Special transverse slices and their enveloping algebras

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2002

MIMS EPrint: **2006.289**

ISSN 1749-9097

SPECIAL TRANSVERSE SLICES AND THEIR ENVELOPING ALGEBRAS

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Abstract. Let G be a simple, simply connected algebraic group over $\mathbb{C}$, $\mathfrak{g} = \operatorname{Lie} G$, $\mathcal{N}(\mathfrak{g})$ the nilpotent cone in $\mathfrak{g}$, and (E, H, F) an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Let $S = E + \operatorname{Ker} \operatorname{ad} F$, the special transverse slice to the adjoint orbit Ω of E, and $S_0 = S \cap \mathcal{N}(\mathfrak{g})$. The coordinate ring $\mathbb{C}[S_0]$ is naturally graded (see [35]). Let $Z(\mathfrak{g})$ be the centre of the enveloping algebra $U(\mathfrak{g})$ and $\eta : Z(\mathfrak{g}) \to \mathbb{C}$ an algebra homomorphism. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via a Killing isomorphism and let χ denote the linear function on $\mathfrak{g}$ corresponding to E. Following [32] we attach to χ a nilpotent subalgebra $\mathfrak{m}_\chi \subset \mathfrak{g}$ of dimension $(\dim \Omega)/2$ and a 1-dimensional $\mathfrak{m}_\chi$-module $\mathbb{C}_\chi$. Let $\tilde{H}_\chi$ denote the algebra opposite to $\operatorname{End}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{m}_\chi)} \mathbb{C}_\chi)$ and $\tilde{H}_{\chi,\eta} = \tilde{H}_\chi \otimes_{Z(\mathfrak{g})} \mathbb{C}_\eta$. It is proved in the paper that the algebra $\tilde{H}_{\chi,\eta}$ has a natural filtration such that $\operatorname{gr}(\tilde{H}_{\chi,\eta})$, the associated graded algebra, is isomorphic to $\mathbb{C}[S_0]$. This construction yields natural noncommutative deformations of all singularities associated with the adjoint quotient map of $\mathfrak{g}$.

1. Introduction

1.1. Let V be a 2-dimensional vector space over $\mathbb{C}$ with basis $\{u, v\}$ and Γ a finite subgroup of $\mathrm{SL}(V)$. In [4], Crawley-Boevey and Holland constructed a family of noncommutative filtered deformations of the graded coordinate ring $\mathbb{C}[V]^\Gamma$ of the Kleinian singularity V/Γ (for Γ cyclic this was done earlier in [13]). To deform $\mathbb{C}[V]^\Gamma$ Crawley-Boevey and Holland pick λ in the centre of the group algebra $\mathbb{C}\Gamma$, let Γ act on the tensor algebra $T(V)$ as homogeneous automorphisms, form the skew group algebra $T(V) * \Gamma$, consider its quotient $\mathcal{S}^\lambda = (T(V) * \Gamma)/(uv - vu - \lambda)$, and then define $\mathcal{O}^\lambda := e\mathcal{S}^\lambda e$ where e is the average of the group elements. The $\mathbb{C}$-algebra $\mathcal{O}^\lambda$ is naturally filtered and the associated graded algebra is isomorphic to $\mathbb{C}[V]^\Gamma$ (see [4, Theorem 1.6]).

1.2. By Brieskorn's theorem, any Kleinian singularity arises in Lie theory as the intersection of the nilpotent cone of a simple Lie algebra $\mathfrak{g}$ with a "good" transverse slice to the subregular nilpotent orbit in $\mathfrak{g}$ (see [1], [35]). Applying the same recipe to the other (nonregular) nilpotent orbits in $\mathfrak{g}$ yields more complicated singularities playing an important rôle in representation theory. The goal of this paper is to prove that the singularities thus obtained *all* admit natural noncommutative deformations similar to those constructed by Crawley-Boevey and Holland in the subregular case. To describe these deformations in detail we need some notation.

1.3. Let G be a simple, simply connected algebraic group over $\mathbb{C}$ and $\mathfrak{g} = \operatorname{Lie} G$. Let $\mathcal{N} = \mathcal{N}(\mathfrak{g})$ denote the nilpotent cone of $\mathfrak{g}$. The affine variety $\mathcal{N}$ is irreducible and G acts on $\mathcal{N}$ with finitely many orbits. The unique open orbit in $\mathcal{N}$ coincides with

1991 *Mathematics Subject Classification.* Primary 20G05; Secondary 17B20.

$\mathcal{N}_{\rm reg}$, the set of all regular nilpotent elements in $\mathfrak{g}$. It is well-known that the closed set $\mathcal{N}' = \mathcal{N} \setminus \mathcal{N}_{\rm reg}$ is irreducible and has codimension 2 in $\mathcal{N}$. The elements in the unique open orbit of $\mathcal{N}'$ are called *subregular* nilpotent elements in $\mathfrak{g}$.

Let (E, H, F) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, $\mathfrak{c} = \operatorname{Ker} \operatorname{ad} F$, and $r = \dim \mathfrak{c}$. It follows from the $\mathfrak{sl}_2$-theory that $\mathfrak{c} \cap [E, \mathfrak{g}] = 0$. So the affine space $S = E + \mathfrak{c}$ is a transverse slice to the adjoint orbit $\Omega = (\operatorname{Ad} G) \cdot E$. It is called the *special* transverse slice to Ω. There is a 1-dimensional torus $\lambda = \lambda_E$ in G such that $E \in \mathfrak{g}(\lambda, 2)$, $F \in \mathfrak{g}(\lambda, -2)$, $\operatorname{Ker} \operatorname{ad} E \subset \oplus_{i \geq 0} \mathfrak{g}(\lambda, i)$, and $\mathfrak{c} \subset \oplus_{i \leq 0} \mathfrak{g}(\lambda, i)$, where $\mathfrak{g}(\lambda, k) = \{\operatorname{Ad}(\lambda(t))x = t^k x \text{ for all } t \in \mathbf{G}_m\}$. Composing the adjoint action of λ with the scalar $\mathbf{G}_m$-action $(t, v) \mapsto \sigma(t)v := tv$ on $\mathfrak{g}$ induces an additional rational action $\rho : \mathbf{G}_m \longrightarrow \operatorname{GL}(\mathfrak{c}),\ t \mapsto \sigma(t^2)\lambda(t^{-1})$ (see [35] for more detail).

1.4. Let $m_1, \ldots, m_l$ denote the exponents of the Weyl group of $\mathfrak{g}$, and let $f_1, \ldots, f_l$ be algebraically independent homogeneous generators of the invariant algebra $\mathbb{C}[\mathfrak{g}]^G$ such that $\deg f_i = m_i + 1$ for $1 \leq i \leq l$. Let φ_S denote the restriction to S of the adjoint quotient $\mathfrak{g} \longrightarrow \mathbb{A}^l, \quad x \mapsto (f_1(x), \ldots, f_l(x))$. According to [35] the morphism φ_S is faithfully flat. In particular, φ_S is surjective and each fibre S_ξ of φ_S has dimension $r - l$. Moreover, each S_ξ is a normal affine variety and the smooth points of S_ξ are exactly the regular elements of $\mathfrak{g}$ contained in S_ξ (see [35]). It is well-known that $S_0 = S \cap \mathcal{N}$. Let τ denote the affine translation $\mathfrak{c} \longrightarrow S,\ x \mapsto E + x$, and $\psi = \varphi_S \circ \tau$. Clearly,

$$\psi : \mathfrak{c} \longrightarrow \mathbb{A}^l, \quad x \mapsto (\psi_1(x), \ldots, \psi_l(x)),$$

is a faithfully flat morphism and $\psi^{-1}(\xi) \cong S_\xi$ for any $\xi = (\xi_1, \ldots, \xi_l) \in \mathbb{A}^l$. Since the null-fibre $\psi^{-1}(0)$ is ρ-stable the coordinate ring $\mathbb{C}[\psi^{-1}(0)]$ has a natural $\mathbb{N}_0$-grading. The zero part of this grading is $\mathbb{C}$.

It is proved in Section 5 of this paper that all fibres of ψ are irreducible and the ideal of regular functions on $\mathfrak{c}$ vanishing on $\psi^{-1}(\xi)$ is generated by $\psi_1 - \xi_1, \ldots, \psi_l - \xi_l$ (the second half of this statement was known to the experts but missing in the literature). In particular, $\psi^{-1}(0)$ is an irreducible, normal complete intersection of dimension $r - l$ in $\mathfrak{c}$. The set of smooth points of $\psi^{-1}(0)$ coincides with $(-E + \mathcal{N}_{\rm reg}) \cap \mathfrak{c}$.

1.5. Let κ denote the Killing form on $\mathfrak{g}$. By the $\mathfrak{sl}_2$-theory, $\kappa(E, F) \neq 0$. Set $\Phi = \kappa(E, F)^{-1} \cdot \kappa$ and define $\chi \in \mathfrak{g}^*$ by letting $\chi(X) = \Phi(E, X)$ for all $X \in \mathfrak{g}$. Set $\mathfrak{z}_\chi = \operatorname{Ker} \operatorname{ad} E$, $\mathfrak{g}(i) = \mathfrak{g}(\lambda, i)$, and $\mathfrak{p}_+ = \oplus_{i \geq 0} \mathfrak{g}(i)$. Let $X_1, \ldots, X_m$ be a basis of $\mathfrak{p}_+$ satisfying $X_i \in \mathfrak{g}(n_i)$ for some n_i, where $1 \leq i \leq r$, and such that $X_1, \ldots, X_r$ is a basis of $\mathfrak{z}_\chi \subset \mathfrak{p}_+$.

Define the skew-symmetric bilinear form ψ_E on $\mathfrak{g}(-1)$ by setting $\psi_E(X, Y) = \Phi(E, [X, Y])$ for all $X, Y \in \mathfrak{g}$. Since $\mathfrak{z}_\chi \subset \mathfrak{p}_+$ this form is nondegenerate. Let $\{Z'_1, \ldots, Z'_s, Z_1, \ldots Z_s\}$ be a Witt basis of $\mathfrak{g}(-1)$ relative to ψ_E and $\mathfrak{g}(-1)^0$ the subspace of $\mathfrak{g}(-1)$ spanned by the Z'_i. Define $\mathfrak{m}_\chi := \mathfrak{g}(-1)^0 \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i)$. By construction, $\mathfrak{m}_\chi$ is a nilpotent subalgebra of dimension $(\dim \Omega)/2$. Let N_χ be the left ideal of the enveloping algebra $U(\mathfrak{m}_\chi)$ generated by all $X - \chi(X)$ with $X \in \mathfrak{m}_\chi$, and $\mathbb{C}_\chi = U(\mathfrak{m}_\chi)/N_\chi$, a 1-dimensional left $U(\mathfrak{m}_\chi)$-module. Let $\tilde{1}_\chi$ be the image of 1 in $\mathbb{C}_\chi$.

Let $\tilde{Q}_\chi = U(\mathfrak{g}) \otimes_{U(\mathfrak{m}_\chi)} \mathbb{C}_\chi$, an induced $\mathfrak{g}$-module, and $\tilde{H}_\chi = \tilde{H}_\chi(\mathfrak{g}) = \operatorname{End}_\mathfrak{g}(\tilde{Q}_\chi)^{\rm op}$, an associative algebra over $\mathbb{C}$. The representation $\tilde{\rho}_\chi : U(\mathfrak{g}) \longrightarrow \operatorname{End}(\tilde{Q}_\chi)$ is injective on the centre of $U(\mathfrak{g})$ (see (6.1)). Given a pair $(\mathbf{a}, \mathbf{b}) \in \mathbb{N}_0^m \times \mathbb{N}_0^s$ we denote by $X^\mathbf{a} Z^\mathbf{b}$ the monomial $X_1^{a_1} \cdots X_m^{a_m} Z_1^{b_1} \cdots Z_s^{b_s}$ in $U(\mathfrak{g})$. By the PBW theorem, the vectors

$\{X^{\mathbf{i}}Z^{\mathbf{j}}\otimes\tilde{1}_\chi\,|\,(\mathbf{i},\mathbf{j})\in\mathbb{N}_0^m\times\mathbb{N}_0^s\}$ form a basis of $\tilde{Q}_\chi$ over $\mathbb{C}$. For $k\in\mathbb{N}_0$ we denote by $\tilde{H}^k$ the linear span of all $h\in\tilde{H}_\chi$ such that $h(\tilde{1}_\chi)$ is a linear combination of $X^{\mathbf{a}}Z^{\mathbf{b}}\otimes\tilde{1}_\chi$ with

$$\sum_{i=1}^m a_i(n_i+2)+\sum_{i=1}^s b_i\le k.$$

It is proved in the paper that the subspaces $\{\tilde{H}^i\,|\,i\in\mathbb{N}_0\}$ form a filtration of the algebra $\tilde{H}_\chi$ and the associated graded algebra $\mathrm{gr}(\tilde{H}_\chi)$ is isomorphic to a graded polynomial algebra in r variables with free homogeneous generators of degree $n_1+2,\dots,n_r+2$ (Theorem 4.6).

1.6. For $k\ge 0$ let U^k denote kth component of the standard filtration of $U(\mathfrak{g})$. It is well-known that the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is generated by algebraically independent elements $\tilde{f}_1,\dots,\tilde{f}_l$ satisfying $\tilde{f}_i\in Z(\mathfrak{g})\cap U^{m_i+1}$ for all i. Since the restriction of $\tilde{\rho}_\chi$ to $Z(\mathfrak{g})$ is injective we can identify $Z(\mathfrak{g})$ with its image in $\mathrm{End}(\tilde{Q}_\chi)$. Under this identification, $\tilde{f}_i\in\tilde{H}^{2m_i+2}\setminus\tilde{H}^{2m_i+1}$ for $1\le i\le r$ (see (6.1)). We denote by $\tilde{\psi}_i$ the image of $\tilde{f}_i$ in $\mathrm{gr}_{2m_i+2}(\tilde{H}_\chi)$. The Killing isomorphism $x\mapsto\Phi(x,\,\cdot\,)$ induces a natural isomorphism, $\tilde{\kappa}$, between $\mathbb{C}[\mathfrak{c}]$ and $S(\mathfrak{z}_\chi)$. For $1\le k\le r$ we set $\xi_k=\tilde{\kappa}(X_k)|_{\mathfrak{c}}$ and view ξ_k as a homogeneous polynomial function of degree n_k+2 on $\mathfrak{c}$. In (6.3), we prove that there is an isomorphism of graded algebras $\delta:\,\mathrm{gr}(\tilde{H}_\chi)\xrightarrow{\sim}\mathbb{C}[\mathfrak{c}]$ such that $\delta(\bar{\Theta}_k)=\xi_k$ for $1\le k\le r$ and $\delta(\tilde{\psi}_i)=\psi_i$ for $1\le i\le l$.

1.7. Let $\eta:\,Z(\mathfrak{g})\longrightarrow\mathbb{C}$ be an algebra homomorphism, $J_\eta=\tilde{H}_\chi\cdot\mathrm{Ker}\,\eta$, a two-sided ideal of $\tilde{H}_\chi$, and $\mathbb{C}_\eta=Z(\mathfrak{g})/\mathrm{Ker}\,\eta$. Define

$$\tilde{H}_{\chi,\eta}:=\tilde{H}_\chi\otimes_{Z(\mathfrak{g})}\mathbb{C}_\eta\cong\tilde{H}_\chi/J_\eta.$$

The subspaces $\{(\tilde{H}^k+J_\eta)/J_\eta\,|\,k\ge 0\}$ form a filtration of the algebra $\tilde{H}_{\chi,\eta}$. We denote by $\mathrm{gr}(\tilde{H}_{\chi,\eta})$ the associated graded algebra. Our main result is the following:

Theorem 6.4. *The graded algebras* $\mathrm{gr}(\tilde{H}_{\chi,\eta})$ *and* $\mathbb{C}[\psi^{-1}(0)]$ *are isomorphic.*

In (6.5), we prove that the Poisson bracket on $\mathbb{C}[\mathfrak{c}]$ induced by the isomorphism δ and multiplication in $\tilde{H}_\chi$ is nonzero for any $E\in\mathcal{N}'$. We also show that if $[\mathfrak{z}_\chi,[\mathfrak{z}_\chi,\mathfrak{z}_\chi]]\ne 0$ then the product in $\tilde{H}_{\chi,\eta}$ induces a nonzero Poisson bracket on the coordinate ring $\mathbb{C}[\psi^{-1}(0)]$ (Theorem 6.5). In Section 7, we compute this Poisson bracket in the case where E is a subregular nilpotent element in $\mathfrak{g}$.

1.8. To obtain the results described above we first establish their finite dimensional analogues. To that end, we assume in Sections 2 and 3 that $\mathfrak{g}$ is the Lie algebra of a reductive algebraic group G over an algebraically closed field K of characteristic $p>0$. Given a finite dimensional restricted Lie algebra $\mathcal{L}$ over K with $[p]$th power map $x\mapsto x^{[p]}$ and a linear function $\xi\in\mathcal{L}^*$ we denote by $U_\xi(\mathcal{L})$ the reduced enveloping algebra of $\mathcal{L}$ associated with ξ (recall that $U_\xi(\mathcal{L})=U(\mathcal{L})/I_\xi$ where I_ξ is the two-sided ideal of $U(\mathcal{L})$ generated by all $x^p-x^{[p]}-\xi(x)^p$ with $x\in\mathcal{L}$). Following [32] we attach to $\chi\in\mathfrak{g}^*$ a $[p]$-nilpotent subalgebra $\mathfrak{m}_\chi\subset\mathfrak{g}$ of dimension $(\dim G\cdot\chi)/2$. Because $U_\chi(\mathfrak{m}_\chi)$ is a local algebra the left $U_\chi(\mathfrak{m}_\chi)$-module $K_\chi:=U_\chi(\mathfrak{m}_\chi)/\mathrm{Jac}(U_\chi(\mathfrak{m}_\chi))$ is 1-dimensional. In Section 3 we investigate the induced $U_\chi(\mathfrak{g})$-module $Q_\chi=U_\chi(\mathfrak{g})\otimes_{U_\chi(\mathfrak{m}_\chi)}K_\chi$, a finite dimensional analogue of $\tilde{Q}_\chi$. It follows from a Morita theorem proved in Section 2 that Q_χ is a projective generator for $U_\chi(\mathfrak{g})$ and

$$U_\chi(\mathfrak{g}) \cong \mathrm{Mat}_{d(\chi)}(H_\chi)$$

where $d(\chi) = p^{\frac{1}{2}\dim G\cdot\chi}$ and $H_\chi = \mathrm{End}_{\mathfrak{g}}(Q_\chi)^{\mathrm{op}}$ (see Theorems 2.3 and 2.4 and Proposition 2.6). The projectivity of Q_χ implies that there are $\theta_1, \dots, \theta_r$ in H_χ such that the monomials $\theta_1^{a_1}\cdots\theta_r^{a_r}$ with $0 \le a_i \le p-1$ form a K-basis of H_χ (Theorem 3.4). We show that, to some extent, these generators are independent of p and can be lifted to characteristic 0. This yields a very nice generating set, $\Theta_1, \dots, \Theta_r$, in the $\mathbb{C}$-algebra $\tilde{H}_\chi$ (see the proof of Theorem 4.6 for more detail).

1.9. The modular setting of Sections 2 and 3 is reinstated in the last section of the paper where we use the results of [8] to prove that for any $\chi \in \mathfrak{g}^*$ the image of the centre of $U(\mathfrak{g})$ under the canonical homomorphism $U(\mathfrak{g}) \twoheadrightarrow U_\chi(\mathfrak{g})$ has dimension p^l where $l = \mathrm{rk}\, G$ (Theorem 8.2). Combining this with the main result of [28] we show that the image of the centre of $U(\mathfrak{g})$ in the restricted enveloping algebra $U^{[p]}(\mathfrak{g})$ is isomorphic to a direct sum of coinvariant algebras for the Weyl group of $\mathfrak{g}$ (see Proposition 8.3 for more detail).

1.10. We wish to finish the introduction with a (nonrigorous) discussion on possible applications of the algebras $\tilde{H}_\chi$ in the theory of W-algebras. Recall that Poisson Reduction is a method used in physics to construct new Poisson algebras $(B, \{\,\cdot\,,\,\cdot\,\}')$ from a known Poisson algebra $(A, \{\,\cdot\,,\,\cdot\,\})$. One starts by fixing a finite set S in A called the set of constraints. If the set S is second class relative to $\{\,\cdot\,,\,\cdot\,\}$ it gives rise to a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}'$ on a quotient algebra B of A, the Dirac bracket associated with S. If some constraints in S are first class the Dirac bracket is not well-defined. This can be resolved by adding gauge fixing constraints to S in such a way that the total set of constraints $\tilde{S}$ is second class (this will force B to shrink further).

In [5], de Boer and Tjin take as a Poisson algebra A the polynomial algebra $\mathbb{C}[\mathfrak{g}] \cong \mathrm{gr}(U(\mathfrak{g}))$ with its standard Poisson bracket (induced by multiplication in $U(\mathfrak{g})$) and observe that any nilpotent element in $\mathfrak{g}$ yields a nice set of first class constraints in $\mathbb{C}[\mathfrak{g}]$. They then determine the group of gauge transformations generated by these constraints and choose the so-called lowest weight gauges to fix gauge invariances. They argue that any nilpotent element $E \in \mathfrak{g}$ gives rise to a Poisson algebra $(\mathcal{W}_E, \{\,\cdot\,,\,\cdot\,\}_E)$ called the finite W-algebra associated with E.

The process bringing $\mathcal{W}_E$ to life is natural. As an algebra, $\mathcal{W}_E$ is nothing but the ring of polynomial functions on the centraliser $\mathfrak{c}_{\mathfrak{g}}(E)$ of E in $\mathfrak{g}$. The new feature of $\mathcal{W}_E$ is its highly nontrivial Poisson structure: the bracket $\{\,\cdot\,,\,\cdot\,\}_E$ often takes nonlinear values on linear generators of $\mathbb{C}[\mathfrak{g}]$.

Let $\chi = \Phi(E, \cdot\,)$. It seems likely that $\mathcal{W}_E$ and $\mathrm{gr}(\tilde{H}_\chi)$ are isomorphic as Poisson algebras.

Since deformation quantisation amounts to replacing Poisson brackets by commutators the question arises: is it always possible to deform $\mathcal{W}_E$ to give a finite *quantum* W-algebra? This question is addressed in [5, Theorem 4] under the assumption that the nilpotent element E is even. The authors of [5] set up the BRST complex $(U(\mathfrak{g}) \otimes \mathbf{C}, d)$ associated with the constraints imposed by E and then show that its only nonvanishing cohomology is $H^0(U(\mathfrak{g}) \otimes \mathbf{C}, d)$. Here $\mathbf{C}$ is the graded Clifford algebra generated by ghost variables and d is the BRST differential, a degree 1 superderivation of the graded algebra $U(\mathfrak{g})\otimes\mathbf{C}$. They argue that the associative algebra $H^0(U(\mathfrak{g})\otimes\mathbf{C}, d)$ is a quantisation of $\mathcal{W}_E$.

It seems likely that the BRST quantisation of $\mathcal{W}_E$ is isomorphic to $H^0(U(\mathfrak{g})\otimes\mathbf{C}, d)$.

Acknowledgement. I am most grateful to P. Slodowy for his comments on a preliminary version of this paper. The idea of using a contracting $\mathbb{C}^*$-action in the proof of Proposition 5.2 is due to him. I would like to thank J.E. Humphreys, S. Skryabin and I. Gordon for helpful discussions and email correspondence, and D. Rumynin and D. Panyushev for bringing [5] and [42] to my attention.

2. Projective generators

2.1. Due to the Kac-Weisfeiler conjecture, confirmed in [30], the following useful result is applicable to reduced enveloping algebras.

Proposition. *Let A be a finite dimensional associative algebra with 1 over an algebraically closed field, and d a positive integer. Suppose that any simple left A-module has dimension divisible by d. Then there exists a projective A-module P such that ${}_AA \cong P\oplus\cdots\oplus P$ (d times), where ${}_AA$ denotes the left regular A-module. Moreover,*

$$A \cong \mathrm{Mat}_d(B^{\mathrm{op}}) \quad \textit{where} \quad B = \mathrm{End}_A(P).$$

Proof. Let $\mathcal{J}$ be the Jacobson radical of A. Let $V_1, \ldots, V_l$ be all simple left A-modules (up to isomorphism). Let $a_i = (\dim V_i)/d$ where $1 \le i \le l$. By our assumption, each a_i is a positive integer. Let P_i denote a projective cover of V_i. Given a left A-module M and a positive integer r let M^r denote the direct sum of r copies of M. Define $P := P_1^{a_1}\oplus\cdots\oplus P_l^{a_l}$. Clearly, P is a projective A-module and

$$P^d/\mathcal{J}P^d \cong \bigoplus_{i=1}^{l}(P_i/\mathcal{J}P_i)^{da_i} \cong \bigoplus_{i=1}^{l} V_i^{\dim V_i}$$

as left A-modules. So it follows from the Wedderburn theorem that the left A-modules ${}_AA/(\mathcal{J}\cdot {}_AA)$ and $P^d/\mathcal{J}P^d$ are isomorphic. Therefore, ${}_AA \cong P^d$ (see, e.g., [29, Corollary 6.2]). Also,

$$A \cong \mathrm{End}_A({}_AA)^{\mathrm{op}} \cong \mathrm{End}_A(P^d)^{\mathrm{op}} \cong (\mathrm{Mat}_d(\mathrm{End}_A(P))^{\mathrm{op}} \cong \mathrm{Mat}_d(B^{\mathrm{op}}),$$

where $B = \mathrm{End}_A(P)$ (see [29, Proposition 1.3 and Corollary 3.4a]). This finishes the proof. □

2.2. Let $\mathcal{L}$ be a finite dimensional restricted Lie algebra over K with pth power map $x \mapsto x^{[p]}$. We denote by $\mathcal{N}(\mathcal{L})$ the set of all nilpotent elements of $\mathcal{L}$, i.e., $\mathcal{N}(\mathcal{L}) = \{x\in\mathcal{L} \mid x^{[p]^e} = 0 \text{ for } e \gg 0\}$. We let $\mathcal{N}_p(\mathcal{L})$ denote the set of all $x \in \mathcal{L}$ with $x^{[p]} = 0$.

Fix a linear function ξ on $\mathcal{L}$ and let $U_\xi(\mathcal{L})$ denote the corresponding reduced enveloping algebra. In [11], Friedlander and Parshall generalised to the context of $U_\xi(\mathcal{L})$ the notion of a *support variety* as studied for $U^{[p]}(\mathcal{L})$ in [10] and [16]. Given $x\in\mathcal{L}$ let $U_\xi(x)$ denote the subalgebra with 1 of $U_\xi(\mathcal{L})$ generated x. Recall that for any $U_\xi(\mathcal{L})$-module M, the support variety $\mathcal{V}_{\mathcal{L}}(M)$ of M consists of 0 and all those $x \in \mathcal{N}_p(\mathcal{L})$ for which M is not a free $U_\xi(x)$-module. The set $\mathcal{V}_{\mathcal{L}}(M)$ is a Zariski closed, conical subset of $\mathcal{N}_p(\mathcal{L})$. One knows that M is a projective $U_\xi(\mathcal{L})$-module if and only if $\mathcal{V}_\mathcal{L}(M) = \{0\}$ (see [11, Proposition 6.2]).

Let $E_1, \ldots, E_s$ be representatives of the isomorphism classes of simple $U_\xi(\mathcal{L})$-modules. Define

$$\mathcal{V}_\mathcal{L}(\xi) := \bigcup_{i=1}^{s} \mathcal{V}_\mathcal{L}(E_i).$$

By [33, Proposition 2.2], $\mathcal{V}_\mathcal{L}(\xi) = \mathcal{V}_\mathcal{L}(U_\xi(\mathcal{L}))$ where $U_\xi(\mathcal{L})$ is viewed as the adjoint $U^{[p]}(\mathcal{L})$-module. Any Zariski closed conical subset of $\mathcal{V}_\mathcal{L}(\xi)$ is of the form $\mathcal{V}_\mathcal{L}(W)$ for some finite dimensional $U_\xi(\mathcal{L})$-module W and conversely, given a finite dimensional $U_\xi(\mathcal{L})$-module M one has $\mathcal{V}_\mathcal{L}(M) \subseteq \mathcal{V}_\mathcal{L}(\xi)$ (see [33, (2.4)] for more detail).

