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Introduction

The Yang-Baxter relation plays a central role in two-dimensional Quantum Field Theory. This relation involves a linear operator $R : V \otimes V \to V \otimes V$, where V is a vector space, and has the form

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

in $End(V \otimes V \otimes V)$, where R^{ij} means R acting in the i-th and j-th components. In the paper [12] V. Drinfeld suggested to study set-theoretical solutions of this relation, i.e. solutions given by a map $R : X \times X \to X \times X$, where X is a given set. Moreover, if X is an algebraic manifold, then R may be a rational map. The general theory of set-theoretical solutions to the quantum Yang-Baxter relation was developed in [11, 13]. Various examples were constructed in [10, 11, 13]. In this paper we construct such solutions from decompositions of matrix polynomials and θ-functions. These solutions arise from the decompositions "in different order". We also construct a "local action of the symmetric group" in these cases, generalizations of the action of the symmetric group SN on X^N given by the set-theoretical solution. The structure of the paper is as follows. In §**1** we give basic definitions. In §**2** we introduce a set-theoretical solution arising from the factorization of matrix polynomials. In §**3** we introduce a set-theoretical solution arising from matrix θ-functions.

For a given set-theoretical solution of the quantum Yang-Baxter relation one can define a twisted Yang-Baxter relation with the set of spectral parameters X (see [14] and (3) of this paper). The corresponding twisted R-matrix describes a scattering of two "particles" such that the spectral parameters change after scattering according to a given set-theoretical solution. Moreover, one can define a generalized star-triangle relation for a given local action of the symmetric group (see [14]). The examples of twisted R-matrices as well as the solutions of the generalized star-triangle relation were found in [4] as intertwiners of cyclic representations and their tensor products of the algebra of monodromy matrices of the six-vertex model at roots of unity [3]. These solutions are natural generalizations of the one from the chiral Potts model [1,2,3]. Other examples were found in [9] for the relativistic Toda chain. One can obtain various solutions of the twisted Yang-Baxter and star-triangle relations by calculating the intertwiners of the representations of the algebras of monodromy matrices at roots of unity for other trigonometric and elliptic R-matrices.

§1. Basic definitions

Let U be a complex manifold, $\mu : U\times U \to U\times U$ be a birational automorphism of $U\times U$. We will use a notation: $\mu(u,v) = (\varphi(u,v), \psi(u,v))$ where $u,v\in U$. Here φ and ψ are meromorphic functions from $U\times U$ to U.

Let us introduce the following birational automorphisms of $U\times U\times U$: $\sigma_1 = \mu\times \mathrm{id}$ and $\sigma_2 = \mathrm{id}\times\mu$. We have: $\sigma_1(u,v,w) = (\varphi(u,v), \psi(u,v), w)$ and $\sigma_2(u,v,w) = (u, \varphi(v,w), \psi(v,w))$.

Definition *We call a map μ a twisted transposition if the automorphisms σ_1 and σ_2 satisfy the following relations:*

$$\sigma_1^2 = \sigma_2^2 = \mathrm{id},\ \ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \tag{1}$$

If μ is a twisted transposition, then for each $N\in\mathbb{N}$ we have a birational action of the symmetric group S_N on the manifold U^N such that the transposition $(i,i+1)$ acts by an automorphism $\sigma_i = \mathrm{id}^{i-1}\times\mu\times\mathrm{id}^{N-i-1}$. So we have $\sigma_i(u_1,\dots,u_N) = (u_1,\dots,\varphi(u_i,u_{i+1}),\psi(u_i,u_{i+1}),\dots,u_N)$. It is clear that the relations (1) are equivalent to the defining relations in the group S_N: $\sigma_i^2 = e, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j|>1$.

