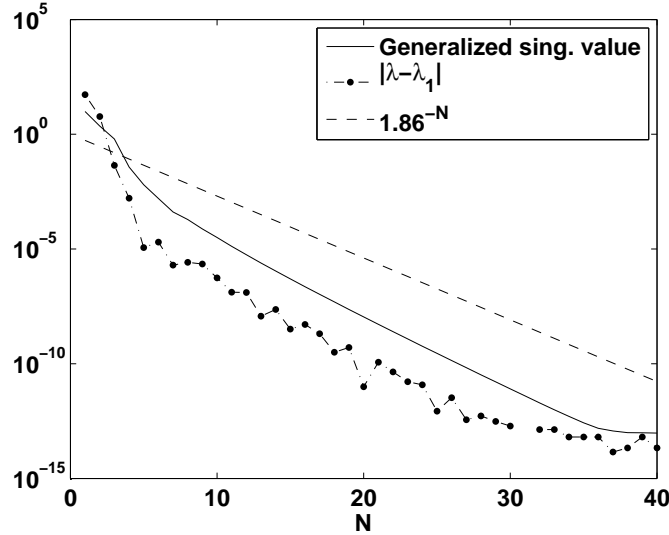


FIG. 9. Comparison of estimated and measured convergence for the decomposition of the domain from Figure 4 into two subdomains.

we obtain an overall convergence rate of $O(1.05^{-N})$ for the domain decomposition method, while our analysis leads to a rate of $O(1.86^{-N})$, which is the asymptotically exact rate of approximating ϕ_k on G_2 .

We can also answer the question now of why the convergence in Figure 5 is much slower if we use a domain decomposition into four subdomains as shown in Figure 4. The responsible subdomain is Ω_3 . The closest singularity to Ω_3 is z_4 leading to a convergence rate of $\rho_3 \approx 1.28$ on Ω_3 , which is much slower than the rate of 1.86 for the approximation on G_2 if we use only two subdomains. Hence, in contrast to standard Finite Element Methods a finer decomposition does not necessarily guarantee a better approximation quality. More important is the distance of the singularities to the subdomains. It is interesting to note that with the decomposition into four subdomains power series estimates are not possible. Since the distance between z_4 and z_3 is smaller than the radius of Ω_3 there exists no series of the form (5.2), which converges in the whole of Ω_3 to u_k .

The convergence analysis is not restricted to simply connected domains. With a suitable domain decomposition we can also apply the results to certain multiply connected domains if all subdomains are simply connected and do not violate the restriction on the domain decomposition.

Consider the domain shown in Figure 10. The inner boundary is a square with side lengths 1 and lower left corner at 0 while the outer boundary is a square with side length 3 and lower left corner at $-.5 - .5i$. The dashed lines show the interfaces between the four subdomains. Since every subdomain is simply connected we can easily compute the maximum convergence rate ρ_j on each subdomain Ω_j . It turns out that the rate of convergence is dominated by Ω_3 , where we have $\rho_3 \approx 1.16$. The smallest singular value $\sigma_1(\lambda_1)$ for a growing number N of basis functions in each subdomain is plotted in Figure 11. Eventually the slope of the observed curve (solid line)

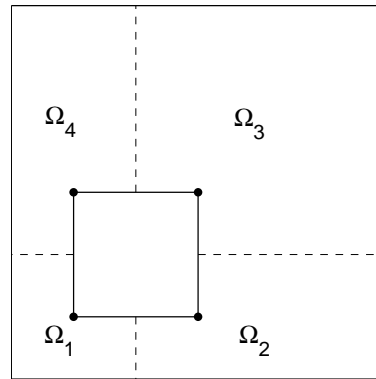


FIG. 10. A multiply connected domain, on which eigenfunctions can have singularities at the interior corners. The dashed lines are the interfaces between the subdomains.