2.3. A restricted subalgebra $\mathfrak{q}$ of $\mathcal{L}$ is called $[p]$-*nilpotent* if $\mathfrak{q} \subseteq \mathcal{N}(\mathcal{L})$. By Engel's theorem, $\mathfrak{q}$ is a nilpotent subalgebra of $\mathcal{L}$. It follows that $\mathfrak{z}(\mathfrak{q}) \neq 0$. A straightforward induction argument based on this inequality shows that the $[p]$-closure $\overline{[\mathfrak{q},\mathfrak{q}]}$ of the derived subalgebra $[\mathfrak{q},\mathfrak{q}]$ is a proper ideal of $\mathfrak{q}$.

Definition. A restricted subalgebra $\mathfrak{n}$ of $\mathcal{L}$ is called ξ-*admissible* if it satisfies the following three conditions:

- **(ξ1):** the subalgebra $\mathfrak{n}$ is a $[p]$-nilpotent;
- **(ξ2):** the linear function ξ vanishes on $\overline{[\mathfrak{n},\mathfrak{n}]}$;
- **(ξ3):** the intersection $\mathcal{V}_\mathcal{L}(\xi) \cap \mathfrak{n}$ is zero.

Let $U_\xi(\mathfrak{n})$ denote the unital subalgebra of $U_\xi(\mathcal{L})$ generated by $\mathfrak{n}$ (it is isomorphic to the reduced enveloping algebra of $\mathfrak{n}$ associated with $\xi_{|\mathfrak{n}}$). Due to Jacobson's formula [14, Ch. V, Sect. 7] condition (ξ2) is equivalent to saying that $\xi(x) = 0$ for all $x \in \mathfrak{n}^{[p]} \cup [\mathfrak{n},\mathfrak{n}]$. It follows that the subspace $\mathfrak{n}' := \mathfrak{n} \cap \operatorname{Ker} \xi$ is a restricted ideal of codimension ≤ 1 in $\mathfrak{n}$. By Engel's theorem, $\mathfrak{n}'$ acts trivially on any simple $\mathfrak{n}$-module with p-character $\xi_{|\mathfrak{n}}$ (see [32, P. 248] for more detail). But then $U_\xi(\mathfrak{n})$ has a unique simple module which is 1-dimensional. In other words, there is a canonical augmentation map $U_\xi(\mathfrak{n}) \to K$ whose kernel $N_\mathfrak{n}$ coincides with the Jacobson radical of $U_\xi(\mathfrak{n})$. We denote by K_ξ the 1-dimensional left $U_\xi(\mathfrak{n})$-module $U_\xi(\mathfrak{n})/N_\mathfrak{n}$.

Condition (ξ3) and our discussion in (2.2) show that $\mathfrak{n} \cap \mathcal{V}_\mathcal{L}(M) \subseteq \mathfrak{n} \cap \mathcal{V}_\mathcal{L}(\xi) = \{0\}$ for any finite dimensional $U_\xi(\mathcal{L})$-module M. This implies that all finite dimensional $U_\xi(\mathcal{L})$-modules are projective over $U_\xi(\mathfrak{n})$. By the above, $U_\xi(\mathfrak{n})$ is a local algebra. Therefore, any finite dimensional $\mathcal{L}$-module with p-character ξ is a free $U_\xi(\mathfrak{n})$-module (see [17, Corollary 3.4] for more detail).

Theorem. *Let $\xi \in \mathcal{L}^*$ and let $\mathfrak{n}$ be a ξ-admissible subalgebra of $\mathcal{L}$ of dimension m. Set $d = p^m$ and denote by $Q_\mathfrak{n}$ the induced $U_\xi(\mathcal{L})$-module $U_\xi(\mathcal{L}) \otimes_{U_\xi(\mathfrak{n})} K_\xi$. Define $H_\mathfrak{n} := \operatorname{End}_\mathcal{L}(Q_\mathfrak{n})^{\mathrm{op}}$. Then the following hold.*

(i) $Q_\mathfrak{n}$ is a projective $U_\xi(\mathcal{L})$-module. Moreover, $Q_\mathfrak{n}^d$ is a free $U_\xi(\mathcal{L})$-module of rank 1.

(ii) $U_\xi(\mathcal{L}) \cong \operatorname{Mat}_d(H_\mathfrak{n})$ as K-algebras.

(iii) The adjoint $U^{[p]}(\mathfrak{n})$-module $U_\xi(\mathcal{L})N_\mathfrak{n}$ is free.

(iv) $H_\mathfrak{n} \cong U_\xi(\mathcal{L})^\mathfrak{n}/U_\xi(\mathcal{L})^\mathfrak{n} \cap U_\xi(\mathcal{L})N_\mathfrak{n}$ where $U_\xi(\mathcal{L})^\mathfrak{n}$ is the centraliser of $\mathfrak{n}$ in $U_\xi(\mathcal{L})$.

(v) $U_\xi(\mathcal{L}) \cong Q_\mathfrak{n} \otimes U_\xi(\mathfrak{n})$ as $(U_\xi(\mathcal{L}), U_\xi(\mathfrak{n}))$-bimodules.

(vi) There is a (noncanonical) isomorphism of associative algebras

$$H_\mathfrak{n} \otimes U_\xi(\mathfrak{n}) \xrightarrow{\sim} U_\xi(\mathcal{L})^\mathfrak{n}$$

which maps $H_{\mathfrak{n}}\otimes N_{\mathfrak{n}}$ onto $U_\xi(\mathcal{L})^{\mathfrak{n}}\cap U_\xi(\mathcal{L})N_{\mathfrak{n}}$.
(vii) $U_\xi(\mathcal{L})^{\mathfrak{n}}\cap U_\xi(\mathcal{L})N_{\mathfrak{n}}$ is contained in the Jacobson radical of $U_\xi(\mathcal{L})^{\mathfrak{n}}$.
(viii) $Q_{\mathfrak{n}}$ is a free module over $\mathrm{End}_{U_\xi(\mathcal{L})}(Q_{\mathfrak{n}})$.
(ix) $U_\xi(\mathcal{L})$ is free as a right $U_\xi(\mathcal{L})^{\mathfrak{n}}$-module.

Proof. It is immediate from the PBW theorem that $Q_{\mathfrak{n}}$ is a free $U_\xi(x)$-module whenever $x\in\mathcal{N}_p(\mathcal{L})\backslash\mathfrak{n}$. It follows that $\mathcal{V}_{\mathcal{L}}(Q_{\mathfrak{n}})\subseteq\mathcal{V}_{\mathcal{L}}(\xi)\cap\mathfrak{n}$. Now ($\xi 3$) forces $\mathcal{V}_{\mathcal{L}}(Q_{\mathfrak{n}})=\{0\}$. In other words, $Q_{\mathfrak{n}}$ is a projective $U_\xi(\mathcal{L})$-module (see (2.2)). Let P_i denote the projective cover of E_i where $1\le i\le s$. Since each E_i is free over $U_\xi(\mathfrak{n})$ one has $\dim E_i=dr_i$ for some positive integers r_i. So Proposition 2.1 applies yielding an algebra isomorphism

$$U_\xi(\mathcal{L})\cong\mathrm{Mat}_d(\mathrm{End}_{\mathcal{L}}(P)^{\mathrm{op}})$$

where $P=P_1^{r_1}\oplus\cdots\oplus P_s^{r_s}$. We claim that $P\cong Q_{\mathfrak{n}}$ as $U_\xi(\mathcal{L})$-modules. To prove the claim it suffices to show that $r_i=\dim\mathrm{Hom}_{\mathcal{L}}(P,E_i)=\dim\mathrm{Hom}_{\mathcal{L}}(Q_{\mathfrak{n}},E_i)$ for all i. Since each E_i is free over $U_\xi(\mathfrak{n})$, Frobenius reciprocity yields

$$\dim\mathrm{Hom}_{\mathcal{L}}(U_\xi(\mathcal{L})\otimes_{U_\xi(\mathfrak{n})}K_\xi,E_i)=\dim\mathrm{Hom}_{\mathfrak{n}}(K_\xi,E_i)=\mathrm{rk}_{U_\xi(\mathfrak{n})}E_i=r_i.$$

So the claim follows proving statements (i) and (ii) of the theorem.

Next observe that the support variety of the adjoint $U^{[p]}(\mathfrak{n})$-module $U_\xi(\mathcal{L})$ equals $\mathcal{V}_{\mathcal{L}}(U_\xi(\mathcal{L}))\cap\mathfrak{n}=\mathcal{V}_{\mathcal{L}}(\xi)\cap\mathfrak{n}=\{0\}$ (by [11, Proposition 7.1(a)], [33, Proposition 2.2] and ($\xi 3$)). So the adjoint $U^{[p]}(\mathfrak{n})$-module $U_\xi(\mathcal{L})$ is projective, hence free (for the algebra $U^{[p]}(\mathfrak{n})$ is local). As $N_{\mathfrak{n}}$ is a two–sided ideal of $U_\xi(\mathfrak{n})$ the left ideal $I:=U_\xi(\mathcal{L})N_{\mathfrak{n}}$ is (ad $\mathfrak{n}$)-stable. The left $U_\xi(\mathfrak{n})$-module $\overline{U}:=U_\xi(\mathcal{L})/I$ is free (see our discussion above). Given $x\in\mathfrak{n}$ and $u\in U_\xi(\mathcal{L})$ one has

$$x(u+I)=(\xi(x)u+[x,u])+I$$

(because $x-\xi(x)\in N_{\mathfrak{n}}$ and $\mathrm{ad}\,x=\mathrm{ad}\,(x-\xi(x))$. It follows that for any $x\in\mathcal{N}_p(\mathcal{L})\cap\mathfrak{n}$, the endomorphism $\mathrm{ad}\,x$ acts on $\overline{U}$ as a direct sum of Jordan blocks of length p. This shows that the support variety of the adjoint $U^{[p]}(\mathfrak{n})$-module $\overline{U}$ is zero. So the adjoint $U^{[p]}(\mathfrak{n})$-module $\overline{U}$ is projective, hence free. Thus the short exact sequence of (ad $\mathfrak{n}$)-modules

$$0\to I\to U_\xi(\mathcal{L})\to\overline{U}\longrightarrow 0$$

splits. In other words, there is a subspace $V\subset U_\xi(\mathcal{L})$ such that $[\mathfrak{n},V]\subseteq V$ and $U_\xi(\mathcal{L})\cong V\oplus I$ as (ad $\mathfrak{n}$)-modules. Thus the adjoint $U^{[p]}(\mathfrak{n})$-module I is projective, hence free, proving (iii). It also follows that

$$B:=\{u\in U_\xi(\mathcal{L})\,|\,Iu\subseteq I\}\;=\;\{u\in U_\xi(\mathcal{L})\,|\,[\mathfrak{n},u]\subset I\}\;=\;V^{\mathfrak{n}}\oplus I.$$

This gives $B/I\cong U_\xi(\mathcal{L})^{\mathfrak{n}}/U_\xi(\mathcal{L})^{\mathfrak{n}}\cap I$. Since $Q_{\mathfrak{n}}\cong\overline{U}$ as $U_\xi(\mathcal{L})$-modules we have $\mathrm{End}_{\mathcal{L}}(Q_{\mathfrak{n}})\cong(B/I)^{\mathrm{op}}$ as algebras (see, e.g., [29, Exercise 2.1.2(c)]). This proves (iv).

To establish (v) we first show that $U_\xi(\mathcal{L})$ is projective as a $(U_\xi(\mathcal{L}),U_\xi(\mathfrak{n}))$-bimodule. Let $\tilde{\mathcal{L}}$ denote the direct sum $\mathcal{L}\oplus\mathfrak{n}$ of restricted Lie algebras. Define $\tilde{\xi}\in\tilde{\mathcal{L}}^*$ by setting $\tilde{\xi}(l+n)=\xi(l)-\xi(n)$ for all $l\in\mathcal{L}$, $n\in\mathfrak{n}$. By construction, $U_{\tilde{\xi}}(\tilde{\mathcal{L}})\cong U_\xi(\mathcal{L})\otimes U_{-\xi}(\mathfrak{n})$ as algebras. On the other hand, the antipode of $U(\mathfrak{n})$ maps the defining ideal of $U_\xi(\mathfrak{n})$ onto that of $U_{-\xi}(\mathfrak{n})$ and induces an algebra isomorphism $U_\xi(\mathfrak{n})^{\mathrm{op}}\cong U_{-\xi}(\mathfrak{n})$. Thus the $(U_\xi(\mathcal{L}),U_\xi(\mathfrak{n}))$-bimodule $U_\xi(\mathcal{L})$ is just a $U_{\tilde{\xi}}(\tilde{\mathcal{L}})$-module and it suffices to show that its support variety $\mathcal{V}$ is zero. Let $z+z'\in\mathcal{V}$, where $z\in\mathcal{L}$ and $z'\in\mathfrak{n}$, and $a=z-z'$. Let

$\lambda = \tilde{\xi}(z+z')$. Suppose $a \neq 0$. Choose a basis $x_1, \ldots, x_n$ of $\mathcal{L}$ with $x_1 = a$. Let W be the subspace of $U_\xi(\mathcal{L})$ spanned by the monomials $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with $0 \le a_1 \le p-2$ and $a_i \in \{0, 1, \ldots, p-1\}$ for $i > 1$. Clearly, W has codimension p^{n-1} in $U_\xi(\mathcal{L})$. Let ρ denote the representation of $\tilde{\mathcal{L}}$ in $U_\xi(\mathcal{L})$. Then $\rho(z+z') = l_z - r_{z'}$ where l_x (respectively, r_x) denotes the left (respectively, right) multiplication by $x \in \mathcal{L}$. This yields

$$\rho(z+z') - \lambda \cdot id = (l_a - \lambda \cdot id) + \mathrm{ad}\ z'.$$

Note that $\mathrm{ad}\, z'$ respects the standard filtration of $U_\xi(\mathcal{L})$ (induced by expanding vectors via the monomial basis $\{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \,|\, 0 \le i \le p-1\}$ and counting degrees). This observation shows that

$$\mathrm{Ker}\,(\rho(z+z') - \lambda \cdot id) \cap W = 0 \quad \text{and} \quad \dim\,(\mathrm{Ker}\,(\rho(z+z') - \lambda \cdot id) \le p^{n-1}.$$

Since $(z+z')^{[p]} = 0$ we must have $(\rho(z+z') - \tilde{\xi}(z+z')\, id)^p = 0$. Hence

$$\dim \mathrm{Ker}\,(\rho(z+z') - \lambda \cdot id) \ge p^{n-1}.$$

Combining the two inequalities we obtain that $\rho(z+z') - \lambda \cdot id$ acts on $U_\xi(\mathcal{L})$ as a direct sum of nilpotent Jordan blocks of length p. This, however, contradicts the fact that $U_\xi(\mathcal{L})$ is not a free $U_{\tilde{\xi}}(z+z')$-module. Thus $a = 0$ yielding $\rho(z+z') = \mathrm{ad}\, z'$. But then z' belongs to the support variety of the adjoint $\mathcal{L}$-module $U_\xi(\mathcal{L})$. The latter coincides with $\mathcal{V}_\mathcal{L}(\xi)$ ([33, Proposition 2.2]). So $z' \in \mathcal{V}_\mathcal{L}(\xi) \cap \mathfrak{n} = \{0\}$ as $\mathfrak{n}$ is ξ-admissible. We deduce $\mathcal{V} = \{0\}$ as desired.

Let $K_{-\xi}$ be the unique simple module over the local algebra $U_{-\xi}(\mathfrak{n})$. Clearly, the modules $\{E_i \otimes K_\xi \,|\, 1 \le i \le s\}$ form a set of representatives of the isomorphism classes of simple $U_\xi(\mathcal{L}) \otimes U_\xi(\mathfrak{n})$-modules. For any $i \le s$,

$$\begin{aligned} \mathrm{Hom}_{U_\xi(\mathcal{L}) \otimes U_{-\xi}(\mathfrak{n})}(U_\xi(\mathcal{L}), E_i \otimes K_{-\xi}) &= \{\phi \in \mathrm{Hom}_{U_\xi(\mathcal{L})}(U_\xi(\mathcal{L}), E_i) \,|\, \phi(I) = 0\} \\ = \mathrm{Hom}_{U_\xi(\mathcal{L})}(\overline{U}, E_i) &\cong \mathrm{Hom}_{U_\xi(\mathcal{L})}(Q_\mathfrak{n}, E_i). \end{aligned}$$

By (i), $(Q_\mathfrak{n} \otimes U_\xi(\mathfrak{n}))^d$ is a free $U_\xi(\mathcal{L}) \otimes U_{-\xi}(\mathfrak{n})$-module of rank 1. It follows that

$$\dim \mathrm{Hom}_{U_\xi(\mathcal{L}) \otimes U_{-\xi}(\mathfrak{n})}(Q_\mathfrak{n} \otimes U_\xi(\mathfrak{n}), E_i \otimes K_{-\xi}) = (\dim E_i)/d = \dim \mathrm{Hom}_{U_\xi(\mathcal{L})}(Q_\mathfrak{n}, E_i).$$

As both $U_\xi(\mathcal{L})$ and $Q_\mathfrak{n} \otimes U_\xi(\mathfrak{n})$ are projective over $U_\xi(\mathcal{L}) \otimes U_{-\xi}(\mathfrak{n})$ we get (v) (see [29, Corollary 6.2]).

For (vi), consider the endomorphism algebras A and A' of the $(U_\xi(\mathcal{L}), U_\xi(\mathfrak{n}))$-bimodules $U_\xi(\mathcal{L})$ and $Q_\mathfrak{n} \otimes U_\xi(\mathfrak{n})$. By a standard argument, for any $\beta \in A$ there is a unique $b \in U_\xi(\mathcal{L})^\mathfrak{n}$ such that $\beta(x) = x \cdot b$ for all $x \in U_\xi(\mathcal{L})$. In other words, $A^{\mathrm{op}} \cong U_\xi(\mathcal{L})^\mathfrak{n}$. On the other hand, (v) implies that

$$A \cong A' \cong \mathrm{End}_{U_\xi(\mathcal{L})}(Q_\mathfrak{n}) \otimes U_{-\xi}(\mathfrak{n})^{\mathrm{op}} \cong H_\mathfrak{n}^{\mathrm{op}} \otimes U_\xi(\mathfrak{n})$$

as algebras. Since $Q_\mathfrak{n}$ is a projective left $U_\xi(\mathcal{L})$-module there is an idempotent $e \in U_\xi(\mathcal{L})$ such that $Q_\mathfrak{n} \cong U_\xi(\mathcal{L})e$ as modules and $H_\mathfrak{n} \cong eU_\xi(\mathcal{L})e$ as algebras ([29, Sect. 6.4]). Fix a bimodule isomorphism $\alpha \colon U_\xi(\mathcal{L}) \longrightarrow Q_\mathfrak{n} \otimes U_\xi(\mathfrak{n})$ and express the image of the unity element as $\alpha(1) = q_1 \otimes u_1 + \cdots + q_d \otimes u_d$ with $q_i \in U_\xi(\mathcal{L})e$ and $u_i \in U_\xi(\mathfrak{n})$. The isomorphism $A' \xrightarrow{\sim} A$ induced by α can be described as follows. First we identify $Q_\mathfrak{n}$ with $U_\xi(\mathcal{L})e$ and $H_\mathfrak{n}^{\mathrm{op}}$ with $eU_\xi(\mathcal{L})e$. Given $h \in H_\mathfrak{n}$ and

$u \in U_\xi(\mathfrak{n})$ there is a unique $\kappa_{h,u} \in A'$ such that $\kappa_{h,u}(q \otimes u') = qh \otimes uu'$ for all $q \in Q_\mathfrak{n}$ and $u' \in U_\xi(\mathfrak{n})$. Then

$$(\alpha^{-1} \circ \kappa_{h,u} \circ \alpha)(1) = \alpha^{-1}(\sum\nolimits_{i=1}^{d} q_i h \otimes uu_i) = \sum\nolimits_{i=1}^{d} (\alpha^{-1}(q_i h)) \cdot uu_i =: \eta(h,u).$$

Hence $\alpha^{-1} \circ \kappa_{h,u} \circ \alpha = r_{\eta(h,u)}$ where r_x denotes the right multiplication by $x \in U_\xi(\mathcal{L})$. It is immediate from our discussion that the map $(h,u) \mapsto \eta(h,u)$ extends uniquely to an algebra isomorphism $\eta\colon H_\mathfrak{n} \otimes U_\xi(\mathfrak{n}) \xrightarrow{\sim} U_\xi(\mathcal{L})^\mathfrak{n}$. If $u \in N_\mathfrak{n}$ then $\eta(h,u) \in I$ for any $h \in H_\mathfrak{n}$. So $\eta(H_\mathfrak{n} \otimes N_\mathfrak{n}) \subseteq U_\xi(\mathcal{L})^\mathfrak{n} \cap I$. By (iv), $\eta(H_\mathfrak{n} \otimes N_\mathfrak{n}) = U_\xi(\mathcal{L})^\mathfrak{n} \cap I$. Since $H_\mathfrak{n} \otimes N_\mathfrak{n}$ is a nilpotent ideal of $H_\mathfrak{n} \otimes U_\xi(\mathfrak{n})$, statements (vi) and (vii) follow.

By (i) and (ii), there are idempotents $e = e_1, e_2 \dots, e_d \in U_\xi(\mathcal{L})$ such that $e_1 + \cdots + e_d = 1$, $e_i \cdot e_j = 0$ for $i \neq j$, and $e_i U_\xi(\mathcal{L}) \cong eU_\xi(\mathcal{L})$ as right $U_\xi(\mathcal{L})$-modules. Then $e_i U_\xi(\mathcal{L})e \cong eU_\xi(\mathcal{L})e$ as right $eU_\xi(\mathcal{L})e$-modules, hence

$$Q_\mathfrak{n} \cong U_\xi(\mathcal{L})e \;=\; \bigoplus_{i=1}^{d} e_i U_\xi(\mathcal{L})e \cong (eU_\xi(\mathcal{L})e)^d \cong (H_\mathfrak{n}^{\mathrm{op}})^d$$

as H_n^{op}-modules. We get (viii). To obtain (ix) we need to show that the $(U_\xi(\mathcal{L}), U_\xi(\mathfrak{n}))$-bimodule $U_\xi(\mathcal{L})$ is free over its endomorphism algebra. This follows from (vi) and (viii) completing the proof of the theorem. □

2.4. Theorem 2.3 provides a perfect setting for a Morita equivalence. Given a K-algebra $\mathcal{A}$ we denote by $\mathcal{A}$-**mod** the category of all finite dimensional left $\mathcal{A}$-modules.

Let $\mathcal{L}$, ξ, $\mathfrak{n}$, $H_\mathfrak{n}$ and $N_\mathfrak{n}$ be as in (2.3). Given a left $U_\xi(\mathcal{L})$-module M define $M^\mathfrak{n} = \{v \in M \mid N_\mathfrak{n} \cdot v = 0\}$. Identify $H_\mathfrak{n}$ with $U_\xi(\mathcal{L})^\mathfrak{n}/U_\xi(\mathcal{L})^\mathfrak{n} \cap U_\xi(\mathcal{L})N_\mathfrak{n}$ in accordance with Theorem 2.3(iv) and view any left $H_\mathfrak{n}$-module as a $U_\xi(\mathcal{L})^\mathfrak{n}$-module with the trivial action of the ideal $U_\xi(\mathcal{L})^\mathfrak{n} \cap U_\xi(\mathcal{L})N_\mathfrak{n}$.

Theorem. *The functors*

$$U_\xi(\mathcal{L})\text{-}\mathbf{mod} \rightsquigarrow H_\mathfrak{n}\text{-}\mathbf{mod}, \quad M \longmapsto M^\mathfrak{n},$$

and

$$H_\mathfrak{n}\text{-}\mathbf{mod} \rightsquigarrow U_\xi(\mathcal{L})\text{-}\mathbf{mod}, \quad V \longmapsto U_\xi(\mathcal{L}) \otimes_{U_\xi(\mathcal{L})^\mathfrak{n}} V,$$

are mutually inverse category equivalences.

Proof. Let M be a finite dimensional left $U_\xi(\mathcal{L})$-module and

$$\operatorname{ind} M^\mathfrak{n} \;=\; U_\xi(\mathcal{L}) \otimes_{U_\xi(\mathcal{L})^\mathfrak{n}} M^\mathfrak{n}.$$

We need to show that $M \cong \operatorname{ind} M^\mathfrak{n}$ as $\mathcal{L}$-modules. By Theorem 2.3, $\dim(\operatorname{ind} M^\mathfrak{n}) = d \cdot \dim M^\mathfrak{n}$. By (2.2), any finite dimensional $U_\xi(\mathcal{L})$-module is free over $U_\xi(\mathfrak{n})$. Since

$$\dim(\operatorname{ind} M^\mathfrak{n})^\mathfrak{n} = \mathrm{rk}_{U_\xi(\mathfrak{n})}(\operatorname{ind} M^\mathfrak{n}) = (\dim \operatorname{ind} M^\mathfrak{n})/d = \dim M^\mathfrak{n}$$

we have $(\operatorname{ind} M^\mathfrak{n})^\mathfrak{n} = 1 \otimes M^\mathfrak{n}$. Let $\psi_M\colon \operatorname{ind} M^\mathfrak{n} \longrightarrow M$ denote the $\mathcal{L}$-module homomorphism such that $\psi_M(1 \otimes v) = v$ for any $v \in M^\mathfrak{n}$. It suffices to show that ψ_M is an isomorphism. As $\operatorname{Ker} \psi_M$ is a $U_\xi(\mathcal{L})$-submodule of $\operatorname{ind} M^\mathfrak{n}$, it intersects with $(\operatorname{ind} M^\mathfrak{n})^\mathfrak{n} = 1 \otimes M^\mathfrak{n}$. Since ψ_M is injective on $1 \otimes M^\mathfrak{n}$ we have $\operatorname{Ker} \psi_M = 0$. As M is free over $U_\xi(\mathfrak{n})$, we also have that $\dim M = d \cdot \dim M^\mathfrak{n}$. Then $\dim M = \dim \operatorname{ind} M^\mathfrak{n}$, hence ψ_M is an isomorphism as desired. □

2.5. In general, it is very difficult to obtain a satisfactory description of $H_{\mathfrak{n}}$ as an algebra. However, there is an important special case where this problem can be solved. For Lie algebras of reductive groups, this case includes the Kac-Weisfeiler theorem [44, 20] and the Morita theorem obtained by Friedlander and Parshall in [11].

