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The Quadratic Eigenvalue Problem*

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Abstract. We survey the quadratic eigenvalue problem, treating its many applications, its mathematical properties, and a variety of numerical solution techniques. Emphasis is given to exploiting both the structure of the matrices in the problem (dense, sparse, real, complex, Hermitian, skew-Hermitian) and the spectral properties of the problem. We classify numerical methods and catalogue available software.

Key words. quadratic eigenvalue problem, eigenvalue, eigenvector, λ-matrix, matrix polynomial, second-order differential equation, vibration, Millennium footbridge, overdamped system, gyroscopic system, linearization, backward error, pseudospectrum, condition number, Krylov methods, Arnoldi method, Lanczos method, Jacobi–Davidson method

AMS subject classification. 65F30

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1. Introduction.

On its opening day in June 2000, the 320-meter-long Millennium footbridge over the river Thames in London (see Figure 1.1) started to wobble alarmingly under the weight of thousands of people; two days later the bridge was closed. To explain the connection between this incident and the quadratic eigenvalue problem (QEP), the subject of this survey, we need to introduce some ideas from vibrating systems. A natural frequency of a structure is a frequency at which the structure prefers to vibrate. When a structure is excited by external forces whose frequencies are close to the natural frequencies, the vibrations are amplified and the system becomes unstable. This is the phenomenon of resonance. The external forces on the Millennium Bridge were pedestrian-induced movements. On the opening day, a large crowd of people initially walking randomly started to adjust their balance to the bridge movement, probably due to high winds on that day. As they walked, they became more synchronized with each other and the bridge started to wobble even more. The lateral vibrations experienced on the bridge occurred because some of its natural modes of vibration are similar in frequency to the sideways component of pedestrian footsteps on the bridge.[1] The connection with this survey is that the natural modes and frequencies of a structure are the solution of

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[1]More details on the Millennium Bridge, its construction, and the wobbling problem can be found at http://www.arup.com/MillenniumBridge.

Fig. 1.1 *The Millennium footbridge over the river Thames.*

an eigenvalue problem that is quadratic when damping effects are included in the model.

The QEP is currently receiving much attention because of its extensive applications in areas such as the dynamic analysis of mechanical systems in acoustics and linear stability of flows in fluid mechanics. The QEP is to find scalars λ and nonzero vectors x, y satisfying

$$(\lambda^2 M + \lambda C + K)x = 0, \quad y^*(\lambda^2 M + \lambda C + K) = 0,$$

where M, C, and K are $n \times n$ complex matrices and x, y are the right and left eigenvectors, respectively, corresponding to the eigenvalue λ. A major algebraic difference between the QEP and the standard eigenvalue problem (SEP),

$$Ax = \lambda x,$$

and the generalized eigenvalue problem (GEP),

$$Ax = \lambda Bx,$$

is that the QEP has $2n$ eigenvalues (finite or infinite) with up to $2n$ right and $2n$ left eigenvectors, and if there are more than n eigenvectors they do not, of course, form a linearly independent set.

QEPs are an important class of nonlinear eigenvalue problems that are less familiar and less routinely solved than the SEP and the GEP, and they need special attention. A major complication is that there is no simple canonical form analogous to the Schur form for the SEP or the generalized Schur form for the GEP.

A large literature exists on QEPs, spanning the range from theory to applications, but the results are widely scattered across disciplines. Our goal in this work is to gather together a collection of applications where QEPs arise, to identify and classify their characteristics, and to summarize current knowledge of both theoretical and algorithmic aspects. The type of problems we will consider in this survey and their spectral properties are summarized in Table 1.1.

The structure of the survey is as follows. In section 2 we discuss a number of applications, covering the motivation, characteristics, theoretical results, and algorithmic issues. Section 3 reviews the spectral theory of QEPs and discusses important classes of QEPs coming from overdamped systems and gyroscopic systems. Several

Table 1.1 *Matrix properties of QEPs considered in this survey with corresponding spectral properties. The first column refers to the section where the problem is treated. Properties can be added: QEPs for which M, C, and K are real symmetric have properties $P3$ and $P4$ so that their eigenvalues are real or come in pairs $(\lambda, \bar{\lambda})$ and the sets of left and right eigenvectors coincide. Overdamped systems yield QEPs having the property $P6$. In $P6$, $\gamma(M,C,K) = \min\{(x^*Cx)^2 - 4(x^*Mx)(x^*Kx) : ||x||_2 = 1\}$. Gyroscopic systems yield QEPs having at least the property $P7$.*

	Matrix properties	Eigenvalue properties	Eigenvector properties
P1 §3.1	M nonsingular	$2n$ finite eigenvalues	
P2 §3.1	M singular	Finite and infinite eigenvalues	
P3 §3.1	M, C, K real	Eigenvalues are real or come in pairs $(\lambda, \bar{\lambda})$	If x is a right eigenvector of λ then $\bar{x}$ is a right eigenvector of $\bar{\lambda}$
P4 §3.8	M, C, K Hermitian	Eigenvalues are real or come in pairs $(\lambda, \bar{\lambda})$	If x is a right eigenvector of λ then x is a left eigenvector of $\bar{\lambda}$
P5 §3.8	M Hermitian positive definite, C, K Hermitian positive semidefinite	$\mathrm{Re}(\lambda) \leq 0$	
P6 §3.9	M, C symmetric positive definite, K symmetric positive semidefinite, $\gamma(M,C,K) > 0$	λs are real and negative, gap between n largest and n smallest eigenvalues	n linearly independent eigenvectors associated with the n largest (n smallest) eigenvalues
P7 §3.10	M, K Hermitian, M positive definite, $C = -C^*$	Eigenvalues are purely imaginary or come in pairs $(\lambda, -\bar{\lambda})$	If x is a right eigenvector of λ then x is a left eigenvector of $-\bar{\lambda}$
P8 §3.10	M, K real symmetric and positive definite, $C = -C^T$	Eigenvalues are purely imaginary	

linearizations are introduced and their properties discussed. In this paper, the term *linearization* means that the nonlinear QEP is transformed into a linear eigenvalue problem with the same eigenvalues. Our treatment of the large existing body of theory is necessarily selective, but we give references where further details can be found. Section 4 deals with tools that offer an understanding of the sensitivity of the problem and the behavior of numerical methods, covering condition numbers, backward errors, and pseudospectra.

A major division in numerical methods for solving the QEP is between those that treat the problem in its original form and those that linearize it into a GEP of twice the dimension and then apply GEP techniques. A second division is between methods for dense, small- to medium-size problems and iterative methods for large-scale problems. The former methods compute all the eigenvalues and are discussed in section 5, while the latter compute only a selection of eigenvalues and eigenvectors and are reviewed in section 6. The proper choice of method for particular problems is discussed. Finally, section 7 catalogues available software, and section 8 contains further discussion and related problems.

We shall generally adopt the Householder notation: capital letters A, B, C, ... denote matrices, lower case roman letters denote column vectors, Greek letters denote scalars, $\bar{\alpha}$ denotes the conjugate of the complex number α, A^T denotes the transpose of the complex matrix A, A^* denotes the conjugate transpose of A, and $\|\cdot\|$ is any vector norm and the corresponding subordinate matrix norm. The values taken by

any integer variable are described using the colon notation: "$i = 1{:}n$" means the same as "$i = 1, 2, \ldots, n$." We write $A > 0$ ($A \geq 0$) if A is Hermitian positive definite (positive semidefinite) and $A < 0$ ($A \leq 0$) if A is Hermitian negative definite (negative semidefinite). A definite pair (A, B) is defined by the property that $A, B \in \mathbb{C}^{n\times n}$ are Hermitian and

$$\min_{\substack{z\in\mathbb{C}^n \\ \|z\|_2=1}} \sqrt{(z^*Az)^2 + (z^*Bz)^2} > 0,$$

which is certainly true if $A > 0$ or $B > 0$.

2. Applications of QEPs. A wide variety of applications require the solution of a QEP, most of them arising in the dynamic analysis of structural mechanical, and acoustic systems, in electrical circuit simulation, in fluid mechanics, and, more recently, in modeling microelectronic mechanical systems (MEMS) [28], [155]. QEPs also have interesting applications in linear algebra problems and signal processing. The list of applications discussed in this section is by no means exhaustive, and, in fact, the number of such applications is constantly growing as the methodologies for solving QEPs improve.

2.1. Second-Order Differential Equations. To start, we consider the solution of a linear second-order differential equation

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t), \tag{2.1}$$

where M, C, and K are $n \times n$ matrices and $q(t)$ is an nth-order vector. This is the underlying equation in many engineering applications. We show that the solution can be expressed in terms of the eigensolution of the corresponding QEP and explain why eigenvalues and eigenvectors give useful information.

Two important areas where second-order differential equations arise are the fields of mechanical and electrical oscillation. The left-hand picture in Figure 2.1 illustrates a single mass-spring system in which a rigid block of mass M is on rollers and can move only in simple translation. The (static) resistance to displacement is provided by a spring of stiffness K, while the (dynamic) energy loss mechanism is represented by a damper C; $f(t)$ represents an external force. The equation of motion governing this system is of the form (2.1) with $n = 1$.

As a second example, we consider the flow of electric current in a simple RLC circuit composed of an inductor with inductance L, a resistor with resistance R, and a capacitor with capacitance C, as illustrated on the right of Figure 2.1; $E(t)$ is the input voltage. The Kirchhoff loop rule requires that the sum of the changes in potential around the circuit must be zero, so

$$L\frac{di(t)}{dt} + Ri(t) + \frac{q(t)}{C} - E(t) = 0, \tag{2.2}$$

where $i(t)$ is the current through the resistor, $q(t)$ is the charge on the capacitor, and t is the elapsed time. The charge $q(t)$ is related to the current $i(t)$ by $i(t) = dq(t)/dt$. Differentiation of (2.2) gives the second-order differential equation

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dE(t)}{dt}, \tag{2.3}$$

which is of the same form as (2.1), again with $n = 1$.

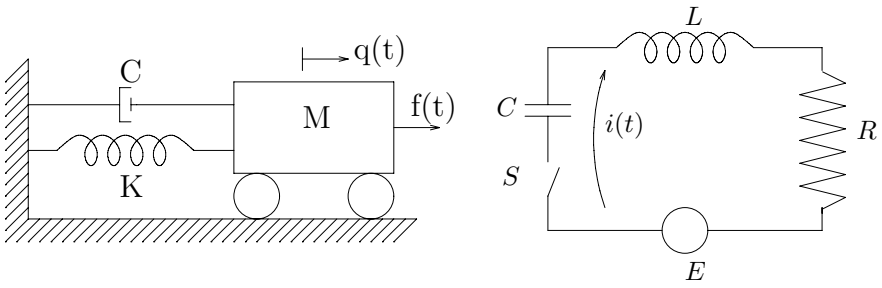

Fig. 2.1 *Left: Single mass-spring system. Right: A simple electric circuit.*

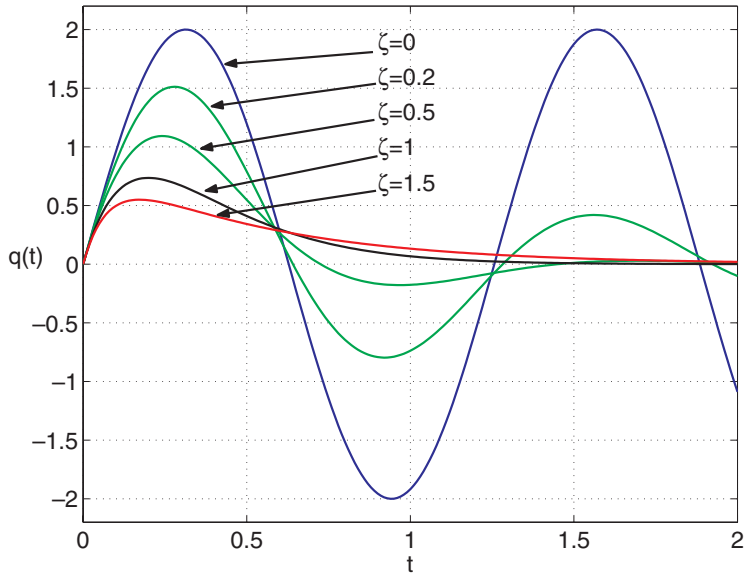

Fig. 2.2 *Response of the single degree of freedom mass-spring system of Figure* 2.1 *for different values of the viscous damping factor* $\zeta = C/2M\omega$ *and initial conditions* $q(0) = 0$, $\dot{q}(0) = 10$.

Let us briefly discuss the solution of the homogeneous equation (2.1) with $f = 0$ and $n = 1$, rewritten in the form, for $M \neq 0$,

$$\ddot{q}(t) + 2\zeta\omega\dot{q}(t) + \omega^2 q(t) = 0, \quad \omega = \sqrt{\frac{K}{M}}, \quad \zeta = \frac{C}{2M\omega}. \tag{2.4}$$

In engineering, ω is called the *natural frequency* and ζ is a dimensionless quantity known as the *viscous damping factor*. The general solution to (2.4), also called the *free response*, is

$$q(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t},$$

where $\lambda_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega$ are the roots of $\lambda^2 + 2\zeta\omega\lambda + \omega^2 = 0$ and α_1, α_2 are constants determined by the initial conditions $q(t_0)$, $\dot{q}(t_0)$. Figure 2.2 illustrates the behavior of the solution for different values of the viscous damping factor ζ. For $\zeta = 0$, the system governed by (2.4) is *undamped* and executes simple harmonic oscillation

with natural frequency ω. For $0 < \zeta < 1$, referred to as the *underdamped* case, the solution is a sine wave with a decaying amplitude. For $\zeta > 1$ the solution is a decaying exponential; this is the *overdamped* case. The case $\zeta = 1$ is known as *critical damping* and corresponds to a double root ($\lambda_1 = \lambda_2$). This is the smallest amount of damping for which no oscillation occurs in the free response $q(t)$.

The properties of systems of second-order differential equations (2.1) for $n \geq 1$ have been analyzed in some detail by Lancaster [87] and more recently by Gohberg, Lancaster, and Rodman [62]. For simplicity, we examine the case where all the eigenvalues λ_i, $i = 1\colon 2n$, of $(\lambda^2 M + \lambda K + C)x = 0$ are distinct. The case of nondistinct eigenvalues is considered in section 3.7. We write

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n}), \quad X = [x_1, x_2, \ldots, x_{2n}], \quad Y = [y_1, y_2, \ldots, y_{2n}], \tag{2.5}$$

where x_i and y_i are right and left eigenvectors, respectively, corresponding to λ_i. In this case the general solution to the homogeneous equation (2.1) ($f(t) = 0$) can be written as

$$q(t) = \sum_{k=1}^{2n} \alpha_k x_k e^{\lambda_k t} = X e^{\Lambda t} a, \tag{2.6}$$

where $a = [\alpha_1, \ldots, \alpha_{2n}]^T$ is a vector of arbitrary constants.

If an eigenvalue has positive real part, $\|q(t)\|$ can grow exponentially toward infinity. If the $2n$ eigenvalues satisfy $\operatorname{Re}(\lambda_k) < 0$, $k = 1\colon 2n$, then the homogeneous second-order differential equation is stable, that is, $\lim_{t\to\infty} \|q(t)\| = 0$ (see section 3.7 for a weaker definition of stability when $\operatorname{Re}(\lambda_k) \leq 0, k = 1\colon 2n$).

Suppose now that we are interested in the response $q(t)$ of a system (2.1) excited by a time harmonic force $f(t) = f_0 e^{i\omega_0 t}$ with frequency ω_0. This can be, for instance, a structure vibrated by a shaker on a test bank. In this case, a particular solution $q_p(t)$ is given by

$$q_p(t) = e^{i\omega_0 t} \sum_{j=1}^{2n} \frac{y_j^* f_0}{i\omega_0 - \lambda_j} x_j, \tag{2.7}$$

provided $i\omega_0$ is not an eigenvalue of $\lambda^2 M + \lambda C + K$ [87, sect. 6.3]. As $i\omega_0$ approaches an eigenvalue λ_j, the jth coefficient $y_j^* f_0/(i\omega_0 - \lambda_j)$ in the expansion (2.7) increases indefinitely provided that $y_j^* f_0 \neq 0$. In such a case, the system is said to approach a *resonance* condition. Depending on the application, the resonance phenomenon might be wanted or unwanted. For the Millennium Bridge it is unwanted. On the other hand, in many electrical engineering applications, such as the tuning of a radio, the interest is in finding an amplitude at resonance that is as large as possible.