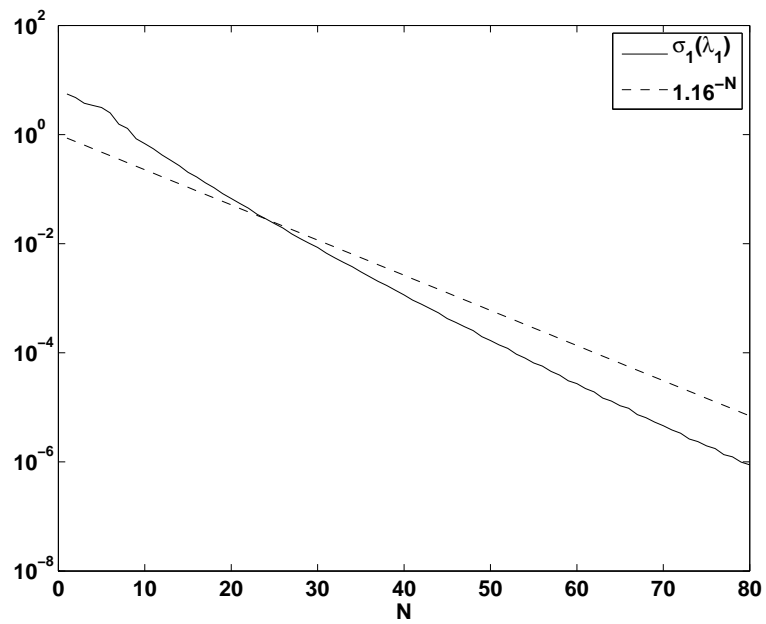


FIG. 11. The convergence of $\sigma_1(\lambda)$ for a growing number of N basis functions in each subdomain of the multiply connected domain. The solid curve shows the computed values and the dashed curve shows the estimated rate of convergence.

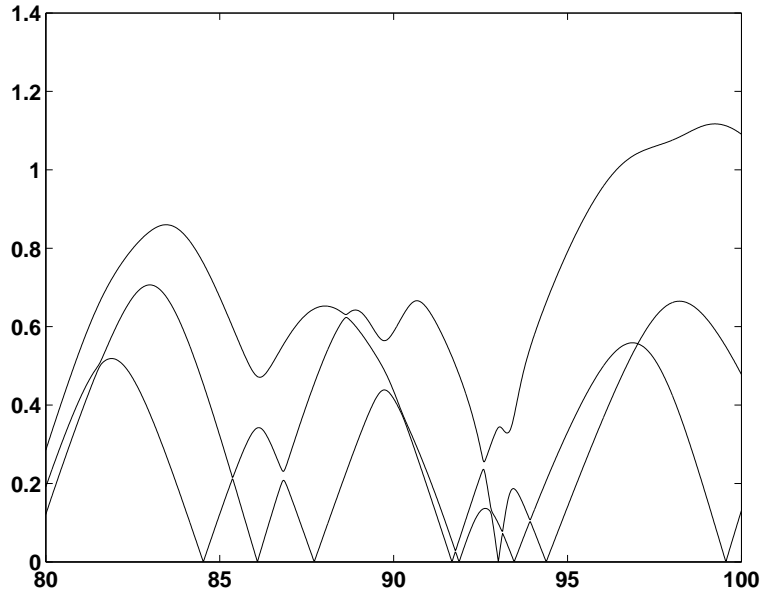


FIG. 12. The smallest three generalized singular values $\sigma_1(\lambda), \dots, \sigma_3(\lambda)$ around the eigenvalue λ_{50} of the square shaped hole. The minima of $\sigma_1(\lambda)$ point to the eigenvalues on this domain.

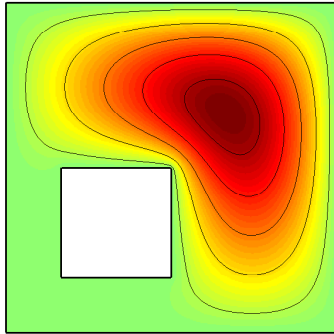
decreases and slowly approaches the slope of the estimated rate (dashed line). In Figure 13 we show plots of some eigenfunctions on this domain computed with the domain decomposition GSVD method. Note that only for higher eigenvalues do the corresponding eigenfunctions fully penetrate the lower left part of the domain.

The numbers of the eigenvalues can be determined by counting the minima of the smallest generalized singular value $\sigma_1(\lambda)$ in dependence on λ . If $\sigma_1(\lambda)$ is evaluated on a sufficient number of points and enough basis functions are chosen this works reliably. Eigenvalue clusters can be spotted by looking at the higher generalized singular values. If for example the smallest two generalized singular values become small we expect a cluster of two eigenvalues nearby. This is shown in Figure 12, where the smallest three generalized singular values are plotted around the eigenvalue $\lambda_{50} \approx 94.38700$ for the multiply connected domain. Manual counting of eigenvalues becomes infeasible for high wavenumbers. In that case estimates for the numbers of eigenvalues can be obtained by Weyl's law (see for example Kuttler & Sigillito (1984)).