Proposition. *Let* $\mathfrak{n}$ *be a* ξ*-admissible subalgebra of* $\mathcal{L}$ *and* $\mathfrak{p}$ *the normaliser of* $\mathfrak{n}$ *in* $\mathcal{L}$*. Suppose* $\dim\mathfrak{p} \geq \dim\mathcal{L} - \dim\mathfrak{n}$ *and* $\xi_{|\mathfrak{n}} = 0$ *(so that* ξ *induces a linear function on* $\mathfrak{p}/\mathfrak{n}$*). Then* $\dim\mathfrak{p} = \dim\mathcal{L} - \dim\mathfrak{n}$ *and* $H_{\mathfrak{n}} \cong U_\xi(\mathfrak{p}/\mathfrak{n})$ *as algebras. In particular,* $U_\xi(\mathcal{L})$ *and* $U_\xi(\mathfrak{p}/\mathfrak{n})$ *are Morita equivalent.*

Proof. As ξ vanishes on $\mathfrak{n}$ the ideal $N_{\mathfrak{n}}$ of $U_\xi(\mathfrak{n})$ is generated by the image of $\mathfrak{n}$ in $U_\xi(\mathfrak{n})$. This yields $[\mathfrak{p}, N_{\mathfrak{n}}] \subseteq N_{\mathfrak{n}}$. Let $U_\xi(\mathfrak{p})$ denote the unital subalgebra of $U_\xi(\mathcal{L})$ generated by $\mathfrak{p}$ (it is canonically isomorphic to the reduced enveloping algebra of $\mathfrak{p}$ associated with $\xi_{|\mathfrak{p}}$). In view of our previous remark, $U_\xi(\mathfrak{p})N_{\mathfrak{n}}$ is a two-sided ideal of $U_\xi(\mathfrak{p})$. By the PBW theorem, $U_\xi(\mathfrak{p})/U_\xi(\mathfrak{p})N_{\mathfrak{n}} \cong U_\xi(\mathfrak{p}/\mathfrak{n})$ as algebras. Let $1_\xi = 1 + N_{\mathfrak{n}}$, the image of 1 in K_ξ, and $Q^0_{\mathfrak{n}} = U_\xi(\mathfrak{p}) \cdot 1_\xi$. Since $Q_{\mathfrak{n}} \cong U_\xi(\mathcal{L})/U_\xi(\mathcal{L})N_{\mathfrak{n}}$ as left $U_\xi(\mathcal{L})$-modules, the PBW theorem and the discussion above imply that $\dim Q^0_{\mathfrak{n}} = \dim U_\xi(\mathfrak{p}/\mathfrak{n})$. Given $q \in Q^0_{\mathfrak{n}}$ there is $u \in U_\xi(\mathfrak{p})$ such that $q = u \cdot 1_\xi$. By Jacobi identity, $[n, u] \in U_\xi(\mathfrak{p})N_{\mathfrak{n}}$ for any $n \in N_{\mathfrak{n}}$, forcing $N_{\mathfrak{n}} \cdot Q^0_{\mathfrak{n}} = 0$. The universality property of induced modules implies that for any $q \in Q^0_{\mathfrak{n}}$ there is a unique $h_q \in \mathrm{End}_{U_\xi(\mathcal{L})}(Q_{\mathfrak{n}})$ such that $h_q(1_\xi) = q$. This means that $\dim H_{\mathfrak{n}} \geq \dim U_\xi(\mathfrak{p}/\mathfrak{n})$. On the other hand,

$$\dim H_{\mathfrak{n}} = p^{\dim\mathcal{L} - 2\dim\mathfrak{n}} \quad \text{and} \quad \dim U_\xi(\mathfrak{p}/\mathfrak{n}) = p^{\dim\mathfrak{p} - \dim\mathfrak{n}} \geq p^{\dim\mathcal{L} - 2\dim\mathfrak{n}}$$

(by Theorem 2.3(i) and our assumption). It follows that $\dim\mathfrak{p} = \dim\mathcal{L} - \dim\mathfrak{n}$ and $\mathrm{End}_{U_\xi(\mathcal{L})}(Q_{\mathfrak{n}}) = \{\, h_q \mid q \in Q^0_{\mathfrak{n}} \}$. Using this equality it is not hard to deduce that $\mathrm{End}_{U_\xi(\mathcal{L})}(Q_{\mathfrak{n}}) \cong U_\xi(\mathfrak{p}/\mathfrak{n})^{\mathrm{op}}$ as K-algebras. Then $H_{\mathfrak{n}} \cong U_\xi(\mathfrak{p}/\mathfrak{n})$, completing the proof. □

2.6. Let G be a reductive algebraic group over K. We assume that the derived subgroup $G^{(1)}$ of G is simply connected and p is a good prime for the root system R of G. Given a closed subgroup $H \subseteq G$ we denote by $X_*(H)$ the set of all 1-dimensional tori contained in H. The adjoint action of $\nu \in X_*(G)$ turns $\mathfrak{g} = \mathrm{Lie}\, G$ into a $\mathbb{Z}$-graded Lie algebra:

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(\nu, i), \qquad [\mathfrak{g}(\nu, i), \mathfrak{g}(\nu, j)] \subseteq \mathfrak{g}(\nu, i+j) \text{ for all } i, j \in \mathbb{Z},$$

where $\mathfrak{g}(\nu, i)$ denotes the weight space of $\mathfrak{g}$ corresponding to weight $i \in X^*(\nu) \cong \mathbb{Z}$. By [30, Sect. 3], G possesses a finite dimensional semisimple rational representaton ρ such that the trace form

$$\Phi\colon \mathfrak{g} \times \mathfrak{g} \to K, \quad (X, Y) \mapsto \mathrm{tr}\, d\rho(X)\, d\rho(Y),$$

has the property that $\mathrm{Rad}\,\Phi \subseteq \mathfrak{z}(\mathfrak{g})$.

The Lie algebra $\mathfrak{g}$ carries a natural pth power map $x \mapsto x^{[p]}$ invariant under the adjoint action of G. Given a linear function l on $\mathfrak{g}$ we denote by $\mathfrak{z}_l = \mathfrak{z}_{\mathfrak{g}}(l)$ the stabiliser of l in $\mathfrak{g}$. Obviously, $\mathfrak{z}_{\mathfrak{g}}(l)$ is a restricted subalgebra of even codimension in $\mathfrak{g}$. Let $Z_G(l)$ denote the isotropy subgroup of l relative to the coadjoint action of G. By [20, Sect. 3] and [30, Lemma 3.1], there exist unique $l_s \in \mathfrak{g}^*$ and $e_l \in \mathcal{N}(\mathfrak{z}_{\mathfrak{g}}(l_s))$ such that

(1) $l(x) = l_s(x) + \Phi(x, e_l)$ for all $x \in \mathfrak{g}$;
(2) $Z_G(l_s)$ is a Levi subgroup of G and $\mathfrak{z}_{\mathfrak{g}}(l_s) = \mathrm{Lie}\ Z_G(l)$;

(3) $\mathfrak{z}_{\mathfrak{g}}(l) = \{x \in \mathfrak{z}_{\mathfrak{g}}(l_s) \mid [e,x] \in \operatorname{Rad}\Phi\}$;
(4) $Z_G(l) = \{g \in Z_G(l) \mid (\operatorname{Ad} g)\cdot e_l = e_l\}$.

The decomposition $l = l_s + \Phi(\cdot\,, e_l)$ is called the *Jordan decomposition* of l.

Fix a nonzero $\chi \in \mathfrak{g}^*$ and set $e = e_\chi$, $L = Z_G(\chi_s)$, $\mathfrak{l} = \operatorname{Lie} Z_G(\chi_s)$. Let P_+ be a parabolic subgroup of G such that $P_+ = L \cdot N_+$ where $N_+ = R_u(P_+)$. Let P_- be a parabolic subgroup conjugate to P_+ and such that $P_+ \cap P_- = L$. Set $N_- = R_u(P_-)$ and $\mathfrak{n}_\chi^\pm = \operatorname{Lie} N_\pm$. Then $\mathfrak{g} = \mathfrak{n}_\chi \oplus \mathfrak{l} \oplus \mathfrak{n}_\chi^+$.

First suppose that $e \neq 0$. By [21], [37], [31], there exists a 1-dimensional torus $\lambda_e \in X_*(L^{(1)})$ such that $e \in \mathfrak{g}(\lambda_e, 2)$ and $\operatorname{Ker} \operatorname{ad}_{\mathfrak{l}} e \subseteq \oplus_{i\geq 0} \mathfrak{l}(\lambda_e, i)$ In [31], such a torus is called a *Dynkin torus* for e. Put $\mathfrak{l}(i) = \mathfrak{l}(\lambda_e, i)$ and let ψ_e denote the skew-symmetric form on $\mathfrak{l}(-1)$ such that $\psi_e(u,v) = \Phi([u,v],e)$ for all $u, v \in \mathfrak{l}(-1)$. If $p = 2$ define a quadratic form Q on $\mathfrak{l}(-1)$ by letting $Q(u) = \Phi(u^{[2]}, e)$ for all $u \in \mathfrak{l}(-1)$. Let $\mathfrak{l}(-1)^0$ denote a maximal totally isotropic subspace of $\mathfrak{l}(-1)$ relative to ψ_e. If $p = 2$ suppose in addition that Q vanishes on $\mathfrak{l}(-1)^0$ (such a subspace exists by [30, P. 97]). Set

$$\mathfrak{m}_\chi = \mathfrak{l}(-1)^0 \oplus \big(\textstyle\sum_{i\leq -2} \mathfrak{l}(i)\big) \oplus \mathfrak{n}_\chi^-.$$

If $e = 0$ set $\mathfrak{m}_\chi = \mathfrak{n}_\chi^-$.

Proposition. *Suppose $\chi \in \mathfrak{g}^*$ is such that $[e,\mathfrak{l}] \cap \mathfrak{z}(\mathfrak{g}) = 0$. Then $\mathfrak{m}_\chi$ is a χ-admissible subalgebra of dimension $(\dim G\cdot\chi)/2$ in $\mathfrak{g}$.*

Proof. By [30, Lemmas 3.2 and 3.6], $\mathfrak{m}_\chi$ is a restricted $[p]$-nilpotent subalgebra of $\mathfrak{g}$, $\dim \mathfrak{m}_\chi = (\dim G\cdot\chi)/2$, and χ vanishes on a restricted ideal of $\mathfrak{m}_\chi$ containing $[\mathfrak{m}_\chi, \mathfrak{m}_\chi]$. Thus to finish the proof it suffices to show that $\mathfrak{m}_\chi$ satisfies (ξ3). By [33, Proposition 2.4], $\mathfrak{m}_\chi \cap \mathcal{V}_{\mathfrak{g}}(\chi) \subseteq \mathfrak{m}_\chi \cap \mathfrak{z}_{\mathfrak{g}}(\chi)$ (see also [32, Corollary 1.2]). On the other hand, it follows from (2.6(3)) and our assumption that $\mathfrak{z}_{\mathfrak{g}}(\chi) = \operatorname{Ker} \operatorname{ad}_{\mathfrak{l}} e \subseteq \sum_{i\geq 0} \mathfrak{l}(i)$. So we get $\mathfrak{m}_\chi \cap \mathcal{V}_{\mathfrak{g}}(\chi) \subseteq \mathfrak{m}_\chi \cap \big(\sum_{i\geq 0} \mathfrak{l}(i)\big) = \{0\}$, as desired. □

Proposition 2.6 shows that if $[e,\mathfrak{l}] \cap \mathfrak{z}(\mathfrak{g}) = 0$ then $\mathfrak{g}$ contains a χ-admissible subalgebra $\mathfrak{m}$ of dimension $d(\chi) = (\dim G\cdot\chi)/2$. In view of Theorem 2.3(i) this subalgebra induces an algebra isomorphism $U_\chi(\mathfrak{g}) \cong \operatorname{Mat}_{q(\chi)}(H_{\mathfrak{m}})$ where $q(\chi) = p^{d(\chi)}$. On the other hand, if $\operatorname{Mat}_r(A) \cong \operatorname{Mat}_r(B)$, where A and B are finite dimensional associative algebras over a field and $r \in \mathbb{N}$, then $A \cong B$ (this is not hard to deduce from the Krull-Remak-Schmidt theorem). As a consequence, the isomorphism classes of the algebra $H_{\mathfrak{m}}$ and the projective generator $Q_{\mathfrak{m}}$ do not depend on the choice of a χ-admissible subalgebra $\mathfrak{m}$ of dimension $d(\chi)$.

For χ satisfying the assumption of Proposition 2.6 we set $Q_\chi = Q_{\mathfrak{m}_\chi}$, $H_\chi(\mathfrak{g}) = H_{\mathfrak{m}_\chi}$ and let ρ_χ denote the representation of $U_\chi(\mathfrak{g})$ in $\operatorname{End}(Q_\chi)$. As D. Kazhdan pointed out to me, there is a striking resemblance between the representations $\{\rho_\chi \mid \chi \in \mathfrak{g}^*\}$ and the generalised Gelfand-Graev representations of finite Chevalley groups introduced by Kawanaka (see, e.g., [21]).

3. PBW bases in $H_\chi(\mathfrak{g})$

3.1. In this section, we retain the assumptions of (2.6) and take a closer look at the algebras $H_\chi(\mathfrak{g})$. What we do here will be crucial for constructing noncommutative filtered deformations of the graded coordinate rings $\mathbb{C}[\psi^{-1}(0)]$.

Henceforth we assume that χ is *nilpotent* that is $\chi_s = 0$ and $\chi = \Phi(\,\cdot\,, e)$ where $e = e_\chi$. The general case can be reduced to the nilpotent case by applying Proposition 2.5 to an appropriate parabolic decomposition of $\mathfrak{g}$. Thus from now on $\mathfrak{l} = \mathfrak{g}$ and $\mathfrak{m}_\chi = \mathfrak{g}(-1)^0 \oplus \sum_{i\le -2} \mathfrak{g}(i)$. We also assume that $[e, \mathfrak{g}] \cap \mathfrak{z}(\mathfrak{g}) = 0$. Then $\mathfrak{z}_\chi = \mathfrak{c}_\mathfrak{g}(e)$, in view of (2.6(3)) and the inclusion $\operatorname{Rad}\Phi \subseteq \mathfrak{z}(\mathfrak{g})$. As $\mathfrak{c}_\mathfrak{g}(e)$ is $(\operatorname{Ad}\lambda)$-stable, $\mathfrak{z}_\chi = \sum_{i\ge 0} \mathfrak{z}_\chi(i)$ where $\mathfrak{z}_\chi(i) = \mathfrak{z}_\chi \cap \mathfrak{g}(i)$.

For $x \in \mathfrak{g}$ we write $\mathrm{wt}(x) = k$ if and only if $x \in \mathfrak{g}(k)$. We choose a homogeneous basis $x_1 \ldots, x_r$ of $\mathfrak{z}_\chi$ and extend it up to a homogeneous basis $x_1, \ldots, x_r, x_{r+1}, \ldots, x_m$ of the graded parabolic subalgebra $\mathfrak{p}_e = \sum_{i\ge 0} \mathfrak{g}(i)$. Let $\mathrm{wt}(x_i) = n_i$, $1 \le i \le m$. Since $\mathfrak{z}_\chi \subseteq \mathfrak{p}_e$ and $\operatorname{Rad}\Phi \subseteq \mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}(0)$ (see, e.g., [30, Sect. 3]), the equality $[e, \mathfrak{p}_e] = \sum_{i\ge 2} \mathfrak{g}(i)$ holds. It follows that $r = \dim \mathfrak{g}(0) + \dim \mathfrak{g}(1)$ and there exist homogeneous $y_{r+1}, \ldots, y_m \in \sum_{i\le -2} \mathfrak{g}(i)$ such that

$$\Phi([y_i, x_j], e) = \delta_{ij} \qquad (r+1 \le i, j \le m).$$

Fix a Witt basis $z_1', \ldots, z_s', z_1, \ldots z_s$ of $\mathfrak{g}(-1)$ relative to ψ_e such that $z_1', \ldots, z_s' \in \mathfrak{g}(-1)^0$. Using the injectivity of $\operatorname{ad} e$ on $\sum_{i\le -1} \mathfrak{g}(i)$ it is easy to deduce that $s = 0$ if and only if $\mathfrak{g}(k) = 0$ for all odd k. If $s = 0$ one says that e is *even*.

For $k \in \mathbb{N}$ define $\Lambda_k := \{(l_1, \ldots, l_k) \mid l_i \in \mathbb{Z},\ 0 \le l_i \le p-1\}$. Set $\mathbf{e}_i = (\delta_{i1}, \ldots, \delta_{ik})$ where $1 \le i \le k$. For $\mathbf{l} = (l_1, \ldots, l_k) \in \Lambda_k$ set $|\mathbf{l}| = l_1 + \cdots + l_k$. Given $(\mathbf{a}, \mathbf{b}) = (a_1, \ldots, a_m; b_1, \ldots, b_s) \in \Lambda_m \times \Lambda_s$ define

$$|(\mathbf{a}, \mathbf{b})|_e := \sum_{i=1}^m a_i(n_i + 2) + \sum_{i=1}^s b_i$$

and denote by $x^\mathbf{a} z^\mathbf{b}$ the monomial $x_1^{a_1} \cdots x_m^{a_m} z_1^{b_1} \cdots z_s^{b_s}$ in $U_\chi(\mathfrak{g})$. We say that $x^\mathbf{a} z^\mathbf{b}$ has *e-degree* $|(\mathbf{a}, \mathbf{b})|_e$ and write $\deg_e(x^\mathbf{a} z^\mathbf{b}) = |(\mathbf{a}, \mathbf{b})|_e$. Note that

$$\deg_e(x^\mathbf{a} z^\mathbf{b}) = \mathrm{wt}\,(x^\mathbf{a} z^\mathbf{b}) + 2\deg\,(x^\mathbf{a} z^\mathbf{b}) \tag{1}$$

where $\mathrm{wt}\,(x^\mathbf{a} z^\mathbf{b}) = (\sum_{i\le m} n_i a_i) - |\mathbf{b}|$ and $\deg\,(x^\mathbf{a} z^\mathbf{b}) = |\mathbf{a}| + |\mathbf{b}|$ are the *weight* and the *standard degree* of $x^\mathbf{a} z^\mathbf{b}$, respectively.

Set $1_\chi := 1 + N_{\mathfrak{m}_\chi} \in K_\chi$. The vectors $x^\mathbf{a} z^\mathbf{b} \otimes 1_\chi$ with $(\mathbf{a}, \mathbf{b}) \in \Lambda_m \times \Lambda_s$ form a basis of Q_χ over K. It is well-known (see, e.g., [40, (5.7)]) that

$$u \cdot x^\mathbf{a} z^\mathbf{b} = \sum_{\mathbf{i} \in \Lambda_m} \binom{\mathbf{a}}{\mathbf{i}} x^{\mathbf{a}-\mathbf{i}} \cdot [u x^\mathbf{i}] \cdot z^\mathbf{b} \tag{2}$$

for any $u \in U_\chi(\mathfrak{g})$, where $[u x^\mathbf{i}] = (-1)^{|\mathbf{i}|} (\operatorname{ad} x_m)^{i_m} \cdots (\operatorname{ad} x_1)^{i_1}(u)$ and $\binom{\mathbf{a}}{\mathbf{i}} = \prod_{k=1}^m \binom{a_k}{i_k}$.

Lemma. *Let $(\mathbf{a}, \mathbf{b}), (\mathbf{c}, \mathbf{d}) \in \Lambda_m \times \Lambda_s$ be such that $|(\mathbf{a}, \mathbf{b})|_e = A$ and $|(\mathbf{c}, \mathbf{d})|_e = B$. Then*

$$\big(\rho_\chi(x^\mathbf{a} z^\mathbf{b})\big)(x^\mathbf{c} z^\mathbf{d} \otimes \mathbf{1}_\chi) = (x^{\mathbf{a}+\mathbf{c}} z^{\mathbf{b}+\mathbf{d}} + \text{ terms of } e\text{-degree } \le A + B - 2) \otimes 1_\chi.$$

The first summand on the right is interpreted as 0 if $(\mathbf{a}+\mathbf{c}, \mathbf{b}+\mathbf{d}) \notin \Lambda_m \times \Lambda_s$.

Proof. First suppose that $\mathbf{a} = \mathbf{0}$ and $|\mathbf{b}| = 1$, so that $A = 1$. Then $z^\mathbf{b} = z_k$ for some k. Applying (3.1(2)) one obtains

$$\big(\rho_\chi(z_k)\big)(x^\mathbf{c} z^\mathbf{d} \otimes 1_\chi) = (x^\mathbf{c} \cdot \rho_\chi(z_k)\, z^\mathbf{d} + \sum_{\mathbf{i} \ne \mathbf{0}} \alpha_\mathbf{i}\, x^{\mathbf{c}-\mathbf{i}} \cdot \rho_\chi([z_k x^\mathbf{i}])\, z^\mathbf{d}) \otimes 1_\chi$$

for some $\alpha_{\mathbf{i}} \in K$. Since $\rho_\chi(\mathfrak{m}_\chi)$ stabilises the line $K1_\chi$, the first summand on the right equals $x^{\mathbf{c}} z^{\mathbf{d}+\mathbf{e}_k} \otimes 1_\chi$ modulo terms of lower e-degree (if $d_k+1=p$ then $z^{\mathbf{d}+\mathbf{e}_k}$ is interpreted as 0). Suppose $\mathbf{i} \neq \mathbf{0}$ is such that $\mathrm{wt}([z_k x^{\mathbf{i}}]) \leq -1$. Then $\rho_\chi([z_k x^{\mathbf{i}}])\, z^{\mathbf{d}} \otimes 1_\chi$ is a linear combination of $z^{\mathbf{j}} \otimes 1_\chi$ with $|\mathbf{j}| \leq |\mathbf{d}|+1$ (because $\rho_\chi(\mathfrak{m}_\chi)$ stabilises the line $K1_\chi$). As a consequence, $x^{\mathbf{c}-\mathbf{i}} \cdot \rho_\chi([z_k x^{\mathbf{i}}])\, z^{\mathbf{d}} \otimes 1_\chi$ is a linear combination of $x^{\mathbf{c}-\mathbf{i}} z^{\mathbf{j}} \otimes 1_\chi$ with

$$\deg_e(x^{\mathbf{c}-\mathbf{i}} z^{\mathbf{j}}) = \sum_{k=1}^{m} (c_k - i_k)(n_k+2) + |\mathbf{j}| \leq \sum_{k=1}^{m} c_k(n_k+2) + (|\mathbf{j}| - 2|\mathbf{i}|) \leq A+B-2$$

(one should take into account that $\mathbf{i} \neq \mathbf{0}$ and all n_k are nonnegative).

Now suppose $\mathbf{i} \neq \mathbf{0}$ is such that $\mathrm{wt}([z_k x^{\mathbf{i}}]) \geq 0$. Since χ vanishes on $\mathfrak{p}_e$, the reduced enveloping algebra $U_\chi(\mathfrak{p}_e)$ is canonically isomorphic to the restricted enveloping algebra $U^{[p]}(\mathfrak{p}_e)$. It follows that $\mathrm{Ad}\,\lambda$ acts on $U_\chi(\mathfrak{p}_e)$ as algebra automorphisms. This, in turn, implies that $x^{\mathbf{c}-\mathbf{i}} \cdot [z_k x^{\mathbf{i}}]$ is a linear combination of $x^{\mathbf{l}}$ with $\mathrm{wt}(x^{\mathbf{l}}) = \mathrm{wt}(x^{\mathbf{c}}) - 1$ and $|\mathbf{l}| \leq |\mathbf{c}| - |\mathbf{i}| + 1$. Therefore, $x^{\mathbf{c}-\mathbf{i}} \cdot \rho_\chi([z_k x^{\mathbf{i}}])\, z^{\mathbf{d}} \otimes 1_\chi$ is a linear combination of $x^{\mathbf{l}} z^{\mathbf{d}} \otimes 1_\chi$ with

$$\deg_e(x^{\mathbf{l}} z^{\mathbf{d}}) = -1 + \mathrm{wt}(x^{\mathbf{c}}) + 2|\mathbf{l}| + |\mathbf{d}| \leq \mathrm{wt}(x^{\mathbf{c}} z^{d}) + 2\deg(x^{\mathbf{c}} z^{\mathbf{d}}) - 2|\mathbf{i}| + 1 \leq A+B-2.$$

Thus in all cases,

$$\big(\rho_\chi(z_k)\big)(x^{\mathbf{c}} z^{\mathbf{d}} \otimes 1_\chi) = (x^{\mathbf{c}} z^{\mathbf{d}+\mathbf{e}_k} + \text{ terms of } e\text{-degree } \leq A+B-2) \otimes 1_\chi.$$

Iduction on $|\mathbf{b}| = |(\mathbf{0},\mathbf{b})|_e = B$ now shows that

$$\big(\rho_\chi(z^{\mathbf{b}})\big)(x^{\mathbf{c}} z^{\mathbf{d}} \otimes 1_\chi) = (x^{\mathbf{c}} z^{\mathbf{b}+\mathbf{d}} + \text{ terms of } e\text{-degree } \leq A+B-2) \otimes 1_\chi \qquad (3)$$

for all $\mathbf{b} \in \Lambda_s$. Since $\mathrm{Ad}\,\lambda$ acts on $U_\chi(\mathfrak{p}_e)$ as algebra automorphisms the PBW theorem implies that

$$x^{\mathbf{a}} \cdot x^{\mathbf{c}} = x^{\mathbf{a}+\mathbf{c}} + \sum_{|\mathbf{i}|<|\mathbf{a}|+|\mathbf{c}|} \beta_{\mathbf{i}}\, x^{\mathbf{i}}$$

where $\beta_{\mathbf{i}} = 0$ unless $\mathrm{wt}(x^{\mathbf{i}}) = \mathrm{wt}(x^{\mathbf{a}}) + \mathrm{wt}(x^{\mathbf{c}})$ (the first summand should be interpreted as 0 if $\mathbf{a}+\mathbf{c} \notin \Lambda_m$). Combining this equality with (3.1(3)) and (3.1(1)) it is now easy to derive that

$$\big(\rho_\chi(x^{\mathbf{a}} z^{\mathbf{b}})\big)(x^{\mathbf{c}} z^{\mathbf{d}} \otimes 1_\chi) = (x^{\mathbf{a}+\mathbf{c}} z^{\mathbf{b}+\mathbf{d}} + \text{ terms of } e\text{-degree } \leq A+B-2) \otimes 1_\chi$$

for all admissible $(\mathbf{a},\mathbf{b})$ and $(\mathbf{c},\mathbf{d})$. □

3.2. Recall that any $0 \neq h \in H_\chi(\mathfrak{g})$ is uniquely determined by its value $h(1_\chi) \in Q_\chi$. Write $h(1_\chi) = (\sum_{|(\mathbf{i},\mathbf{j})|_e \leq n} \lambda_{\mathbf{i},\mathbf{j}}\, x^{\mathbf{i}} z^{\mathbf{j}}) \otimes 1_\chi$, where $n = n(h)$ and $\lambda_{\mathbf{i},\mathbf{j}} \neq 0$ for at least one $(\mathbf{i},\mathbf{j})$ with $|(\mathbf{i},\mathbf{j})|_e = n$. For $k \in \mathbb{N}_0$ put $\Lambda_h^k = \{(\mathbf{i},\mathbf{j}) \mid \lambda_{\mathbf{i},\mathbf{j}} \neq 0 \ \& \ |(\mathbf{i},\mathbf{j})|_e = k\}$ and let $\Lambda_h^{\max}$ denote the set of all $(\mathbf{p},\mathbf{q}) \in \Lambda_h^n$ for which the quantity $\mathrm{wt}(x^{\mathbf{p}} z^{\mathbf{q}})$ assumes its maximum value. This maximum value will be denoted by $N = N(h)$.

For $a \in \mathbb{Z}$ we let $\bar{a}$ denote the residue of a in $\mathbb{F}_p \subset K$.

Lemma. *Let $h \in H_\chi(\mathfrak{g}) \setminus \{0\}$ and $(\mathbf{p},\mathbf{q}) \in \Lambda_h^{\max}$. Then $\mathbf{q} = \mathbf{0}$ and $\mathbf{p} \in \Lambda_r \times \{\mathbf{0}\}$.*

Proof. Suppose the contrary. Then $(p_{r+1},\dots,p_m,q_1,\dots,q_s) \neq \mathbf{0}$. If $p_k \neq 0$ for some $k > r$ set $y = y_k \in \mathfrak{g}(-n_k-2)$. If all p_i's are zero for $i > r$ then $q_j \neq 0$ for some $j \leq s$. In this case set $y = z'_j$. Let $w = \mathrm{wt}\,(y)$.

Let $(\mathbf{a},\mathbf{b})\in\Lambda_h^d$. It is immediate from (3.1(2)) and the definition of Q_χ that

$$\big(\rho_\chi(y)\big)(x^{\mathbf{a}}z^{\mathbf{b}}\otimes 1_\chi)=\Big(\sum_{\mathbf{i}}\binom{\mathbf{a}}{\mathbf{i}}x^{\mathbf{a}-\mathbf{i}}\cdot\rho_\chi([yx^{\mathbf{i}}])\,z^{\mathbf{b}}\Big)\otimes 1_\chi$$

where the summation runs over all $\mathbf{i}\in\Lambda_m$ such that $[yx^{\mathbf{i}}]$ is nonzero and has weight ≥ -2.