It is easy to check that the relations (1) are equivalent to the following functional equations for φ and ψ:

$$\varphi(\varphi(u,v),\psi(u,v)) = u,\ \ \psi(\varphi(u,v),\psi(u,v)) = v$$

$$\varphi(u,\varphi(v,w)) = \varphi(\varphi(u,v),\varphi(\psi(u,v),w))$$

$$\varphi(\psi(u,\varphi(v,w)),\psi(v,w)) = \psi(\varphi(u,v),\varphi(\psi(u,v),w)) \tag{2}$$

$$\psi(\psi(u,v),w) = \psi(\psi(u,\varphi(v,w)),\psi(v,w))$$

Remarks 1. From (2) it follows that for each N the functions $\varphi(u_1,\varphi(u_2,\dots,\varphi(u_N,w)\dots)$ and $\psi(\dots(\psi(w,u_1),u_2)\dots,u_N)$ are invariant with respect to the action of the group S_N on the variables $u_1,\dots,u_N$.

2. Let $\sigma : U\times U\to U\times U$ be the map given by $\sigma(u,v) = (v,u)$. Then $\sigma\mu$ is a set-theoretical solution to the quantum Yang-Baxter relation.

3. Informally one can consider $\sigma\mu$ as an infinite dimensional R-matrix in the space of functions. Namely, if we consider the space of meromorphic functions $\{f : U\times U\to\mathbb{C}\}$ as an "extended tensor square" of the space of meromorphic functions $\{f : U\to\mathbb{C}\}$, then the linear operator $R_{\sigma\mu} : f \to f\sigma\mu$ (that is $R_{\sigma\mu}f(u,v) = f(\sigma(\mu(u,v))))$ satisfies the usual Yang-Baxter relation.

Examples 1. Let $q, q^{-1} : U\to U$ be birational automorphisms such that $qq^{-1} = q^{-1}q = \mathrm{id}$. Then

$$\mu(u,v) = (q(v), q^{-1}(u))$$

is a twisted transposition.

2. Let $U = \mathbb{C}$, then the following formula gives a twisted transposition:

$$\mu(u, v) = (1 - u + uv, \frac{uv}{1 - u + uv})$$

3. Let U be a finite dimensional associative algebra with a unity $1 \in U$, for example $U = \mathrm{Mat}_m$. Then the following formula gives a twisted transposition:

$$\mu(u, v) = (1 - u + uv, (1 - u + uv)^{-1} uv)$$

Let V be a n-dimensional vector space. For each $u \in U$ we denote by $V(u)$ a vector space canonically isomorphic to V. Let R be a meromorphic function from $U \times U$ to $End(V \otimes V)$. We will consider $R(u, v)$ as a linear operator

$$R(u, v) : V(u) \otimes V(v) \to V(\varphi(u, v)) \otimes V(\psi(u, v))$$

Definition *We call R a twisted R-matrix (with respect to the twisted transposition μ) if it satisfies the following properties:*

1. The composition

$$V(u) \otimes V(v) \to V(\varphi(u, v)) \otimes V(\psi(u, v)) \to V(u) \otimes V(v)$$

is equal to the identity, that is $R(\varphi(u, v), \psi(u, v))R(u, v) = 1$.

2. The following diagram is commutative:

$$\begin{array}{ccc}
V(\varphi(u,v)) \otimes V(\psi(u,v)) \otimes V(w) & \xrightarrow{1 \otimes R(\psi(u,v),w)} & V(\varphi(u,v)) \otimes V(\varphi(\psi(u,v)),w)) \otimes V(\psi(\psi(u,v),w)) \\
\Big\uparrow {\scriptstyle R(u,v) \otimes 1} & & \Big\downarrow {\scriptstyle R(\varphi(u,v),\varphi(\psi(u,v),w)) \otimes 1} \\
V(u) \otimes V(v) \otimes V(w) & & \widetilde{V} \\
\Big\downarrow {\scriptstyle 1 \otimes R(v,w)} & & \Big\uparrow {\scriptstyle 1 \otimes R(\psi(u,\varphi(v,w)),\psi(v,w))} \\
V(u) \otimes V(\varphi(v,w)) \otimes V(\psi(v,w)) & \xrightarrow{R(u,\varphi(v,w)) \otimes 1} & V(\varphi(u,\varphi(v,w))) \otimes V(\psi(u,\varphi(v,w))) \otimes V(\psi(v,w))
\end{array}$$

Here $\widetilde{V} = V(\varphi(u, \varphi(v, w))) \otimes V(\psi(\varphi(u, v), \varphi(\psi(u, v), w))) \otimes V(\psi(\psi(u, v), w))$.