How can instability and unwanted resonance be avoided for a given system? The system (2.1) can be controlled with the application of a forcing function or state feedback controller of the form

$$f(t) = -B\big(F_C \dot{q}(t) + F_K q(t) + r(t)\big),$$

where $F_C, F_K \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$, and $r(t) \in \mathbb{C}^m$ with $m \leq n$. In this case, (2.1) is replaced by the closed loop system

$$M\ddot{q}(t) + (C + BF_C)\dot{q}(t) + (K + BF_K)q(t) = -Br(t)$$

and corresponding QEP

$$\big(\lambda^2 M + \lambda(C + BF_C) + K + BF_K\big)x = 0. \tag{2.8}$$

The behavior of the new system can be shaped by selecting the feedback gain matrices F_C and F_K to assign a set of desired eigenvalues to the QEP (2.8) [109]. This is an inverse eigenvalue problem. In the case $m = 1$, the partial pole assignment problem by state feedback control is to find feedback vectors f_C^T and f_K^T such that some of the eigenvalues of (2.8) are prescribed and the others are in the spectrum of $\lambda^2 M + \lambda C + K$. By this means, eigenvalues corresponding to unstable modes or yielding large vibrations can be relocated or damped. This problem has an explicit solution in the case where M, C, and K are symmetric, M is positive definite, and all the eigenvalues are distinct [33].

2.2. Vibration Analysis of Structural Systems—Modal Superposition Method. As mentioned previously, structural mechanics is one of the major areas where second-order differential equations arise. The ultimate goals of vibration analysis are to determine the effect of vibration on the performance and safety of systems and to control this effect. In this section we give some insight into the physics of the problem, the techniques used by engineers to obtain the solution, and the problems engineers are facing. We emphasize the structure and properties of the matrices defining the QEP, as they are important when designing a numerical method for the solution of the eigenvalue problem.

The equations of motion arising in the dynamic analysis of structures discretized by the finite element method are of the form considered in the previous section, namely,

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t), \tag{2.9}$$

where M is now called the *mass matrix*, C is the *viscous damping matrix*, K is the *stiffness matrix*, and $f(t)$ is a time-dependent external force vector. The matrices M and K are related to the kinetic and strain energy, respectively, by a quadratic form, which makes them symmetric; furthermore, for most structures, M and K are positive definite and sparse. Unfortunately, the damping properties of a system are rarely known in the same way as those of the inertia and stiffness, making the damping matrix sometimes difficult to evaluate precisely [29], [30], [134].

In section 2.1 we showed that the general solution to the homogeneous equation is a vital preliminary to subsequent stability and dynamic response analysis of the system and that this solution is given in terms of the solution of the QEP

$$(\lambda^2 M + \lambda C + K)x = 0. \tag{2.10}$$

We assume that M and K are real and $M > 0$ or $K > 0$ (see section 1 for the definition of $M > 0$), so that (M, K) is a real definite pair, which implies the existence of a matrix $X \in \mathbb{R}^{n\times n}$ of eigenvectors such that

$$X^T M X = I, \qquad X^T K X = \Omega^2,$$

where Ω^2 is a nonnegative diagonal matrix containing the inverse of the n eigenvalues of (M, K). Then, premultiplying (2.9) by X^T and letting $q(t) = Xy(t)$ gives the *modal equation*

$$\ddot{y}(t) + X^T C X \dot{y}(t) + \Omega^2 y(t) = X^T f(t). \tag{2.11}$$

When the system is undamped ($C = 0$), (2.11) is a system of uncoupled second-order differential equations that can be solved separately. The complete solution is then obtained by superposing the modal solutions of (2.11) using $q(t) = Xy(t)$. This is referred to as the *modal superposition method*.

When damping is present, the differential equations in (2.11) are coupled unless the damping matrix satisfies the orthogonality conditions

$$x_j^T C x_i = 0, \quad j \neq i,$$

in which case the damping is called *proportional damping*. A general class of matrices giving proportional damping, for nonsingular M, is [30, sect. 13.3]

$$C = \sum_j \alpha_j M(M^{-1}K)^j = \alpha_0 M + \alpha_1 K + \alpha_2 K M^{-1} K + \cdots,$$

in which as many terms can be included as desired. Note that *Rayleigh damping*, where $C = \alpha_0 M + \alpha_1 K$, is a particular case of proportional damping commonly used in practice. Proportional damping assumes that the energy loss mechanism is distributed over the structure in the same way as the mass and the stiffness. With proportional damping the solution is again easily determined by the modal superposition method. Another major advantage of the modal superposition method is that, in general, the response $q(t)$ of a system is largely determined by just a few of the eigenvalues nearest to the real axis (lower modes) so that the superposition process does not require knowledge of all the modes to get a useful estimate of the response $q(t)$. However, there are situations for which ignoring higher modes introduces large error in the calculated response $q(t)$. There is a need for efficient techniques that take into account the effects of higher modes without having to compute all the modes and for estimates of the error due to the truncation.

Assuming proportional damping greatly simplifies the analysis, but this assumption is usually not valid and does not lead to a satisfactory estimate of the response of structures when dynamic effects dominate. Nonproportionally damped systems arise, for example, when new energy-dissipative materials and concentrated damping devices cause a significant variation between energy absorption rates of materials in different parts of the structure. Other nonproportionally damped systems come from soil-structure and fluid-structure interaction problems that involve subsystems with heterogeneous dissipation properties. A specific example of a system that is not classically damped is the idealized nuclear power plant of Figure 2.3, which consists of four elastically interconnected rigid structures that have their own physical properties.

To solve (2.9) for nonproportionally damped systems, one can use a direct time-history integration method [128], but this does not provide the insight into the resonant behavior that is fundamentally important to the design of a vibrating system.

To uncouple the equations in (2.9) and use the modal superposition technique, one needs the eigensolution of the damped system (2.10) [51], [67]. The differential equation (2.9) is reformulated into a first-order $2n$-dimensional equation

$$Ap(t) + B\dot{p}(t) = g(t), \tag{2.12}$$

where, for instance,

$$A = \begin{bmatrix} 0 & I \\ K & C \end{bmatrix}, \quad B = \begin{bmatrix} -I & 0 \\ 0 & M \end{bmatrix}, \quad g(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \quad p(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}. \tag{2.13}$$

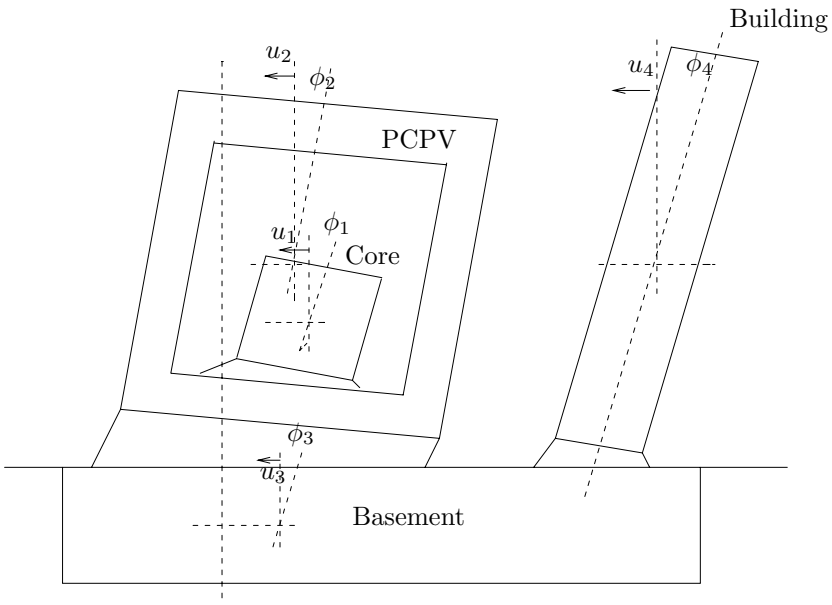

Fig. 2.3 *Nuclear power plant simplified into an eight-degrees-of-freedom system, from* [79]. *The system is composed of four elastically interconnected different types of rigid structures: core, prestressed concrete pressure vessel (PCPV), building, and basement. Each structure has two degrees of freedom that correspond to sway direction u and rocking direction ϕ.*

The solution of the homogeneous equation $Ap(t) + B\dot{p}(t) = 0$ is sought in the form $p(t) = ue^{\lambda t}$, yielding the GEP $Au = \lambda Bu$. If all the eigenvalues are distinct, the pair (A, B) is diagonalizable [125, Prop. 9.2],

$$W^*AU = \operatorname{diag}(\alpha_1, \ldots, \alpha_{2n}), \qquad W^*BU = \operatorname{diag}(\beta_1, \ldots, \beta_{2n}),$$

where W and U are $2n \times 2n$ matrices containing the left and right eigenvectors of (A, B) and $W^* = U^T$ if A and B are real. Let $p(t) = Uz(t)$, $\alpha_i = w_i^*Au_i$, and $\beta_i = w_i^*Bu_i$. Then (2.12) can be reduced to $2n$ decoupled equations

$$\alpha_i z_i(t) + \beta_i \dot{z}_i(t) = w_i^* g(t).$$

The solution $q(t)$ is recovered from $q(t) = [\, I \quad 0 \,]\, Uz(t)$. From a numerical standpoint, since A and B in (2.13) are $2n \times 2n$ matrices, the computation, execution time, and also storage requirement increase compared with the $n \times n$ GEP-based solution of proportionally damped problems. Fortunately, as the response is typically dominated by a relatively small number of the lowest modes, it is not necessary to compute the complete eigensystem of (A, B).

Gyroscopic systems are another important class of nonproportionally damped systems. They correspond to spinning structures where the Coriolis inertia forces are taken into account. These forces are represented by a term $G\dot{q}(t)$ added to (2.9), where G is real and skew-symmetric ($G = -G^T$). Examples of such systems include helicopter rotor blades and spin-stabilized satellites with flexible elastic appendages such as solar panels or antennas (see Figure 2.4). Gyroscopic systems are widely known to exhibit instabilities whose analysis is nontrivial [44], [87], [89].

When the system is subjected to certain forces such as friction or follower forces, a constraint or structural damping matrix D that can be unsymmetric [102] is added to

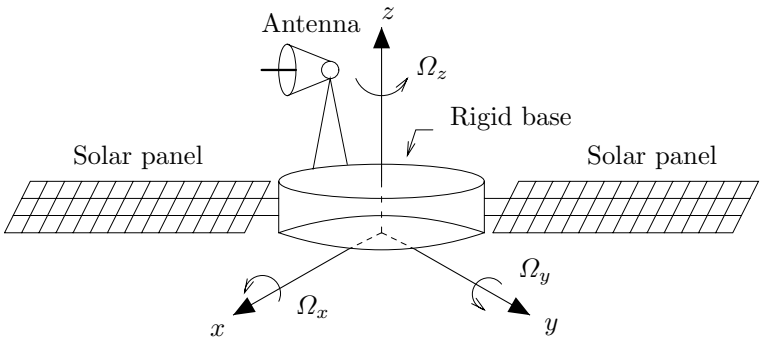

Fig. 2.4 *Spacecraft with deployable antenna and two solar panels attached to a rigid base undergoing angular motion of angular velocity Ω.*

the symmetric stiffness matrix K. Also affecting K is any *hysteretic damping*, which may be modeled by adding a pure imaginary but symmetric matrix. If both Coriolis forces and damping effects are present, the equation of motion takes the general form [154]

$$M\ddot{q}(t) + (C+G)\dot{q}(t) + (K+D)q(t) = f(t). \tag{2.14}$$

How does one prevent resonant oscillation in a building? The basic method is to change its period by reinforcing its structure to make it stiffer [126]. This is often a costly remedy. For the Millennium Bridge, this solution has been discarded by Ove Arup, the engineers who built the bridge, because the lateral stiffness of the bridge would need to be increased by a factor of around 9, which would significantly alter the character of the bridge.[2] The solution to be implemented uses tuned dynamic dampers, which oscillate in the opposite direction to the bridge [34]:

> *The futuristic footbridge, spanning the Thames by St Paul's Cathedral, will be fitted with shock absorbers to reduce the alarming* 10*cm swaying first noticed when a huge crowd surged across at its opening in June. The bridge, designed by Lord Foster and closed three days after its opening, will be fitted with "viscous dampers" and "tuned mass dampers"—likened to shock absorbers—by the bridge's engineers, Ove Arup.*

2.3. Vibro-Acoustics. A great deal of attention has recently been given to fluid-structure interaction problems with the major goal of decreasing the level of noise in aircraft and cars. In this section, we consider the linear oscillations of an acoustic (i.e., inviscid, compressible, barotropic) fluid in a cavity, with reflecting walls and absorbing walls able to dissipate acoustic energy. This can be, for instance, the propagation of sound in a room where one of the walls is coated with an absorbing material (see Figure 2.5). The equations characterizing the wave motion in $\Omega \subset \mathbb{R}^2$ can be taken to be [18], [118]

$$\rho \frac{\partial^2 U}{\partial t^2} + \nabla P = 0, \quad P = -\rho c^2 \operatorname{div} U, \tag{2.15}$$

[2] http://www.arup.com/MillenniumBridge

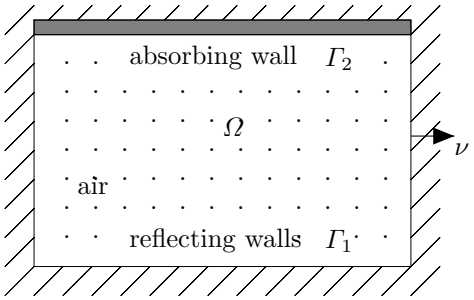

Fig. 2.5 *Fluid in a cavity with one absorbing wall.*

where the acoustic pressure P and the fluid displacement U depend on space x and time t, ρ is the fluid density, and c is the speed of sound in air. The boundary conditions are

$$\text{(2.16)} \qquad \begin{cases} U \cdot \nu = 0 \text{ on } \Gamma_1 & \text{(reflecting boundary)}, \\ \alpha U \cdot \nu + \beta \dfrac{\partial U}{\partial t} \cdot \nu = P \text{ on } \Gamma_2 & \text{(absorbing boundary)}, \end{cases}$$

where ν is the normal to the boundary, and α, β are coefficients related to the normal acoustic impedance (see [18]). Let

$$\mathcal{V} = \left\{ v \in H(\operatorname{div}, \Omega) : \ v \cdot \nu \in L^2(\partial\Omega) \text{ and } v \cdot \nu = 0 \text{ on } \Gamma_1 \right\}.$$

A variational formulation of (2.15)–(2.16) involving only displacement variables, with P and U of the form $P(x,t) = p(x)e^{\lambda t}$ and $U(x,t) = u(x)e^{\lambda t}$, yields the following problem: find $\lambda \in \mathbb{C}$ and $u \in \mathcal{V}$, $u \neq 0$, such that

$$\text{(2.17)} \quad \lambda^2 \int_\Omega \rho u \cdot v + \lambda \int_{\Gamma_2} \beta \ u \cdot \nu \ v \cdot \nu + \int_{\Gamma_2} \alpha \ u \cdot \nu \ v \cdot \nu + \int_\Omega \rho c^2 \operatorname{div} u \operatorname{div} v$$

for all $v \in \mathcal{V}$. This is a quadratic eigenvalue problem for operators. $\lambda = 0$ is one eigenvalue with corresponding eigenspace

$$\mathcal{X} = \{ u \in \mathcal{V} : \operatorname{div} u = 0 \text{ in } \Omega \text{ and } u \cdot \nu = 0 \text{ on } \partial\Omega \}.$$

In the displacement formulation for the fluid, zero eigenvalues are expected and correspond to modes with no physical meaning. After a discretization of (2.17), these modes may not correspond to zero eigenvalues and they are mixed among modes that have a physical meaning. In [18] a discretization based on Raviart–Thomas finite elements is used. This numerical scheme is proved to converge and to be free of spurious modes.

This application raises the problem of truncation error due to the discretization of operators in an infinite-dimensional space. We will not treat this aspect in this paper but will restrict ourselves to the study of QEPs of finite dimension.