Suppose $\mathbf{i}$ is such that $\mathrm{wt}([yx^{\mathbf{i}}])\geq 0$. Then $|\mathbf{i}|\geq 1$. Recall that $\mathrm{Ad}\,\lambda$ acts on $U_\chi(\mathfrak{p}_e)$ as algebra automorphisms. This implies that $x^{\mathbf{a}-\mathbf{i}}\cdot\rho_\chi([yx^{\mathbf{i}}])\,z^{\mathbf{b}}\otimes 1_\chi$ is a linear combination of $x^{\mathbf{j}}z^{\mathbf{b}}\otimes 1_\chi$ with

$$\mathrm{wt}(x^{\mathbf{j}}z^{\mathbf{b}})=w+\mathrm{wt}(x^{\mathbf{a}}z^{\mathbf{b}})\tag{4}$$

and

$$\deg_e(x^{\mathbf{j}}z^{\mathbf{b}})=w+\mathrm{wt}(x^{\mathbf{a}}z^{\mathbf{b}})+2|\mathbf{j}|+2|\mathbf{b}|\leq 2+d+w-2|\mathbf{i}|.\tag{5}$$

Now suppose $\mathbf{i}$ is such that $\mathrm{wt}([yx^{\mathbf{i}}])=-1$. Then $\sum_{k\leq m}i_kn_k=-w-1$. Since $\rho_\chi(\mathfrak{m}_\chi\cap\mathfrak{g}(-1))$ annihilates 1_χ, the vector $x^{\mathbf{a}-\mathbf{i}}\cdot\rho_\chi([yx^{\mathbf{i}}])\,z^{\mathbf{b}}\otimes 1_\chi$ is a linear combination of $x^{\mathbf{a}-\mathbf{i}}z^{\mathbf{l}}\otimes 1_\chi$ with $|\mathbf{l}|=|\mathbf{b}|\pm 1$. Moreover, if $|\mathbf{l}|=|\mathbf{b}|+1$ then $|\mathbf{i}|\geq 1$,

$$\mathrm{wt}(x^{\mathbf{a}-\mathbf{i}}z^{\mathbf{l}})=\mathrm{wt}(x^{\mathbf{a}-\mathbf{i}}z^{\mathbf{b}})-1=w+\mathrm{wt}(x^{\mathbf{a}}z^{\mathbf{b}}),\tag{6}$$

and

$$\deg_e(x^{\mathbf{a}-\mathbf{i}}z^{\mathbf{l}})=2+d+w-2|\mathbf{i}|.\tag{7}$$

If $|\mathbf{l}|=|\mathbf{b}|-1$ then

$$\mathrm{wt}(x^{\mathbf{a}-\mathbf{i}}z^{\mathbf{l}})=2+w+\mathrm{wt}(x^{\mathbf{a}}z^{\mathbf{b}});\quad \deg_e(x^{\mathbf{a}-\mathbf{i}}z^{\mathbf{l}})=d+w-2|\mathbf{i}|.\tag{8}$$

Finally, suppose $\mathbf{i}$ is such that $\mathrm{wt}([yx^{\mathbf{i}}])=-2$. Then

$$x^{\mathbf{a}-\mathbf{i}}\cdot\rho_\chi([yx^{\mathbf{i}}])\,z^{\mathbf{b}}\otimes 1_\chi=\chi([yx^{\mathbf{i}}])\cdot x^{\mathbf{a}-\mathbf{i}}z^{\mathbf{b}}\otimes 1_\chi.\tag{9}$$

As $\sum_{k\leq m}i_kn_k=-2-w$ we have

$$\mathrm{wt}(x^{\mathbf{a}-\mathbf{i}}z^{\mathbf{b}})=2+w+\mathrm{wt}(x^{\mathbf{a}}z^{\mathbf{b}});\quad \deg_e(x^{\mathbf{a}-\mathbf{i}}z^{\mathbf{b}})=2+d+w-2|\mathbf{i}|.\tag{10}$$

For $i,j\in\mathbb{Z}$ let $\pi_{i,j}$ denote the endomorphism of Q_χ such that $\pi_{i,j}(x^{\mathbf{a}}z^{\mathbf{b}}\otimes 1_\chi)=x^{\mathbf{a}}z^{\mathbf{b}}\otimes 1_\chi$ if $\deg_e(x^{\mathbf{a}}z^{\mathbf{b}})=i$ and $\mathrm{wt}(x^{\mathbf{a}}z^{\mathbf{b}})=j$, and 0 otherwise.

Suppose $y=y_k$. Then $w\leq -2$ and

$$0=(\rho_\chi(y)-\chi(y)\,id)\cdot h(1_\chi)=\Big(\sum_{\mathbf{a},\mathbf{b}}\lambda_{\mathbf{a},\mathbf{b}}\sum_{|\mathbf{i}|\geq 1}\binom{\mathbf{a}}{\mathbf{i}}x^{\mathbf{a}-\mathbf{i}}\cdot\rho_\chi([yx^{\mathbf{i}}])\,z^{\mathbf{b}}\Big)\otimes 1_\chi.$$

Applying (3.2(4))–(3.2(10)) we get $0=\pi_{n+w,\,N+w+2}((\rho_\chi(y)-\chi(y)\,id)\cdot h(1_\chi))=$

$$\Big(\sum_{(\mathbf{a},\mathbf{b})\in\Lambda_h^{\max}}\lambda_{\mathbf{a},\mathbf{b}}\sum_{i=1}^{m}\bar{a}_i\,\Phi([y,x_i],e)\cdot x^{\mathbf{a}-\mathbf{e}_i}z^{\mathbf{b}}\Big)\otimes 1_\chi=\sum_{(\mathbf{a},\mathbf{b})\in\Lambda_h^{\max}}\lambda_{\mathbf{a},\mathbf{b}}\,\bar{a}_k\,x^{\mathbf{a}-\mathbf{e}_k}z^{\mathbf{b}}\otimes 1_\chi\neq 0,$$

a contradiction. Thus $y=z_j$ whence $w=-1$ and $\chi(y)=0$. By (3.2(5)) and (3.2(7)), $0=\rho_\chi(y)\cdot h(1_\chi)=$

$$\Big(\sum_{(\mathbf{a},\mathbf{b})\in\Lambda_h^n}\big(\lambda_{\mathbf{a},\mathbf{b}}\,\bar{b}_j\,x^{\mathbf{a}}z^{\mathbf{b}-\mathbf{e}_j}+\sum_{i=1}^{m}x^{\mathbf{a}-\mathbf{e}_i}\cdot\rho_\chi([y,x_i])\,z^{\mathbf{b}}\big)+\sum_{|(\mathbf{i},\mathbf{j})|_e\leq n-2}\beta_{\mathbf{i},\mathbf{j}}\,x^{\mathbf{i}}z^{\mathbf{j}}\Big)\otimes 1_\chi$$

for some $\beta_{\mathbf{i},\mathbf{j}} \in K$. But then

$$\pi_{n-1,\,N+1}((\rho_\chi(y)\cdot h(1_\chi)) = \sum_{(\mathbf{a},\mathbf{b})\in\Lambda_h^{\max}} \lambda_{\mathbf{a},\mathbf{b}}\,\bar{b}_j\,x^{\mathbf{a}} z^{\mathbf{b}-\mathbf{e}_j} \otimes 1_\chi \neq 0$$

(in view of (3.2(4)), (3.2(6)) and (3.2(8))). This contradiction completes the proof of the lemma. □

3.3. For $k \in \mathbb{N}_0$ let H^k denote the linear span of all $0 \neq h \in H_\chi(\mathfrak{g})$ with $n(h) \leq k$. It follows readily from Lemma 3.1 that $H^i \cdot H^j \subseteq H^{i+j}$ for all $i, j \in \mathbb{N}_0$. In other words, $\{H^i \mid i \in \mathbb{N}_0\}$ is a filtration of the algebra $H_\chi(\mathfrak{g})$ (obviously, $H_\chi(\mathfrak{g}) = H^k$ for all $k \gg 0$). We set $H^{-1} = 0$ and let $\mathrm{gr}\,(H_\chi(\mathfrak{g})) = \bigoplus_{i\geq 0} H^i/H^{i-1}$ denote the corresponding graded algebra. Lemma 3.1 implies that the algebra $\mathrm{gr}\,(H_\chi(\mathfrak{g}))$ is commutative.

Proposition. *For any* $\mathbf{i} \in \Lambda_r$ *there is* $h_{\mathbf{i}} \in H_\chi(\mathfrak{g})$ *such that* $\Lambda_{h_{\mathbf{i}}}^{\max} = \{\mathbf{i}\}$. *The vectors* $\{h_{\mathbf{i}} \mid \mathbf{i} \in \Lambda_r\}$ *form a basis of* $H_\chi(\mathfrak{g})$ *over* K.

Proof. Given $(a, b) \in \mathbb{N}_0^2$ let $H^{a,b}$ denote the subspace of $H_\chi(\mathfrak{g})$ spanned by H^{a-1} and all $h \in H_\chi(\mathfrak{g})$ such that $n(h) = a$ and $N(h) \leq b$. Order the elements in $\mathbb{N}_0^2$ lexicographically. By construction, $H^{a,b} \subseteq H^{c,d}$ whenever $(a, b) \prec (c, d)$. Applying the Basis Extension Theorem to the (finite) chain of subspaces just defined we obtain that $H_\chi(\mathfrak{g})$ has basis $B = \bigsqcup_{(i,j)} B_{i,j}$ such that $n(v) = i$ and $N(v) = j$ whenever $v \in B_{i,j}$.

Define the linear map $\pi_B\colon H_\chi(\mathfrak{g}) \to Q_\chi$ by setting $\pi_B(v) = \pi_{i,j}(v(1_\chi))$ for any $v \in B_{i,j}$ and extending to $H_\chi(\mathfrak{g})$ by linearity (the idempotents $\pi_{i,j} \in \mathrm{End}\,(Q_\chi)$ are defined in the course of the proof of Lemma 3.2). By Lemma 3.2, π_B maps $H_\chi(\mathfrak{g})$ into the subspace $U_\chi(\mathfrak{z}_\chi) \otimes 1_\chi$ of Q_χ. By construction, π_B is injective. On the other hand, $\dim H_\chi(\mathfrak{g}) = p^r = \dim U_\chi(\mathfrak{z}_\chi) \otimes 1_\chi$ due to Theorem 2.3(ii) and Proposition 2.6. Thus $\pi_B\colon H_\chi(\mathfrak{g}) \to U_\chi(\mathfrak{z}_\chi) \otimes 1_\chi$ is a linear isomorphism. For $\mathbf{a} = (a_1, \dots, a_r) \in \Lambda_r$ set $h_{\mathbf{a}} = \pi_B^{-1}(x_1^{a_1} \cdots x_r^{a_r} \otimes 1_\chi)$. By the bijectivity of π_B and the PBW theorem (applied to $U_\chi(\mathfrak{z})$), the vectors $h_{\mathbf{i}}$ with $\mathbf{i} \in \Lambda_r$ form a basis of $H_\chi(\mathfrak{g})$, while from the definition of π_B it follows that $\Lambda_{h_{\mathbf{i}}}^{\max} = \{\mathbf{i}\}$ for any $\mathbf{i} \in \Lambda_r$. □

3.4. As an immediate consequence of Proposition 3.3 we obtain that there exist $\theta_1, \dots, \theta_r \in H_\chi(\mathfrak{g})$ such that

$$\theta_k(1_\chi) = \big(x_k + \sum_{|(\mathbf{i},\mathbf{j})|_e = n_k+2,\ |\mathbf{i}|+|\mathbf{j}|\geq 2} \lambda_{\mathbf{i},\mathbf{j}}^k\, x^{\mathbf{i}} z^{\mathbf{j}} + \sum_{|(\mathbf{i},\mathbf{j})|_e < n_k+2} \lambda_{\mathbf{i},\mathbf{j}}^k\, x^{\mathbf{i}} z^{\mathbf{j}}\big) \otimes 1_\chi \qquad (11)$$

where $\lambda_{\mathbf{i},\mathbf{j}}^k \in K$ and $\lambda_{\mathbf{a},\mathbf{b}}^k = 0$ if $(\mathbf{a}, \mathbf{b})$ is such that $a_{r+1} = \cdots = a_m = b_1 = \cdots = b_s = 0$. Let $\bar{\theta}_i$ denote $\theta_i + H^{n_i+1}$, the image of θ_i in $\mathrm{gr}\,(H_\chi(\mathfrak{g}))$.

Theorem. *(i) The monomials* $\bar{\theta}_1^{a_1} \cdots \bar{\theta}_r^{a_r}$ *and* $\theta_1^{a_1} \cdots \theta_r^{a_r}$ *with* $0 \leq a_i \leq p-1$ *form bases of* $\mathrm{gr}\,(H_\chi(\mathfrak{g}))$ *and* $H_\chi(\mathfrak{g})$, *respectively.*

(ii) Let $1 \leq i, j \leq r$. *Then* $[\theta_i, \theta_j] = \theta_j \circ \theta_i - \theta_i \circ \theta_j \in H^{n_i+n_j+2}$. *Moreover, if* $[x_i, x_j] = \sum_{k=1}^r \alpha_{ij}^k\, x_k$ *in* $\mathfrak{z}_\chi$ *then*

$$[\theta_i, \theta_j] \equiv \sum_{k=1}^r \alpha_{ij}^k\, \theta_k + q_{ij}(\theta_1, \dots, \theta_r) \pmod{H^{n_i+n_j+1}}$$

where q_{ij} is a truncated polynomial in r variables whose constant term and linear part are both zero.

Proof. (i) Easy induction on $|\mathbf{a}|$ shows that $(\theta_r^{a_r} \circ \cdots \circ \theta_1^{a_1})(1_\chi) =$

$$= \big(x_1^{a_1} \cdots x_r^{a_r} + \sum_{|(\mathbf{i},\mathbf{j})|_e = |(\mathbf{a},\mathbf{0})|_e,\ |\mathbf{i}|+|\mathbf{j}|>|\mathbf{a}|} \lambda_{\mathbf{i},\mathbf{j}}^{\mathbf{a}}\, x^{\mathbf{i}} z^{\mathbf{j}} + \text{ terms of lower } e\text{-degree}\big) \otimes 1_\chi$$

for any $\mathbf{a} = (a_1, \ldots, a_r) \in \Lambda_r$ (the induction step is based on (3.4(11)) and Lemma 3.1). Due to Proposition 3.3 we have that

$$\theta_1^{a_1} \cdots \theta_r^{a_r} = \mu_{\mathbf{a}} h_{\mathbf{a}} + \sum_{\mathbf{i}\in\Lambda_r} \mu_{\mathbf{i}} h_{\mathbf{i}}, \quad \mu_{\mathbf{a}} \neq 0, \tag{12}$$

where $\mu_{\mathbf{i}} = 0$ unless $(n(h_{\mathbf{i}}), N(h_{\mathbf{i}})) \prec (n(h_{\mathbf{a}}), N(h_{\mathbf{a}}))$. Since this holds for any $\mathbf{a} \in \Lambda_r$, the monomials $\theta_1^{a_1} \cdots \theta_r^{a_r}$ with $\mathbf{a} \in \Lambda_r$ form a basis of $H_\chi(\mathfrak{g}) = \mathrm{End}_{\mathfrak{g}}(Q_\chi)^{\mathrm{op}}$. It follows from (the proof of) Proposition 3.3 that for any $k \geq 0$, the cosets $h_{\mathbf{i}} + H^{k-1}$ with $\mathbf{i} \in \Lambda_r$ and $\sum_{j\leq r} i_j(n_j + 2) = k$ form a basis of $\mathrm{gr}_k(H_\chi(\mathfrak{g}))$. Due to (3.4(12)) and Lemma 3.1 the cosets $\theta_r^{i_r} \cdots \theta_1^{i_1} + H^{k-1}$ with $\mathbf{i} \in \Lambda_r$ and $\sum_{j\leq r} i_j(n_j + 2) = k$ have the same property. To complete the proof of part (i) it remains to note that $\bar{\theta}_1^{i_1} \cdots \bar{\theta}_r^{i_r} = \theta_1^{i_1} \cdots \theta_r^{i_r} + H^{k-1}$ for any $\mathbf{i} \in \Lambda_r$ with $\sum_{j\leq r} i_j(n_j + 2) = k$.

(ii) Combining (3.4.(11)) with Lemma 3.1 we deduce that $[\theta_i, \theta_j] \in H^{n_i+n_j+2}$. As in the proof of Lemma 3.1, induction on $|\mathbf{b}|$ yields $\big(\rho_\chi(z^{\mathbf{b}})\big)(x^{\mathbf{c}} z^{\mathbf{d}} \otimes 1_\chi) =$

$$(x^{\mathbf{c}} z^{\mathbf{b}+\mathbf{d}} + \sum_{i,j} \beta_{ij} x^{\mathbf{c}-\mathbf{e}_i} \rho_\chi([z_i, x_j]) z^{\mathbf{b}+\mathbf{d}-\mathbf{e}_j} + \text{ terms of } e\text{-degree } \leq |(\mathbf{c}, \mathbf{b}+\mathbf{d})|_e - 3) \otimes 1_\chi.$$

It follows that $\big(\rho_\chi(x^{\mathbf{a}} z^{\mathbf{b}})\big)(x^{\mathbf{c}} z^{\mathbf{d}}) =$

$$\begin{aligned} &= (x^{\mathbf{a}+\mathbf{c}} z^{\mathbf{b}+\mathbf{d}} + \sum_{i<j} \alpha_{ij} x^{\mathbf{a}+\mathbf{c}-\mathbf{e}_i-\mathbf{e}_j} [x_i, x_j] z^{\mathbf{b}+\mathbf{d}} + \sum_{i,j} \beta_{ij} x^{\mathbf{a}+\mathbf{c}-\mathbf{e}_i} \rho_\chi([z_i, x_j]) z^{\mathbf{b}+\mathbf{d}-\mathbf{e}_j} \\ &+ \text{ terms of } e\text{-degree } \leq |(\mathbf{a}+\mathbf{c}, \mathbf{b}+\mathbf{d})|_e - 3) \otimes 1_\chi. \end{aligned}$$

Together with (3.4(11)) this shows that $(\theta_j \circ \theta_i - \theta_i \circ \theta_j)(1_\chi) =$

$$= \big([x_i, x_j] + \sum_{|(\mathbf{i},\mathbf{j})|_e = n_i+n_j+2,\ |\mathbf{i}|+|\mathbf{j}|\geq 2} \mu_{\mathbf{i},\mathbf{j}} x^{\mathbf{i}} z^{\mathbf{j}} + \sum_{|(\mathbf{i},\mathbf{j})|_e < n_i+n_j+2} \mu_{\mathbf{i},\mathbf{j}} x^{\mathbf{i}} z^{\mathbf{j}}\big) \otimes 1_\chi$$

for some $\mu_{\mathbf{i},\mathbf{j}} \in K$. As a consequence, $\pi_{n_i+n_j+2,\, n_i+n_j}\big([\theta_i, \theta_j] - \sum_{k=1}^r \alpha_{ij}^k\, \theta_k\big) = 0$. On the other hand, part (i) of this proof shows that there exists a unique truncated polynomial $\tilde{q}_{ij}$ in $x_1, \ldots, x_r$ such that $[\theta_i, \theta_j] - \sum_{k=1}^r \alpha_{ij}^k\, \theta_k = \tilde{q}_{ij}(\theta_1, \ldots, \theta_r)$. Moreover, it follows from the preceding remark that the linear part of $\tilde{q}_{ij}$ involves only those x_k for which $\mathrm{wt}(x_k) < n_i + n_j$. So there exists a truncated polynomial q_{ij} in r variables with initial form of degree at least 2 such that

$$[\theta_i, \theta_j] - \sum_{k=1}^r \alpha_{ij}^k\, \theta_k - q_{ij}(\theta_1, \ldots, \theta_r) \in H^{n_i+n_j+1}.$$

This completes the proof of the theorem. □

4. Complex analogues of Q_χ and $H_\chi(\mathfrak{g})$

4.1. Let $G_{\mathbb{C}}$ be a simple, simply connected algebraic group over $\mathbb{C}$, and H a connected subgroup of $G_{\mathbb{C}}$. Given a 1-dimensional torus $\nu \subseteq H$ we denote by $\mathfrak{h}(\nu, i)$ the weight space of weight i in $\mathfrak{h} = \operatorname{Lie} H$ relative to the adjoint action of ν.

Let E be a nonzero nilpotent element of $\mathfrak{g}_{\mathbb{C}} = \operatorname{Lie} G_{\mathbb{C}}$. By the Bala–Carter theory [3, Ch. 5], there exist a Levi subgroup $L_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ and a 1-dimensional torus $\lambda = \lambda_E$ in $L_{\mathbb{C}}^{(1)}$ such that $\bigoplus_{i\ge 0} \mathfrak{l}_{\mathbb{C}}^{(1)}(\lambda, i)$ is a distinguished parabolic subalgebra of $\mathfrak{l}_{\mathbb{C}}^{(1)} = \operatorname{Lie} L_{\mathbb{C}}^{(1)}$, $E \in \mathfrak{l}_{\mathbb{C}}(\lambda, 2)$, $\operatorname{Ker} \operatorname{ad} E \subseteq \bigoplus_{i\ge 0} \mathfrak{g}_{\mathbb{C}}(\lambda, i)$, and $[E, \mathfrak{g}_{\mathbb{C}}(\lambda, i)] = \mathfrak{g}_{\mathbb{C}}(\lambda, i+2)$ for $i \ge 0$.

There exists $F \in \mathfrak{l}_{\mathbb{C}}(\lambda, -2)$ such that $(E, [E, F], F)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$ and $\operatorname{Lie} \nu = \mathbb{C}\,[E, F]$. Let $t = \operatorname{rk} L_{\mathbb{C}}^{(1)}$. By a result of Dynkin (or by the Bala–Carter theory), there exist a maximal torus T of $G_{\mathbb{C}}$ contained in $L_{\mathbb{C}}$, a Chevalley system $\mathcal{S} = \{E_\alpha\,, H_\alpha \mid \alpha \in R = R(\mathfrak{g}_{\mathbb{C}}, T)\}$, and root vectors $E_{\gamma_1}, \dots, E_{\gamma_t} \in \mathcal{S} \cap \mathfrak{l}_{\mathbb{C}}(\lambda, 2)$ such that $\nu \subseteq T$ and $E = E_{\gamma_1} + \cdots + E_{\gamma_t}$ (see, e.g., [39, (III, 4.29)]). The roots $\gamma_1, \dots, \gamma_t$ are $\mathbb{Q}$-independent in $\mathbb{Q}R$.

4.2. For $i \in \mathbb{Z}$ set $\mathfrak{g}_{\mathbb{C}}(i) = \mathfrak{g}_{\mathbb{C}}(\lambda, i)$. Let N be a large positive integer and $\mathcal{A} = \mathbb{Z}[1/N!]$. We denote by $\mathfrak{g}_{\mathcal{A}}$ the $\mathcal{A}$-submodule of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathcal{S}$. This is an $\mathcal{A}$-form in $\mathfrak{g}_{\mathbb{C}}$ (in particular, a free $\mathcal{A}$-module) and a Lie algebra over $\mathcal{A}$. Clearly, $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$ as Lie algeras. Put $\mathfrak{g}_{\mathbb{Q}} = \mathfrak{g}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Q}$. Since $\gamma_1, \dots, \gamma_t$ are $\mathbb{Q}$-independent and $t = \dim T \cap L_{\mathbb{C}}^{(1)}$ we have $[E, F] \in \mathfrak{g}_{\mathbb{Q}}$ (for $[E, F] \in \operatorname{Lie} T \cap L_{\mathbb{C}}^{(1)}$ and $[[E, F], E] = 2E$). Since $\sum_{i\ge 0} \mathfrak{l}_{\mathbb{C}}^{(1)}(\lambda, i)$ is distinguished in $\mathfrak{l}^{(1)}$, the map $\operatorname{ad} E\colon \mathfrak{l}_{\mathbb{C}}(\lambda, -2) \to \mathfrak{l}_{\mathbb{C}}^{(1)}(\lambda, 0)$ is a linear isomorphism. Thus $F \in \mathfrak{g}_{\mathbb{Q}}$ as well. Enlarging N if necessary we may assume that $F \in \mathfrak{g}_{\mathcal{A}}$.

Let κ denote the Killing form of $\mathfrak{g}_{\mathbb{C}}$. Obviously, $\kappa(\mathfrak{g}_{\mathcal{A}}, \mathfrak{g}_{\mathcal{A}}) \subseteq \mathcal{A}$. Representation theory of $\mathfrak{sl}_2$ implies that $\kappa(E, F)$ is a positive integer. In what follows we assume that $N > \kappa(E, F)$ and denote by $\Phi_{\mathbb{C}}$ the bilinear form $\kappa(E, F)^{-1} \cdot \kappa$ on $\mathfrak{g}_{\mathbb{C}}$. We let $\Phi_{\mathcal{A}}$ be the restriction of $\Phi_{\mathbb{C}}$ to $\mathfrak{g}_{\mathcal{A}}$. By our assumption, this form is $\mathcal{A}$-valued.

Let p be a prime with $p \gg N$, K an algebraically closed field of characteristic p, and $\mathfrak{g}_K = \mathfrak{g}_{\mathcal{A}} \otimes_{\mathcal{A}} K \cong (\mathfrak{g}_{\mathcal{A}}/p\,\mathfrak{g}_{\mathcal{A}}) \otimes_{\mathbb{F}_p} K$. Let G_K denote a simple, simply connected algebraic K-group such that $\mathfrak{g}_K = \operatorname{Lie} G_K$. The form $\Phi_{\mathcal{A}}$ induces a G_K-invariant bilinear form on $\mathfrak{g}_K$, Φ_K say. Given $X \in \mathfrak{g}_{\mathcal{A}}$ we denote by x the image of X under the canonical homomorphism $\mathfrak{g}_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}/p\,\mathfrak{g}_{\mathcal{A}}$ and identify x with $x \otimes 1$ in $\mathfrak{g}_K$. Note that $\Phi_K(e, f) = 1$. Since $p \gg 0$ the form Φ_K is nondegenerate. Define $\chi \in \mathfrak{g}_{\mathbb{C}}^*$ by setting $\chi(X) = \Phi_{\mathbb{C}}(E, X)$ for all $X \in \mathfrak{g}_{\mathbb{C}}$. Since χ is $\mathcal{A}$-valued on $\mathfrak{g}_{\mathcal{A}}$, it induces a linear form on $\mathfrak{g}_K$, χ_K say.

By construction, $\mathfrak{g}_{\mathcal{A}} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\mathcal{A}}(i)$ where $\mathfrak{g}_{\mathcal{A}}(i){=}\mathfrak{g}_{\mathcal{A}}(i){\cap}\mathfrak{g}_{\mathcal{A}}$. Let $\mathfrak{g}_K(i) = \mathfrak{g}_{\mathcal{A}}(i){\otimes_{\mathcal{A}}}K$. It follows from (4.1) that for any $i \ge -1$, the $\mathcal{A}$-module $[E, \mathfrak{g}_{\mathcal{A}}(i)]$ is an $\mathcal{A}$-lattice in $\mathfrak{g}_{\mathbb{C}}(\lambda, i+2)$ (one should take into account that $\dim \mathfrak{g}_{\mathbb{C}}(-1) = \dim \mathfrak{g}_{\mathbb{C}}(1)$). Enlarging N if necessary we may assume that $[E, \mathfrak{g}_{\mathcal{A}}(i)] = \mathfrak{g}_{\mathcal{A}}(i+2)$ for all $i \ge -1$. Then $\operatorname{ad} e\colon \mathfrak{g}_K(i) \to \mathfrak{g}_K(i+2)$ is surjective for all $i \ge -1$. Since Φ_K is nondegenerate, a standard duality argument shows that $\mathfrak{c}_{\mathfrak{g}_K}(e) \subseteq \bigoplus_{i\ge 0} \mathfrak{g}_K(i)$.

4.3. Let ψ_E denote the skew-symmetric form on $\mathfrak{g}_{\mathbb{C}}(-1)$ such that $\psi_E(X, Y) = \Phi_{\mathbb{C}}(E, [X, Y])$ for all $X, Y \in \mathfrak{g}_{\mathbb{C}}(-1)$. Since ψ_E is $\mathcal{A}$-valued on $\mathfrak{g}_{\mathcal{A}}(-1)$ it induces a skew-symmetric bilinear form on $\mathfrak{g}_K(-1)$ denoted by ψ_e. By our discussion in (4.2), both ψ_E and ψ_e are nondegenerate. Also, $\psi_e(x, y) = \Phi_K(e, [x, y])$ for all $x, y \in \mathfrak{g}_K(-1)$.