In other words,

$$R^{12}(\varphi(u, v), \varphi(\psi(u, v), w)))R^{23}(\psi(u, v), w)R^{12}(u, v) =$$
$$R^{23}(\psi(u, \varphi(v, w)), \psi(v, w))R^{12}(u, \varphi(v, w))R^{23}(v, w) \qquad (3)$$

Here $R^{12} = R \otimes 1$ and $R^{23} = 1 \otimes R$ are linear operators in $V \otimes V \otimes V$.

We call (3) a twisted Yang-Baxter relation.

Let $\{x_i, i = 1 \ldots, n\}$ be a basis of the linear space V, $\{x_i(u)\}$ be the corresponding basis of the linear space $V(u)$. It is clear that the following two linear operators are twisted R-matrices for each μ:

$$x_i(u) \otimes x_j(v) \to x_i(\varphi(u,v)) \otimes x_j(\psi(u,v))$$

$$x_i(u) \otimes x_j(v) \to x_j(\varphi(u,v)) \otimes x_i(\psi(u,v))$$

§2. Set-theoretical solution from factorization of matrix polynomials

For the general theory of matrix polynomials and factorizations see [7]. For our purposes we state results, which may be well known to the experts.

We denote by $S(a)$ the set of eigenvalues of a matrix $a \in Mat_m$. More generally, we denote by $S(a_1, \ldots, a_d)$, $a_1, \ldots, a_d \in Mat_m$, the set of roots of a polynomial $f(t) = \det(t^d - a_1 t^{d-1} + \cdots + (-1)^d a_d)$. We will consider polynomials with generic coefficients only, so $\# S(a_1, \ldots, a_d) = md$.

Proposition 1. *Let*

$$t^d - a_1 t^{d-1} + \cdots + (-1)^d a_d = (t - b_1) \ldots (t - b_d) \tag{4}$$

for generic matrices $a_1, \ldots, a_d \in Mat_m$, then $S(b_i) \cap S(b_j) = \emptyset$ for $i \neq j$ and $S(b_1) \cup \cdots \cup S(b_d) = S(a_1, \ldots, a_d)$. For each decomposition $S(a_1, \ldots, a_d) = A_1 \cup \cdots \cup A_d$, such that $\# A_i = m$, $A_i \cap A_j = \emptyset$ $(i \neq j)$ there exists a unique factorization (4) with $S(b_i) = A_i$.

Proof The first statement follows from the equation $\det(t^d - a_1 t^{d-1} + \cdots + (-1)^d a_d) = \det(t - b_1) \ldots \det(t - b_d)$.

On the other hand, if we know eigenvalues of $b_1, \ldots, b_d$ then we can calculate eigenvectors of them. For $\lambda \in S(b_d)$ the corresponding eigenvector is a vector v_λ, such that $(\lambda^d - a_1 \lambda^{d-1} + \cdots + (-1)^d a_d) v_\lambda = 0$. If we know all eigenvectors of b_d, then we can calculate all eigenvectors of b_{d-1} similarly and so on. This implies the uniqueness. By our construction of $b_1, \ldots, b_d$ the determinants of the matrix polynomials in the right hand side and the left hand side of (4) have the same sets of roots. Moreover, for each root λ the operators represented by these matrix polynomials have the same kernel if we set $t = \lambda$. It implies that these polynomials are equal.

Proposition 2. *Let $a_1, a_2 \in Mat_m$ be generic matrices. Then there exists a unique pair of matrices $b_1, b_2 \in Mat_m$ such that $(t - a_1)(t - a_2) = (t - b_1)(t - b_2)$ and $S(b_1) = S(a_2), S(b_2) = S(a_1)$. We have $b_1 = a_1 + \Lambda^{-1}, b_2 = a_2 - \Lambda^{-1}$ where $a_2 \Lambda - \Lambda a_1 = 1$.*

Proof If $S(b_2) \cap S(a_2) \neq \emptyset$, then $\det(a_2 - b_2) = 0$, because a_2 and b_2 have a common eigenvector. Otherwise, we can put $\Lambda = (a_2 - b_2)^{-1}$.