2.4. Fluid Mechanics. Eigenvalue problems arise in fluid mechanics in the study of the linear stability of flows. Temporal stability analysis leads to generalized eigenvalue problems [103], [119], whereas spatial stability analysis leads to polynomial eigenvalue problems of degree 2 or higher [22], [71]. The aim in this type of analysis

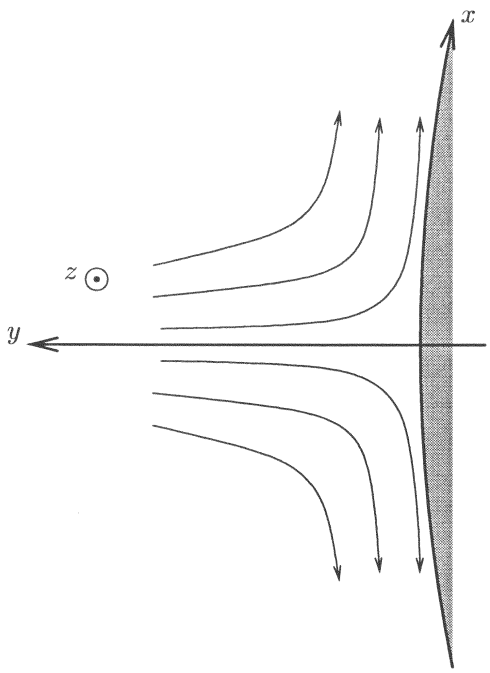

Fig. 2.6 *Coordinate system for attachment-line boundary layer flow.*

is to understand the instability mechanisms that cause transition from laminar to turbulent flows.

As an example, we consider the stability of incompressible attachment-line boundary layers, which have some practical importance in the design of laminar-flow wings [70], [138]. An attachment line is formed on the surface of any cylindrical object immersed in fluid flows, as shown in Figure 2.6. The x-axis is taken to be the chordwise direction, the y-axis is the direction normal to the surface, and the z-axis is in the spanwise direction. For viscous incompressible flow over a swept body, the local solution in the vicinity of the attachment line can be represented by the Hiemenz flow, which is an exact solution of the Navier–Stokes equations given by

$$\bar{q}(x,y) = (x\bar{u}(y)/R,\ \bar{v}(y)/R,\ \bar{w}(y),\ \bar{p}(x))\,,$$

where R is the Reynolds number, $(\bar{u},\bar{v},\bar{w})$ is the flow velocity vector in the xyz-coordinates, and $\bar{p} = -x^2/2R$ denotes the pressure.

To derive the linear stability equations we consider small disturbances propagating along the attachment line. The perturbed Hiemenz flow can be expressed as

$$\tilde{q}(x,y,z,t) = \bar{q}(x,y) + q'(x,y,z,t), \tag{2.18}$$

where $q'(x,y,z,t)$ is the disturbance quantity. From the special structure of the Hiemenz flow, we can assume the separation in the variables z and t so that

$$q'(x,y,z,t) \approx q(x,y)e^{i(\beta z-\omega t)}, \quad q(x,y) = (u,v,w,p). \tag{2.19}$$

Substituting (2.18) together with (2.19) into the incompressible Navier–Stokes equations and linearizing with respect to the small perturbations gives a set of partial

differential equations that describes the stability characteristics of small perturbations:

$$\begin{aligned} -i\omega u + \bar{u}u_x + \bar{v}u_y + \bar{u}_y v + p_x - \frac{1}{R}(u_{xx} + u_{yy}) + i\bar{w}\beta u + \frac{\beta^2}{R}u = 0, \\ -i\omega v + \bar{u}v_x + \bar{v}v_y + \bar{v}_y v + p_y - \frac{1}{R}(v_{xx} + v_{yy}) + i\bar{w}\beta v + \frac{\beta^2}{R}v = 0, \\ -i\omega w + \bar{u}w_x + \bar{v}w_y + \bar{w}_y v - \frac{1}{R}(w_{xx} + w_{yy}) + i\bar{w}\beta w + i\beta p + \frac{\beta^2}{R}w = 0, \\ u_x + v_y + i\beta w = 0. \end{aligned} \tag{2.20}$$

Appropriate boundary conditions must be added [71], [98]. The stability analysis is pursued by discretizing the equations (2.20) using a suitable numerical method such as a spectral collocation method [98], [144] or a finite difference scheme [71]. The discretization leads to

$$(\omega A_1 + A_2 + \beta A_3 + \beta^2 A_4)q = 0, \tag{2.21}$$

with complex coefficient matrices A_i, $i = 1\colon 4$. Note that if a spectral collocation method is used, the resulting matrices are dense, whereas if finite differences are used, the matrices are larger but sparse.

When studying the temporal stability, β is a real and fixed wavenumber. Equation (2.21) is then a GEP with eigenvalue ω. The imaginary part of ω represents the temporal growth rate. For spatial stability, ω is a known, real parameter. Since the unknown β appears quadratically in (2.21), we now have a QEP in β. $\mathrm{Im}(\beta) < 0$ signifies spatially amplified disturbances in the positive direction of z.

2.5. Constrained Least Squares Problem. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and $b \in \mathbb{R}^n$ and consider the constrained least squares problem

$$\min\left\{x^T Ax - 2b^T x : x^T x = \alpha^2\right\}. \tag{2.22}$$

This problem can be reduced to a QEP, as we show now, by applying the method of Lagrange multipliers. Let

$$\phi(x, \lambda) = x^T Ax - 2b^T x - \lambda(x^T x - \alpha^2).$$

Differentiating ϕ with respect to x and then λ yields the equations

$$Ax - \lambda x = b, \quad \alpha^2 = x^T x. \tag{2.23}$$

In [57] it is shown that the smallest solution λ of these equations is needed to solve (2.22). Assuming that λ is not an eigenvalue of A, and setting $y = (A - \lambda I)^{-2}b = (A - \lambda I)^{-1}x$, (2.23) is equivalent to

$$b^T y - \alpha^2 = 0, \tag{2.24}$$

$$(A - \lambda I)^2 y = b. \tag{2.25}$$

From (2.24) we have $(b^T y)/\alpha^2 = 1$, and by expanding (2.25) we get the symmetric QEP

$$\left(\lambda^2 I - 2\lambda A + (A^2 - \alpha^{-2}bb^T)\right) y = 0. \tag{2.26}$$

The solution of (2.22) is $x = (A - \lambda I)^{-1}b$, where λ is the smallest eigenvalue of (2.26).

2.6. Signal Processing. In the linear predictive coding of speech, biomedical signal processing, and time series forecasting, the signal is modeled as an autoregressive (AR) process of the form

$$x_t = -\sum_{k=1}^{p} \alpha_k x_{t-k} + \epsilon_t, \quad t = 1\colon n,$$

where n is the number of data points, p is the order of the process, ϵ_t is zero-mean white noise with variance σ_ϵ, and the x_t, $t = -p+1\colon 0$, are given. The parameters α_k can be estimated by taking $n = p$ and solving the Yule–Walker equations:

$$R\alpha = -r^x,$$

where $r^x = [r_1^x, \ldots, r_p^x]^T$ is a vector whose components $r_k^x = E(x_t, x_{t+k})$ are the autocorrelation function of x_t, $R \in \mathbb{R}^{p\times p}$ is a symmetric Toeplitz matrix with coefficients $R_{ij} = r^x_{|j-i|}$, and $\alpha = [\alpha_1, \ldots, \alpha_p]^T$ is the AR parameter vector. The solution can be computed in $2n^2$ flops using Durbin's algorithm [64].

Often, the signal to be modeled is observed with noise [36], [80]. The observed time series are

$$y_t = x_t + w_t, \quad t = 1\colon n,$$

where w_t is an uncorrelated white observation noise with unknown variance σ_w. In this case the Yule–Walker estimates of the parameters of the model are biased and can produce misleading results. A way to remedy this problem is Davila's subspace approach [36]. Davila used the noise-compensated Yule–Walker equations defined by

$$(S - \lambda B)v = 0, \tag{2.27}$$

where $S \in \mathbb{R}^{(p+n)\times(p+1)}$ with $n \geq p$ is now rectangular, with coefficients $S_{ij} = r^y_{i-j+1}$ defined from the autocorrelation functions of y_t, and

$$B = \begin{bmatrix} 0 & I_p \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(p+n)\times(p+1)},$$

with I_p the $p \times p$ identity matrix. The unknowns are $v = [1, \alpha_1, \ldots, \alpha_p]^T$ and λ, which is an estimate of the unknown variance σ_w^2. Premultiplying (2.27) on the left by $(S - \lambda B)^T$ leads to the $(p+1)\times(p+1)$ symmetric QEP

$$(A_0 + \lambda A_1 + \lambda^2 A_2)v = 0,$$

where $A_0 = S^T S$, $A_1 = -(S^T B + B^T S)$, and $A_2 = B^T B$. The parameter estimates of the AR process are obtained from the eigenvector corresponding to the smallest eigenvalue in modulus.

2.7. MIMO Systems. Suppose that the system (2.9) is controlled by some input function $u(t) \in \mathbb{C}^m$, $m \leq n$, and that there is an output vector $y(t) \in \mathbb{C}^r$, $r \leq n$, that depends linearly on $q(t)$. This dynamical system is modeled by

$$\begin{cases} M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = Bu(t), \\ y(t) = Lq(t), \end{cases} \tag{2.28}$$

where $B \in \mathbb{C}^{n\times m}$ and $L \in \mathbb{C}^{r\times n}$ are input and output influence matrices, respectively. The vector $q(t)$ describes the state of the system. Taking Laplace transforms of

the multiple input, multiple output (MIMO) system (2.28) and assuming zero initial conditions gives

$$\begin{cases} s^2M\bar{q}(s) + sC\bar{q}(s) + K\bar{q}(s) = B\bar{u}(s), \\ \bar{y}(s) = L\bar{q}(s). \end{cases}$$

Clearly $\bar{y}(s) = G(s)\bar{u}(s)$, where $G(s) = L(s^2M + sC + K)^{-1}B$ is called the transfer function matrix since it relates the Laplace transform of the output vector to that of the input vector. When M is nonsingular and all the eigenvalues are distinct, $(s^2M + sC + K)^{-1} = X(sI - \Lambda)^{-1}Y^*$ (see section 3.5), where Λ, X, and Y are the matrices of eigenvalues and right and left eigenvectors, respectively. The eigenvalues of the quadratic polynomial $s^2M + sC + K$ are the poles of $G(s)$. In the case of a single input, single output (SISO) system, $B = b$ and $L^* = l$ are vectors ($r = 1$, $m = 1$) and the transfer function has the form

$$G(s) = \sum_{i=1}^{2n} \frac{(l^*x_i)(y_i^*b)}{s - \lambda_i}. \tag{2.29}$$

Classical control theory is based on the study of $G(s)$. For the frequency response analysis of (2.28), engineers are interested in the values of the transfer function $G(s)$, with s on the imaginary axis, say, $s = i\omega$, in a certain range $\omega \in [a, b]$. In practice, it is not advisable to compute $G(s)$ using the partial fraction representation (2.29) as this form is ill conditioned. When the problem is of reasonable size, a number of methods proposed for the linear case [92], [104] can be extended to the quadratic case with the use of an appropriate linearization of $Q(s) = s^2M + sC + K$.

A typical example of very large problems is in the analysis of interconnected RLC circuits for a very large scale integration (VLSI) chip. Before any numerical model becomes feasible, the original model (2.28) needs to be replaced with a reduced mathematical description that retains sufficient information about the original circuit. This is the aim of model reduction. The dynamical system (2.28) is replaced by the system

$$\begin{cases} \widetilde{M}\ddot{\widetilde{q}}(t) + \widetilde{C}\dot{\widetilde{q}}(t) + \widetilde{K}\widetilde{q}(t) = \widetilde{B}u(t), \\ \widetilde{y}(t) = \widetilde{L}\widetilde{q}(t), \end{cases} \tag{2.30}$$

whose constituent matrices are of smaller dimension and are such that for any admissible input $u(t)$, $\widetilde{y}(t)$ is a good approximation of $y(t)$. Note that the poles of the reduced transfer function $\widetilde{G}(s)$ provide approximations of the eigenvalues of $Q(s)$.

Several approaches can be taken to the model reduction problem, notably model reduction via balanced realization and Hankel norm error estimates [61], [106], Krylov subspace projection by Padé approximation via the Lanczos method (PVL) [46], [9], or multipoint rational interpolation [56], [65], [129], [149]. In contrast with the linear case, we are not aware of any model reduction techniques that actually build the dynamical system (2.30).

3. Spectral Theory. In this section we give a general description of the spectral theory associated with QEPs, beginning with general structure and then moving on to QEPs in which the coefficient matrices have certain properties such as symmetry or skew-symmetry.

3.1. General Background. Throughout the rest of the paper

$$Q(\lambda) = \lambda^2 M + \lambda C + K$$

is an $n \times n$ *matrix polynomial* of degree 2, where M, C, and K are $n \times n$ matrices with complex entries. In other words, the coefficients of the matrix $Q(\lambda)$ are quadratic polynomials in the scalar λ. We will often call $Q(\lambda)$ a λ-*matrix*. We denote by $\Lambda(Q)$ the *spectrum* of $Q(\lambda)$,

$$\Lambda(Q) = \{\lambda \in \mathbb{C} : \det Q(\lambda) = 0\},$$

that is, the set of eigenvalues of $Q(\lambda)$.

The λ-matrix $Q(\lambda)$ is called *regular* when $\det Q(\lambda)$ is not identically zero for all values of λ, and *nonregular* otherwise. Unless otherwise specified we assume that $Q(\lambda)$ is regular.

The characteristic polynomial is $\det Q(\lambda) = \det(M)\lambda^{2n} + \text{lower order terms}$, so when M is nonsingular, $Q(\lambda)$ is regular and has $2n$ finite eigenvalues. When M is singular the degree of $\det Q(\lambda)$ is $r < 2n$ and $Q(\lambda)$ has r finite eigenvalues, to which we add $2n - r$ infinite eigenvalues. Infinite eigenvalues correspond to the zero eigenvalues of the reverse polynomial $\lambda^2 Q(\lambda^{-1}) = \lambda^2 K + \lambda C + M$. A regular $Q(\lambda)$ may have two distinct eigenvalues having the same eigenvector.

To illustrate, the λ-matrix defined by

$$Q(\lambda) = \begin{bmatrix} \lambda + 1 & 6\lambda^2 - 6\lambda & 0 \\ 2\lambda & 6\lambda^2 - 7\lambda + 1 & 0 \\ 0 & 0 & \lambda^2 + 1 \end{bmatrix},$$

or equivalently by

$$M = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -6 & 0 \\ 2 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K = I,$$

is regular because

$$\det Q(\lambda) = -6\lambda^5 + 11\lambda^4 - 12\lambda^3 + 12\lambda^2 - 6\lambda + 1 \not\equiv 0.$$

There are six eigenpairs (λ_k, x_k), $k = 1\colon 6$, given by

k	1	2	3	4	5	6
λ_k	$1/3$	$1/2$	1	i	$-i$	∞
x_k	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Five of the eigenvalues are finite (they are the roots of $\det Q(\lambda)$) and one of them is infinite. We see that while $\lambda_1 \neq \lambda_2$ we have $x_1 = x_2$. This example illustrates the fact that if a regular $Q(\lambda)$ has $2n$ distinct eigenvalues, then there exists a set of n linearly independent eigenvectors [62, Thm. 3.21], which is a nontrivial generalization of standard results for the SEP and the GEP.

When the coefficient matrices are real, the spectrum of $Q(\lambda)$ is symmetric with respect to the real axis of the complex plane, and therefore the eigenvalues are either

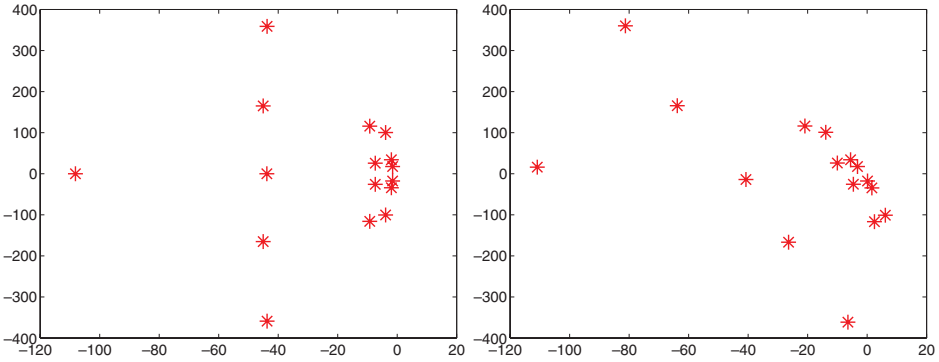

Fig. 3.1 *Spectrum of the eight-degrees-of-freedom simplified nuclear power plant system illustrated in Figure* 2.3. *Left: M, C, and K are real. Right: Hysteretic damping is added so that K is complex.*

real or occur in complex conjugate pairs. For example, the matrices defining $Q(\lambda)$ for the nuclear power plant model in Figure 2.3 as defined in [79] are all real. The symmetry of the spectrum is illustrated on the left side of Figure 3.1. The plot on the right shows the spectrum for the same problem where hysteretic damping is added: the stiffness matrix K is replaced by $(1+i\mu)K$ with hysteretic damping factor $\mu = 0.2$ and is now complex.