Let $\mathcal{W} = \{Z'_1, \dots, Z'_s, Z_1 \dots, Z_s\}$ be a Witt basis of $\mathfrak{g}_{\mathbb{C}}(-1)$ relative to ψ_E contained in $\mathfrak{g}_{\mathbb{Q}}$. Enlarging N if necessary we may assume that $\mathcal{W}$ is a free basis of the $\mathcal{A}$-module $\mathfrak{g}_{\mathcal{A}}(-1)$ (in particular, $\mathcal{W} \subset \mathfrak{g}_{\mathcal{A}}(-1)$). Then $\{z'_1, \dots, z'_s, z_1, \dots, z_s\}$ is a Witt basis of $\mathfrak{g}_K(-1)$ relative to ψ_e. Let $\mathfrak{g}_{\mathbb{C}}(-1)^0$ (respectively, $\mathfrak{g}_K(-1)^0$) denote the subspace of $\mathfrak{g}_{\mathbb{C}}$ (respectively, $\mathfrak{g}_K$) spanned by the Z'_i (respectively, z'_i). Let $\mathfrak{m}_{\mathbb{C},\chi} = \mathfrak{g}_{\mathbb{C}}(-1)^0 \oplus \bigoplus_{i\leq -2} \mathfrak{g}_{\mathbb{C}}(i)$ and $\mathfrak{m}_{\chi K} = \mathfrak{g}_K(-1)^0 \oplus \bigoplus_{i\leq -2} \mathfrak{g}_K(i)$, nilpotent subalgebras in $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_K$. Choose a free homogeneous basis $X_1, \dots, X_r, X_{r+1}, \dots, X_m$ of the $\mathcal{A}$-module $\bigoplus_{i\geq 0} \mathfrak{g}_{\mathcal{A}}(i)$ such that $X_1, \dots, X_r$ is a basis of $\mathfrak{z}_{\mathbb{C},\chi} = \operatorname{Ker} \operatorname{ad} E$ over $\mathbb{C}$ (it exists because $\operatorname{ad} E\colon \mathfrak{g}_{\mathcal{A}}(i) \to \mathfrak{g}_{\mathcal{A}}(i+2)$ is surjective for all $i \geq 0$ and $\mathcal{A}$ is a principal ideal domain). Then $x_1, \dots, x_m$ form a basis of $\bigoplus_{i\geq 0} \mathfrak{g}_K(i)$ and $x_1, \dots, x_r$ span $\mathfrak{c}_{\mathfrak{g}_K}(e)$.

4.4. Given a Lie algebra L over a commutative ring and $k \in \mathbb{N}_0$, we denote by $U^k(L)$ the kth component of the standard filtration of $U(L)$, the enveloping algebra of L. Let N_χ be the left ideal of $U(\mathfrak{m}_{\mathbb{C},\chi})$ generated by all $X - \chi(X)\,1$ with $X \in \mathfrak{m}_{\mathbb{C},\chi}$, and $\mathbb{C}_\chi$ the 1-dimensional $U(\mathfrak{m}_{\mathbb{C},\chi})$-module $U(\mathfrak{m}_{\mathbb{C},\chi})/N_\chi$. The image of 1 in $\mathbb{C}_\chi$ is denoted by $\tilde{1}_\chi$. Let $\tilde{Q}_\chi = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{m}_{\mathbb{C},\chi})} \mathbb{C}_\chi$ and $\tilde{H}_\chi(\mathfrak{g}_{\mathbb{C}}) = \operatorname{End}_{\mathfrak{g}_{\mathbb{C}}}(\tilde{Q}_\chi)^{\mathrm{op}}$. The representation of $U(\mathfrak{g}_{\mathbb{C}})$ in $\operatorname{End}(\tilde{Q}_\chi)$ is denoted by $\tilde{\rho}_\chi$.

Given $(\mathbf{a}, \mathbf{b}) \in \mathbb{N}_0^m \times \mathbb{N}_0^s$ we denote by $X^{\mathbf{a}}Z^{\mathbf{b}}$ the monomial $X_1^{a_1} \cdots X_m^{a_m} Z_1^{b_1} \cdots Z_s^{b_s}$ in $U(\mathfrak{g}_{\mathbb{C}})$. By the PBW theorem, the vectors $X^{\mathbf{i}}Z^{\mathbf{j}} \otimes \tilde{1}_\chi$ with $(\mathbf{i}, \mathbf{j}) \in \mathbb{N}_0^m \times \mathbb{N}_0^s$ form a basis of $\tilde{Q}_\chi$ over $\mathbb{C}$. We assume that $\mathrm{wt}(X_i) = n_i$, i.e., $X_i \in \mathfrak{g}_{\mathbb{C}}(n_i)$ where $1 \leq i \leq m$, and adopt for our new setting the notation of Section 3. For example, given $(\mathbf{a}, \mathbf{b}) \in \mathbb{N}_0^m \times \mathbb{N}_0^s$, we say that $X^{\mathbf{a}}Z^{\mathbf{b}}$ has e-degree k, written $\deg_e(X^{\mathbf{a}}Z^{\mathbf{b}}) = k$, if

$$|(\mathbf{a}, \mathbf{b})|_e = \sum_{i=1}^{m} a_i(n_i + 2) + \sum_{i=1}^{s} b_i = k.$$

.

Lemma. *Let $(\mathbf{a}, \mathbf{b})$, $(\mathbf{c}, \mathbf{d}) \in \mathbb{N}_0^m \times \mathbb{N}_0^s$ be such that $|(\mathbf{a}, \mathbf{b})|_e = A$ and $|(\mathbf{c}, \mathbf{d})|_e = B$. Then*

$$\big(\tilde{\rho}_\chi(X^{\mathbf{a}}Z^{\mathbf{b}})\big)(X^{\mathbf{c}}Z^{\mathbf{d}} \otimes \tilde{1}_\chi) = (X^{\mathbf{a}+\mathbf{c}}Z^{\mathbf{b}+\mathbf{d}} + \text{ terms of } e\text{-degree } \leq A + B - 2) \otimes \tilde{1}_\chi.$$

Proof. It is well-known that

$$u \cdot X^{\mathbf{a}}Z^{\mathbf{b}} = \sum_{\mathbf{i}\in\mathbb{N}_0^m} \binom{\mathbf{a}}{\mathbf{i}} x^{\mathbf{a}-\mathbf{i}} \cdot [uX^{\mathbf{i}}] \cdot Z^{\mathbf{b}} \tag{13}$$

for any $u \in U(\mathfrak{g}_{\mathbb{C}})$ (see [40, (5.7)]). Now repeat the proof of Lemma 3.1 applying (4.4(13)) in place of (3.1(2)). □

4.5. Any $h \in \tilde{H}_\chi(\mathfrak{g}_{\mathbb{C}})$ is uniquely determined by its value $h(\tilde{1}_\chi) \in \tilde{Q}_\chi$. For $h \neq 0$ we let $n(h)$, $N(h)$ and $\Lambda_h^{\max}$ have the same meaning as in (3.2).

Lemma. *Let $h \in \tilde{H}_\chi(\mathfrak{g}_{\mathbb{C}})$ and $(\mathbf{p}, \mathbf{q}) \in \Lambda_h^{\max}$. Then $\mathbf{q} = \mathbf{0}$ and $\mathbf{p} \in \mathbb{N}_0^r \times \{\mathbf{0}\}$.*

Proof. Repeat verbatim the proof of Lemma 3.2 but apply (4.4(13)) in place of (3.1(2)). □

4.6. For $k \in \mathbb{N}_0$ we denote by $\tilde{H}^k$ the linear span of all $0 \neq h \in \tilde{H}_\chi(\mathfrak{g}_{\mathbb{C}})$ with $n(h) \leq k$. Put $\tilde{H}^{-1} = 0$. It follows from Lemma 4.4 that the subspaces $\{\tilde{H}^i \,|\, i \in \mathbb{N}_0\}$

form a filtration of the algebra $\tilde{H}_\chi(\mathfrak{g}_\mathbb{C})$. Moreover, Lemma 4.4 implies that the graded algebra $\mathrm{gr}\,(\tilde{H}_\chi(\mathfrak{g}_\mathbb{C})) = \bigoplus_{i\geq 0} \tilde{H}^i/\tilde{H}^{i-1}$ is commutative.

Theorem. *(i) There exist* $\Theta_1, \dots, \Theta_r \in \tilde{H}_\chi(\mathfrak{g}_\mathbb{C})$ *such that*

$$\Theta_k(\tilde{1}_\chi) = \big(X_k + \sum_{|(\mathbf{i},\mathbf{j})|_e = n_k+2,\ |\mathbf{i}|+|\mathbf{j}|\geq 2} \lambda^k_{\mathbf{i},\mathbf{j}}\, X^{\mathbf{i}} Z^{\mathbf{j}} + \sum_{|(\mathbf{i},\mathbf{j})|_e < n_k+2} \lambda^k_{\mathbf{i},\mathbf{j}}\, X^{\mathbf{i}} Z^{\mathbf{j}}\big) \otimes \tilde{1}_\chi$$

where $\lambda^k_{\mathbf{i},\mathbf{j}} \in \mathbb{Q}$ *and* $\lambda^k_{\mathbf{a},\mathbf{b}} = 0$ *if* $\mathbf{b} = \mathbf{0}$ *and* $a_{r+1} = \cdots = a_m = 0$.

(ii) The monomials $\Theta_1^{a_1} \dots \Theta_r^{a_r}$ *with* $a_i \in \mathbb{N}_0$ *form a basis of* $\tilde{H}_\chi(\mathfrak{g}_\mathbb{C})$ *over* $\mathbb{C}$.

(iii) The elements $\bar{\Theta}_i = \Theta_i + \tilde{H}^{n_i+1} \in \mathrm{gr}\,(\tilde{H}_\chi(\mathfrak{g}_\mathbb{C}))$ *are algebraically independent and generate* $\mathrm{gr}\,(\tilde{H}_\chi(\mathfrak{g}_\mathbb{C}))$. *In particular,* $\mathrm{gr}\,(\tilde{H}_\chi(\mathfrak{g}_\mathbb{C}))$ *is a graded polynomial algebra with homogeneous generators of degrees* $n_1+2,\ n_2+2,\ \dots,\ n_r+2$.

(iv) Let $1 \leq i,j \leq r$. *Then* $[\Theta_i, \Theta_j] = \Theta_j \circ \Theta_i - \Theta_i \circ \Theta_j \in \tilde{H}^{n_i+n_j+2}$. *Moreover, if* $[X_i, X_j] = \sum_{k=1}^r \alpha^k_{ij}\, X_k$ *in* $\mathfrak{z}_{\mathbb{C},\chi}$ *then*

$$[\Theta_i, \Theta_j] \equiv \sum_{k=1}^r \alpha^k_{ij}\, \Theta_k + q_{ij}(\Theta_1, \dots, \Theta_r) \pmod{\tilde{H}^{n_i+n_j+1}}$$

where q_{ij} *is a polynomial in* r *variables whose constant term and linear part are both zero.*

Proof. Let $U^k_{\chi_K}(\mathfrak{g}_K)$ denote the kth component of the standard filtration of the reduced enveloping algebra $U_{\chi_K}(\mathfrak{g}_K)$. For $k < p$ the canonical homomorphism $U(\mathfrak{g}_K) \to U_{\chi_K}(\mathfrak{g}_K)$ induces a linear isomorphism between $U^k(\mathfrak{g}_K)$ and $U^k_{\chi_K}(\mathfrak{g}_K)$. Let $\tilde{Q}^k_\chi$ (respectively, $Q^k_{\chi_K}$) denote the subspace of $\tilde{Q}_\chi$ (respectively, Q_{χ_K}) spanned by all $X^{\mathbf{i}} Z^{\mathbf{j}} \otimes \tilde{1}_\chi$ (respectively, all $x^{\mathbf{i}} z^{\mathbf{j}} \otimes 1_\chi$) with $|(\mathbf{i},\mathbf{j})|_e \leq k$. Note that $|(\mathbf{i},\mathbf{j})|_e > |\mathbf{i}| + |\mathbf{j}|$. So it follows from our preceding remark that $\dim_\mathbb{C} \tilde{Q}^k_\chi = \dim_K Q^k_{\chi_K}$ provided that $p > k$.

Recall that $p \gg 0$, in particular, $p > \max\{n_i + 2 \,|\, 1 \leq i \leq m\}$. Let $C_1, \dots, C_{m+s-r}$ be a homogeneous basis of the free $\mathcal{A}$-module $\mathfrak{g}_\mathcal{A}(-1)^0 \oplus \bigoplus_{i\leq -2} \mathfrak{g}_\mathcal{A}(i)$ where $\mathfrak{g}_\mathcal{A}(-1)^0$ is the $\mathcal{A}$-span of $Z'_1, \dots, Z'_s$. The universality property of induced modules implies that in order to construct $\Theta_k \in \tilde{H}_\chi(\mathfrak{g}_\mathbb{C})$ it suffices to find a collection $\{\lambda^k_{\mathbf{i},\mathbf{j}}\} \subset \mathbb{Q}$ satisfying certain linear conditions (like $\lambda^k_{\mathbf{i},\mathbf{j}} = 0$ whenever $|(\mathbf{i},\mathbf{j})|_e = n_k + 2$ and $|\mathbf{i}| + |\mathbf{j}| = 1$) and such that

$$\tilde{\rho}_\chi(C_i)\big((X_k + \sum_{\mathbf{i},\mathbf{j}} \lambda^k_{\mathbf{i},\mathbf{j}}\, X^{\mathbf{i}} Z^{\mathbf{j}}) \otimes \tilde{1}_\chi\big) = \chi(C_i) \cdot (X_k + \sum_{\mathbf{i},\mathbf{j}} \lambda^k_{\mathbf{i},\mathbf{j}}\, X^{\mathbf{i}} Z^{\mathbf{j}}) \otimes \tilde{1}_\chi$$

for all i. Now (4.4(13)) and our discussion in (4.1)–(4.3) show that the left-hand side of each of these vector equations can be expressed as a linear combinaion of $X^{\mathbf{i}} Z^{\mathbf{j}} \otimes \tilde{1}_\chi$, where $|\mathbf{i}| + |\mathbf{j}| < p$, with coefficients in $\mathcal{A}$. So $m + s - r$ vector equations above together with the linear conditions imposed on $\{\lambda^k_{\mathbf{i},\mathbf{j}}\}$ are equivalent to a system of linear equations

$$D \cdot \mathbf{x} = \mathbf{d}$$

over $\mathcal{A}$. Let $\tilde{D} = (D \,|\, \mathbf{d})$ denote the augmented matrix of the system. We denote by $\tilde{D}_p = (D_p \,|\, \mathbf{d}_p)$ the matrix whose entries are the residues of the entries of $\tilde{D}$ in $\mathcal{A}/p\,\mathcal{A}$. Obviously, $\tilde{D}_p$ is the augmented matrix of the linear system

$$D_p \cdot \mathbf{x} = \mathbf{d}_p$$

over $\mathbb{F}_p$. Since $\rho_{\chi_K}(c_i)\cdot\theta_k(1_{\chi_K}) = \chi_K(c_i)\cdot\theta_k(1_{\chi_K})$ for all $i \le m+s-r$, our discussion in (4.1)–(4.3) in conjunction with (3.4(11)) and (3.1(2)) shows that the latter system has a solution over K. This, in turn, yields $\operatorname{rk}\tilde{D}_p = \operatorname{rk} D_p$. Since this holds for almost all primes we must have $\operatorname{rk}\tilde{D} = \operatorname{rk} D$. But then the former system has a solution over $\mathbb{Q}$ proving (i).

For $\mathbf{a} \in \mathbb{N}_0$ we denote by $\Theta^{\mathbf{a}}$ the monomial $\Theta_1^{a_1}\cdots\Theta_r^{a_r} = \Theta_r^{a_r}\circ\cdots\circ\Theta_1^{a_1}$ in $\tilde{H}_\chi(\mathfrak{g}_\mathbb{C}) = \operatorname{End}_{\mathfrak{g}_\mathbb{C}}(\tilde{Q}_\chi)^{\mathrm{op}}$. Straightforward induction on $|\mathbf{a}|$ shows that $\Theta^{\mathbf{a}}(\tilde{1}_\chi) =$

$$= \big(X_1^{a_1}\cdots X_r^{a_r} + \sum_{|(\mathbf{i},\mathbf{j})|_e=|(\mathbf{a},\mathbf{0})|_e,\ |\mathbf{i}|+\mathbf{j}|>|\mathbf{a}|} \lambda_{\mathbf{i},\mathbf{j}}^{\mathbf{a}}\, X^{\mathbf{i}}Z^{\mathbf{j}} + \text{ terms of lower } e\text{-degree}\big)\otimes\tilde{1}_\chi$$

for some $\lambda_{\mathbf{i},\mathbf{j}}^{\mathbf{a}} \in \mathbb{Q}$ (the induction step relies on Lemma 4.4 and the form of the Θ_i's). As a consequence, $\Delta_{\Theta^{\mathbf{a}}}^{\max} = \{\mathbf{a}\}$. From this it is immediate the monomials $\Theta^{\mathbf{a}}$ with $\mathbf{a} \in \mathbb{N}_0$ are linearly independent.

For $(a,b) \in \mathbb{N}_0$ we let $\tilde{H}^{a,b}$ be the subspace of $\tilde{H}_\chi(\mathfrak{g}_\mathbb{C})$ spanned by H^{a-1} and all $h \in \tilde{H}_\chi(\mathfrak{g}_\mathbb{C})$ with $n(h) = a$ and $N(h) \le b$. Let ι denote the unique bijection between $\mathbb{N}_0^2$ and $\mathbb{N}$ with the property that $\iota(a,b) < \iota(c,d)$ whenever $(a,b) \prec (c,d)$. Suppose that for all (i,i') with $\iota(i,i') \le k$ the subspace $\tilde{H}^{i,i'}$ is spanned by the $\Theta^{\mathbf{a}}$ satisfying $\sum_{j=1}^r a_j(n_j+2) \le i$ (this is true for $k = 1$). Let (a,b) be such that $\iota(a,b) = k+1$, and $h \in H^{a,b}\setminus\{0\}$. Lemma 4.5 says that $\Lambda_h^{\max} \subseteq \{(\mathbf{i},0,\dots,0)\times\mathbf{0}\,|\,\mathbf{i}\in\mathbb{N}_0^r\}$. By our remarks earlier in the proof, there exist $\mathbf{a}_1,\dots,\mathbf{a}_t \in \mathbb{N}_0^r$ with $|(\mathbf{a}_i,\mathbf{0})|_e = a$ and $\mu_1,\dots,\mu_t \in \mathbb{C}$ such that $h \in \sum_{i=1}^t \mu_i\Theta^{\mathbf{a}_i} + \sum_{\iota(i,i')\le k}\tilde{H}^{i,i'}$. Note that $\iota(i,i') \le k$ forces $i \le a$. Our earlier remarks now ensure that for any $i \ge 0$, the monomilas $\Theta^{\mathbf{a}}$ with $\sum_{j=1}^r a_j(n_j+2) \le i$ form a basis of $\tilde{H}^i$.

As an immediate consequence we obtain that the $\Theta^{\mathbf{a}}$ with $\mathbf{a} \in \mathbb{N}_0^r$ form a basis of $\tilde{H}_\chi(\mathfrak{g}_\mathbb{C})$. As a second consequence we derive that the monomials $\bar{\Theta}_1^{a_1}\cdots\bar{\Theta}_r^{a_r} = \Theta_1^{a_1}\cdots\Theta_r^{a_r} + \tilde{H}^{i-1}$ with $\sum_{j=1}^r a_j(n_j+2) = i$ form a basis of $\operatorname{gr}_i(\tilde{H}_\chi(\mathfrak{g}_\mathbb{C}))$ for all $i \ge 0$. This implies that $\bar{\Theta}_1,\dots,\bar{\Theta}_r$ are algebraically independent and generate $\operatorname{gr}(\tilde{H}_\chi(\mathfrak{g}_\mathbb{C}))$. Since each $\bar{\Theta}_k$ is homogeneous of degree n_k+2 we get (iii).

To obtain (iv) one argues as in the proof of Theorem 3.4(ii) applying (4.4(13)) and part (i) of this theorem in place of (3.1(2)) and 3.4(11), respectively. □

5. The adjoint quotient and the special transverse slices

5.1. We retain the assumptions of Section 4. To ease notation we drop the subscript $\mathbb{C}$ throughout this section and write G, $\mathfrak{g}$, Φ, etc. in place of $G_\mathbb{C}$, $\mathfrak{g}_\mathbb{C}$, $\Phi_\mathbb{C}$, etc. Set $\mathfrak{c} = \mathfrak{c}_\mathfrak{g}(F)$ and $\mathfrak{c}(i) = \mathfrak{c}\cap\mathfrak{g}(\lambda,i)$ (recall that $\mathfrak{c}(i) = 0$ for all positive i). Identify $S(\mathfrak{g}^*)$, the symmetric algebra of $\mathfrak{g}^*$, with the algebra of regular functions on $\mathfrak{g}$. Let $f_1,\dots,f_l$ denote algebraically independent homogeneous generators of the invariant algebra $S(\mathfrak{g}^*)^G$. Recall that $\deg f_i = m_i+1$ where $m_1,\dots,m_l$ are the exponents of the Weyl group of $\mathfrak{g}$.

In this section, we are concerned with geometric properties of the restriction, φ_S, of the adjoint quotient

$$\varphi:\ \mathfrak{g}\longrightarrow\mathbb{A}^l,\qquad x\mapsto(f_1(x),\dots,f_l(x)),$$

to the special transverse slice $S = E + \operatorname{Ker}\operatorname{ad}F$. By [35, Corollary 7.4.1], the morphism φ_S is faithfully flat. As a consequence, φ_S is surjective and all its fibres have dimension $r-l$. According to [36] (see also [38]) the fibres of φ_S are *generically* smooth and irreducible. Given $\xi\in\mathbb{A}^l$ we denote by S_ξ the fibre of φ_S above ξ. It follows from [23] that

$$S_0 = S\cap\mathcal{N}(\mathfrak{g}) = (E+\operatorname{Ker}\operatorname{ad}F)\cap\mathcal{N}(\mathfrak{g}).$$

Let τ denote the translation $\mathfrak{c}\xrightarrow{\sim}S,\ x\mapsto E+x$, an isomorphism of affine varieties, and $\psi=\varphi_S\circ\tau$. Clearly,

$$\psi:\ \mathfrak{c}\longrightarrow\mathbb{A}^l,\qquad x\mapsto(\psi_1(x),\dots,\psi_l(x)),$$

is faithfully flat and $\psi^{-1}(\xi)\cong S_\xi$ for any $\xi\in\mathbb{A}^l$. The scalar action of $\mathbf{G}_m$ on $\mathfrak{g}$, given by $(t,v)\mapsto\sigma(t)v:=tv$, commutes with the adjoint action of the 1-parameter subgroup λ. Following [35, (7.4)] we consider the $\mathbf{G}_m$-action

$$\rho:\ \mathbf{G}_m\longrightarrow\mathrm{GL}(\mathfrak{g}),\qquad t\mapsto\sigma(t^2)\lambda(t^{-1}).$$

Each $x\in\mathfrak{g}$ decomposes uniquely as $x=\sum_i x_i$ with $x_i\in\mathfrak{g}(i)$. By construction, $\rho(t)x=\sum_i t^{2-i}x_i$, hence both S and $\mathfrak{c}$ are ρ-stable. Arguing as in [35, Proposition 7.4.1] we get

$$\begin{aligned}\psi_i(x) &= f_i(\lambda(t^{-1})(E+x))\\ &= f_i(t^{-2}E+\sum_i t^{-i}x_i)=t^{-2m_i-2}f_i(E+\sum_i t^{2-i}x_i)=t^{-2m_i-2}\psi_i(\rho(t)x).\end{aligned}$$

Therefore, $\psi_i(\rho(t)x)=t^{2m_i+2}\psi_i(x)$ for all $x\in\mathfrak{c}$. In other words, the morphism $\psi:\ \mathfrak{c}=\bigoplus_i\mathfrak{c}(i)\to\mathbb{A}^l$ is quasihomogeneous relative to ρ of type

$$(2m_1+2,\dots,2m_l+2;n_1+2,\dots,n_r+2)$$

(see [35, (7.4)] for more detail). As a consequence, both S_0 and $\psi^{-1}(0)$ are ρ-stable.

5.2. Recall that an element $x\in\mathfrak{g}$ is called *regular* if $\dim\mathfrak{c}_\mathfrak{g}(x)=l$. It is well-known that the set of all regular elements, $\mathfrak{g}_{\rm reg}$, is nonempty and Zariski open in $\mathfrak{g}$.

Proposition. *Let $x\in\mathfrak{c}$ be such that $E+x\in\mathfrak{g}_{\rm reg}$. Then $(\mathrm{d}\psi)_x$ is surjective.*

Proof. Our proof will consist of three steps.

(a) By the differential criterion for regularity [23], the linear map

$$(\mathrm{d}\varphi)_{E+x}:\mathfrak{g}\longrightarrow\mathbb{C}^n$$

is surjective. Let $y,z\in\mathfrak{g}$ and $\lambda\in\mathbb{C}$. Since each f_i is $(\operatorname{Ad}G)$-invariant we have that $f_i((\exp\lambda\operatorname{ad}z).y)=f_i(y)$. This forces $(\mathrm{d}f_i)_y([z,y])=0$ implying that $(\mathrm{d}\varphi)_{E+x}$ vanishes on $\operatorname{Im}\operatorname{ad}(E+x)$.

(b) Pick a basis $v_1,\dots,v_r$ of $\mathfrak{c}$ and extend it to a basis $v_1,\dots,v_r,v_{r+1},\dots,v_d$ of $\mathfrak{g}$. For $x=t_1v_1+\dots+t_rv_r$ we let $\mathcal{M}(t_1,\dots,t_r)$ denote the $(d+r)\times d$ matrix whose rows are the coordinate vectors of $[E+x,v_1],\dots,[E+x,v_d],v_1,\dots,v_r$ relative to the basis $\{v_i\,|\,1\le i\le d\}$. Let $\Delta_1(x),\dots,\Delta_N(x)$ be the $d\times d$ minors of $\mathcal{M}(t_1,\dots,t_r)$ and let Y denote the set of all $y\in\mathfrak{c}$ for which $\Delta_i(y)=0$ where $1\le i\le N$. Clearly, $Y=\{y\in\mathfrak{c}\,|\,\mathfrak{c}+\operatorname{Im}\operatorname{ad}(E+y)\subsetneq\mathfrak{g}\}$. Since the Δ_i's are polynomials in $t_1,\dots,t_r$, the set Y is Zariski closed in $\mathfrak{c}$. It is not hard to see that Y is ρ-stable. If $Y\ne\emptyset$ then $0\in Y$ (for all weights of ρ on $\mathfrak{c}$ are positive). But then $\mathfrak{c}+[E,\mathfrak{g}]\ne\mathfrak{g}$ contrary to the fact that S is a transverse slice to the adjoint orbit of E. Thus $Y=\emptyset$, that is

$$\mathfrak{c}+\operatorname{Im}\operatorname{ad}(E+y)=\mathfrak{g}\qquad(\forall y\in\mathfrak{c}).$$

(c) Now let $x \in \mathfrak{c}$ be such that $E+x \in \mathfrak{g}_{\rm reg}$. Let W be a subspace of $\mathfrak{c}$ complementary to $\mathfrak{c} \cap \operatorname{Im}\operatorname{ad}(E+x)$. Then, naturally, $W \cap \operatorname{Im}\operatorname{ad}(E+x) = 0$. By part (b) of this proof, $\dim W = l$. So we must have $\mathfrak{g} = W \oplus \operatorname{Im}\operatorname{ad}(E+x)$. By part (a), the linear map $(\mathrm{d}\varphi)_{E+x}$ vanishes on $\operatorname{Im}\operatorname{ad}(E+x)$. Applying the differential criterion for regularity [23] we deduce that the restriction of $(\mathrm{d}\varphi)_{E+x}$ to $\mathfrak{c}$ is surjective. But then $(\mathrm{d}\psi)_x : \mathfrak{c} \longrightarrow \mathbb{C}^l$ is surjective, too. □

5.3. In order to prove the main result of this section we need to generalise the method of associated cones [26, (II.4.2)] to the case of a nonscalar $\mathbf{G}_m$-action. We follow [26, (II.4.2)] closely. Let V be a finite dimensional vector space over an algebraically closed field K and

$$\mu : \mathbf{G}_m \longrightarrow \mathrm{GL}(V), \qquad v \mapsto \mu(t)v,$$

a rational $\mathbf{G}_m$-action. Then V decomposes into weight spaces with respect to μ, that is $V = \bigoplus_{k\in\mathbb{Z}} V(k)$ and $\mu(t)v_k = t^k v_k$ for all $t \in \mathsf{K}^*$ and all $v_k \in V(k)$. We assume that all weights of μ on V are *nonnegative*, i.e., $V(i) = 0$ for all $i < 0$. By our discussion in (5.1), the $\mathbf{G}_m$-action $\rho : \mathbf{G}_m \longrightarrow \mathrm{GL}(\mathfrak{c})$ satisfies this assumption.