Let $U = Mat_m$. From the propositions 1 and 2 it follows that the formula $\mu(a_1, a_2) = (b_1, b_2)$ gives a twisted transposition, where $b_1 + b_2 = a_1 + a_2$, $b_1 b_2 = a_1 a_2, S(b_1) = S(a_2), S(b_2) = S(a_1)$. We have $\mu(a_1, a_2) = (a_1 + \Lambda^{-1}, a_2 - \Lambda^{-1})$, where Λ is the solution of the linear matrix equation $a_2 \Lambda - \Lambda a_1 = 1$.

Let $\overline{U} = \overline{Mat_m}$ be the set of $m \times m$ matrices with different eigenvalues and fixed order of eigenvalues. The proposition 1 gives an action of the symmetric group S_{mN} on the space $\overline{U}^N$ by birational automorphisms. By definition, for $\sigma \in S_{mN}$, $b_1, \ldots, b_N \in \overline{U}$ we have $\sigma(b_1, \ldots, b_N) = (b'_1, \ldots, b'_N)$ where $(t - b_1) \ldots (t - b_N) = (t - b'_1) \ldots (t - b'_N)$ and $\overline{S}(b'_i) = \sigma \overline{S}(b_i)$, $\overline{S}$ stands for the ordered set of eigenvalues.

This action is local in the following sense. The transposition $(i, i+1)$ for $\alpha m < i < (\alpha+1)m$ acts only inside the $\alpha+1$-th factor of $\bar{U}^N$ and the transposition $(\alpha m, \alpha m+1)$ acts only inside the product of the α-th and the $\alpha+1$-th factors. We have also the twisted transposition $\mu : \overline{U} \times \overline{U} \to \overline{U} \times \overline{U}$ in this case which is the action of the element $(1, m+1)(2, m+2)\ldots(m-1, 2m-1) \in S_{2m}$.

Remark Let U be the set of matrix polynomials of the form $at+b$, where $a, b \in Mat_m, a = (a_{ij}), b = (b_{ij})$ and $a_{ij} = 0$ for $i < j$, $b_{ij} = 0$ for $i > j$. It is possible to define a twisted transposition μ such that for $\mu(f(t), g(t)) = (f_1(t), g_1(t))$ we have $f(t)g(t) = f_1(t)g_1(t)$, $\det f(t)$ and $\det g_1(t)$ have the same sets of roots and the first coefficients of $f(t)$ and $g_1(t)$ have the same diagonal elements. In [4] we found the solutions of the corresponding twisted Yang-Baxter relation (for $m = 2$), which is a generalization of the R-matrix from chiral Potts model.

§3. Set-theoretical solution from factorization of matrix θ-functions

Let $\Gamma \subset \mathbb{C}$ be a lattice generated by 1 and τ where $\mathrm{Im}\tau > 0$. We have $\Gamma = \{\alpha + \beta\tau; \alpha, \beta \in \mathbb{Z}\}$. Let $\varepsilon \in \mathbb{C}$ be a primitive root of unity of degree m. Let $\gamma_1, \gamma_2 \in Mat_m$ be $m \times m$ matrices such that $\gamma_1^m = \gamma_2^m = 1, \gamma_2\gamma_1 = \varepsilon\gamma_1\gamma_2$. We have $\gamma_1 v_\alpha = \varepsilon^\alpha v_\alpha, \gamma_2 v_\alpha = v_{\alpha+1}$ in some basis $\{v_\alpha; \alpha \in \mathbb{Z}/m\mathbb{Z}\}$ of $\mathbb{C}^m$. Let us assume that $\{v_1, \ldots, v_m\}$ is the standard basis of $\mathbb{C}^m$.