The *algebraic multiplicity* of an eigenvalue λ_0 is the order, α, of the corresponding zero in $\det Q(\lambda)$. The *geometric multiplicity* of λ_0, γ, is the dimension of $\mathrm{Ker}(Q(\lambda_0))$. For a *simple* eigenvalue we have $\alpha = \gamma = 1$, and for a *semisimple* eigenvalue we have $\alpha = \gamma$. A *defective* eigenvalue is an eigenvalue that is not semisimple. An eigenvalue of multiplicity $k > n$ is necessarily defective. We say that x_1 is a *generalized eigenvector* associated with λ_0 if x_1 is a solution of the equation [91, Chap. 14]

$$Q(\lambda_0)x_1 = -Q'(\lambda_0)x_0$$

for some eigenvector x_0, where $Q'(\lambda_0) = 2\lambda_0 M + C$. A semisimple eigenvalue is an eigenvalue for which there is no generalized eigenvector. More generally, we say that $x_0, \ldots, x_{m-1}$ form a *Jordan chain* of length m for $Q(\lambda)$ associated with the eigenvalue λ_0 if the following m relations hold:

$$\begin{aligned}
&Q(\lambda_0)x_0 = 0,\\
&Q(\lambda_0)x_1 + Q'(\lambda_0)x_0 = 0,\\
&Q(\lambda_0)x_2 + Q'(\lambda_0)x_1 + \tfrac{1}{2}Q''(\lambda_0)x_0 = 0,\\
&\qquad\qquad\vdots\\
&Q(\lambda_0)x_{m-1} + Q'(\lambda_0)x_{m-2} + \tfrac{1}{2}Q''(\lambda_0)x_{m-3} = 0.
\end{aligned}$$

$x_0 \neq 0$ is an eigenvector and the subsequent vectors $x_1, \ldots, x_{m-1}$ are generalized eigenvectors. In contrast to the Jordan chain of a single matrix A, the vectors x_i, $i = 0\colon m-1$, need not be linearly independent. Furthermore, it is possible for a generalized eigenvector to be the zero vector.

3.2. Division and Factorization. Define

$$Q(S) = MS^2 + CS + K, \qquad S \in \mathbb{C}^{n\times n}. \tag{3.1}$$

Then

$$\begin{aligned} Q(\lambda) - Q(S) &= M(\lambda^2 I - S^2) + C(\lambda I - S) \\ &= (\lambda M + MS + C)(\lambda I - S), \end{aligned} \tag{3.2}$$

which is the *generalized Bézout theorem* [58, p. 81] for matrix polynomials of degree 2. A solution $S \in \mathbb{C}^{n\times n}$ of the quadratic matrix equation $Q(S) = 0$ (if one exists) is called a *solvent* (more precisely, S is a right solvent to distinguish it from a left solvent, which is a solution of $S^2M + SC + K = 0$). Theory on the existence of solvents is given in [40], [73], [74], [91, pp. 520–526]. If S is a solvent, the generalized Bézout theorem gives

$$Q(\lambda) = (\lambda M + MS + C)(\lambda I - S), \tag{3.3}$$

which provides a factorization of $Q(\lambda)$. Note that we cannot interchange the order of the two factors on the right of (3.3) in general, since matrix multiplication is not commutative. Equation (3.3) shows that the eigenvalues of $Q(\lambda)$ are those of the pair $(MS + C, -M)$ together with those of the matrix S, an observation that is the basis of a numerical method described in section 5.

3.3. Canonical Form. Two λ-matrices $P(\lambda)$ and $Q(\lambda)$ of the same dimension are *equivalent* if

$$P(\lambda) = E(\lambda)Q(\lambda)F(\lambda),$$

where $E(\lambda), F(\lambda)$ are λ-matrices with constant nonzero determinants. It follows that the zeros of $\det P(\lambda)$ and $\det Q(\lambda)$ coincide. If $E(\lambda)$ and $F(\lambda)$ are independent of λ then $P(\lambda)$ and $Q(\lambda)$ are said to be *strictly equivalent*.

The Smith theorem provides a canonical (simplest possible) λ-matrix equivalent to $Q(\lambda)$ that displays the invariants that are common to all the λ-matrices equivalent to $Q(\lambda)$. The Smith theorem says that $Q(\lambda)$ is equivalent to a diagonal matrix,

$$Q(\lambda) = E(\lambda)\Gamma(\lambda)F(\lambda), \tag{3.4}$$

where $\Gamma(\lambda) = \operatorname{diag}\big(e_1(\lambda), e_2(\lambda), \ldots, e_n(\lambda)\big)$ and $e_i(\lambda)$ is a monic polynomial such that $e_i(\lambda)$ divides $e_{i+1}(\lambda)$. The diagonal matrix $\Gamma(\lambda)$ is called the Smith form or canonical form of $Q(\lambda)$, and it is unique, though $E(\lambda)$ and $F(\lambda)$ are not. For a proof of the decomposition (3.4) we refer to [58], [62], [150, pp. 19–20]. The polynomials $e_1(\lambda), \ldots, e_n(\lambda)$ are called the *invariant polynomials* of $Q(\lambda)$. To gain some insight into the decomposition (3.4) we consider the case where $Q(\lambda)$ has $2n$ distinct eigenvalues. In this case, one can show (see [87, sect. 3.3], for instance) that $e_i(\lambda) = 1$, $i = 1\colon n-1$, and $e_n(\lambda) = \prod_{i=1}^{2n}(\lambda - \lambda_i)$.

3.4. Linearization. In this section we consider a transformation of a λ-matrix $Q(\lambda)$ analogous to the linearization of a second-order differential equation, that is, its reduction to a first-order equation. On defining $q_0 = q$, $q_1 = \dot{q}_0$, (2.1) is replaced by an equivalent system that has twice as many unknowns $[q_0^T, q_1^T]^T$:

$$\begin{aligned} M\dot{q}_1 + Cq_1 + Kq_0 &= f, \\ q_1 &= \dot{q}_0. \end{aligned}$$

The corresponding transformation for $Q(\lambda) = \lambda^2 M + \lambda C + K$ is to find an equivalent linear λ-matrix $A - \lambda B$. We say that a $2n \times 2n$ linear λ-matrix $A - \lambda B$ is a *linearization*

of $Q(\lambda)$ [62], [91] if

$$\begin{bmatrix} Q(\lambda) & 0 \\ 0 & I_n \end{bmatrix} = E(\lambda)(A - \lambda B)F(\lambda), \tag{3.5}$$

where $E(\lambda)$ and $F(\lambda)$ are $2n \times 2n$ λ-matrices with constant nonzero determinants. Clearly, the eigenvalues of $Q(\lambda)$ and $A - \lambda B$ coincide. A linearization is not unique, and it is important to choose one that respects the symmetry and other structural properties of $Q(\lambda)$, if possible.

Most of the linearizations used in practice are of the first companion form,

$$\text{L1:} \qquad \begin{bmatrix} 0 & N \\ -K & -C \end{bmatrix} - \lambda \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}, \tag{3.6}$$

or the second companion form,

$$\text{L2:} \qquad \begin{bmatrix} -K & 0 \\ 0 & N \end{bmatrix} - \lambda \begin{bmatrix} C & M \\ N & 0 \end{bmatrix}, \tag{3.7}$$

where N can be any nonsingular $n \times n$ matrix. To show that (3.6) is a linearization of $Q(\lambda)$, just multiply (3.6) on the left by $E(\lambda)$ and on the right by $F(\lambda)$, where

$$E(\lambda) = \begin{bmatrix} -(C + \lambda M)N^{-1} & -I \\ N^{-1} & 0 \end{bmatrix}, \quad F(\lambda) = \begin{bmatrix} I & 0 \\ \lambda I & I \end{bmatrix}. \tag{3.8}$$

The easiest way to construct a linearization is to use a substitution such as $u = \lambda x$ in $(\lambda^2 M + \lambda C + K)x = 0$ and rewrite the equation as $\lambda M u + C u + K x = 0$. This yields the GEP

$$\begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0,$$

which corresponds to the first companion form, with $N = I$. The second companion form is obtained by rewriting the equation as $\lambda M u + \lambda C x + K x = 0$.

The choice between (3.6) and (3.7) may depend on the nonsingularity of M and K. In general N is chosen to be the identity matrix or a multiple of the identity matrix such as $\|M\|I$ or $\|K\|I$ [1].

3.5. Inverse of $Q(\lambda)$. This section treats a special case of the general theory of the realization of rational matrix functions [121]. The matrix $Q(\lambda)^{-1}$ is called the *resolvent* of $Q(\lambda)$ and is closely related to the transfer function for a time-invariant linear system.

Let $A - \lambda B$ be a linearization of $Q(\lambda)$. Then there exist nonsingular matrices $E(\lambda)$ and $F(\lambda)$ such that (3.5) holds. Hence,

$$Q(\lambda)^{-1} = [\, I \quad 0 \,]\, F(\lambda)^{-1}(A - \lambda B)^{-1} E(\lambda)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{3.9}$$

For the rest of this section, we take $A - \lambda B$ to be the first companion linearization (3.6) of $Q(\lambda)$ with $N = -K$ for which, using (3.8), we find that (3.9) simplifies to

$$Q(\lambda)^{-1} = -[\, I \quad 0 \,]\, (A - \lambda B)^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Next we obtain an explicit expression for the resolvent in terms of the eigenpairs of $Q(\lambda)$. First, we note that the equations $Q(\lambda_i)x_i = 0$ and $y_i^*Q(\lambda_i) = 0$ are equivalent to $(A - \lambda_i B)\phi_i = 0$ and $\psi_i^*(A - \lambda_i B) = 0$ with

$$\phi_i = \begin{bmatrix} x_i \\ \lambda_i x_i \end{bmatrix} \quad \text{and} \quad \psi_i = \begin{bmatrix} y_i \\ \bar{\lambda}_i y_i \end{bmatrix}. \tag{3.10}$$

We assume that M is nonsingular and that all the eigenvalues are semisimple. The case of infinite and/or defective eigenvalues is considered in section 3.6. Let $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_{2n})$ be the diagonal matrix of eigenvalues of $Q(\lambda)$ and $X = [x_1, \ldots, x_{2n}]$, $Y = [y_1, \ldots, y_{2n}]$ be the corresponding sets of right and left eigenvectors. As the eigenvalues of $Q(\lambda)$ are semisimple, the eigenvalues of $A - \lambda B$ are semisimple too, and the pencil (A, B) is diagonalizable. From the structure of the right and left eigenvectors in (3.10) we see that

$$\Phi = \begin{bmatrix} X \\ X\Lambda \end{bmatrix} \quad \text{and} \quad \Psi = \begin{bmatrix} Y \\ Y\bar{\Lambda} \end{bmatrix}$$

are $2n \times 2n$ matrices containing the corresponding right and left eigenvectors of $A - \lambda B$. We normalize Φ and Ψ so that

$$\Psi^* A \Phi = \Lambda, \quad \Psi^* B \Phi = I.$$

If λ is not an eigenvalue of (A, B) or, equivalently, of $Q(\lambda)$, then it follows that

$$(A - \lambda B)^{-1} = \Phi(\Lambda - \lambda I)^{-1}\Psi^*,$$

so that, from (3.9),

$$Q(\lambda)^{-1} = X(\lambda I - \Lambda)^{-1}Y^* = \sum_{i=1}^{2n} \frac{x_i y_i^*}{\lambda - \lambda_i}. \tag{3.11}$$

3.6. Jordan Triples for $Q(\lambda)$. A Jordan decomposition $A = XJX^{-1}$ of a single matrix A provides a complete and explicit description of a monic polynomial of degree 1,

$$P(\lambda) = \lambda I + A = X(\lambda I + J)X^{-1}.$$

The matrix J is defined by $\operatorname{diag}(J_1, \ldots, J_t)$, where

$$J_k = \begin{bmatrix} \lambda_k & 1 & \cdots & 0 \\ 0 & \lambda_k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_k \end{bmatrix}$$

is $m_k \times m_k$ and $m_1 + \cdots + m_t = n$. The J_k are referred to as *Jordan blocks*. The nonsingular matrix X contains the eigenvectors and generalized eigenvectors of A. The integer m_k is known as a *partial multiplicity* of λ_k. A multiple eigenvalue may have several partial multiplicities. The algebraic multiplicity of an eigenvalue λ is the sum of its partial multiplicities. The geometric multiplicity of λ is the number of partial multiplicities.

The generalization of the Jordan form of a single matrix A to a matrix polynomial of degree 2 includes a Jordan matrix J of size $2n \times 2n$ that contains the eigenvalues and their multiplicities and a matrix X of size $n \times 2n$ that contains the corresponding Jordan chains. In the rest of the section we give a brief summary of Jordan triples for matrix polynomials of degree 2 and their properties; for full details see [62, Chap. 7], [91, Chap. 14].

We assume that M is nonsingular. Let $J = \mathrm{diag}(J_1, \ldots, J_t)$, where J_k, $k = 1\colon t$, is a Jordan block of size m_k and $m_1 + \cdots + m_t = 2n$. Partition the $n \times 2n$ matrix X comformably with J, that is, $X = [X_1, \ldots, X_t]$, where X_k, $k = 1\colon t$, is $n \times m_k$. The columns of $X_k = [x_0^k, \ldots, x_{m_k-1}^k]$ form a Jordan chain of length m_k for $Q(\lambda)$ corresponding to the eigenvalue λ_k. The pair (X, J) is referred to as a *Jordan pair* for $Q(\lambda)$ and is such that $\left[\begin{smallmatrix} X \\ XJ \end{smallmatrix}\right]$ is nonsingular and

$$MXJ^2 + CXJ + KX = 0. \tag{3.12}$$

The left Jordan chains can be obtained from the $2n \times n$ matrix Y defined by

$$Y = \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} M^{-1}.$$

When all the eigenvalues are semisimple, the conjugates of the rows of Y form a set of left eigenvectors for $Q(\lambda)$. The three matrices (X, J, Y) form a *Jordan triple*. X and Y satisfy the *biorthogonality* condition [91, p. 499]

$$\begin{bmatrix} Y & JY \end{bmatrix} \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X \\ XJ \end{bmatrix} = I,$$

or, equivalently,

$$YCX + YMXJ + JYMX = I.$$

We also have

$$XYM = 0 \quad \text{and} \quad XJYM = I. \tag{3.13}$$

The coefficient matrices M, C, and K of $Q(\lambda)$ can be expressed in terms of the Jordan triple (X, J, Y). We have

$$\begin{aligned} M &= (XJY)^{-1}, \\ \begin{bmatrix} K & C \end{bmatrix} &= -MXJ^2 \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1}. \end{aligned}$$

When M is singular, the Jordan pair (X, J) is decomposed into a finite Jordan pair (X_F, J_F) corresponding to the finite eigenvalues and an infinite Jordan pair (X_∞, J_∞) corresponding to the infinite eigenvalues, where J_∞ is a Jordan matrix formed of Jordan blocks with eigenvalue $\lambda = 0$. If $Q(\lambda)$ has r finite eigenvalues, $X_F \in \mathbb{C}^{n\times r}$, $J_F \in \mathbb{C}^{r\times r}$, $X_\infty \in \mathbb{C}^{n\times(2n-r)}$, and $J_\infty \in \mathbb{C}^{(2n-r)\times(2n-r)}$. The matrix

$$\begin{bmatrix} X_F & X_\infty J_\infty \\ X_F J_F & X_\infty \end{bmatrix}$$

is nonsingular and

$$MX_F J_F^2 + CX_F J_F + KX_F = 0, \quad KX_\infty J_\infty^2 + CX_\infty J_\infty + MX_\infty = 0.$$

As an example, consider the quadratic matrix polynomial

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for which $\det Q(\lambda) = (\lambda-1)^3(\lambda+1)$. The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = \lambda_4 = 1$, and $\lambda_5 = \lambda_6 = \infty$. In this case,

$$X_F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J_F = \operatorname{diag}\left(-1, 1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right),$$

and

$$X_\infty = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad J_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In [62, Chap. 7], it was shown that

$$Q(\lambda)^{-1} = [\, X_F \quad X_\infty \,] \begin{bmatrix} \lambda I - J_F & 0 \\ 0 & \lambda J_\infty - I \end{bmatrix}^{-1} \begin{bmatrix} Z_F \\ Z_\infty \end{bmatrix}, \tag{3.14}$$

where

$$\begin{bmatrix} Z_F \\ Z_\infty \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & J_\infty \end{bmatrix} \begin{bmatrix} X_F & X_\infty \\ MX_FJ_F & -KX_\infty J_\infty - CX_\infty \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{3.15}$$

Equation (3.14) reduces to (3.11) when M is nonsingular and the eigenvalues are semisimple.