Identify $V(i)^*$ with with the subspace of V^* consisting of all linear functions ξ with $\xi(V(j)) = 0$ for all $j \neq i$. Clearly $V^* = \bigoplus_{i>0} V(i)^*$. This gives $S(V^*) \cong \mathsf{K}[V]$ a graded algebra structure, $S(V^*) = \bigoplus_{i\geq 0} S(V^*)_i$. The torus μ acts on $S(V^*)$ as algebra automorphisms.

Given a subspace M of $S(V^*)$ we let $\mathrm{gr}_\mu M$ denote the homogeneous subspace of $S(V^*)$ with the property that $g \in \mathrm{gr}_\mu M \cap S(V^*)_r$ if and only if there is $\tilde{g} \in M$ such that $\tilde{g} - g \in \bigoplus_{j<r} S(V^*)_j$. Obviously, the subspace $\mathrm{gr}_\mu M$ is μ-invariant. If M is an ideal of $S(V^*)$ then so is $\mathrm{gr}_\mu M$.

Given a subset $X \subseteq V$ we set $I_X = \{g \in S(V^*) \mid g(X) = 0\}$ and define

$$\mathbb{K}_\mu X := \{v \in V \mid f(v) = 0 \ \text{ for all } \ f \in \mathrm{gr}_\mu I_X\}.$$

By construction, $\mathbb{K}_\mu X$ is a Zariski closed μ-stable subset of V. We call $\mathbb{K}_\mu X$ the *μ-cone associated with X*.

The operations gr_μ and $\mathbb{K}_\mu$ have the following properties:

(1) If I and J are two ideals of $S(V^*)$ satisfying $I \subsetneq J$ then $\mathrm{gr}_\mu I \subsetneq \mathrm{gr}_\mu J$.
(2) $\mathrm{gr}_\mu \sqrt{I} \subseteq \sqrt{\mathrm{gr}_\mu I}$ for any ideal I of $S(V^*)$.
(3) $\mathrm{gr}_\mu I \cdot \mathrm{gr}_\mu J \subseteq \mathrm{gr}_\mu IJ \subseteq \mathrm{gr}_\mu I \cap \mathrm{gr}_\mu J$ for any two ideals I, J of $S(V^*)$.
(4) The correspondence $X \mapsto \mathbb{K}_\mu X$ respects inclusions and has the property that $\mathbb{K}_\mu(X \cup Y) = \mathbb{K}_\mu X \cup \mathbb{K}_\mu Y$ for all $X, Y \subseteq V$.
(5) For any subset $X \subseteq V$ the μ-cone $\mathbb{K}_\mu X$ is contained in $\overline{\mu(\mathsf{K}^*)X}$ and $\dim \mathbb{K}_\mu X = \dim \bar{X}$. Moreover, if X is Zariski closed and irreducible then all irreducible components of $\mathbb{K}_\mu X$ have the same dimension.

Here (5.3.1)–(5.3.3) are obvious and (5.3.4) is a direct consequence of (5.3.3). The first part of (5.3.5) is clear: since $I_{\mu(\mathsf{K}^*)X}$ is μ-stable it is a homogeneous ideal contained in I_X, hence also in $\mathrm{gr}_\mu I_X$. The second part of (5.3.5) is proved below by using a straightforward modification of the argument from [26, Ch. II, Theorem 4.2.2].

We may assume (without loss ot generality) that X is Zariski closed. Define a $\mathbf{G}_m$-action

$$\tilde{\mu} : \mathbf{G}_m \longrightarrow \mathrm{GL}(V \oplus \mathsf{K})$$

on $V \oplus \mathsf{K}$ by setting $\tilde{\mu}(t)(v,1) = (\mu(t)v, t)$ for all $v \in V$. Let

$$X' = \tilde{\mu}(\mathsf{K}^*)(X \times \{1\}) = \{(\mu(\lambda)x, \lambda) \in V \oplus \mathsf{K} \,|\, \lambda \in \mathsf{K}^*,\ x \in X\}.$$

Let Z be the Zariski closure of X' in $V \oplus \mathsf{K}$ and $\eta :\ Z \to \mathsf{K}$ the (nonzero) regular function on Z induced by the projection $V \oplus \mathsf{K} \twoheadrightarrow \mathsf{K}$. Let $R = S(V^*)$. Then

$$\mathsf{K}[V \oplus \mathsf{K}] = R[t] = \bigoplus_{d \geq 0} R[t]_d, \qquad R[t]_d = \sum_{i=0}^{d} R_i\, t^{d-i}.$$

For a homogeneous $f = \sum_{i=0}^{d} f_i t^{d-i} \in R[t]_d$ and $\lambda \in \mathsf{K}$ we have that $f(\mu(\lambda)v, \lambda) = \lambda^d \sum_{i=0}^{d} f_i$. Therefore, $f \in I_Z$ if and only if $\sum_{i=0}^{d} f_i \in I_X$. Since the ideal I_Z is $\tilde{\mu}$-stable we deduce that $(z, \lambda) \in Z$ for $\lambda \neq 0$ if and only if $z = \mu(\lambda)x$ for some $x \in X$. It follows that

$$\eta^{-1}(\lambda) = \mu(\lambda)X \times \{\lambda\} \cong X \quad \text{if} \quad \lambda \neq 0.$$

For $g = \sum_{i=0}^{d} g_i \in R,\ g_d \neq 0$, set $\tilde{g} = \sum_{i=0}^{d} g_i t^{d-i}$. Clearly, $\tilde{g}(v, 0) = g_d(v)$ and $I_{X'} = \{\tilde{g} \,|\, g \in I_X\}$. Therefore, $(v, 0) \in Z$ if and only if $v \in \mathbb{K}_\mu X$. In other words,

$$Z \cap (V \times \{0\}) = \eta^{-1}(0) = \mathbb{K}_\mu X \times \{0\} \cong \mathbb{K}_\mu X.$$

By construction, $X' \cong X \times \mathsf{K}^*$. Thus if X is irreducible then so is Z, and $\dim Z = \dim X + 1$. From this it is immediate that all irreducible components of $\mathbb{K}_\mu X \cong \eta^{-1}(0)$ have dimension equal to $\dim X$. Using (5.3.4) we derive that $\dim \mathbb{K}_\mu X = \dim X$ for reducible X, too.

5.4. We are now in a position to prove the main result of this section.

Theorem. *Let $\xi = (\xi_1, \ldots, \xi_l) \in \mathbb{A}^l$ and $\psi^{-1}(\xi) = \{x \in \mathfrak{c} \,|\, \psi(x) = \xi\}$.*

(i) The ideal $I_{\psi^{-1}(\xi)}$ is generated by $\psi_1 - \xi_1, \ldots, \psi_l - \xi_l$.

(ii) The closed set $\psi^{-1}(\xi)$ is an irreducible, normal complete intersection of dimension $r - l$ in $\mathfrak{c}$.

(iii) Let $z \in \psi^{-1}(\xi)$. Then $E + z \in \mathfrak{g}_{\mathrm{reg}}$ if and only if z is a smooth point of $\psi^{-1}(\xi)$.

Proof. (1) According to [35, Lemma 5.2] the fibre $\varphi_S^{-1}\varphi_S(E+z)$ is normal and $E+z \in S$ is a smooth point of the fibre $\varphi_S^{-1}\varphi_S(E+z)$ if and only if $E + z \in \mathfrak{g}_{\mathrm{reg}}$. By (5.1), φ_S is surjective and $\varphi_S^{-1}\varphi_S(E+z) = E + \psi^{-1}\psi(z)$. Hence $\psi^{-1}\psi(z)$ is normal and z is a smooth point of $\psi^{-1}\psi(z)$ if and only if $E + z \in \mathfrak{g}_{\mathrm{reg}}$, proving (iii).

(2) Let $R = \mathbb{C}[\mathfrak{c}]$. Clearly, $\psi_1, \ldots, \psi_l \in R$. Let $\mathfrak{a}_\xi$ denote the linear span of $\psi_i - \xi_i$ where $1 \leq i \leq l$. By our discussion in (5.1), $\dim \psi^{-1}(\xi) = \dim \mathfrak{c} - l$. Since $\psi^{-1}(\xi)$ is the set of all common zeros of the ideal

$$\mathfrak{a}_\xi R \,=\, \langle \psi_1 - \xi_1, \ldots, \psi_l - \xi_l \rangle$$

and R is a polynomial ring, the ring $R/\mathfrak{a}_\xi R$ is Cohen-Macaulay (see, e.g., [9, Proposition 18.13]).

Let J denote the ideal of $R/\mathfrak{a}_\xi R$ generated by all $l \times l$ minors of the Jacobian matrix $\mathcal{J} = (\partial\psi_i/\partial x_j)$, taken modulo $\mathfrak{a}_\xi R$, and

$$\psi^{-1}(\xi)^\circ = \{x \in \psi^{-1}(\xi) \,|\, g(x) = 0 \ \text{ for all } \ g \in J\}.$$

Since $\psi^{-1}(\xi)$ is normal $\mathrm{Sing}(\psi^{-1}(\xi))$, the set of all singular points in $\psi^{-1}(\xi)$, has codimension ≤ 2 in $\psi^{-1}(\xi)$ (see, e.g., [34, Ch. II, Theorem 5.3]). By part (1) of this proof,

$$\psi^{-1}(\xi) - \mathrm{Sing}(\psi^{-1}(\xi)) \;=\; (-E + \mathfrak{g}_{\mathrm{reg}}) \cap \psi^{-1}(\xi).$$

So Proposition 5.2 yields that $\psi^{-1}(\xi)^{\circ}$ has codimension ≥ 2 in $\psi^{-1}(\xi)$. Applying [9, Theorem 18.15] we now deduce that $R/\mathfrak{a}_\xi R$ is a direct product of domains. As a consequence, the scheme-theoretic fibre of ψ above ξ is reduced, proving (i).

(3) It remains to show that each $\psi^{-1}(\xi)$ is irreducible. We first consider the null-fibre of ψ. By (5.1), ρ acts on $\psi^{-1}(0)$ and gives $\mathbb{C}[\psi^{-1}(0)] = \mathbb{C}[\mathfrak{c}]/(\psi_1, \dots, \psi_l)$ a graded algebra structure. Since all weights of ρ on $\mathfrak{c}$ are positive the zero part of this grading is $\mathbb{C}$. But then 1 is the only idempotent of $\mathbb{C}[\psi^{-1}(0)]$ (because the algebra $\mathbb{C}[\psi^{-1}(0)]$ is reduced). So it follows from (2) that $\mathbb{C}[\psi^{-1}(0)]$ is a domain, i.e., the null-fibre $\psi^{-1}(0)$ is irreducible.

(4) We now consider an arbitrary fibre of ψ making use of the operation gr_ρ introduced in (5.3). Clearly, $\mathrm{gr}_\rho \mathfrak{a}_\xi = \mathfrak{a}_0$. Let C be an irreducible component of $\psi^{-1}(\xi)$. Due to (5.3.2), the irreducibility of $\psi^{-1}(0)$, part (2) of this proof, and (5.3.5) we get

$$\mathfrak{a}_0 R \subseteq \mathrm{gr}_\rho \mathfrak{a}_\xi R \subseteq \mathrm{gr}_\rho I_C \subseteq \sqrt{\mathrm{gr}_\rho I_C} = I_{\psi^{-1}(0)} = \mathfrak{a}_0 R.$$

Then $\mathrm{gr}_\rho \mathfrak{a}_\xi R = \mathrm{gr}_\rho I_C$ hence $\mathfrak{a}_\xi R = I_C$ (by (5.3.1)). We deduce that $\mathfrak{a}_\xi R$ is a prime ideal, completing the proof. □

Remark. The above argument can be carried out over K if $p = \mathrm{char}\, K$ is not too small. To get a modular version of Theorem 5.4 valid for all very good primes one has to replace the special transverse slice $e + \mathfrak{c}_{\mathfrak{g}_K}(f)$ by a *good* transverse slice to the adjoint orbit of e (see [37, 38]).

6. Noncommutative deformations of the graded algebra $\mathbb{C}[\psi^{-1}(0)]$

6.1. In this section, our ground field is $\mathbb{C}$ and we retain the assumptions and conventions of Sections 5 and 6. We denote by $Z(\mathfrak{g})$ the centre of the universal enveloping algebra $U = U(\mathfrak{g})$. Recall that U^k denotes the kth component of the standard filtration of U. By the PBW theorem, $\mathrm{gr}(U) \cong S(\mathfrak{g})$ as graded algebras. The Killing isomorphism $x \mapsto \Phi(x, \cdot\,)$ enables us to identify the graded $(\mathrm{Ad}\, G)$-algebras $S(\mathfrak{g})$ and $S(\mathfrak{g}^*)$. Since $\mathfrak{g} = \mathfrak{c} \oplus [E, \mathfrak{g}]$, by the $\mathfrak{sl}(2)$-theory, and $[E, \mathfrak{g}]$ is orthogonal to $\mathfrak{z}_\chi$ under Φ, the Killing isomorphism induces an isomorphism, $\tilde{\kappa}$, between the $(\mathrm{Ad}\,\lambda)$-algebras $\mathbb{C}[\mathfrak{c}]$ and $S(\mathfrak{z}_\chi)$.

There exist algebraically independent $\tilde{f}_i \in Z(\mathfrak{g}) \cap U^{m_i+1}$, $1 \le i \le l$, such that $Z(\mathfrak{g}) = \mathbb{C}[\tilde{f}_1, \dots, \tilde{f}_l]$ and $\mathrm{gr}\, \tilde{f}_i = f_i$ for all i (see, e.g., [7, (7.4)]).

6.2. Let $T \subset G$ be as in (4.1) and $\mathfrak{t} = \mathrm{Lie}\, T$. Since the Harish-Chandra homomorphism $Z(\mathfrak{g}) \longrightarrow U(\mathfrak{t})$ is injective so is the restriction of $\tilde{\rho}_\chi :\, U \longrightarrow \mathrm{End}(\tilde{Q}_\chi)$ to $Z(\mathfrak{g})$. In what follows we identify $Z(\mathfrak{g})$ with a central subalgebra of $\tilde{H}_\chi$.

Since $\tilde{f}_i \in Z(\mathfrak{g})$ is fixed by the adjoint action of $\lambda = \lambda_E$ on U it is not hard to see, using a suitable PBW basis of U, that $\tilde{f}_i(\tilde{1}_\chi)$ is a linear combination of $X^{\mathbf{a}} Z^{\mathbf{b}} F^c(\tilde{1}_\chi)$ with $|(\mathbf{a}, \mathbf{b})|_e - 2c = 2|\mathbf{a}| + 2|\mathbf{b}|$ and $|\mathbf{a}| + |\mathbf{b}| + c \le m_i + 1$. Moreover, since $\mathrm{gr}\, \tilde{f}_i$ has degree $m_i + 1$ in $S(\mathfrak{g})$ at least one vector $X^{\mathbf{a}} Z^{\mathbf{b}} F^c(\tilde{1}_\chi)$ with $|\mathbf{a}| + |\mathbf{b}| + c = m_i + 1$ occurs in $\tilde{f}_i(\tilde{1}_\chi)$. It follows that $\tilde{f}_i(\tilde{1}_\chi)$ is a linear combination of $X^{\mathbf{a}} Z^{\mathbf{b}} \otimes \tilde{1}_\chi$ with

$|(\mathbf{a},\mathbf{b})|_e \le 2m_i+2$ and at least one basis vector $X^{\mathbf{a}}Z^{\mathbf{b}} \otimes \tilde{1}_\chi \in \tilde{Q}_\chi$ with $|(\mathbf{a},\mathbf{b})|_e = 2m_i+2$ occurs in $\tilde{f}_i(\tilde{1}_\chi)$. This implies that

$$\tilde{f}_i \in \tilde{H}^{2m_i+2} \setminus \tilde{H}^{2m_i+1}, \qquad 1 \le i \le l.$$

6.3. We denote by $\tilde{\psi}_i$ the image of $\tilde{f}_i$ in $\mathrm{gr}_{2m_i+2}(\tilde{H}_\chi)$. The dth homogeneous component of the polynomial algebra $\mathbb{C}[\mathfrak{c}]$ viewed with the grading induced by the action of ρ is denoted by $\mathbb{C}[\mathfrak{c}]_d$. For $1 \le k \le r$ set $\xi_k = \tilde{\kappa}(X_k)|_{\mathfrak{c}}$ and view ξ_k as a homogeneous polynomial function on $\mathfrak{c}$ of degree n_k+2.

Proposition. *There exists an isomorphism of graded algebras* $\delta :\ \mathrm{gr}(\tilde{H}_\chi) \xrightarrow{\sim} \mathbb{C}[\mathfrak{c}]$ *such that* $\delta(\bar{\Theta}_k) = \xi_k$ *for* $1 \le k \le r$ *and* $\delta(\tilde{\psi}_i) = \psi_i$ *for* $1 \le i \le l$.

Proof. Adopt the notation of (4.1) and let M denote the subspace of $\mathfrak{g}$ spanned by $Z_1, \dots, Z_s$ and $X_1, \dots, X_r, X_{r+1}, \dots, X_m$. The e-degree of monomials $X^{\mathbf{a}}Z^{\mathbf{b}} \in S(M)$ is defined as in (4.4) and gives $S(M)$ a graded algebra structure. In view of Lemma 4.4 there is an embedding of graded algebras $\delta' :\ \mathrm{gr}(\tilde{H}_\chi) \hookrightarrow S(M)$. Define an algebra homomorphism $\nu :\ S(M) \longrightarrow S(\mathfrak{z}_\chi)$ by letting $\nu(Z_i) = 0$ for all i, $\nu(X_i) = 0$ for $i > r$, $\nu(X_i) = X_i$ for $1 \le i \le r$, and extending algebraically. Let δ'' denote the restriction of $\nu \circ \delta'$ to $\mathrm{gr}(\tilde{H}_\chi)$. Using Theorem 4.6(i) it is not hard to observe that $\delta''(\bar{\Theta}_k) = X_k$ for all k. So it follows from Theorem 4.6(iii) that δ'' is an isomorphism of graded algebras.

Let $\mathfrak{g}(-2)' = \{x \in \mathfrak{g}(-2) \,|\, \Phi(x, E) = 0\}$. By our discussion in (4.2), $\mathfrak{g}(-2) = \mathfrak{g}(-2)' \oplus \mathbb{C}F$. Extend ν to an algebra homomorphism $\tilde{\nu} :\ S(\mathfrak{g}) \longrightarrow S(\mathfrak{z}_\chi)$ by letting $\tilde{\nu}(Z_j') = \tilde{\nu}(\mathfrak{g}(-2)') = \tilde{\nu}(\mathfrak{g}(i)) = 0$ for all $i < -2$ and $j \le s$, and $\tilde{\nu}(F) = 1$. From our discussion in (6.2) it follows that

$$\delta''(\tilde{\psi}_i) = \tilde{\nu}(\mathrm{gr}\tilde{f}_i), \qquad 1 \le i \le l,$$

where $\mathrm{gr}\tilde{f}_i$ denotes the image of $\tilde{f}_i$ in $S^{m_i+1}(\mathfrak{g}) = U^{m_i+1}/U^{m_i}$. On the other hand, $(\tilde{\kappa} \circ \tilde{\nu})(\mathrm{gr}\tilde{f}_i) = \psi_i$, by the choice of $\tilde{f}_i$. Let $\delta = \tilde{\kappa} \circ \delta''$. Then $\delta :\ \mathrm{gr}(\tilde{H}_\chi) \longrightarrow \mathbb{C}[\mathfrak{c}]$ is an isomorphism of graded algebras and $\delta(\tilde{\psi}_i) = \psi_i$ for all i, as desired. $\square$

6.4. Let $\eta :\ Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ be an algebra homomorphism and $\mathbb{C}_\eta = Z(\mathfrak{g})/\mathrm{Ker}\,\eta$. Define

$$\tilde{H}_{\chi,\eta}(\mathfrak{g}) = \tilde{H}_{\chi,\eta} := \tilde{H}_\chi \otimes_{Z(\mathfrak{g})} \mathbb{C}_\eta \cong \tilde{H}_\chi / \tilde{H}_\chi \cdot \mathrm{Ker}\,\eta.$$

Set $J_\eta = \tilde{H}_\chi \cdot \mathrm{Ker}\,\eta$, $J_\eta^k = J_\eta \cap \tilde{H}^k$, and $\tilde{H}_{\chi,\eta}^k = (\tilde{H}^k + J_\eta)/J_\eta \cong \tilde{H}^k/J_\eta^k$ where $k \in \mathbb{N}_0$. Notice that $\mathrm{gr}(J_\eta) = \bigoplus_{k \ge 0} (J_\eta^k + \tilde{H}^{k-1})/\tilde{H}^{k-1}$ is a graded ideal of $\mathrm{gr}(\tilde{H}_\chi)$ and $\{\tilde{H}_{\chi,\eta}^k \,|\, k \in \mathbb{N}_0\}$ is a filtration of the algebra $\tilde{H}_{\chi,\eta}(\mathfrak{g})$. We denote by $\mathrm{gr}(\tilde{H}_{\chi,\eta})$ the associated graded algebra. It is easy to see that

$$\mathrm{gr}(\tilde{H}_{\chi,\eta}) \cong \mathrm{gr}(\tilde{H}_\chi)/\mathrm{gr}(J_\eta)$$

as graded algebras. Recall that the coordinate ring $\mathbb{C}[\psi^{-1}(0)] \cong \mathbb{C}[S_0]$ is naturally graded by the action of the 1-dimensional torus ρ (see (5.1)).

Theorem. *The graded algebras* $\mathrm{gr}(\tilde{H}_{\chi,\eta})$ *and* $\mathbb{C}[\psi^{-1}(0)]$ *are isomorphic.*

Proof. (a) By Theorem 5.4(i), the ideal $I_{\psi^{-1}(0)}$ is generated by $\psi_1, \dots, \psi_l$. Therefore, in view of Proposition 6.3, it suffices to show that the ideal $\mathrm{gr}(J_\eta)$ is generated by $\tilde{\psi}_1, \dots, \tilde{\psi}_l$. Note that $\tilde{\psi}_i \in \mathrm{gr}(J_\eta)$ for all i. Let $\mathcal{Z}$ denote the subalgebra of $\mathrm{gr}(\tilde{H}_\chi)$ generated by the $\tilde{\psi}_i$'s. Combining Proposition 6.3 with [35, Corollary 7.4.1] and our discussion in (5.1) we observe that $\mathrm{gr}(\tilde{H}_\chi)$ is a flat $\mathcal{Z}$-module. Since S is a transverse slice to the orbit $(\mathrm{Ad}\, G)\cdot E$, the polynomials $\psi_1, \dots, \psi_l$ are algebraically independent. Using Proposition 6.3 we now obtain that so are $\tilde{\psi}_1, \dots, \tilde{\psi}_l$.

For $k \in \mathbb{N}_0$, set $Z^k(\mathfrak{g}) = Z(\mathfrak{g}) \cap \tilde{H}^k$. Let $\eta_i = \eta(\tilde{f}_i)$ where $1 \le i \le l$. By our final remark in (6.2), $\tilde{f}_i - \eta_i \in Z^{2m_i+2}(\mathfrak{g})$. It follows from our discussion in (a) that there exists an algebra isomorphism $\xi : Z(\mathfrak{g}) \xrightarrow{\sim} \mathcal{Z}$ such that $\xi(\tilde{f}_i - \eta_i) = \tilde{\psi}_i$ for all i. It sends $Z^k(\mathfrak{g})$ onto $\mathcal{Z} \cap \bigoplus_{i \le k} \mathrm{gr}_i(\tilde{H}_\chi)$.

(b) Let $x \in J_\eta^d \setminus J_\eta^{d-1}$. Then $x = \sum_{i=1}^l u_i(\tilde{f}_i - \eta_i)$ where $u_i \in \tilde{H}^{k_i} \setminus \tilde{H}^{k_i-1}$ for some $k_i \in \mathbb{N}_0$. Of course, such a presentation of x is not unique, and we are going to minimise

$$N_0 := \max\{k_i + 2m_i + 2 \mid 1 \le i \le l\}$$

which depends on the presentation. By our assumption on x we have that $N_0 \ge d$. Suppose the presentation is such that $N_0 > d$ and let $\mathcal{I}_0 = \{i \le l \mid k_i + 2m_i + 2 = N_0\}$. Then $\mathcal{I}_0 \ne \emptyset$ and

$$\sum_{i \in \mathcal{I}_0} \mathrm{gr}(u_i) \cdot \mathrm{gr}(\tilde{f}_i - \eta_i) = \sum_{i \in \mathcal{I}_0} \mathrm{gr}(u_i)\tilde{\psi}_i = 0$$

is a nontrivial homogeneous relation in $\mathrm{gr}(\tilde{H}_\chi)$. By our discussion in (a), $\mathrm{gr}(\tilde{H}_\chi)$ is a flat $\mathcal{Z}$-module. Applying the Equational Criterion for Flatness (see, e.g., [9, Corollary 6.5]) we deduce that there are $a_{ij} \in \mathcal{Z}$ and $u'_j \in \mathrm{gr}(\tilde{H}_\chi)$, homogeneous with $\deg a_{ij} + \deg u'_j = k_i$, such that

$$\mathrm{gr}(u_i) = \sum_j a_{ij} u'_j \quad \text{and} \quad \sum_{i \in \mathcal{I}_0} a_{ij}\tilde{\psi}_i = 0 \tag{14}$$

for all $i \in I_0$ and all j. Choose $\tilde{u}'_j \in \tilde{H}_\chi$ and $\tilde{a}_{ij} \in Z(\mathfrak{g})$ with $\mathrm{gr}(\tilde{u}'_j) = u'_j$ and $\xi(\tilde{a}_{ij}) = a_{ij}$. Since ξ is an isomorphism of algebras the second part of (14) yields

$$\sum_{i \in \mathcal{I}_0} \tilde{a}_{ij}(\tilde{f}_i - \eta_i) = 0 \qquad (\forall\, j). \tag{15}$$

It follows from (15) that

$$\begin{aligned} x = \sum_{i=1}^{l} u_i(\tilde{f}_i - \eta_i) &= \sum_{i=1}^{l} u_i(\tilde{f}_i - \eta_i) - \sum_{i \in \mathcal{I}_0} \Big(\sum_j \tilde{a}_{ij}\tilde{u}'_j\Big)(\tilde{f}_i - \eta_i) \\ &= \sum_{i \notin \mathcal{I}_0} u_i(\tilde{f}_i - \eta_i) + \sum_{i \in \mathcal{I}_0} \Big(u_i - \sum_j \tilde{a}_{ij}\tilde{u}'_j\Big)(\tilde{f}_i - \eta_i). \end{aligned}$$

The first part of (14) shows that

$$u_i - \sum_j \tilde{a}_{ij}\tilde{u}'_j \in \tilde{H}^{k_i-1}$$

for all $i \in \mathcal{I}_0$. It follows that N_0 of the new presentation of x is smaller than that of the initial one. Continuing this process we eventually arrive at a presentation of x with $N_0 = d$.