We denote by $M\Theta_{n,m,c}(\Gamma)$ for $n, m \in \mathbb{N}, c \in \mathbb{C}$ the space of everywhere holomorphic functions $f : \mathbb{C} \to Mat_m$, which satisfy the following equations:

$$f(z + \frac{1}{m}) = \gamma_1^{-1} f(z)\gamma_1$$

$$f(z + \frac{1}{m}\tau) = e^{-2\pi i(mnz-c)}\gamma_2^{-1} f(z)\gamma_2 \tag{5}$$

Proposition 3. $\dim M\Theta_{n,m,c}(\Gamma) = m^2 n$ *and for each element* $f \in M\Theta_{n,m,c}(\Gamma)$ *the equation* $\det f(z) = 0$ *has exactly* mn *zeros modulo* $\frac{1}{m}\Gamma$*. The sum of these zeros is equal to* $mc + \frac{mn}{2}$ *modulo* Γ*.*

Proof For $m = 1$ we have the usual θ-functions $\Theta_{n,c}(\Gamma) = M\Theta_{n,1,c}(\Gamma)$ and all these statements are well known in this case ([8]). One has a basis $\{\theta_\alpha(z); \alpha \in \mathbb{Z}/n\mathbb{Z}\}$ in the space $\Theta_{n,c}(\Gamma)$ such that $\theta_\alpha(z + \frac{1}{n}) = e^{2\pi i\frac{\alpha}{n}}\theta_\alpha(z), \theta_\alpha(z + \frac{1}{n}\tau) = e^{-2\pi i(z-\frac{n-1}{2n}\tau-\frac{1}{n}c)}\theta_{\alpha+1}(z)$ ([8]). From (5) it follows that $f(z+1) = f(z)$ and $f(z + \tau) = e^{-2\pi i(m^2nz-c_1)}f(z)$ for some $c_1 \in \mathbb{C}$. So the matrix elements of $f(z)$ are θ-functions from the space $\Theta_{m^2n,c_1}(\Gamma)$. We have decomposition $f(z) = \sum_\alpha \varphi_\alpha\theta_\alpha(z)$, where $\varphi_\alpha \in Mat_m$ are constant matrices, $\{\theta_\alpha\}$ is a basis in the space $\Theta_{m^2n,c_1}(\Gamma)$. Substituting this decomposition in (6) one can calculate the dimension of the space $M\Theta_{n,m,c}(\Gamma)$. We have also $\det f(z + \frac{1}{m}) = \det f(z)$ and $\det f(z + \frac{1}{m}\tau) = e^{-2\pi i(m^2nz-mc)} \det f(z)$. From this follows the statement about zeros of the equation $\det f(z) = 0$.

Proposition 4. *For generic complex numbers* $\lambda_1, \ldots, \lambda_{mn}$ *such that* $\lambda_1 + \cdots + \lambda_{mn} \equiv mc + \frac{mn}{2}$ $mod\frac{1}{m}\Gamma$ *and nonzero vectors* $v_1, \ldots, v_{mn} \in \mathbb{C}^m$ *there exists a unique up to proportionality element* $f(z) \in M\Theta_{n,m,c}(\Gamma)$ *such that* $\det f(\lambda_\alpha) = 0, f(\lambda_\alpha)v_\alpha = 0$ *for* $1 \leqslant \alpha \leqslant mn$*.*

Proof Considering the decomposition $f(z) = \sum_\alpha \varphi_\alpha\theta_\alpha(z)$, one has the system of linear equations $\{\sum_\alpha \theta_\alpha(\lambda_\beta)\varphi_\alpha v_\beta = 0; \beta = 1, \ldots, mn\}$ for matrix elements of $\{\varphi_\alpha\}$. One can see that this system defines $\{\varphi_\alpha\}$ uniquely up to proportionality for generic $\lambda_1, \ldots, \lambda_{mn}, v_1, \ldots, v_{mn}$.

We denote by $S(f)$ the set of zeros of the equation $\det f(z) = 0$ modulo $\frac{1}{m}\Gamma$.

Proposition 5. *Assume that* $f(z) \in M\Theta_{n,m,c}(\Gamma)$ *is a generic element and we have a factorization* $f(z) = f_1(z) \ldots f_n(z)$*, where* $f_\alpha(z) \in M\Theta_{1,m,c_\alpha}(\Gamma)$*,* $c_1 + \cdots + c_n = c$*. Then* $S(f_\alpha) \cap S(f_\beta) = \emptyset$ *for* $\alpha \neq \beta$ *and* $S(f) = S(f_1) \cup \cdots \cup S(f_n)$*. For each decomposition* $S(f) = A_1 \cup \cdots \cup A_n$ *such that* $A_\alpha \cap A_\beta = \emptyset$ *for* $\alpha \neq \beta$ *and* $\#A_\alpha = m$

there exists a unique factorization $f(z) = f_1(z) \dots f_n(z)$ *up to proportionality of* f_α *such that* $S(f_\alpha) = A_\alpha$ *for* $\alpha = 1, \dots, n$.