3.7. Solution of Second-Order Differential Equations. The homogeneous differential equation (2.1) ($f(t) = 0$) or the corresponding λ-matrix $Q(\lambda)$ is said to be *stable* (that is, all solutions $q(t)$ decrease exponentially to *zero* as $t \to \infty$) if and only if $\operatorname{Re}(\lambda) < 0$ for all $\lambda \in \Lambda(Q)$. If $\operatorname{Re}(\lambda) \le 0$ for all $\lambda \in \Lambda(Q)$ and if the eigenvalues λ for which $\operatorname{Re}(\lambda) = 0$ are semisimple, then (2.1) or $Q(\lambda)$ is said to be *weakly stable* [88] in the sense that the solutions are bounded as $t \to \infty$.

Assume that M is nonsingular and let (X, J, Y) be a Jordan triple for $Q(\lambda)$. Using (3.12) it is easy to verify that

$$q(t) = Xe^{Jt}a,$$

where $a \in \mathbb{C}^{2n}$ is a vector of arbitrary constants, is the general solution to the homogeneous differential equation (2.1). We show by differentiation and direct verification that

$$q_p(t) = Xe^{Jt} \int_0^t e^{-Js} Y f(s) ds$$

is a particular solution of (2.1). The Leibniz integral rule and the normalization conditions (3.13) give

$$\dot{q}_p(t) = XJe^{Jt} \int_0^t e^{-Js} Y f(s) ds,$$

$$\ddot{q}_p(t) = XJ^2e^{Jt} \int_0^t e^{-Js} Y f(s) ds + XJY f(t).$$

Hence

$$M\ddot{q}_p(t) + C\dot{q}_p(t) + Kq_p(t) = (MXJ^2 + CXJ + KX)e^{Jt}\int_0^t e^{-Js}Yf(s)ds + MXJYf(t)$$
$$= f(t),$$

using (3.12) and (3.13). We conclude that, when M is nonsingular, every solution of (2.1) can be written in the form

$$q(t) = Xe^{Jt}\Big(a + \int_0^t e^{-Js}Yf(s)ds\Big).$$

When M is singular, the general solution to (2.1) can be given in terms of the finite Jordan pair (X_F, J_F) and infinite Jordan pair (X_∞, J_∞) of $Q(\lambda)$ by

$$q(t) = X_F e^{J_F t}\Big(a + \int_0^t e^{-J_F s}Z_F f(s)ds\Big) + \sum_{k=0}^{\nu-1} X_\infty J_\infty^k Z_\infty f^{(k)}(t),$$

where $Z = \left[\begin{smallmatrix} Z_F \\ Z_\infty \end{smallmatrix}\right]$ is defined as in (3.15), ν is such that $J_\infty^\nu = 0$, the function f is assumed to be $\nu + 1$ times continuously differentiable, and a is an arbitrary vector. For more details on how to derive the general solution of pth-order linear differential equations we refer to [62, Chap. 8].

3.8. Self-Adjoint $Q(\lambda)$. We say that the λ-matrix $Q(\lambda)$ is *self-adjoint* if $Q(\lambda) = Q(\bar{\lambda})^*$ for all $\lambda \in \mathbb{C}$ or, equivalently, if M, C, and K are Hermitian.

3.8.1. Spectrum Location. The eigenvalues of a self-adjoint $Q(\lambda)$ are real or arise in complex conjugate pairs:

$$Q(\lambda)x = 0 \quad \Longleftrightarrow \quad x^*Q(\bar{\lambda}) = 0.$$

Hence x is a right eigenvector of λ and a left eigenvector of $\bar{\lambda}$. It follows that if the matrices are real, then the sets of left and right eigenvectors coincide.

Let $m(x) = x^*Mx$, $c(x) = x^*Cx$, and $k(x) = x^*Kx$, where $x \in \mathbb{C}^n$ is nonzero. We assume that $M > 0$ so that $m(x) > 0$. If x is an eigenvector, the roots of $x^*Q(\lambda)x = 0$ are given by

$$\lambda = \Big(-c(x) \pm \sqrt{c(x)^2 - 4m(x)k(x)}\Big)\Big/2m(x), \tag{3.16}$$

and in general only one of them is an eigenvalue of $Q(\lambda)$.

We observe that for $C > 0$ and $K > 0$, $\mathrm{Re}(\lambda) < 0$, so all the eigenvalues lie in the left half-plane and the system is stable. This is, for instance, the case for the system describing the free vibrations of the nuclear plant problem in Figure 2.3: the matrices M, C, and K are symmetric positive definite and all the eigenvalues have negative real part (see the left side of Figure 3.1).

Following Lancaster [88], we introduce two classes of Hermitian QEPs corresponding to the cases $\Lambda(Q) \subset \mathbb{R}$ and $\Lambda(Q) \cap \mathbb{R} = \emptyset$. A QEP is said to be *hyperbolic* if $c(x)^2 > 4m(x)k(x)$ for all nonzero $x \in \mathbb{C}^n$ and *quasi-hyperbolic* if $c(x)^2 > 4m(x)k(x)$ for all eigenvectors of $Q(\lambda)$ (but not necessarily all nonzero $x \in \mathbb{C}^n$). Immediately, it follows from (3.16) that hyperbolic and quasi-hyperbolic QEPs have real eigenvalues. These QEPs have been thoroughly analyzed [43], [44], [99]. In particular, Duffin showed that the eigenvalues of hyperbolic QEPs are not only real but also semisimple.

A QEP is said to be *elliptic* if $c(x)^2 < 4m(x)k(x)$ for all nonzero $x \in \mathbb{C}^n$ or, equivalently [88], if $c(x)^2 < 4m(x)k(x)$ for all eigenvectors of $Q(\lambda)$. There is no notion of quasi-elliptic in this case. It follows from (3.16) that elliptic QEPs have nonreal eigenvalues.

3.8.2. Linearization. We now consider linearizations for Hermitian QEPs. In many engineering applications, $M > 0$ or $K > 0$. In this case we have the choice between an unsymmetric linearization (A, B) with A non-Hermitian and B Hermitian positive definite and a symmetric linearization with A and B Hermitian but indefinite.

To be more specific, if M is definite, then (3.6) with $N = \pm I$ gives a pair (A, B) with A non-Hermitian and B Hermitian definite; if K is definite an analogous property holds for (3.7). The symmetry of $Q(\lambda)$ is not preserved.

If K (respectively, M) is nonsingular, we obtain a symmetric linearization by taking $N = -K$ in (3.6) (respectively, $N = M$ in (3.7)). Symmetry alone does not guarantee that the eigenvalues are real and that the pencil is diagonalizable by congruence transformations. Veselić [147] shows that the overdamping condition (3.17) is equivalent to the definiteness of the symmetric linearization obtained from L1 or L2. Hence, for overdamped problems (see section 3.9), the symmetric linearization fully reflects the spectral properties of the QEP (real eigenvalues).

3.8.3. Self-Adjoint Triple, Sign Characteristic, and Factorization. There exist many interesting results on self-adjoint matrix polynomials. Their derivation involves matrix analysis in indefinite scalar product spaces [63], [84]. For self-adjoint $Q(\lambda)$, there exist particular Jordan triples (X, J, Y) called *self-adjoint triples*. For such triples, there exists a simple relation between X and Y:

$$Y^* = XP_{\epsilon,J}, \quad X^* = P_{\epsilon,J}^{-1}Y,$$

where $P_{\epsilon,J}$ is a nonsingular matrix whose entries are 0, 1, or -1 and depend on an invariant called the sign characteristic of $Q(\lambda)$ and the Jordan blocks in J [62].

Also, if $M > 0$, $Q(\lambda)$ can always be factorized into a pair of linear factors [62, Thm. 11.2]

$$Q(\lambda) = M(\lambda I - U)(\lambda I - V),$$

with $V = U^*$ when $Q(\lambda)$ is elliptic [91, sect. 13.2].

3.9. Overdamped Systems. In structural mechanics, the differential system (2.1), where M, C, and K are real symmetric, $M > 0$, $C > 0$, and $K \geq 0$, is said to be *overdamped* when the *overdamping condition*

$$\min_{||x||_2=1} \left[(x^*Cx)^2 - 4(x^*Mx)(x^*Kx)\right] > 0 \tag{3.17}$$

is satisfied [43]. Note that if a system is overdamped, the corresponding QEP is hyperbolic. In this case, it is easy to verify from (3.16) that all the eigenvalues are not only real but also nonpositive. This ensures that the general solution to the equation of motion (2.1) is a sum of bounded exponentials. Lancaster [87, sect. 7.6] proves several other properties of overdamped problems:

- there is a gap between the n largest eigenvalues (the primary eigenvalues) and the n smallest eigenvalues (the secondary eigenvalues);
- there are n linearly independent eigenvectors associated with the primary eigenvalues and likewise for the secondary eigenvalues;

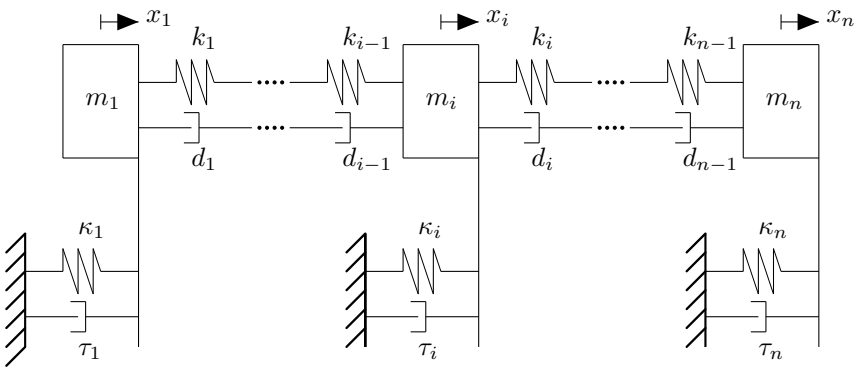

Fig. 3.2 *An n-degrees-of-freedom damped mass-spring system.*

- $Q(S)$ in (3.1) has at least two real solvents, having as their eigenvalues the primary eigenvalues and the secondary eigenvalues, respectively.

As an illustration, we consider the connected damped mass-spring system illustrated in Figure 3.2. The ith mass of weight m_i is connected to the $(i+1)$st mass by a spring and a damper with constants k_i and d_i, respectively. The ith mass is also connected to the ground by a spring and a damper with constants κ_i and τ_i, respectively. The vibration of this system is governed by a second-order differential equation of the form (2.1), where the mass matrix $M = \operatorname{diag}(m_1, \ldots, m_n)$ is diagonal and the damping matrix C and stiffness matrix K are symmetric tridiagonal and defined by

$$C = P \operatorname{diag}(d_1, \ldots, d_{n-1}, 0) P^T + \operatorname{diag}(\tau_1, \ldots, \tau_n),$$
$$K = P \operatorname{diag}(k_1, \ldots, k_{n-1}, 0) P^T + \operatorname{diag}(\kappa_1 \ldots, \kappa_n),$$

with $P = (\delta_{ij} - \delta_{i,j+1})$, where δ_{ij} is the Kronecker delta.

In the following example we take all the springs (respectively, dampers) to have the same constant κ (respectively, τ), except the first and last ones for which $\kappa_1 = \kappa_n = 2\kappa$ and $\tau_1 = \tau_n = 2\tau$, and we take $m_i \equiv 1$. Then

$$M = I, \quad C = \tau \operatorname{tridiag}(-1, 3, -1), \quad K = \kappa \operatorname{tridiag}(-1, 3, -1).$$

We take $n = 50$ degrees of freedom and first choose $\kappa = 5$ and $\tau = 3$. The problem is not overdamped, but, as $M > 0$, $C > 0$, and $K > 0$, the system is stable. All the eigenvalues lie in the left half-plane as shown in Figure 3.3. Second, we take $\kappa = 5$ and $\tau = 10$. Since $\lambda_{\min}(C)^2 - 4\|M\|_2\|K\|_2 = 1.9 \times 10^{-2} > 0$, the system is overdamped, and so all the eigenvalues are real and nonpositive. Figure 3.4 displays the eigenvalue distribution, with the characteristic gap between the n smallest eigenvalues and the n largest.

3.10. Gyroscopic Systems. The λ-matrix associated with a gyroscopic system is of the form

$$G(\lambda) = \lambda^2 M + \lambda C + K \tag{3.18}$$

with M and K Hermitian, $M > 0$, and $C = -C^*$ skew-Hermitian.

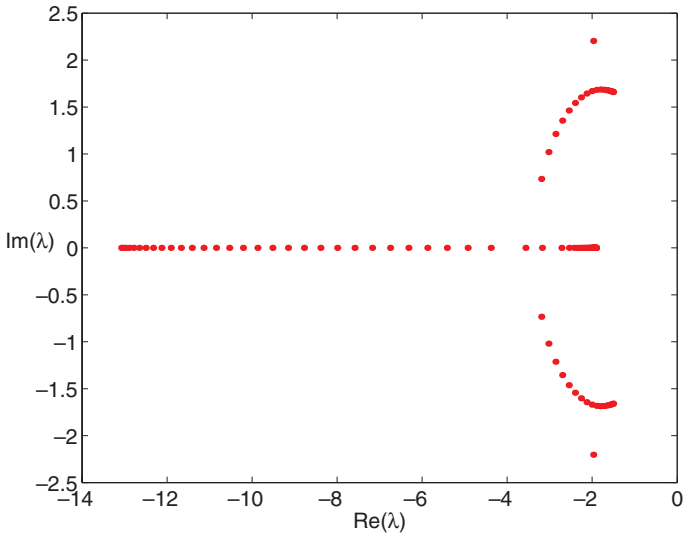

Fig. 3.3 *Eigenvalues in the complex plane of the QEP for the nonoverdamped mass-spring system with* $n = 50$.

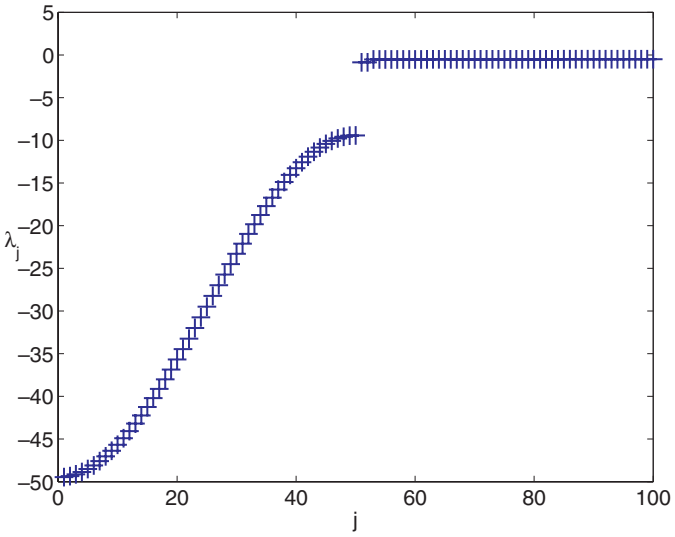

Fig. 3.4 *Eigenvalue distribution of the QEP for the overdamped mass-spring system with* $n = 50$. *Now all the eigenvalues are real, so they are plotted against the index rather than as points in the complex plane.*

3.10.1. Spectrum Location and Stability.

As

$$G(\lambda)^* = G(-\bar{\lambda}), \tag{3.19}$$

the distribution of the eigenvalues of $G(\lambda)$ in the complex plane is symmetric with respect to the imaginary axis. If x is a right eigenvector associated with the eigenvalue λ, then x is a left eigenvector associated with the eigenvalue $-\bar{\lambda}$.

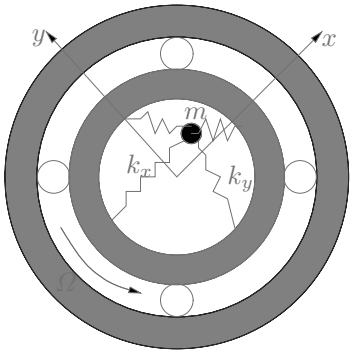

Fig. 3.5 *A two-degrees-of-freedom model of a shaft rotating with angular velocity* Ω.

If M, C, and K are real, then to the property (3.19) is added the property

$$G(\lambda)^T = G(-\lambda).$$

In this case, the eigenvalues of $G(\lambda)$ have Hamiltonian properties; that is, they are symmetrically placed with respect to both the real and imaginary axes. Immediately we conclude that a necessary but not sufficient condition for a real gyroscopic system

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0 \tag{3.20}$$

to be stable is that all its eigenvalues are on the imaginary axis.