(c) As a consequence, we can assume that $x = \sum_{i=1}^{l} u_i(\tilde{f}_i - \eta_i)$ where $k_i + 2m_i + 2 \leq d$ for all i. Then

$$\mathrm{gr}(x) = \sum_{i \in \mathcal{I}_0} \mathrm{gr}(u_i)\tilde{\psi}_i \in \mathrm{gr}(J_\eta),$$

implying that $\mathrm{gr}(J_\eta)$ is generated by $\tilde{\psi}_1, \ldots, \tilde{\psi}_l$ and thereby completing the proof. □

6.5. Recall that Proposition 6.3 identifies the graded algebras $\mathrm{gr}(\tilde{H}_\chi)$ and $\mathbb{C}[\mathfrak{c}]$.

Proposition. *(i) The product in $\tilde{H}_\chi$ induces a Poisson bracket $\{\cdot, \cdot\}$ on $\mathbb{C}[\mathfrak{c}]$ such that*

$$\{\xi_i, \xi_j\} = \sum_{k=1}^{r} \alpha_{ij}^k \, \xi_k + q'_{ij}(\xi_1, \ldots, \xi_r) \in \mathbb{C}[\mathfrak{c}]_{n_i+n_j+2},$$

where α_{ij}^k are as in Theorem 4.6(iv) and q'_{ij} is a polynomial in r variables whose constant term and linear part are both zero.

(ii) If the Lie algebra $\mathfrak{z}_\chi$ is nonabelian then there is a homomorphism $\eta : Z(\mathfrak{g}_\mathbb{C}) \longrightarrow \mathbb{C}$ such that the algebra $\tilde{H}_{\chi,\eta}$ is noncommutative.

(iii) If $[\mathfrak{z}_\chi, [\mathfrak{z}_\chi, \mathfrak{z}_\chi]] \neq 0$ then for any $\eta : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ the product in $\tilde{H}_{\chi,\eta}$ induces a nonzero Poisson bracket on $\mathbb{C}[\psi^{-1}(0)]$.

Proof. Let $h \in \tilde{H}^a$ and $h' \in \tilde{H}^b$. Then $[h, h'] \in \tilde{H}^{a+b-2}$, by Theorem 4.6 and Leibniz rule. As a consequence, we can define a graded $\mathbb{C}$-bilinear skew-symmetric bracket $\{\cdot, \cdot\} : \mathrm{gr}(\tilde{H}_\chi) \times \mathrm{gr}(\tilde{H}_\chi) \longrightarrow \mathrm{gr}(H_\chi)$ by setting

$$\{h + \tilde{H}^{a-1}, \, h' + \tilde{H}^{b-1}\} := hh' - h'h + \tilde{H}^{a+b-3}.$$

It is straightforward to check that the bracket $\{\cdot, \cdot\}$ satisfies Jacobi identity and Leibniz rule, hence is a Poisson bracket on $\mathrm{gr}(\tilde{H}_\chi)$. Part (i) now follows from Theorem 4.6(iv), Proposition 6.3 and the definition of $\{\cdot, \cdot\}$.

Now suppose $\tilde{H}_{\chi,\eta}$ is commutative for any $\eta : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$. Let V be any simple $\tilde{H}_\chi$-module. Since $\mathrm{gr}(\tilde{H}_\chi) \cong \mathbb{C}[\mathfrak{c}]$ is finitely generated and commutative, Quillen's lemma shows that $\mathrm{End}_{\tilde{H}_\chi}(V)$ consists of scalar operators (see, e.g., [7, Lemma 2.6.4]). It follows that $Z(\mathfrak{g}) \subseteq \tilde{H}_\chi$ acts on V via a central character $\eta' : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$. Then $J_{\eta'}$ annihilates V. Since V is an arbitrary simple $\tilde{H}_\chi$-module and $[\tilde{H}_\chi, \tilde{H}_\chi] \subseteq J_{\eta'}$ by our assumption, we deduce that $[\tilde{H}_\chi, \tilde{H}_\chi]$ is contained in $J(\tilde{H}_\chi)$, the Jacobson radical of $\tilde{H}_\chi$. For any $x \in J(\tilde{H}_\chi)$ the element $1+x$ is invertible in $\tilde{H}_\chi$ (see, e.g., [7, Proposition 3.1.12]). Suppose $x \in J(\tilde{H}_\chi)$ is such that $x \in \tilde{H}^k \setminus \tilde{H}^{k-1}$ for some $k > 0$. Then $1 + x \in \tilde{H}^k \setminus \tilde{H}^{k-1}$ also. Let $y \in \tilde{H}_\chi$ be such that $(1+x)y = 1$. There is $t \geq 0$ such that $y \in \tilde{H}^t \setminus \tilde{H}^{t-1}$. Since $\mathrm{gr}(\tilde{H}_\chi) \cong \mathbb{C}[\mathfrak{c}]$ we must have $(1+x)y \in \tilde{H}^{k+t} \setminus \tilde{H}^{k+t-1}$. But then $(1+x)y \neq 1$, a contradiction. Since $\tilde{H}^0 = \mathbb{C}\,1$ we derive $J(\tilde{H}_\chi) = 0$. Therefore, $\tilde{H}_\chi$ is itself commutative, which forces the Poisson bracket $\{\cdot, \cdot\}$ to be identically zero on $\mathbb{C}[\mathfrak{c}]$. Part (i) of this proof now shows that all structure constants of the Lie algebra $\mathfrak{z}_\chi$ vanish, hence (ii).

For (iii), let I denote the ideal of $\mathbb{C}[\mathfrak{c}]$ generated by $\xi_1, \ldots, \xi_r$. Clearly, I^k is a Poisson ideal of $\mathbb{C}[\mathfrak{c}]$ for any $k > 0$. Combining Theorem 5.4(i), Proposition 6.3 and Theorem 6.4 we obtain that the image of $\mathrm{gr}(J_\eta)$ in $\mathbb{C}[\mathfrak{c}]$ coincides with $I_{\psi^{-1}(0)} = \langle \psi_1, \ldots, \psi_l \rangle$. It follows that $I_{\psi^{-1}(0)}$ is a Poisson ideal of $\mathbb{C}[\mathfrak{c}]$ and the natural Poisson bracket on $\mathbb{C}[\psi^{-1}(0)]/I_{\psi^{-1}(0)}$ coincides with that induced by multiplication in $\tilde{H}_{\chi,\eta}$. Since $[\Theta_i, \tilde{f}_j] = 0$ we have that $\{\xi_i, \psi_j\} = 0$ for all $1 \le i \le r$ and $1 \le j \le l$ (this is immediate from Proposition 6.3).

The Poisson bracket on $\mathbb{C}[\mathfrak{c}]$ gives the factor space I/I^2 a Lie algebra structure. By part (i), $\mathfrak{z}_\chi \cong I/I^2$ as Lie algebras. Let Ψ denote the linear span of all ψ_i. It follows from our preceding remark that $\Psi + I^2$ is contained in the centre of the Lie algebra I/I^2. If $\{\,\cdot\,,\,\cdot\,\}$ induces the zero bracket on $\mathbb{C}[\psi^{-1}(0)]$ then $\{\xi_i, \xi_j\} \in \Psi + I^2$ for all $1 \le i, j \le r$. But then $[\mathfrak{z}_\chi, \mathfrak{z}_\chi]$ is a central subalgebra of $\mathfrak{z}_\chi$ forcing $[\mathfrak{z}_\chi, [\mathfrak{z}_\chi, \mathfrak{z}_\chi]] = 0$. The proof of the proposition is now complete. □

Remark 1. It is proved in [42] and [27] that $\mathfrak{z}_\chi$ is abelian if and only if E is a regular nilpotent element in $\mathfrak{g}$ (Springer and Steinberg posed this as an open problem in [39, (III, 1.18)]). The proof in [42] is computer-free.

Remark 2. Proposition 6.5(ii) leaves is a lot of room for improvement. The following question seems relevant:

Is it true that for any two-sided ideal $\mathcal{I}$ of $\tilde{H}_\chi$, the centre of the quotient algebra $\tilde{H}_\chi/\mathcal{I}$ coincides with the image of $Z(\mathfrak{g})$ in $\tilde{H}_\chi/\mathcal{I}$?

The answer is positive in the two extremes. If $\chi = 0$ then $\tilde{H}_\chi = U(\mathfrak{g})$. Let $\mathcal{I}$ be a two-sided ideal of $U(\mathfrak{g})$ and $\mathcal{A} = U(\mathfrak{g})/\mathcal{I}$. According to [7, Proposition 4.2.5], the centre of $\mathcal{A}$ coincides with the image of $Z(\mathfrak{g})$ in $\mathcal{A}$ (this follows easily from the semisimplicity of the adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$). For the regular nilpotent case, see our discussion in (7.2).

Remark 3. Proposition 6.5(iii) in conjunction with [42, Lemma 2.4]) shows that the Poisson bracket on $\mathbb{C}[\psi^{-1}(0)]$ induced by $\{\,\cdot\,,\,\cdot\,\}$ is nonzero for all nondistinguished nilpotent elements in $\mathfrak{g}$. Of course, it should be nonzero for all nonregular ones. In the next section we will compute this Poisson bracket in the subregular nilpotent case.

7. Some special cases

7.1. Suppose the nilpotent element E in $\mathfrak{g} = \mathfrak{g}_\mathbb{C}$ (respectively, e in $\mathfrak{g} = \mathfrak{g}_K$) is even. Then $\mathfrak{g}(k) = 0$ for k odd and $\mathfrak{m}_\chi = \sum_{i \le -2} \mathfrak{g}(i)$ (see (3.1) for more detail). Moreover, $\tilde{Q}_\chi \cong U\big(\sum_{i \ge 0} \mathfrak{g}(i)\big)$ and $Q_\chi \cong U_\chi\big(\sum_{i \ge 0} \mathfrak{g}(i)\big) = U^{[p]}\big(\sum_{i \ge 0} \mathfrak{g}(i)\big)$ as vector spaces. It follows from our discussion in (4.5) and (3.2) that $\tilde{H}_\chi$ and H_χ can be identified with subalgebras of $U\big(\sum_{i \ge 0} \mathfrak{g}(i)\big)$ and $U^{[p]}\big(\sum_{i \ge 0} \mathfrak{g}(i)\big)$, respectively. The PBW theorem implies that there exists a natural projection $U\big(\sum_{i \ge 0} \mathfrak{g}(i)\big) \to U(\mathfrak{g}(0))$ (respectively, $U^{[p]}\big(\sum_{i \ge 0} \mathfrak{g}(i)\big) \to U^{[p]}(\mathfrak{g}(0))$), a surjective algebra homomorphism. The restriction of this projection to $\tilde{H}_\chi$ (respectively, to H_χ) induces an algebra homomorphism $\mu : \tilde{H}_\chi \longrightarrow U(\mathfrak{g}(0))$ (respectively, $\mu^{[p]} : H_\chi \longrightarrow U^{[p]}(\mathfrak{g}(0))$). Inspired by similarity between the algebras $\tilde{H}_\chi$ and $BRST$ quantisations of finite W algebras (see, e.g., [5]) we call μ (respectively, $\mu^{[p]}$) the *Miura homomorphism* from $\tilde{H}_\chi$ to $U(\mathfrak{g}(0))$ (respectively, from H_χ to $U^{[p]}(\mathfrak{g}(0))$).

In the modular case, it follows from Engel's theorem that the $[p]$-nilpotent ideal $\sum_{i>0} \mathfrak{g}(i)$ of the restricted Lie algebra $\mathfrak{p}_e = \sum_{i \ge 0} \mathfrak{g}(i)$ acts trivially on any simple $U^{[p]}(\mathfrak{p}_e)$-module. From this it follows that the kernel of the projection $U^{[p]}(\mathfrak{p}_e) \twoheadrightarrow$

$U^{[p]}(\mathfrak{g}(0))$ is contained in the Jacobson radical of $U^{[p]}(\mathfrak{p}_e)$. As a consequence, the kernel of $\mu^{[p]}$ is a nilpotent ideal of H_χ, hence acts trivially on any simple H_χ-module. Thus simple $\mathfrak{g}$-modules with p-character χ are in one-to-one correspondence with irreducible representations of $\mu^{[p]}(H_\chi)$, the image of H_χ under the Miura homomorphism. In the subregular nilpotent case one is essentially reduced to a mysterious subalgebra of $U^{[p]}(\mathfrak{sl}(2))$.

Remark. If $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$ and χ is regular nilpotent then the Miura homomorphism μ is injective (indeed, μ is essentially the Harish-Chandra homomorphism in this case). Probably this holds for any even χ (cf. [5, Theorem 5]).

7.2. Suppose E is a regular nilpotent element in $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$. Then $\dim \mathfrak{c} = l$ and $\psi : \mathfrak{c} \longrightarrow \mathbb{A}^l$ is a ρ-equivariant isomorphism of affine varieties (see [35, Lemma 8.1.1 and Corollary 7.4.2]). This means that $\mathbb{C}[\mathfrak{c}]$ is generated by $\psi_1, \ldots, \psi_l$. By Proposition 6.3, $\mathrm{gr}(\tilde{H}_\chi)$ is generated by $\mathrm{gr}\,\tilde{f}_1, \ldots, \mathrm{gr}\,\tilde{f}_l$. A standard filtration argument now shows that $\tilde{H}_\chi$ is generated by $\tilde{f}_1, \ldots, \tilde{f}_l$, that is $\tilde{H}_\chi = Z(\mathfrak{g})$. It should be mentioned that in the regular case the module $\tilde{Q}_\chi$ is generated by a Whittaker vector (see [24]). So the equality $\tilde{H}_\chi = Z(\mathfrak{g})$ can also be deduced from the main results of [24]. In the modular case, one can use a similar argument to show that if $\mathrm{char}\, K$ is very good for $\mathfrak{g}_K$ and $\chi = \chi_K$ is regular nilpotent then the algebra $H_\chi(\mathfrak{g}_K)$ coincides with Z_χ, the image of the centre of $U(\mathfrak{g}_K)$ in H_χ. The latter also follows from [28, Theorem 12] and Theorem 8.2 of this paper.

7.3. Compared with the regular nilpotent case, the case of a subregular nilpotent χ is much more interesting and leads to a deep modular representation theory (see [18], [19]). Until the end of this section we assume that E is a subregular nilpotent element in $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$. In this case, it was conjectured by Grothendieck and proved by Brieskorn that, for $\mathfrak{g}$ of type A, D, E, the affine variety $\psi^{-1}(0)$ is a surface with an isolated rational double point of the type corresponding to the Lie algebra $\mathfrak{g}$. More precisely, $\psi^{-1}(0)$ is isomorphic to the Kleinian singularity $\mathbb{C}^2/\Gamma$ where $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ is the binary polyhedral group whose Coxeter-Dynkin-Witt diagram $\Delta(\Gamma)$ has the same type as the Dynkin diagram of $\mathfrak{g}$. Detailed proofs of Brieskorn's results, outlined in [1], can be found in [35]. Slodowy also extended Brieskorn's results to include the remaining simple Lie algebras (i.e., those of type B_l, C_l, F_4 and G_2).

Set $w_i = n_i + 2$ and $d_j = 2m_j + 2$ where $1 \le i \le r$ and $1 \le j \le l$ (in the present case $r = l + 2$). According to [35, Proposition 7.4.2], we can choose basis vectors $X_i \in \mathfrak{z}_\chi$ and homogeneous polynomial invariants $f_i \in S(\mathfrak{g}^*)^G$ such that $w_i = d_i$ for $1 \le i \le l - 1$. The ρ-weights $w_1 \ldots, w_{l+2}$ are given in the table below:

Type	d_1	d_2	d_3	$\cdots$	d_{l-3}	d_{l-2}	d_{l-1}	d_l	w_l	w_{l+1}	w_{l+2}
A_l	4	6	8	$\ldots$	$2l-4$	$2l-2$	$2l$	$2l+2$	2	$l+1$	$l+1$
B_l	4	8	12	$\ldots$	$4l-12$	$4l-8$	$4l-4$	$4l$	2	$2l$	$2l$
C_l	4	8	12	$\ldots$	$4l-12$	$4l-8$	$4l-4$	$4l$	4	$2l-2$	$2l$
D_l	4	8	12	$\ldots$	$4l-12$	$4l-8$	$2l$	$4l-4$	4	$2l-4$	$2l-2$
E_6	4	10	12			16	18	24	6	8	12
E_7	4	12	16		20	24	28	36	8	12	18
E_8	4	16	24	28	36	40	48	60	12	20	30
F_4	4	12	16					24	6	8	12
G_2	4							12	4	4	6

which also lists the degrees $d_1, \ldots, d_l$. This table is taken from [35, (7.4)] for reader's convenience.

7.4. According to [35, Lemma 8.3.1], the differential of $\psi : \mathfrak{c} \longrightarrow \mathbb{A}^l \cong \mathbb{C}^l$ has rank $l-1$ at 0. Therefore, by [35, Lemma 8.1.2], there are direct decompositions $\mathbb{C}^l = V \oplus A_0$ and $\mathfrak{c} \cong \mathbb{C}^l \oplus \mathbb{C}^2 = V \oplus \mathfrak{c}_0$, with all summands ρ-stable and with $\dim V = l-1$, as well as a ρ-equivariant polynomial automorphism α of $\mathfrak{c}$ such that $\psi \circ \alpha : V \oplus \mathfrak{c}_0 \longrightarrow V \oplus A_0$ has the form

$$(v, c) \mapsto (v, \psi'(v, c)) \qquad (v \in V,\, c \in \mathfrak{c}_0)$$

where $\psi' : \mathfrak{c} \longrightarrow A_0 \cong \mathbb{A}^1$ is a polynomial function on $\mathfrak{c}$. Moreover, ρ operates on V with weights $d_1 = w_1 \ldots, d_{l-1} = w_{l-1}$ (see [35, (8.2)]).

Looking at the comorphism of $\psi \circ \alpha$ is not hard to observe that one can adjust the homogeneous basis $X_1, \ldots, X_r$ of $\mathfrak{z}_\chi$ in such a way that $\psi_i = \xi_i \circ \alpha^{-1}$ for $1 \le i \le l-1$. Set $u = \alpha^{-1}(\xi_l)$, $v = \alpha^{-1}(\xi_{l+1})$, $w = \alpha^{-1}(\xi_{l+2})$. Since α is ρ-equivariant we have that $u \in \mathbb{C}[\mathfrak{c}]_{w_l}$, $v \in \mathbb{C}[\mathfrak{c}]_{w_{l+1}}$ and $w \in \mathbb{C}[\mathfrak{c}]_{w_{l+2}}$. Since α is a polynomial automorphism of $\mathfrak{c}$ the elements $\psi_1, \ldots, \psi_{l-1}, h, u, w$ form a system of free homogeneous generators for the graded polynomial algebra $\mathbb{C}[\mathfrak{c}]$. Combining Proposition 6.3 with a standard filtration argument we now deduce that there exist $\tilde{u} \in \tilde{H}^{w_l}$, $\tilde{v} \in \tilde{H}^{w_{l+1}}$ and $\tilde{w} \in \tilde{H}^{w_{l+2}}$ such that the elements $\tilde{f}_1, \ldots, \tilde{f}_{l-1}$ thogether with $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ generate $\tilde{H}_\chi$ as an algebra. More precisely, we obtain the following.

Proposition. *If E is subregular nilpotent in $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$ then the algebra $\tilde{H}_\chi$ is a free module over its subalgebra $Z' = \mathbb{C}[\tilde{f}_1, \ldots, \tilde{f}_{l-1}]$. Moreover, the monomials $\tilde{u}^a \tilde{v}^b \tilde{w}^c$ with $a, b, c \in \mathbb{N}_0$ form a free basis of $\tilde{H}_\chi$ over Z'.*

Remark. Proposition 7.4 has a modular analogue valid under the assumption that $p = \operatorname{char} K$ is a very good prime for the root system of G.

7.5. We denote by I_0 the ideal of $\mathbb{C}[\mathfrak{c}]$ generated by $\psi_1, \ldots, \psi_{l-1}$. By our discussion in (6.5), I_0 is a Poisson ideal of $\mathbb{C}[\mathfrak{c}]$. As usual, we identify the graded algebras $\mathbb{C}[\mathfrak{c}]/I_0$ and $\mathbb{C}[\mathfrak{c}_0]$. The rest of this section is devoted to computing the Poisson bracket on $\mathbb{C}[\mathfrak{c}]/I_0$ induced by $\{\,\cdot\,,\,\cdot\,\}$.

Since α preserves both $\mathfrak{c}_0$ and V the cosets $x' = u + I_0$, $y' = v + I_0$ and $z' = w + I_0$ form a free system of homogeneous generators for $\mathbb{C}[\mathfrak{c}]/I_0$. Let $f = \psi_r + I_0$. Recall that ρ acts on $\mathfrak{c}_0$ and f is a quasihomogeneous polynomial relative to ρ of type $(d_l; w_l, w_{l+1}, w_{l+2})$. According to [35, Proposition 8.3.2], there exists a ρ-equivariant polynomial automorphism $\beta : \mathfrak{c}_0 \to \mathfrak{c}_0$ such that $f \circ \beta$ has the normal form of a rational double point. This form is given in the second column of the table below.

Type	f	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$
$A_l,\ l \geq 1$	$x^{l+1} + yz$	y	$-z$	$(l+1)x^l$
$B_l,\ l \geq 2$	$x^{2l} + yz$	y	$-z$	$2lx^{2l-1}$
$C_l,\ l \geq 3$	$x^l + xy^2 + z^2$	$2z$	$-2xy$	$lx^{l-1} + y^2$
$D_l,\ l \geq 4$	$x^{l-1} + xy^2 + z^2$	$2z$	$-2xy$	$(l-1)x^{l-2} + y^2$
E_6	$x^4 + y^3 + z^2$	$2z$	$-3y^2$	$4x^3$
E_7	$x^3y + y^3 + z^2$	$2z$	$-x^3 - 3y^2$	$3x^2y$
E_8	$x^5 + y^3 + z^2$	$2z$	$-3y^2$	$5x^4$
F_4	$x^4 + y^3 + z^2$	$2z$	$-3y^2$	$4x^3$
G_2	$x^3 + xy^2 + z^2$	$2z$	$-2xy$	$3x^2 + y^2$

Since β can be lifted to a ρ-equivariant polynomial automorphism of $\mathfrak{c}$ we may assume without loss of generality that f has the normal form with respect to x', y', z'. There exist homogeneous polynomials $r_1, \dots, r_{l-1} \in \mathbb{C}[\psi_1, \dots, \psi_{l-1}, u, v, w]$ such that

$$\psi_l = f(u, v, w) + \sum_{i=1}^{l-1} \psi_i r_i. \tag{16}$$

7.6. Suppose the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ is identically zero on $\mathbb{C}[\mathfrak{c}]/I_0$ (this implies $I_0 \neq 0$, hence $\mathfrak{g}$ is not of type A_1). Let I denote the ideal of $\mathbb{C}[\mathfrak{c}]$ generated by $\psi_1, \dots, \psi_{l-1}, u, v, w$, and $\bar{I}_0 = \mathbb{C}[\psi_1, \dots, \psi_{l-1}] \cap I_0$. Since $\mathfrak{z}_\chi$ is nonabelian (by [42]) we must have

$$\big\{\{u, v\}, \{u, w\}, \{v, w\}\big\} \not\subset I \cdot I_0 \tag{17}$$

(otherwise $\{p, q\} \in I^2$ for all $p, q \in \mathbb{C}[\mathfrak{c}]$ contrary to Proposition 6.5(i)). Since $\psi_1, \dots, \psi_{l-1}, u, v, w$ are free homogeneous generators of $\mathbb{C}[\mathfrak{c}]$ (see (7.4)),

$$\mathbb{C}[\mathfrak{c}]/I_0^2 \cong \mathbb{C}[u, v, w] \otimes \mathbb{C}[\psi_1, \dots, \psi_{l-1}]/\bar{I}_0^2$$

as graded algebras. From this it is not hard to deduce that, for any $k \geq 1$, the factor algebra $\mathbb{C}[\mathfrak{c}]/(I_0^2 + I^k \cdot I_0)$ has basis consisting of the cosets of $u^a v^b w^c \psi_i^d$ where $1 \leq i \leq l-1$, $0 \leq d \leq 1$ and $a + b + c < k$ if $d = 1$.

7.7. It follows from (7.5(16)) and our assumption on $\{\,\cdot\,,\,\cdot\,\}$ that $\{g, f(u, v, w)\} \in I_0^2$ for any $g \in \mathbb{C}[u, v, w]$.

Suppose $\mathfrak{g}$ is of type A_l where $l \geq 2$. Then $f(u, v, w) = u^{l+1} + vw$ forcing

$$w\{u, v\} + v\{u, w\},\ v\{v, w\} - (l+1)u^l\{u, v\} \in I_0^2.$$

From Table 7.3 we get $\{u, v\} \equiv \lambda\psi_d \pmod{I \cdot I_0}$, $\{u, w\} \equiv \mu\psi_d \pmod{I \cdot I_0}$ and $\{v, w\} \equiv \nu\psi_{l-1} \pmod{I \cdot I_0}$, where $\lambda, \mu, \nu \in \mathbb{C}$ and $d = (l-1)/2$. In particular, $\lambda = \mu = 0$ if l is even. In any event,

$$(\mu v + \lambda w)\psi_d,\ \nu v \psi_{l-1} \in I_0^2 + I^2 \cdot I_0.$$

But then $\lambda = \mu = \nu = 0$ (see our final remark in (7.6)). Since this contradicts (7.6(17)) we deduce that $\mathfrak{g}$ is not of type A_l. A similar argument shows that $\mathfrak{g}$ is not of type B_l.

Suppose $\mathfrak{g}$ is of type C_l where $l \geq 3$. Then $f(u,v,w) = u^l + uv^2 + w^2$ hence

$$2uv\{u,v\} + 2w\{u,w\},\ 2w\{v,w\} - (v^2 + lu^{l-1})\{u,v\} \in I_0^2.$$

From Table 7.3 we get $\{u,v\} \equiv \lambda\psi_{d_1} \pmod{I \cdot I_0}$, $\{u,w\} \equiv \mu\psi_{d_2} \pmod{I \cdot I_0}$ and $\{v,w\} \equiv \nu\psi_{l-1} \pmod{I \cdot I_0}$, where $\lambda, \mu, \nu \in \mathbb{C}$, $d_1 = l/2$ and $d_2 = (l+1)/2$. As a consequence, $\lambda\mu = 0$. Computing modulo $I_0^2 + I^3 \cdot I_0$ and using (7.6) we obtain $\lambda = \mu = 0$. Since $l \geq 3$ we have $2\nu w\psi_{l-1} \in I_0^2 + I^2 \cdot I_0$ yielding $\nu = 0$. Therefore, $\mathfrak{g}$ is not of type C_l. Arguing similarly we deduce that $\mathfrak{g}$ is not of type D_l.

If $\mathfrak{g}$ is of type E_6 then $f(u,v,w) = u^4 + v^3 + w^2$. Also, $\{u,v\} \equiv \lambda\psi_3 \pmod{I \cdot I_0}$, $\{u,w\} \equiv \mu\psi_4 \pmod{I \cdot I_0}$ and $\{v,w\} \equiv \nu\psi_5 \pmod{I \cdot I_0}$ for some $\lambda, \mu, \nu \in \mathbb{C}$ (see Table 7.3). It follows that

$$3v^2\{u,v\} + 2w\{u,w\},\ 2w\{v,w\} - 4u^3\{u,v\} \in I_0^2.$$

Computing modulo $I_0^2 + I^3 \cdot I_0$ (respectively, modulo $I_0^2 + I^2 \cdot I_0$) we derive $\lambda = \mu = 0$ (respectively, $\nu = 0$). Thus $\mathfrak{g}$ is not of type E_6. A similar reasoning shows that $\mathfrak{g}$ is not of type F_4.

If $\mathfrak{g}$ is of type E_7 then $f(u,v,w) = u^3v + v^3 + w^2$. Therefore,

$$(u^3 + 3v^2)\{u,v\} + 2w\{u,w\},\ 2w\{v,w\} - 3u^2\{u,v\} \in I_0^2.$$

A quick look at Table 7.3 yields $\{u,v\} \in I \cdot I_0$, $\{u,w\} \equiv \lambda\psi_5 \pmod{I \cdot I_0}$ and $\{v,w\} \equiv \mu\psi_6 \pmod{I \cdot I_0}$. Computing modulo $I_0^2 + I^2 \cdot I_0$ we get $\lambda = \mu = 0$. Hence $\mathfrak{g}$ is not of type E_7.