Proof is similar to the proof of the proposition 1, we just change polynomials by θ-functions.

Let U_c be the projectivisation of the linear space $M\Theta_{1,m,c}(\Gamma)$ and $U = \bigcup_{c\in\mathbb{C}} U_c$. We have the following twisted transposition $\mu : U \times U \to U \times U$. By definition $\mu(f, g) = (f_1, g_1)$, where $f(z)g(z) = f_1(z)g_1(z)$ and $S(f_1) = S(g), S(g_1) = S(f)$.

Let $\overline{U}$ be the set of elements f from U with a fixed order on $S(f)$. For $f \in \overline{U}$ let $\overline{S}(f)$ be the set $S(f)$ with corresponding order. We have a local action of the symmetric group S_{mN} on the space $\overline{U}^N$. By definition, for $\sigma \in S_{mN}$ we have $\sigma(f_1, \dots, f_N) = (f_1^\sigma, \dots, f_N^\sigma)$ where $f_1(z) \dots f_N(z) = f_1^\sigma(z) \dots f_N^\sigma(z)$ and $\overline{S}(f_\alpha^\sigma) = \sigma\overline{S}(f_\alpha)$ for $1 \leqslant \alpha \leqslant N$.

Remark It is possible to construct twisted R-matrices for this twisted transposition μ as intertwiners of tensor products of cyclic representations of the algebra of monodromy matrices for the elliptic Belavin R-matrix [5] at the point of finite order (see also [6]). It will be the subject of another paper.

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References

1. R.J. Baxter, J.H.H. Perk and H. Au-Yang, New solutions of the star-triangle relations for the chiral Potts model, Phys. Lett. A, 128(1988)138-142.

2. V.V. Bazhanov and Yu.G. Stroganov, Chiral Potts model as a descendant of the six vertex model, J. Stat. Phys. 51(1990)799-817.

3. V.O. Tarasov, Cyclic monodromy matrices for the R-matrix of the six-vertex model and the chiral Potts model with fixed spin boundary conditions, International Jornal of Modern Physics A, Vol.7, Suppl. 1B(1992)963-975.

4. A.A. Belavin, A.V. Odesskii and R.A. Usmanov, New relations in the algebra of the Baxter Q-operators. hep-th/0110126.

5. A.A. Belavin, Discrete groups and integrability of quantum systems, Func. Anal. Appl. 14(1980)18-26.

6. B.L. Feigin and A.V. Odesskii, Sklyanin Elliptic Algebras. The case of point of finite order, Func. Anal. Appl. 29(1995).

7. I. Gelfand, V. Retakh, R. Wilson, Quadratic-linear algebras associated with decompositions of noncommutative polynomials and noncommutative differential polynomials, q-alg/0002238, to appear in Selecta Mathematica.

8. D. Mumford, Tata lectures on theta. I. With the assistance of C. Musili, M. Nory and M. Stillman. Progress in Mathematics, 28. BirkhDuser Boston, Inc., Boston, MA, 1983.

9. S. Pakuliak and S. Sergeev, Relativistic Toda chain at root of unity II. Modified Q-operator, nlin.SI/0107062.

10. V. M. Bukhshtaber, The Yang-Baxter transformation, Russian Math. Surveys 53:6 (1998) 1343-1379.

11. P. Etingof, T. Schedler, A. Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equation, q-alg/9801047, Duke Math. J. 1999.

12. V. Drinfeld, Some unsolved problems in quantum group theory, Lecture Notes in Math, 1510, p.1-8.

13. J-H. Lu, M. Yan, Y-C. Zhu, On set-theoretical Yang-Baxter equation, Duke Math. J. 1999.

14. A. Odesskii, Local action of the symmetric group and the twisted Yang-Baxter relation, q-alg/0110268.