Let

$$Q(\lambda) = -G(-i\lambda) = \lambda^2 M + \lambda(iC) - K.$$

Note that $M^* = M > 0$, $(iC)^* = iC$, and $K^* = K$ so that $Q(\lambda)$ is self-adjoint. The case $K > 0$ is well understood [44], [87]. In this case, the discriminant of $x^*Q(\lambda)x$ is positive for all nonzero $x \in \mathbb{C}^n$, and thus $Q(\lambda)$ is hyperbolic (see section 3.8). The eigenvalues of $Q(\lambda)$ are real and semisimple; that is, the eigenvalues of $G(\lambda)$ are purely imaginary and semisimple. Hence, the system (3.19) is weakly stable.

When $K < 0$, (3.20) corresponds to a system in motion about an unstable equilibrium position. If C is real, and hence skew-symmetric, there is a nonzero vector x_0 such that $x_0^*Cx_0 = 0$. The discriminant of $x_0^*Q(\lambda)x_0$ is $4(x_0^*Mx_0)(x_0^*Kx_0) < 0$ so that $Q(\lambda)$ is not hyperbolic. Hence, the eigenvalues of $G(\lambda)$ are not necessarily purely imaginary and semisimple, and the system (3.20) is not guaranteed to be weakly stable.

Some inequalities involving the matrices M, C, and K have been derived to characterize cases where $Q(\lambda)$ is quasi-hyperbolic and the system (3.20) is weakly stable [10]. Strong stability, which refers to systems that are stable and for which all neighboring systems are stable, has been investigated [89]. In many engineering applications the gyroscopic systems depend on a parameter, and stability criteria for these types of problems have also been derived [76].

As an illustration we consider the two-degrees-of-freedom gyroscopic system containing a mass and four springs illustrated in Figure 3.5. The equation of motion in the rotating reference axes can be written in matrix form as

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 0 & -2\Omega \\ 2\Omega & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} k_x/m - \Omega^2 & 0 \\ 0 & k_y/m - \Omega^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0, \tag{3.21}$$

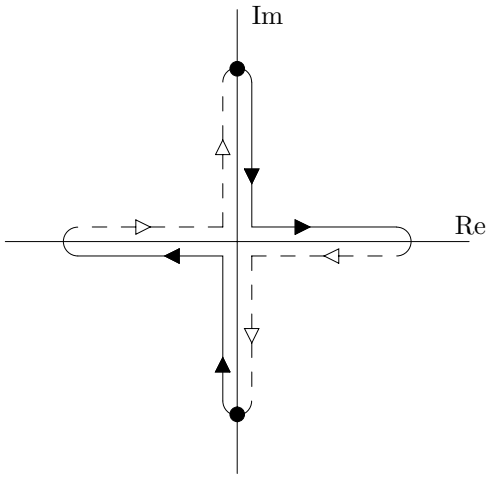

Fig. 3.5 *Location of the spectrum of the gyroscopic system* (3.21) *as the gyroscopic parameter* Ω *increases.*

where m is the mass, k_x, k_y are the stiffnesses of the springs, and Ω is the rotation rate of the system. It is well known that the rotation rate Ω, also called the gyroscopic parameter, characterizes the stability for this class of system. The eigenvalues of the corresponding QEP are given by the zeros of

$$\det Q(\lambda) = \lambda^4 + \lambda^2 \left(2\Omega^2 + \frac{k_x + k_y}{m} \right) + \frac{k_x k_y}{m^2} - \Omega^2 \frac{k_x + k_y}{m} + \Omega^4 = 0,$$

that is,

$$\lambda = \pm \frac{1}{2} \left(-\frac{2(k_x + k_y)}{m} - 4\Omega^2 \pm 2 \left(\frac{(k_x - k_y)^2}{m^2} + 8\Omega^2 \frac{k_x + k_y}{m} \right)^{1/2} \right)^{1/2}.$$

We assume that $k_x \leq k_y$. The two pairs of eigenvalues will be purely imaginary for $0 < \Omega < \sqrt{k_x/m}$ and $\Omega > \sqrt{k_y/m}$. As shown in Figure 3.5, as Ω increases from 0 to $\sqrt{k_x/m}$, the pair of eigenvalues closer to 0 (represented by two black dots in the figure for $\Omega = 0$) coalesces at the origin for $\Omega = \sqrt{k_x/m}$ and splits along the real axis for $\sqrt{k_x/m} < \Omega < \sqrt{k_y/m}$, resulting in instability. However, due to gyroscopic effects, the eigenvalues once again coalesce at the origin for the second critical value of Ω, that is, $\Omega = \sqrt{k_y/m}$. For $\Omega > \sqrt{k_y/m}$ the two pairs of eigenvalues are purely imaginary and the system is said to be *restabilized*.

3.10.2. Linearization. As mentioned previously, the eigenvalues of a real gyroscopic system occur in quadruples $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$, possibly collapsing to real or imaginary pairs or a single zero eigenvalue. A linearization $A - \lambda B$ reflects this property if one of the matrices A or B is Hamiltonian and the other is skew-Hamiltonian. We recall that the matrix A is Hamiltonian if $(AJ)^T = AJ$, where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

and the matrix B is skew-Hamiltonian if $(BJ)^T = -BJ$. A skew-Hamiltonian/Hamiltonian linearization can be obtained from the first companion form L1, and a Hamiltonian/skew-Hamiltonian linearization can be obtained from the second companion form L2 by setting $N = -K$ in (3.6) and $N = M$ in (3.7) and multiplying these linear forms on the right by the anti-identity matrix

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

This results in

$$\text{L3: } \begin{bmatrix} K & 0 \\ C & K \end{bmatrix} - \lambda \begin{bmatrix} 0 & K \\ -M & 0 \end{bmatrix}, \quad \text{L4: } \begin{bmatrix} 0 & -K \\ M & 0 \end{bmatrix} - \lambda \begin{bmatrix} M & C \\ 0 & M \end{bmatrix}. \tag{3.22}$$

If M is singular, L3 is the linearization to use, but if K is singular, L4 is preferred.

4. Perturbation Analysis. The importance of condition numbers for characterizing the sensitivity of solutions to problems and backward errors for assessing the stability and quality of numerical algorithms is widely appreciated. The forward error, condition number, and backward error are related by the inequality (correct to first order in the backward error)

$$\text{forward error} \le \text{condition number} \times \text{backward error}. \tag{4.1}$$

In the following, $\Delta Q(\lambda)$ denotes the perturbation

$$\Delta Q(\lambda) = \lambda^2 \Delta M + \lambda \Delta C + \Delta K$$

of $Q(\lambda)$.

4.1. Conditioning. Let λ be a nonzero simple and finite eigenvalue of a regular $Q(\lambda)$ with corresponding right eigenvector x and left eigenvector y. A normwise condition number of λ can be defined by

$$\kappa(\lambda) = \lim_{\epsilon \to 0} \sup \left\{ \frac{|\Delta\lambda|}{\epsilon|\lambda|} : \big(Q(\lambda+\Delta\lambda) + \Delta Q(\lambda+\Delta\lambda)\big)(x+\Delta x) = 0, \right.$$
$$\left. \|\Delta M\| \le \epsilon\alpha_2,\ \|\Delta C\| \le \epsilon\alpha_1,\ \|\Delta K\| \le \epsilon\alpha_0 \right\}, \tag{4.2}$$

where the α_k are nonnegative parameters that allow freedom in how perturbations are measured—for example, in an absolute sense ($\alpha_k \equiv 1$) or a relative sense ($\alpha_2 = \|M\|$, $\alpha_1 = \|C\|$, $\alpha_0 = \|K\|$). By setting, for instance, $\alpha_2 = 0$ we can force $\Delta M = 0$ and thus keep M unperturbed. In [140] it is shown that

$$\kappa(\lambda) = \frac{(|\lambda|^2\alpha_2 + |\lambda|\alpha_1 + \alpha_0)}{|\lambda|\ |y^* Q'(\lambda) x|} \|y\|\ \|x\|.$$

Condition numbers $\kappa(\lambda)$ are used in eigenstructure assignment problems arising in control design for a second-order system [109]. To measure the robustness of the system, one can take as a global measure

$$\nu^2 = \sum_{k=1}^{2n} \omega_k^2 \kappa(\lambda_k)^2,$$

where the ω_k are positive weights. The control design problem is to select the feedback gains F_C and F_K in (2.8) to assign a given set of $2n$ nondefective eigenvalues to the second-order closed loop system and to minimize ν^2.

A normwise condition number for the eigenvector x corresponding to the simple eigenvalue λ can be defined by

$$\kappa_\lambda(x) = \lim_{\epsilon\to 0}\sup\left\{ \frac{\|\Delta x\|}{\epsilon\|x\|} : \big(Q(\lambda+\Delta\lambda)+\Delta Q(\lambda+\Delta\lambda)\big)(x+\Delta x)=0, \right. \tag{4.3}$$
$$g^*Q'(\lambda)x = g^*Q'(\lambda)(x+\Delta x) \equiv 1,$$
$$\left. \|\Delta M\| \le \epsilon\alpha_2,\ \|\Delta C\| \le \epsilon\alpha_1,\ \|\Delta K\| \le \epsilon\alpha_0 \right\},$$

where we use a linear normalization based on a constant vector g. The normalization in the definition is important, as an eigenvector corresponding to a simple λ is unique only up to a scalar multiple. We can show that for sufficiently small ΔM, ΔC, and ΔK there exists a unique Δx which can be expressed to first order as

$$\Delta x = -V(W^*Q(\lambda)V)^{-1}W^*(\Delta Q(\lambda))x,$$

where the full rank matrices $V, W \in \mathbb{C}^{n\times(n-1)}$ are chosen so that $g^*Q'(\lambda)V = 0$ and $W^*Q'(\lambda)x = 0$. As a consequence,

$$\kappa_\lambda(x) = \|V(W^*Q(\lambda)V)^{-1}W^*\|(|\lambda|^2\alpha_2 + |\lambda|\alpha_1 + \alpha_0). \tag{4.4}$$

The matrices V and W can be explicitly constructed via QR factorizations [72], so $\kappa_\lambda(x)$ in (4.4) is readily computed.

Note that for the perturbation analysis of QEPs it can be more convenient to use the homogeneous form

$$Q(\alpha,\beta) = \alpha^2 M + \alpha\beta C + \beta^2 K,$$

where an eigenvalue is now represented by a pair (α,β) [39]. Infinite eigenvalues correspond to nonzero pairs (α,β) for which $\beta = 0$. With this approach, condition numbers for the eigenvalues can be derived without assuming that the eigenvalue is finite. Moreover, by working in projective spaces the problem of choosing a normalization for the eigenvectors is avoided.

4.2. Backward Error. A natural definition of the normwise backward error of an approximate eigenpair $(\widetilde{x},\widetilde{\lambda})$ of $Q(\lambda)$ is

$$\eta(\widetilde{x},\widetilde{\lambda}) := \min\{\,\epsilon : (Q(\widetilde{\lambda})+\Delta Q(\widetilde{\lambda}))\widetilde{x} = 0, \tag{4.5}$$
$$\|\Delta M\| \le \epsilon\alpha_2,\ \|\Delta C\| \le \epsilon\alpha_1,\ \|\Delta K\| \le \epsilon\alpha_0\,\},$$

and the backward error for an approximate eigenvalue $\widetilde{\lambda}$ is given by

$$\eta(\widetilde{\lambda}) := \min_{\widetilde{x}\neq 0} \eta(\widetilde{x},\widetilde{\lambda}). \tag{4.6}$$

In [140] it is shown that $\eta(\widetilde{x},\widetilde{\lambda})$ is the scaled residual

$$\eta(\widetilde{x},\widetilde{\lambda}) = \frac{\|Q(\widetilde{\lambda})\widetilde{x}\|}{(|\widetilde{\lambda}|^2\alpha_2 + |\widetilde{\lambda}|\alpha_1+\alpha_0)\|\widetilde{x}\|}, \tag{4.7}$$

and if $\widetilde{\lambda}$ is not an eigenvalue of $Q(\lambda)$, then

$$\eta(\widetilde{\lambda})^{-1} = (|\widetilde{\lambda}|^2\alpha_2 + |\widetilde{\lambda}|\alpha_1 + \alpha_0)\|Q(\widetilde{\lambda})^{-1}\|. \tag{4.8}$$

Ideally, a numerical method will provide computed eigenpairs, all of whose backward errors are of the order of the unit roundoff; such a method is called numerically stable.

In most of the numerical methods used to solve the QEP an approximation of x can be recovered from either the first n components or the last n components of a computed $2n$-vector $\widetilde{\xi}^T = (\widetilde{x}_1^T, \widetilde{x}_2^T) \approx (\widetilde{x}^T, \lambda\widetilde{x}^T)$. It is often not appreciated that in finite precision arithmetic these two choices are not equivalent (see the example in section 5.1). One approach, adopted by MATLAB 6's `polyeig` function for solving the polynomial eigenvalue problem and illustrated in Algorithm 5.1 below, is to use whichever part of $\widetilde{\xi}$ yields the smallest backward error (4.7).

4.3. Pseudospectra. Pseudospectra are an established tool for gaining insight into the global sensitivity of the eigenvalues of a matrix to perturbations [142], [143] (see the pseudospectra homepage at http://www.comlab.ox.ac.uk/pseudospectra). The use of pseudospectra is widespread, with applications in areas such as fluid mechanics, Markov chains, and control theory. Most of the existing work is for a single matrix, but the theory can be extended to matrix polynomials [141]. For the QEP, we define the ϵ-pseudospectrum by

$$\begin{aligned}\Lambda_\epsilon(Q) = \{\, \lambda \in \mathbb{C} : (Q(\lambda) + \Delta Q(\lambda))x = 0 \ \text{ for some } x \neq 0 \text{ and } \Delta Q(\lambda) \\ \text{with } \|\Delta M\| \le \epsilon\alpha_2,\ \|\Delta C\| \le \epsilon\alpha_1,\ \|\Delta K\| \le \epsilon\alpha_0 \,\}.\end{aligned} \tag{4.9}$$

In [141] the following equivalent expressions were proved:

$$\begin{aligned}\Lambda_\epsilon(Q) &= \{\, \lambda \in \mathbb{C} : \|Q(\lambda)^{-1}\| \ge (\epsilon\,(|\lambda|^2\alpha_2 + |\lambda|\alpha_1 + \alpha_0))^{-1} \,\} \\ &= \{\, \lambda \in \mathbb{C} : \eta(\lambda) \le \epsilon \,\},\end{aligned}$$

the second equality following from (4.8). $\Lambda_\epsilon(Q)$ can also be related to stability radii [141], which are widely used in control theory to measure the size of the smallest perturbation that changes a stable system into an unstable one. For the computation of $\Lambda_\epsilon(Q)$ we refer to [141], where several numerical methods are proposed.

As an illustration, we consider the eight-degrees-of-freedom ($n = 8$) nuclear power plant system illustrated in Figure 2.3. We use the data given in [79]. Since stability of the system is the key issue, we are interested in the location of the eigenvalues of the perturbed QEP

$$\lambda^2(M + \Delta M)x + \lambda(C + \Delta C)x + (K + \Delta K)x = 0$$

that are the closest to the imaginary axis. For our computation, we consider relative perturbations so that $\alpha_2 = \|M\|_2 \approx 2 \times 10^8$, $\alpha_1 = \|C\|_2 \approx 4 \times 10^{10}$, and $\alpha_0 = \|K\|_2 \approx 2 \times 10^{13}$. Figure 4.1 is a plot of the ϵ-pseudospectra in the form known as a spectral portrait: the inverse of the scaled resolvent norm, $r(z) = \big((|z|^2\alpha_2 + |z|\alpha_1 + \alpha_0)\|Q(z)^{-1}\|_2\big)^{-1}$, is evaluated on a grid of points z in the complex plane, a different color is assigned to each value of $r(z)$, and the resulting array is sent to a color plotter. The eigenvalues of the unperturbed QEP, marked by dots, all lie in the left half-plane, so the unperturbed system is stable. The plot shows that relative perturbations of order 10^{-10} (corresponding to the yellow region of the plot) can move the eigenvalues to the right half-plane, making the system unstable.