If $\mathfrak{g}$ is of type E_8 then $f(u,v,w) = u^5 + v^3 + w^2$ whence

$$3v^2\{u,v\} + 2w\{u,w\},\ 2w\{v,w\} - 5u^4\{u,v\} \in I_0^2.$$

From Table 7.3 we get $\{u,v\} \in I \cdot I_0$. Computing modulo $I_0^2 + I^2 \cdot I_0$ we deduce that $\{u,w\}, \{v,w\} \in I \cdot I_0$ as well. If $\mathfrak{g}$ is of type G_2 then it follows from Table 7.3 that $\{u,v\}, \{u,w\}, \{v,w\} \in I \cdot I_0$.

We have proved that the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ is nonzero on $\mathbb{C}[\mathfrak{c}]/I_0$ in all cases.

7.8. We are now in a position to prove the main result of this section.

Proposition. *The Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ is nonzero on $\mathbb{C}[\mathfrak{c}]/I_0$. Moreover, there exist free homogeneous generators x, y, z of $\mathbb{C}[\mathfrak{c}]/I_0$ such that f has the normal form of a rational double point with respect to x, y, z and the values $\{x,y\}$, $\{x,z\}$, $\{y,z\}$ are as in Table 7.5.*

Proof. According to (7.7), the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ is nonzero on $\mathbb{C}[\mathfrak{c}]/I_0$. Being a polynomial algebra, $\mathbb{C}[\mathfrak{c}]/I_0$ is a unique factorisation domain.

Suppose $\mathfrak{g}$ is of type A_l. Then $x'^{l+1} + y'z' = 0$ hence $z'\{x',y'\} + y'\{x'.z'\} = y'\{y',z'\} - (l+1)x'^l\{x',y'\} = 0$. In particular, $y' \mid \{x',y'\}$. Since y' and $\{x',y'\}$ have the same degree we must have $\{x',y'\} = \lambda y'$ for some $\lambda \in \mathbb{C}^*$. This implies $\{x',z'\} = -\lambda z'$ and $\{y',z'\} = (l+1)\lambda x'^l$. Setting $x = \alpha x'$, $y = \beta y'$, $z = \beta z'$ where $\alpha^{l+1} = \beta^2 = \lambda^{-1}$ we achieve $\lambda = 1$. For $\mathfrak{g}$ of type B_l, one argues similarly to obtain that after a suitable linear substitution, $\{x,y\} = y$, $\{x,z\} = -z$ and $\{y,z\} = (2l+1)\lambda x^{2l}$.

Suppose $\mathfrak{g}$ is of type C_l or D_{l-1} where $l \geq 3$. In this case $x'^l + x'y'^2 + z'^2 = 0$ yielding $2x'y'\{x',y'\} + 2z'\{x',z'\} = 2z'\{y',z'\} - (lx'^{l-1} + y'^2)\{x',y'\} = 0$. It follows that $z' \mid \{x',y'\}$. Since $\{x',y'\}$ and z' have the same degree (see Table 7.3) we get $\{x',y'\} = 2\lambda z'$ for some $\lambda \in \mathbb{C}^*$. Then $\{x',z'\} = -2\lambda x'y'$ and $\{y',z'\} = \lambda(lx'^{l-1} + y'^2)$.

Setting $x = \alpha x'$, $y = \beta y'$, $z = \gamma z'$ with $\alpha = (4\lambda^2)^{-1}$, $\beta^2 = \alpha^{l-1}$ and $\gamma^2 = \alpha^l$ we achieve $\lambda = 1$. For $\mathfrak{g}$ of type G_2, we argue as in C_3-case to obtain that after a suitable linear substitution, $\{x, y\} = 2z$, $\{x, z\} = 2xy$, $\{y, z\} = 3x^2 + y^2$.

Suppose $\mathfrak{g}$ is of type E_6 or F_4. Then $x'^4 + y'^3 + z'^2 = 0$ forcing $3y'^2\{x', y'\} + 2z'\{x', z'\} = 2z'\{y', z'\} - 4x'^3\{x', y'\} = 0$. This implies $z' \mid \{x', y'\}$. According to Table 7.3, z' and $\{x', y'\}$ have the same degree. Therefore, $\{x', y'\} = 2\lambda z'$ for some $\lambda \in \mathbb{C}^*$. This, in turn, yields $\{x', z'\} = -3\lambda y'^2$ and $\{y', z'\} = 4\lambda x'^3$. Setting $x = \alpha x'$, $y = \beta y'$, $z = \gamma z'$ with $\alpha^4 = \beta^3 = \gamma^2$ and $\alpha = (8\lambda^3)^{-1}$ we achieve $\lambda = 1$.

A similar argument shows that for $\mathfrak{g}$ of type E_7, $\{x', y'\} = 2\lambda z'$, $\{x', z'\} = -\lambda(x'^3 + 3y'^2)$, $\{y', z'\} = \{y', z'\} = 3\lambda x'^2 y'$, while for $\mathfrak{g}$ of type E_8, $\{x', y'\} = 2\lambda z'$, $\{x', z'\} = 3\lambda y'^2$, $y', z'\} = 5\lambda x'^4$ where $\lambda \in \mathbb{C}^*$. In both cases, a suitable linear substitution of the form $x = \alpha x'$, $y = \beta y'$, $z = \gamma z'$ yields $\lambda = 1$. This completes the proof. □

7.9. Suppose $\mathfrak{g}$ is not of type A_1. We can combine [27, Theorem B] with information contained in Tables 7.5 and 7.3 to get more insight into the structure of the graded Lie algebra $\mathfrak{z}_\chi$. Indeed, looking at the tables and taking into account Propositions 6.5(i) and 7.8 it is not hard to observe that $\mathfrak{z}_\chi$ contains a graded central Lie subalgebra $\mathfrak{z}'$ of dimension $l - 1$ such that $\mathfrak{h} := \mathfrak{z}_\chi/\mathfrak{z}'$ is solvable. Moreover, if $\mathfrak{g}$ has type A_l or B_l then $\mathfrak{h}^{(1)}$ is abelian and has codimension 1 in $\mathfrak{h}$. If $\mathfrak{g}$ is not of type A_l or B_l then $\mathfrak{h}$ is isomorphic to a three-dimesional Heisenberg Lie algebra. If $\mathfrak{g}$ has type B_l or G_2 then $\mathfrak{z}_\chi \cong \mathfrak{z}' \oplus \mathfrak{h}$ is a split extension. In all other cases the extension is nonsplit. If $\mathfrak{g}$ is not of type G_2 then $\mathfrak{z}'$ coincides with the centre of $\mathfrak{z}_\chi$. The degrees of the graded components of $\mathfrak{z}'$ are $d_1 - 2, \ldots, d_{l-1} - 2$ (see Table 7.3).

8. Properties of the centre: the modular case

8.1. From now on we assume that G is a simple, simply connected algebraic group over K and $\mathfrak{g} = \operatorname{Lie} G$. We assume that $p = \operatorname{char} K$ is a very good prime for the root system $R = R(G, T)$ and adopt the notation introduced in Section 3. As before, U^k stands for the kth component of the standard filtration of $U = U(\mathfrak{g})$, and we denote by $m_1, \ldots, m_l$ are the exponents of the Weyl group $W = N_G(T)/Z_G(T)$. Let $\mathfrak{t} = \operatorname{Lie} T$ and $l = \dim \mathfrak{t}$.

The centre Z of U has two distinguished unital subalgebras: the p-centre Z_p of $\mathfrak{g}$ (generated by all $x^p - x^{[p]}$ with $x \in \mathfrak{g}$) and the invariant algebra U^G. According to Kac and Weisfeiler [20], the Harish-Chandra homomorphism induces an isomorphism $U^G \xrightarrow{\sim} S(\mathfrak{t})^W$ (see also [17, Theorem 9.3]). By [39, (II, 3.17′)], the Chevalley Restriction Theorem holds in our situation. In particular, $S(\mathfrak{t})^W \cong S(\mathfrak{g})^G$ as graded algebras (we identify $K[\mathfrak{g}]$ with $S(\mathfrak{g})$ via the Killing isomorphism induced by the trace form Φ). Combining this isomorphism with the results of Demazure [6] one observes that there exist $\tilde{f}_1 \in U^G \cap U^{m_1+1}, \ldots, \tilde{f}_l \in U^G \cap U^{m_l+1}$ such that the elements $f_i := \operatorname{gr} \tilde{f}_i$ with $1 \le i \le l$ form a free generating set for $S(\mathfrak{g})^G$. Let

$$f : \mathfrak{g}^* \longrightarrow \mathbb{A}^l, \quad \xi \mapsto (f_1(\xi), \ldots, f_l(\xi))$$

denote the adjoint quotient map.

It was discovered by Veldkamp under the assumption that p does not divide the order of W that Z is a free Z_p-module of rank p^l with basis consisting of all $\tilde{f}_1^{a_1} \cdots \tilde{f}_l^{a_l}$ with $0 \le a_k \le p - 1$ for all k (see [43, (3.1)]). This was generalised by Donkin to the case where p is good for R (see [8, (3.3)]). Another proof of Veldkamp's theorem valid under our assumptions on p and G was recently obtained by Mirković and Rumynin in [28].

8.2. Given $\chi \in \mathfrak{g}^*$ we denote by Z_χ the image of Z under the canonical homomorphism $U \twoheadrightarrow U_\chi$. The proof of our next theorem relies on the results of [8].

Theorem. *For any* $\chi \in \mathfrak{g}^*$ *we have* $\dim Z_\chi = p^l$.

Proof. (a) Following [8] we denote by A the coordinate ring of the algebraic group G. Clearly, G acts on A by conjugation. For $1 \le k \le l$ we let ρ_k denote the rational representation of G with highest weight $\varpi_k \in X^+(T)$ (as usual, $X^+(T) = \mathbb{N}_0\varpi_1 \oplus \cdots \oplus \mathbb{N}_0\varpi_l$ stands for the set of all dominant weights of T). Since G is simply connected the trace functions $\mathrm{tr}\rho_1, \ldots \mathrm{tr}\rho_l$ freely generate the invariant algebra $J = A^G$ (see [41, Theorem 3.4.2]).

Given a K-algebra C we denote by $C^{(p)}$ the subalgebra of pth powers of elements of C. Clearly, $A^{(p)} \subset A^{\mathfrak{g}}$, the algebra of $\mathfrak{g}$-invariants. By the main result of [8], multiplication in A induces a G-equivariant isomorphism of K-algebras

$$J \otimes_{J^{(p)}} A^{(p)} \xrightarrow{\sim} A^{\mathfrak{g}}.$$

Together with the preceding remark this implies that $A^{\mathfrak{g}}$ is a free $A^{(p)}$-module of rank p^l. Using earlier results of Koppinen [22, Ch. 4] (see also [15, (11.11)]) Donkin proved in [8, (3.1)] that $A^{\mathfrak{g}}$ is a direct summand of the $A^{(p)}$-module A, that is

$$A = A^{\mathfrak{g}} \oplus A', \qquad A^{(p)} \cdot A' \subseteq A'.$$

Let $\mathfrak{m}$ denote the maximal ideal of A consisting of all regular functions vanishing on $1 \in G$. The function algebra $A_1 = K[G_1]$ on the first Frobenius kernel G_1 of G is nothing but $A \otimes_{A^{(p)}} K \cong A/A\,\mathfrak{m}^{(p)}$. Let $\phi : A \twoheadrightarrow A_1$ denote the canonical homomorphism. Since $A\,\mathfrak{m}^{(p)} = A^{\mathfrak{g}}\,\mathfrak{m}^{(p)} \oplus A'\,\mathfrak{m}^{(p)}$, our discussion above shows that $\dim \phi(J) = p^l$.

(b) Since G is a smooth variety the completion

$$\hat{A} = \varprojlim A/\mathfrak{m}^i$$

with respect to $\mathfrak{m}$ is isomorphic to the formal power series algebra in $\dim \mathfrak{g}$ variables (see [34, Ch. II, (2.2)] for example). Let ι denote the natural map from A into $\hat{A}$. Since A is a domain, ι is an embedding (by the Krull Intersection Theorem). According to [8, (3.4)] the short exact sequence of G-modules $0 \to \mathfrak{m}^2 \to \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 = \mathfrak{g}^* \to 0$ splits. From this it follows that there is a G-equivariant isomorphism $\delta : \hat{A} \xrightarrow{\sim} K[[\mathfrak{g}]]$, where $K[[\mathfrak{g}]]$ stands for the completion of the polynomial algebra $K[\mathfrak{g}] \cong S(\mathfrak{g}^*)$ with respect to its ideal $\mathfrak{n}$ generated by linear forms. We denote by $\hat{\mathfrak{m}}$ and $\hat{\mathfrak{n}}$ the maximal ideals of $\hat{A}$ and $K[[\mathfrak{g}]]$, respectively. The isomorphism δ maps $\hat{\mathfrak{m}}$ onto $\hat{\mathfrak{n}}$, hence $\hat{A}\,\hat{\mathfrak{m}}^{(p)}$ onto $K[[\mathfrak{g}]]\,\hat{\mathfrak{n}}^{(p)}$. As a consequence, δ induces a G-equivariant isomoprphism $\bar{\delta} : \hat{A} \otimes_{\hat{A}^{(p)}} K \xrightarrow{\sim} K[[\mathfrak{g}]] \otimes_{K[[\mathfrak{g}]]^{(p)}} K$.

Let

$$\bar{S} = K[\mathfrak{g}] \otimes_{K[\mathfrak{g}]^{(p)}} K \cong K[\mathfrak{g}]/K[\mathfrak{g}]\,\mathfrak{n}^{(p)},$$

a graded truncated polynomial algebra in $\dim \mathfrak{g}$ variables. It follows from [9, Theorem 7.2(a)] that multiplication induces G-equivariant isomorphisms $A \otimes_{A^{(p)}} \hat{A}^{(p)} \cong \hat{A}$ and

$K[\mathfrak{g}] \otimes_{K[\mathfrak{g}]^{(p)}} K[[\mathfrak{g}]]^{(p)} \cong K[[\mathfrak{g}]]$. As a consequence,

$$\hat{A} \otimes_{\hat{A}^{(p)}} K \cong A \otimes_{A^{(p)}} \hat{A}^{(p)} \otimes_{\hat{A}^{(p)}} K \cong A \otimes_{A^{(p)}} K = A_1$$

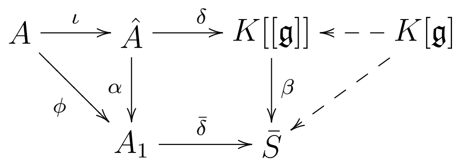
and

$$K[[\mathfrak{g}]] \otimes_{K[[\mathfrak{g}]]^{(p)}} K \cong K[\mathfrak{g}] \otimes_{K[\mathfrak{g}]^{(p)}} K[[\mathfrak{g}]]^{(p)} \otimes_{K[[\mathfrak{g}]]^{(p)}} K \cong K[\mathfrak{g}] \otimes_{K[\mathfrak{g}]^{(p)}} K = \bar{S},$$

yielding G-epimorphisms $\alpha : \hat{A} \to A_1$ and $\beta : K[[\mathfrak{g}]] \to \bar{S}$. By [8, (3.1)], there is a left regular G-submodule Q of A such that $Q \cong A_1$ as left regular G_1-modules and $A \cong Q \otimes A^{(p)}$ as left regular G-modules (in particular, A is a free $A^{(p)}$-module). Moreover, the latter isomorphism is induced by multiplication in A. This yields $\alpha \circ \iota = \phi$.

Let $\hat{\mathfrak{n}}_k$ denote the ideal of $K[[\mathfrak{g}]]$ consisting of all formal power series with initial form of degree at least k. It is not hard to observe that for k sufficiently large, $\hat{\mathfrak{n}}_k \subseteq K[[\mathfrak{g}]]\hat{\mathfrak{n}}^{(p)}$. This implies that the restriction of β to $K[\mathfrak{g}] \subset K[[\mathfrak{g}]]$ is surjective and $\beta(K[[\mathfrak{g}]]^G) = \beta(K[\mathfrak{g}]^G)$.

In view of the above remarks the diagram



is commutative. Since $\dim \phi(J) = p^l$, by (a), we have $\dim \alpha(\hat{A}^G) \geq p^l$. Since δ and $\bar{\delta}$ are G-isomorphisms we also have $\dim \beta(K[[\mathfrak{g}]]^G) \geq p^l$. The equality $\beta(K[[\mathfrak{g}]]^G) = \beta(K[\mathfrak{g}]^G)$ now shows that $\dim \beta(K[\mathfrak{g}]^G) \geq p^l$.

(c) By [8, (3.4)], the algebra $K[\mathfrak{g}]^{\mathfrak{g}}$ is a free $K[\mathfrak{g}]^{(p)}$-module with basis

$$B = \{f_1^{a_1} \cdots f_l^{a_l} \,|\, 0 \leq a_i \leq p-1\}.$$

Therefore, the image of $K[\mathfrak{g}]^G \subset K[\mathfrak{g}]^{\mathfrak{g}}$ in $\bar{S} = K[\mathfrak{g}] \otimes_{K[\mathfrak{g}]^{(p)}} K$ has dimension at most p^l. Combining this with our final remark in (b) we deduce that the image of B in $\bar{S}$ is a linearly independent set.

Let $\tilde{B} = \{\tilde{f}_1^{a_1} \cdots \tilde{f}_l^{a_l} \,|\, 0 \leq a_i \leq p-1\}$, a free basis of the Z_p-module Z (see (8.1)). It follows from the PBW theorem that the standard filtration of U induces a filtration of U_χ such that the associated graded algebra $\mathrm{gr}(U_\chi)$ is isomorphic to $\bar{S}$. If the image of $\tilde{B}$ in U_χ is a linearly dependent set then so is the image of $\mathrm{gr}(\tilde{B})$ in $\bar{S}$. However, $\mathrm{gr}(\tilde{B}) = B$ by our discussion in (8.1). Thus the image of $\tilde{B}$ in U_χ must be linearly independent. On the other hand, Veldkamp's theorem implies that Z_χ is spanned by the image of $\tilde{B}$ in U_χ. So $\dim Z_\chi = p^l$ completing the proof. □

8.3. In this subsection, we combine Theorem 8.2 with [28, Theorem 10] to obtain an explicit description of the algebra Z_χ. Let $\chi = \chi_s + \chi_n$ be the Jordan decomposition of χ (see [20]).

Let R_+ be a positive system in R and $\{h_i \,|\, 1 \leq i \leq l\} \cup \{e_\alpha \,|\, \alpha \in R_+\} \cup \{f_\alpha \,|\, \alpha \in R_+\}$ a Chevalley basis of $\mathfrak{g}$. Note that $\mathfrak{t} = \mathrm{Lie}\, T$ is spanned by $h_1, \ldots, h_l$. Let $\mathfrak{n}_+$ (respectively, $\mathfrak{n}_-$) be the subalgebra of $\mathfrak{g}$ spanned by all e_α (respectively, f_α).

Let $\chi \in \mathfrak{g}^*$ and $g \in G$. As usual, we denote by I_χ the ideal of U generated by all $x^p - x^{[p]} - \chi(x)^p$ with $x \in \mathfrak{g}$. It is well-known (and easy to see) that g sends I_χ onto

$I_{g\cdot\chi}$, hence induces an isomorphism between U_χ and $U_{g\cdot\chi}$ (here $g\cdot\chi = \chi\circ g^{-1}$). Since $g(Z) = Z$ this isomorphism maps Z_χ onto $Z_{g\cdot\chi}$. In particular, $Z_\chi \cong Z_{g\cdot\chi}$ as algebras. By [20], there is $g \in G$ such that $g\cdot\chi$ vanishes on $\mathfrak{n}_+$ and χ_s vanishes on $\mathfrak{n}_\pm$. Thus we may assume without loss of generality that χ vanishes on $\mathfrak{n}_+$ and χ_s vanishes on $\mathfrak{n}_\pm$. Let $\tilde{\chi}_s : S(\mathfrak{t}) \to K$ be the algebra homomorphism such that $\tilde{\chi}_s(h_i) = \chi_s(h_i)^p$ for $1 \le i \le l$.

Set $\Lambda_{\chi_s} = \{\lambda \in \mathfrak{t}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi_s(h)^p \text{ for all } h \in \mathfrak{t}^*\}$. Note that $\Lambda_{\chi_s} = \lambda + \Lambda$ where $\Lambda = \{\lambda \in \mathfrak{t}^* \mid \lambda(h_i) \in \mathbb{F}_p \text{ for } 1 \le i \le l\}$ and λ is an arbitrary element in Λ_{χ_s}. For $\eta \in \Lambda_{\chi_s}$ let W_η, $W_{\eta+\Lambda}$, and $C_{\chi_s,\eta}$ denote the stabiliser of η, the set-wise stabiliser of $\eta + \Lambda$, and the partial coinvariant algebra $S(\mathfrak{t})^{W_\eta} \otimes_{S(\mathfrak{t})^{W_{\eta+\Lambda}}} K_{\tilde{\chi}_s}$, respectively. Let

$$C_{\chi_s} = \bigoplus_{\lambda \in W\Lambda_{\chi_s}/W} C_{\chi_s,\lambda}.$$

Proposition. *(i) For any $\chi \in \mathfrak{g}^*$, $Z_\chi \cong C_{\chi_s}$ as algebras.*

(ii) Let $\chi, \chi' \in \mathfrak{g}^*$ be such that $f(\chi) = f(\chi')$. Then $Z_\chi \cong Z_{\chi'}$.

(iii) The image of Z in the restricted enveloping algebra $U^{[p]}(\mathfrak{g})$ is isomorphic to the coinvariant algebra $C_0 = \bigoplus_{\lambda\in\Lambda/W} S(\mathfrak{t})^{W_\lambda} \otimes_{S(\mathfrak{t})^W} K$.

Proof. It follows from Veldkamp's theorem that the algebra $Z \otimes_{Z_p} K_\chi$ has dimension p^l (the homomorphism $Z_p \to K_\chi$ is induced by the map $x^p - x^{[p]} \mapsto \chi(x)^p$). Since $x^p - x^{[p]} - \chi(x)^p \in I_\chi$ for all $x \in \mathfrak{g}$ the natural homomorphism $Z \to Z_\chi$ induces an epimorphism $Z \otimes_{Z_p} K_\chi \to Z_\chi$. This epimorphism must be injective by Theorem 8.2. Thus $Z_\chi \cong Z \otimes_{Z_p} K_\chi$. So it follows from [28, Theorem 10] that $Z_\chi \cong C_{\chi_s}$ as algebras.

In proving (ii) we may assume that χ and χ' vanish on $\mathfrak{n}_+$ and χ_s and χ'_s vanish on $\mathfrak{n}_\pm$. So we may (and will) identify χ_s and χ'_s with linear functions on $\mathfrak{t}$. There is a 1-dimensional torus $\mu \subset T$ acting on $\mathfrak{n}_+$ with positive weights. Since $f : \mathfrak{g}^* \longrightarrow \mathbb{A}^l$ is homogeneous and μ-equivariant, the equality $f(\chi) = f(\chi')$ implies $f(\chi_s) = f(\chi'_s)$. By the Chevalley Restriction Theorem, χ_s and χ'_s are conjugate under the action of W on $\mathfrak{t}^*$ (see (8.1)). This implies $C_{\chi_s} = C_{\chi'_s}$ yielding $Z_\chi \cong Z_{\chi'}$.

Finally, suppose $\chi_s = 0$. Then $\tilde{\chi}_s = 0$, $\Lambda_{\chi_s} = \Lambda$ and $W_{\eta+\Lambda} = W$ for any $\eta \in \Lambda_{\chi_s}$. As a consequence, $Z_0 \cong \bigoplus_{\lambda\in\Lambda/W} S(\mathfrak{t})^{W_\lambda} \otimes_{S(\mathfrak{t})^W} K$ completing the proof. □

8.4. I would like to finish this paper by sketching an elementary proof of Theorem 8.2 for $\mathfrak{g} = \mathfrak{gl}(n, K)$. In principle, this proof generalises to all types but for exceptional Lie algebras it is more complicated and employs a modular version of [25].

Theorem. *Let $\mathfrak{g} = \mathfrak{gl}(n, K)$ and $\chi \in \mathfrak{g}^*$ (no restriction on $p = \operatorname{char} K$). Let Z_χ be as in (8.2). Then* $\dim Z_\chi = p^n$.

Proof. Let e_{ij} denote the matrix units in $\mathfrak{g}$ and $X = \sum_{i,j}^n x_{ij}e_{ij}$. For $1 \le k \le n$ let σ_k denote the sum of the diagonal $k \times k$ minors of the matrix X. By the Chevalley Restriction Theorem, the invariant algebra $S(\mathfrak{g}^*)^G$ is freely generated by $\sigma_1, \ldots, \sigma_n$ (viewed as polynomial functions on $\mathfrak{g}$). Since the bilinear form $(x, y) \mapsto \operatorname{tr} xy$ on $\mathfrak{g}$ is nondegenerate and G-invariant we may identify $\mathfrak{g}$ with $\mathfrak{g}^*$, hence $S(\mathfrak{g}^*)$ with $S(\mathfrak{g})$. Let I denote the ideal of $S(\mathfrak{g}^*)$ generated by all x_{ij}^p and $\bar{S} = S(\mathfrak{g}^*)/I$, a truncated polynomial algebra in n^2 variables.

It follows from [2, Sect. 3] that Z is a free Z_p-module generated by the set $\{\tilde{f}_1^{a_1}\cdots\tilde{f}_n^{a_n}\,|\,0\le a_i\le p-1\}$ where $\tilde{f}_1,\dots,\tilde{f}_n\in U^G$ are such that $\operatorname{gr}\tilde{f}_i=\sigma_i$ for $1\le i\le n$. Therefore, in order to prove that $\dim Z_\chi=p^n$ it suffices to establish that the image of $\{\sigma_1^{a_1}\dots\sigma_n^{a_n}\,|\,0\le a_i\le p-1\}$ in $\bar{S}$ is a linearly independent set (see part (c) of the proof of Theorem 8.2 for more detail). Since $\sigma_i^p\in I$ for all i the latter is equivalent to showing that the image of $\sigma_1^{p-1}\cdots\sigma_n^{p-1}$ in $\bar{S}$ is nonzero.

Let $\{y_{i,j}\,|\,1\le i,j\le n,\,i+j\ge n+1\}$ be $n(n+1)/2$ indeterminates and Y the truncated polynomial algebra in $y_{i,j}$ over K. Let $\omega:\bar{S}\longrightarrow Y$ denote the algebra homomorphism such that $\omega(x_{ij})=0$ for $i+j\le n$ and $\omega(x_{ij})=y_{i,j}$ for $i+j\ge n+1$ (we identify each x_{ij} with its image under the canonical homomorphism $S(\mathfrak{g}^*)\to\bar{S}$). Since $\sigma_n=\det X$ we have $\omega(\sigma_n^{p-1})=\pm y_{1,n}^{p-1}y_{2,n-1}^{p-1}\cdots y_{n,1}^{p-1}$. Suppose we have already established that

$$\omega(\sigma_{n-k}^{p-1})\cdots\omega(\sigma_n^{p-1})=\pm\prod_{i+j-n-1\le k}y_{i,j}^{p-1}$$

where $0\le k\le n-2$. Using the relations $y_{i,j}^p=0$ and the fact that σ_{n-k-1} is the sum of the diagonal minors of X of order $n-k-1$ we then deduce that

$$\omega(\sigma_{n-k-1}^{p-1})\cdots\omega(\sigma_n^{p-1})=\pm\prod_{i+j-n-1\le k+1}y_{i,j}^{p-1}.$$

Downward induction on k now yields

$$\omega(\sigma_1^{p-1})\cdots\omega(\sigma_n^{p-1})=\pm\prod_{i,j}y_{i,j}^{p-1}\ne 0,$$

showing that the image of $\sigma_1^{p-1}\cdots\sigma_n^{p-1}$ in $\bar{S}$ is nonzero and thereby completing the proof. □

Remark. Combining Theorem 8.4 with [12, Lemma 6.2] and [2, Sect. 3] it is not hard to generalise Theorem 8.2 and Proposition 8.3 to the case where G is as in (2.6) and the trace form Φ is nondegenerate.

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