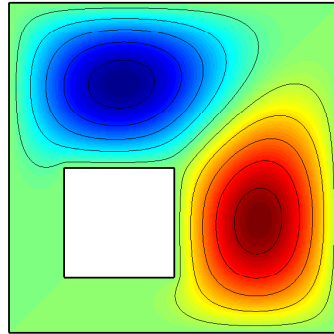
7. Conclusions

In the first part of this paper we presented a domain decomposition method for planar eigenvalue problems that uses the GSVD instead of the generalized eigenvalue decomposition. Since there is no squaring involved in the GSVD formulation it is more accurate and better conditioned than the equivalent GEVD approach. The first part can also be seen as an extension of the GSVD based Method of Particular Solutions discussed in Betcke (2006) to domain decomposition methods.

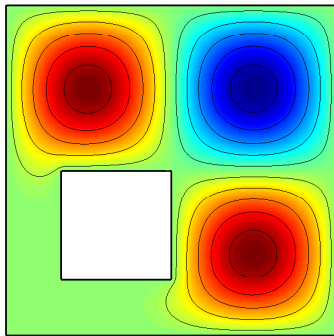
In the second part of this paper we presented based on Vekua's theory an improved conver-



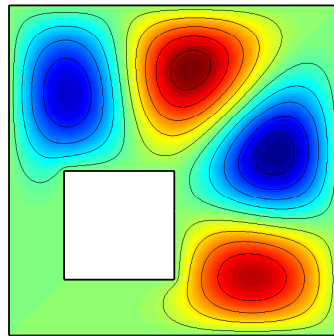
$$\lambda_1 = 4.2786901$$



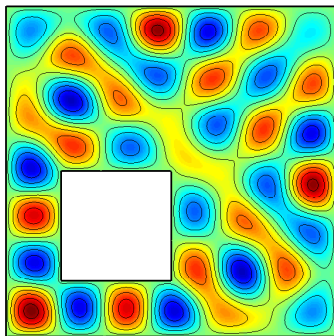
$$\lambda_2 = 6.718692$$



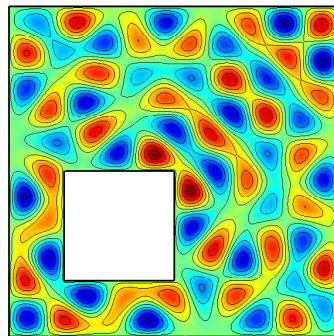
$$\lambda_3 = 8.713789$$



$$\lambda_4 = 13.013411$$



$$\lambda_{50} = 94.38700$$



$$\lambda_{100} = 181.2349$$

FIG. 13. Some eigenfunctions on the square with a square-shaped hole. Only for eigenfunctions belonging to higher eigenvalues is the local wavelength small enough to fully penetrate the lower left part of the region.

gence analysis of such methods that estimates the convergence of approximate eigenfunctions by the rate of a related problem from polynomial approximation in the complex domain. The convergence rates that we obtain are asymptotically optimal for this related complex approximation problem and show a good match with the observed rates for the approximation of the eigenfunctions.

When should one use domain decomposition and when is it more efficient to use particular solutions that live in the whole of Ω ? In Betcke & Trefethen (2005) we present eigenvalue computations on the isospectral drum without domain decomposition by using global particular solutions. There we needed an overall number of 560 Fourier-Bessel basis functions to obtain the first eigenvalue to 13 digits of accuracy. Here we need an overall number of 504 Fourier-Bessel functions for the same accuracy. But for the domain decomposition method we also have to evaluate the derivatives of the Fourier-Bessel functions, which is not necessary in a global approximation method. However, while we can prove exponential convergence for the domain decomposition method if the subdomains are suitably chosen this seems not to be the case for global approximations. In Betcke (2005) we conjecture that the MPS leads to superalgebraic but not exponential convergence on domains with more than one singular corner if the basis functions are chosen to reflect the corner singularities. Hence, for $N \rightarrow \infty$ we can expect the domain decomposition method to outperform global approximations. But in numerical computations we are only interested in results up to the accuracy of machine precision and there it is possible that global approximations need fewer basis functions to achieve this than the exponentially converging domain decomposition method.

The situation is different for certain multiply connected domains. Consider the domain from Figure 10. Fourier-Bessel functions, which are adapted to the singularities at the interior corners always have branch-lines that cross the domain, leading to discontinuities in the approximate eigenfunctions. Certainly, in this case we could use the symmetry of the domain and split it along its symmetry axis to obtain a simply connected domain. But this technique does not work any more if we perturb the symmetry. By a domain decomposition we can overcome this problem and still use basis functions that model the corner singularities to obtain accurate eigenvalue and eigenfunction approximations on this domain.

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