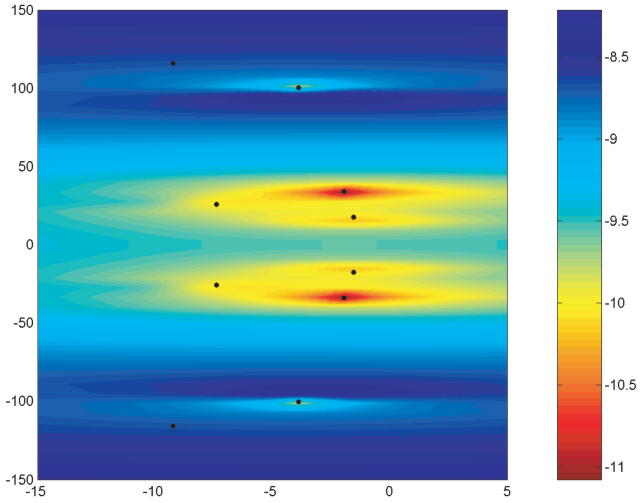

Fig. 4.1 *Spectral portrait of the eight-degrees-of-freedom nuclear power plant system illustrated in Figure* 2.3. *The bar shows the mapping of colors to* $\log_{10}$ *of the reciprocal of the scaled resolvent norm.*

5. Numerical Methods for Dense Problems. The standard approach for the numerical solution of the SEP and the GEP is to reduce the matrices involved to some simpler form that reveals the eigenvalues, for instance, the Schur form for a single matrix A and the generalized Schur form for a pair (A, B). Unfortunately, these canonical forms do not generalize to λ-matrices of degree greater than 1.

Numerical methods break into two types: those that solve the QEP directly and those that work with the linearized form and compute its generalized Schur decomposition or some simple forms revealing eigenvalues and eigenvectors directly. Most of the numerical methods that deal directly with the QEP are variants of Newton's method. These Newton's variants compute one eigenpair at a time and converge as long as the starting guess is close enough to the solution, but in practice even for a good initial guess there is no guarantee that the method will converge to the desired eigenvalue. We refer to Kublanovskaya [85], Peters and Wilkinson [117], and Ruhe [122] and the references therein for a thorough survey of these techniques. Several authors have noted that the factorization (3.3) provides a way to solve the QEP working entirely in terms of $n \times n$ matrices: compute a solvent S of $Q(S) = MS^2 + CS + K = 0$ and then find the eigenvalues of the pencil $\lambda M + MS + C$ and the matrix S. Two obstacles stand in the way of this approach. First, a solvent may not exist, and while existence results are available (see references in section 3.2), their conditions are not easy to check in practice. Second, computation of a solvent is a nontrivial task—one possibly more difficult than that of solving the QEP. Nevertheless, Higham and Kim [74] showed that by using a Bernoulli iteration to compute solvents this technique can be competitive with the others for certain classes of problems, including overdamped problems.

In the rest of this section, we concentrate on methods that compute all the eigenvalues and eigenvectors of $Q(\lambda)$ through one of its linearizations. A drawback of these methods is that they solve a problem of twice the dimension of the original one.

Table 5.1 *Nuclear power plant problem. Backward errors of the computed eigenpairs corresponding to the smallest and largest eigenvalues in modulus.*

$\|\lambda\|$	$\eta_{GEP}(\xi,\lambda)$	$\eta(\xi_1,\lambda)$	$\eta(\xi_2,\lambda)$
17.7	1e-17	3e-5	6e-8
361	2e-17	2e-11	2e-11

5.1. General QEPs. Let $A-\lambda B$ be a linearization of $Q(\lambda)$. A numerically stable reduction is obtained by computing the generalized Schur decomposition

$$W^*AZ = S, \quad W^*BZ = T, \tag{5.1}$$

where W and Z are unitary and S and T are upper triangular. If $Q(\lambda)$ is regular, then $\Lambda(Q) = \{s_{ii}/t_{ii}\}$, with the convention that $s_{ii}/t_{ii} = \infty$ when $t_{ii} = 0$. For real pairs (A, B) there is a real generalized Schur decomposition with Q and Z orthogonal and T upper quasi-triangular. We obtain the following algorithm for the numerical solution of the QEP.

ALGORITHM 5.1.

Form (A, B) such that $A - \lambda B$ is a linearization of $Q(\lambda)$.
Compute the generalized Schur decomposition
$S = W^*AZ$, $T = W^*BZ$.
For $k = 1\colon 2n$
$\lambda_k = s_{kk}/t_{kk}$
Solve $(S - \lambda_k T)\phi = 0$, $\xi = Z\phi$.
$\xi_1 = \xi(1\colon n)$, $\xi_2 = \xi(n+1\colon 2n)$
$r_1 = Q(\lambda_k)\xi_1/\|\xi_1\|$, $r_2 = Q(\lambda_k)\xi_2/\|\xi_2\|$
$x_k = \begin{cases} \xi(1\colon n) \text{ if } \|r_1\| \le \|r_2\|, \\ \xi(n+1\colon 2n) \text{ otherwise.} \end{cases}$
end

This algorithm computes all the eigenpairs. It chooses the part of the eigenvector ξ of $A - \lambda B$ that yields the smallest backward error for the QEP (see section 4.2).

The QZ algorithm [105], [64] computes the decomposition (5.1) and handles the case of infinite eigenvalues well [148]. The QZ algorithm is numerically stable for the solution of the GEP but it is not stable for the solution of the QEP as it does not exploit the special structure of the problem [140]. To illustrate, we consider the QEP associated with the nuclear power system. We use the second companion linearization (3.6) with $N = I$ and the `qz` function of MATLAB 6. Table 5.1 gives the resulting backward errors for the smallest and largest eigenvalues in magnitude (the unit roundoff is of order 10^{-16}), using the 2-norm. Here,

$$\eta_{GEP}(\xi,\lambda) = \|(A - \lambda B)\xi\|/((\|A\| + |\lambda|\|B\|)\|\xi\|)$$

is the backward error of the GEP solution and, as expected, is of the order of the unit roundoff because the QZ algorithm is a backward stable algorithm for the solution of the GEP. The last two columns of the table display the backward errors corresponding to the two possible choices in recovering the eigenvector of the QEP from the eigenvector of the GEP: $\xi_1 = \xi(1\colon n)$, $\xi_2 = \xi(n+1\colon 2n)$. For the smallest eigenvalue in modulus, the second choice ξ_2 yields a smaller backward error. This example shows that even if the algorithm chooses the part of the vector ξ that yields

the smallest backward error, these backward errors can be much larger than the unit roundoff. Even though backward stability is not guaranteed, Algorithm 5.1 is the preferred method if M, C, and K have no particular structure and are not too large.

5.2. Symmetric Linearization. We now assume that the matrices M, C, and K are symmetric and that $A - \lambda B$ is a symmetric linearization of $Q(\lambda)$. Algorithm 5.1 is not attractive, because the QZ algorithm does not respect the symmetry of (A, B).

When B is definite we can compute a Cholesky factorization $B = LL^T$, with L lower triangular, and reduce the symmetric GEP $A\xi = \lambda B\xi$ to the symmetric SEP $L^{-1}AL^{-T}\phi = \lambda\phi$, $\phi = L^T\xi$, which can be solved with the symmetric QR algorithm [64] or any other eigensolver for symmetric matrices. This approach can be unstable when B is ill conditioned, but a few steps of iterative refinement can be used to improve the stability and accuracy [35], [139]. Using complete pivoting in the Cholesky factorization improves the stability properties [35].

If the system is overdamped, the pair (A, B) is symmetric definite and the Jacobi algorithm of Veselić and Slapničar [130], [147] for definite matrix pairs can be used.

If the pencil is indefinite, the HR [21], [23], LR [124], and Falk–Langemeyer [45] algorithms can be employed in order to take advantage of the symmetry of the GEP, but all these methods can be numerically unstable and can even break down completely. There is a need to develop efficient and reliable numerical methods for the solution of symmetric indefinite GEPs.

5.3. Hamiltonian/Skew-Hamiltonian Linearization. Now we assume that the matrices M and K are real symmetric and $C = -C^T$ is skew-symmetric. We recall that such QEPs come from gyroscopic systems. Let (A, B) be a Hamiltonian/skew-Hamiltonian or skew-Hamiltonian/Hamiltonian linearization of $Q(\lambda)$. Instead of using the QZ algorithm we can reduce the pencil (A, B) to a Hamiltonian eigenvalue problem and then use a structure-preserving algorithm for real Hamiltonian matrices, such as the one derived by Benner, Mehrmann, and Xu [17] or the square-reduced algorithm of Van Loan [146]. These two algorithms involve considerably less computation than the QZ and QR algorithms. The structure is preserved by the use of symplectic orthogonal transformations.

The reduction of a Hamiltonian/skew-Hamiltonian or skew-Hamiltonian/Hamiltonian GEP to a Hamiltonian SEP exists as long as the skew-Hamiltonian matrix is nonsingular. For instance, we assume that $A - \lambda B$ is the Hamiltonian/skew-Hamiltonian linearization L4 in (3.22). The reduction to a Hamiltonian eigenproblem uses the fact that when the skew-Hamiltonian matrix B is nonsingular, it can be written in factored form as

$$B = B_1B_2 = \begin{bmatrix} I & \frac{1}{2}C \\ 0 & M \end{bmatrix} \begin{bmatrix} M & \frac{1}{2}C \\ 0 & I \end{bmatrix} \quad \text{with} \quad B_2^T J = JB_1. \tag{5.2}$$

Then $H = B_1^{-1}AB_2^{-1}$ is Hamiltonian. The square-reduced algorithm [146] computes the eigenvalues μ of the skew-symmetric matrix H^2 using stable structure-preserving transformations. The eigenvalues of H are recovered by taking the square roots of μ. As a consequence, there is a loss of accuracy. The algorithm described in [17] is more expensive in both computational cost and workspace but does not suffer from this loss of accuracy in the eigenvalue computation.

To illustrate, we consider a band traveling at speed v between two fixed supports as a model of a band saw or magnetic tape (see Figure 5.1). The band's transverse displacement with no external excitation force is described by the nondimensional

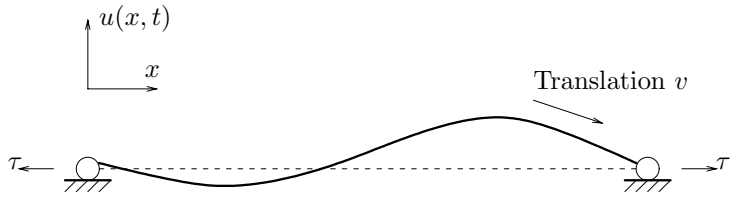

Fig. 5.1 *A schematic view of a traveling band.*

equation

$$\left[\frac{\partial^2}{\partial t^2} + 2v\frac{\partial^2}{\partial x \partial t} + (\kappa v^2 - \tau)\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right] u(x,t) = 0, \quad x \in [0,1],$$

where τ is the dimensionless tension and $\kappa \in [0,1]$ is a constant depending on the pulley mounting system of the band. Approximating

$$u(x,t) = \sum_{k=1}^{n} q_k(t)\sin(k\pi x)$$

and applying the Galerkin method we obtain a second-order differential equation

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0, \tag{5.3}$$

where $q(t) = [\,q_1(t), \ldots, q_n(t)\,]^T$, $M = \frac{1}{2}I_n$, $K = \operatorname{diag}_{1\le j\le n}\left(j^2\pi^2(j^2\pi^2 + \tau - \kappa v^2)/2\right)$, and

$$C = -C^T = (c_{ij}) \quad \text{with} \quad c_{ij} = \begin{cases} \dfrac{4ij}{j^2 - i^2}v & \text{if } i+j \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

We took $n = 20$ and set the band parameters as follows:

$$v = 10, \quad \kappa = 0.8, \quad \tau = 77.9$$

so that as $M > 0$ and $K > 0$ all the eigenvalues are purely imaginary and the gyroscopic system is stable (see section 3.10).

Figure 5.2 compares the spectrum of the QEP associated with (5.3) when computed by the MATLAB 6 function `polyeig` and our implementation of Van Loan's square-reduced algorithm [146]. Not surprisingly, the eigenvalues computed by `polyeig` (which uses a companion linearization and the QZ algorithm) do not have a Hamiltonian structure, and some of them have positive real part, suggesting incorrectly that the system described by (5.3) is unstable. In contrast, Van Loan's square-reduced algorithm together with the Hamiltonian/skew-Hamiltonian linearization preserves the Hamiltonian structure, yielding pure imaginary computed eigenvalues that confirm the system's stability.

5.4. Sensitivity of the Linearization. Condition numbers and backward error are related to the accuracy of the solutions by the inequality (4.1). Most of the algorithms applied to a GEP form of the QEP do not preserve the structure. Hence

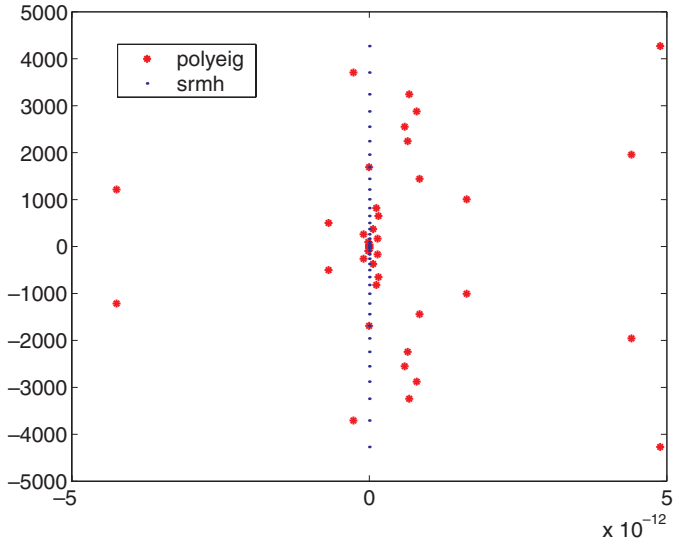

Fig. 5.2 *Spectrum of the moving band illustrated in Figure* 5.1. `polyeig` *uses a companion linearization and the QZ algorithm.* `srmh` *uses a Hamiltonian linearization and the square-reduced algorithm for the Hamiltonian eigenproblem.*

it is the condition number of the GEP form that is relevant. The normwise condition number of a simple and finite eigenvalue of the GEP

$$(A - \lambda B)\xi = 0, \qquad \chi^*(A - \lambda B) = 0 \tag{5.4}$$

given in [72] is

$$\kappa_{GEP}(\lambda) = \frac{\|\chi\|\|\xi\|(\|A\| + |\lambda|\|B\|)}{|\lambda||\chi^* B\xi|}. \tag{5.5}$$

The accuracy with which the eigenvalue λ is computed is affected by the choice of the linearization $A - \lambda B$.

To be more specific, we consider the linearizations L1, L2, and L4 in (3.6), (3.22) with $N = I$. We consider the QEP derived from the mass-spring system described in section 3.9 with $n = 10$, $\tau = 1000$, and $\kappa = 5$. The system is overdamped, with eigenvalues in the range $[-5 \times 10^3, -5 \times 10^{-3}]$. For each linearization, we compute $\kappa_{GEP}(\lambda)$ for $\lambda = \lambda_{10}$ and $\lambda = \lambda_{11}$, where the eigenvalues are ordered so that $\lambda_i > \lambda_{i+1}$. The two eigenvalues λ_{10} and λ_{11} appear on either side of a large gap in the spectrum of the kind seen in Figure 3.4. We used the Symbolic Toolbox of MATLAB to compute the exact solution of (5.4) and the QZ algorithm to compute the approximate eigenvalues $\hat{\lambda}_{10}$ and $\hat{\lambda}_{11}$. The results, displayed in Table 5.2, show that for the same eigenvalue λ, the condition numbers are quite different for the different linearizations. Large condition numbers affect the relative error $|\hat{\lambda} - \lambda|/|\lambda|$ as suggested by the bound (4.1). Tisseur showed [140] that given some information on $\|M\|$, $\|C\|$, and $\|K\|$ and on the structure of ξ and χ, it is possible to compare the condition numbers of different linearizations and identify which formulations are preferred for the large and the small eigenvalues, respectively. These results are of practical relevance, as in applications it is often only the eigenpairs corresponding to small or large eigenvalues that are of interest.

Table 5.2 *Comparison, for three different linearizations, of the condition number and relative error of a small and a large eigenvalue in absolute value of the QEP associated with the mass-spring problem.*

Type of	$\lambda = -8e{-}3$	
linearization	$\kappa(\lambda)$	$\lvert\hat{\lambda}-\lambda\rvert/\lvert\lambda\rvert$
L1	$6e5$	$7e{-}12$
L2	$6e0$	$6e{-}16$
L4	$8e2$	$2e{-}16$

Type of	$\lambda = -1e3$	
linearization	$\kappa(\lambda)$	$\lvert\hat{\lambda}-\lambda\rvert/\lvert\lambda\rvert$
L1	$6e0$	$8e{-}16$
L2	$5e6$	$2e{-}11$
L4	$1e9$	$1e{-}11$

6. Iterative Methods. Methods for dense problems usually destroy any sparsity in A and B. This may lead to large memory requirements and high execution times for QEPs of large dimension. In this section we discuss iterative methods for finding eigenpairs of $Q(\lambda)$ when n is too large to use the methods of section 5. Fortunately, in many large-scale applications, only a few eigenpairs are required, so that computation time and storage can be reduced. For large problems M, C, and K are usually sparse and special data structures are used for their storage. This limits the type of operations one can perform efficiently to matrix-vector products and sparse direct and preconditioned iterative linear solvers. All the methods discussed in this section require the solution of large-scale linear systems, algorithms for which were reviewed in [11], [41], and [42].

Most algorithms for large QEPs proceed by generating a sequence of subspaces $\{\mathcal{K}_k\}_{k\geq 0}$ that contain increasingly accurate approximations to the desired eigenvectors. A projection method is used to extract approximate eigenpairs from the largest $\mathcal{K}_k$.

Let $L(\lambda) \in \mathbb{C}^{N\times N}$ be either $Q(\lambda)$ ($N = n$) or one of its linearizations $A - \lambda B$ ($N = 2n$). The projection method requires a matrix $L(\lambda)$ and a subspace $\mathcal{K}_k$ of dimension $k \leq N$ containing an approximate eigenspace of $L(\lambda)$. It proceeds as follows:

1. Let the columns of $V \in \mathbb{C}^{N\times k}$ be a basis for $\mathcal{K}_k$ and let $W \in \mathbb{C}^{N\times k}$ be such that $W^*V = I$. (V and W are biorthogonal.)
2. Form $L_k = W^*L(\lambda)V$ (projection step).
3. Compute the m desired eigenpairs $(\widetilde{\lambda}_j, \xi_j)$ of L_k, $j = 1\colon m \leq k$.
4. Return $(\widetilde{\lambda}_j, V\xi_j)$ as approximate eigenpairs of $L(\lambda)$ (Ritz pairs).

If the approximate eigenvectors of $L(\lambda)$ and therefore those of $Q(\lambda)$ are not satisfactory, they can be reused in some way to restart the projection method.

Let $\mathcal{L}_k$ be the subspace spanned by the columns of W. If V is orthonormalized, we can take $W = V$ and therefore $\mathcal{L}_k = \mathcal{K}_k$. In this case L_k is the orthogonal projection of $L(\lambda)$ onto $\mathcal{K}_k$. When $W \neq V$, L_k is the oblique projection of $L(\lambda)$ onto $\mathcal{K}_k$ along $\mathcal{L}_k$.

The projection method approximates an eigenvector x of $L(\lambda)$ by a vector $\widetilde{x} = V\xi \in \mathcal{K}_k$ with corresponding approximate eigenvalue $\widetilde{\lambda}$. As $W^*L(\widetilde{\lambda})\widetilde{x} = W^*L(\widetilde{\lambda})V\xi = L_k(\widetilde{\lambda})\xi = 0$, the projection method forces the residual $r = L(\widetilde{\lambda})\widetilde{x}$ to be orthogonal to $\mathcal{L}_k$. This is referred to as the Galerkin condition when $\mathcal{K}_k = \mathcal{L}_k$ and the Petrov–Galerkin condition otherwise. When $L(\lambda)$ is symmetric the projection method is called the Rayleigh–Ritz procedure.

Usually, the projection method does a better job of estimating exterior eigenvalues of $L(\lambda)$ than interior eigenvalues. Prior to any computation one might want to apply spectral transformations that map the desired eigenvalues to the periphery of the

spectrum. For instance, if $L(\lambda) = Q(\lambda)$, the inverted QEP is

$$(\mu^2 K + \mu C + M)x = 0, \quad \text{where} \quad \mu = 1/\lambda = f(\lambda),$$

assuming that 0 is not an eigenvalue of $Q(\lambda)$. This invert spectral transformation is used if eigenvalues with smallest modulus are wanted. The shift-and-invert spectral transformation $f(\lambda) = 1/(\lambda - \sigma)$ and the Cayley spectral transformation $f(\lambda) = (\lambda - \beta)/(\lambda - \sigma)$ (for $\beta \neq \sigma$), used to approximate eigenvalues λ closest to the shift σ, are other possible spectral transformations that are discussed in [7], for example.

In the next section we discuss methods that compute the subspaces $\mathcal{K}_k$ and $\mathcal{L}_k$, beginning with the case where $L(\lambda)$ is linear, i.e., $L(\lambda) = A - \lambda B$, and moving to the case where $L(\lambda) = Q(\lambda)$, i.e., the QEP is tackled directly in its original form.

6.1. Krylov Subspace Methods. When $L(\lambda) = A - \lambda B$ is linear, $\mathcal{K}_k$ and $\mathcal{L}_k$ are often chosen to be Krylov subspaces of the form

$$\mathcal{K}_k(S, v) = \operatorname{span}\{v, Sv, S^2v, S^3v, \ldots, S^{k-1}v\}, \tag{6.1}$$

where v is a given vector and S is a single matrix (for instance, $S = B^{-1}A$) containing the desired eigenvalues in the exterior of its spectrum. This leads to the Lanczos method when S is Hermitian and the Arnoldi or non-Hermitian Lanczos method when S is non-Hermitian. Our aim is to explain how existing Krylov subspace methods can be applied to the QEP. Good background references on Krylov subspace methods are [7], [41], [94], [125].

6.1.1. Arnoldi Method and Two-Sided Lanczos Method for General QEPs. Let $A - \lambda B$ be a linearization of $Q(\lambda)$. The Arnoldi method [4] for the GEP $A - \lambda B$ is an orthogonal projection method onto $\mathcal{K}_k(S, v)$, where $S = B^{-1}A$ if B is nonsingular or $S = A^{-1}B$ if A is nonsingular. The method builds step by step a matrix V_k whose columns are orthonormal and form a basis for the Krylov subspace $\mathcal{K}_k(S, v)$, and an upper Hessenberg projection matrix $H_k = V_k^* S V_k$. Let e_k be the kth column of the identity matrix. The iterations can be represented by

$$SV_k - V_k H_k = v_{k+1} h_{k+1,k} e_k^*, \qquad V_k \in \mathbb{C}^{n\times k}, \quad H_k \in \mathbb{C}^{k\times k}, \tag{6.2}$$

where v_{k+1} is appended to V_k to obtain V_{k+1} and H_k is expanded into H_{k+1} so that its $(k+1, k)$ entry is $h_{k+1,k}$. The columns of V_k are computed by a Gram–Schmidt orthogonalization process. This process does not guarantee orthogonality of the columns of V_k in floating point arithmetic, so reorthogonalization is recommended to improve the numerical stability of the method [32], [135]. Sometimes, B-orthogonalization is used, so that $V_{k+1}^* B V_{k+1} = I$ instead of $V_{k+1}^* V_{k+1} = I$ [94].

The non-Hermitian Lanczos method, also called the two-sided Lanczos method, is an oblique projection method. It produces a non-Hermitian tridiagonal matrix T_k and a pair of matrices V_k and W_k such that $W_k^* V_k = I$ and whose columns form bases for the Krylov subspaces $\mathcal{K}_k(S, v)$ and $\mathcal{K}_k(S^*, w)$, where v and w are starting vectors such that $w^* v = 1$. The matrices V_k, W_k, and T_k satisfy the recurrence relations

$$\begin{aligned} SV_k - V_k T_k &= \gamma_{k+1} v_{k+1} e_k^*, \\ S^* W_k - W_k T_k^* &= \bar{\beta}_{k+1} w_{k+1} e_k^*. \end{aligned}$$

At the next step, we set $V_{k+1} = [V_k, v_{k+1}]$ and $W_{k+1} = [W_k, w_{k+1}]$. The scalars γ_{k+1} and β_{k+1} will be the $(k+1, k)$ and $(k, k+1)$ entries of T_{k+1} [37].

The practical advantage of the two-sided Lanczos method over the Arnoldi method is that the matrix T_k is tridiagonal and we only need to save six vectors, assuming we do not resort to any sort of reorthogonalization. The use of the two-sided Lanczos algorithm to solve the QEP is dealt with in [12], [20], and [26]. A major problem with the Lanczos method is its potential for breakdown, which happens when $w_{k+1}^* v_{k+1} \approx 0$. The look-ahead Lanczos method [53], [116], [152], the block Lanczos method [6], [69], [81], and the nested Lanczos method [38] all offer cures for this problem. Sometimes, B-biorthogonalization is used, for which $W_k^* B V_k = I$ instead of $W_k^* V_k = I$ [6].

The efficiency of these Krylov projection methods depends on the efficiency with which matrix-vector products with the operator S (and with S^* for the two-sided Lanczos method) can be computed. If $A - \lambda B$ is a linearization of $Q(\lambda)$, then $S = B^{-1}A$. For example, if M is nonsingular and $A - \lambda B$ is the first companion linearization (3.6), we have

$$S = B^{-1}A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}.$$

In this case, applying S to a vector requires a (sparse) LU factorization of M. If a shift-and-invert spectral transformation is used with shift σ, then $S = (A - \sigma B)^{-1}B$. If A and B are defined from the first companion linearization (3.6), then, using (3.8), we have

$$S = \begin{bmatrix} I & 0 \\ \sigma I & I \end{bmatrix} \begin{bmatrix} Q(\sigma)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -(C + \lambda M) & -M \\ I & 0 \end{bmatrix}. \tag{6.3}$$

Hence to apply S to a vector we do not need to factorize the $2n \times 2n$ matrix $A - \sigma B$; we need just an LU factorization of the $n \times n$ matrix $Q(\sigma)$.

If M, C, and K are real, a complex shift can be used, but in this case S is complex and loses the spectral properties of $Q(\lambda)$: there is no guarantee that the eigenvalues of S either are real or occur in complex conjugate pairs. Moreover, the Arnoldi or unsymmetric Lanczos procedures have to be carried out in complex arithmetic. We note that each eigenvalue near σ is related to an eigenvalue near $\bar{\sigma}$. To preserve the structure, we need to extract these pairs of eigenvalues together. In this case, the obvious transformation [115] is $f_\sigma(\lambda) = (\lambda - \sigma)^{-1}(\lambda - \bar{\sigma})^{-1}$. The corresponding S is given by

$$\begin{aligned} S &= (A - \sigma B)^{-1} B (A - \bar{\sigma} B)^{-1} B \\ &= \frac{1}{\operatorname{Im}(\sigma)} \operatorname{Im}\big((A - \sigma B)^{-1} B\big). \end{aligned} \tag{6.4}$$

Note that S is now real and the eigenvectors of $A - \lambda B$ are eigenvectors of S. In practice, because of (6.3) this approach requires a complex factorization of $Q(\sigma)$ and, for each multiplication $y = Sv$, the solution of the complex system $(A - \sigma B)y = Bv$. Only the imaginary part of y is kept to perform the rest of the Arnoldi or Lanczos steps. The eigenvalues of $Q(\lambda)$ can be recovered from the eigenvalues μ of S by solving the quadratic equation $(\lambda - \sigma)(\lambda - \bar{\sigma}) = 1/\mu$. As this equation has two solutions λ, instead of solving it, for a given eigenvector ξ the Rayleigh quotient $\lambda = \xi^* A \xi / \xi^* B \xi$ is formed to obtain the eigenvalue corresponding to ξ. If μ is a multiple eigenvalue of S, the λ's are not recovered from the Rayleigh quotient but from the Galerkin projection of $A - \lambda B$ on the eigenspace of μ. If it is not possible to check the multiplicities of the μ's the Galerkin projection on the Krylov space should be used. (Note that the matrix H_k is the Galerkin projection of S and not of $A - \lambda B$.)

A major disadvantage of the shift-and-invert Arnoldi and Lanczos methods is that a change of shift σ requires building a new Krylov subspace: all information built with the old σ is lost. The rational Krylov algorithm [123] is a generalization of the shift-and-invert Arnoldi algorithm where the shift σ can be changed without building a new Krylov subspace. It can be preferable to use shift-and-invert Arnoldi when several eigenvalues are desired.

6.1.2. Symmetric QEPs. When M or K are positive definite we have to choose between a linearization $A-\lambda B$ with A unsymmetric and B symmetric positive definite and a symmetric linearization with A and B symmetric but indefinite. The unsymmetric linearization can be solved via the Arnoldi or two-sided Lanczos process. We can take advantage of the definitiveness of B if no spectral transformation is used.

If the symmetric linearization is used, one can take advantage of the symmetry of A and B in the nonsymmetric Lanczos algorithm [26], [110] and cut the work and storage requirement approximately in half. Parlett and Chen [114] introduced a pseudo-Lanczos algorithm for symmetric pencils that uses an indefinite inner product and respects the symmetry of the problem. Assuming B is nonsingular, the pseudo-Lanczos method builds a basis V_k for $\mathcal{K}_k(B^{-1}A, v)$ which is A-orthogonal (that is, $V_k^* A V_k = \Omega_k$ is diagonal) and a projection matrix $T_k = V_k^* A B^{-1} A V_k$ which is symmetric and tridiagonal. The matrices V_k, T_k, and Ω_k satisfy the recurrence

$$B^{-1}AV_k - V_k\Omega_k^{-1}T_k = \frac{\beta_k}{\omega_k} v_{k+1}e_k^*.$$

This method is discussed in [8] and a thorough analysis is given in [83]. The main drawback of this procedure is that if A is symmetric indefinite, the condition $v_k^* A v_k \neq 0$ may be violated, and there is no assurance that the basis vectors will be linearly independent. The reduced symmetric generalized eigenvalue problem

$$T_k\xi = \lambda\Omega_k\xi$$

inherits the same numerical difficulties as the original pair (A, B). It can be solved by one of the methods discussed in section 5.2.

Kowalski [83] has developed a block version of this pseudosymmetric Lanczos algorithm with constrained look-ahead, combining efficiency and robustness.

6.1.3. Gyroscopic Systems. Let $A - \lambda B$ be a Hamiltonian/skew-Hamiltonian linearization of $Q(\lambda)$ in (3.22). Having a linearization that respects the structure, we need a Krylov method that respects the spectral properties of (A, B) too. As in (5.2), let $B = B_1 B_2$ be such that $H = B_1^{-1} A B_2^{-1}$ is Hamiltonian.

We assume that the matrices are real. The symplectic Lanczos method for Hamiltonians $H \in \mathbb{R}^{2n\times 2n}$ [14], [48], [49] generates a sequence of matrices

$$V_{2k} = [v_1, w_1, v_2, w_2, \ldots, v_k, w_k] \in \mathbb{R}^{2n\times 2k},$$

and a $2k \times 2k$ Hamiltonian J-Hessenberg matrix

$$\widetilde{H}_{2k} = \begin{bmatrix} D_1 & T_k \\ D_2 & -D_1 \end{bmatrix}$$

with D_1, D_2 diagonal and T_k symmetric tridiagonal. The space spanned by the columns of V_{2k} is symplectic, that is, $V_{2k}^T J V_{2k} = J$. The matrices $\widetilde{H}_{2k}$ and V_{2k} satisfy the relation

$$P_{2n} H P_{2n}^T V_{2k} = V_{2k} P_{2k} \widetilde{H}_{2k} P_{2k}^T + \gamma_{k+1} v_{k+1} e_{2k}^T,$$

where $P_{2j} = [e_1, e_3, \ldots, e_{2j-1}, e_2, e_4, \ldots, e_{2j}]$ is a $2j \times 2j$ permutation matrix and γ_{k+1} is the $(k+1, k)$ entry of T_{k+1}. We refer to [14] for a detailed description of the algorithm. The Hamiltonian SR algorithm [24] or the HR algorithm [16] can be used to solve the reduced problem with the Hamiltonian J-Hessenberg matrix $\widetilde{H}_{2k}$. This method can be used on the inverted operator H^{-1}, which is also Hamiltonian, to compute the smallest eigenvalues.

The shift-and-invert operation with shift σ fails to preserve the $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$ spectral structure of H, as $S = (H - \sigma I)^{-1}$ is not Hamiltonian. Each eigenvalue near σ is also related to an eigenvalue near $-\sigma$, $\bar{\sigma}$, and $-\bar{\sigma}$. In the case where the pencil (A, B) is real, Mehrmann and Watkins [101] suggested using all these shifts in one run. The operator is given by

$$S = B_2(A - \sigma B)^{-1}B(A + \sigma B)^{-1}B_1$$

if the shift σ is real and

$$S = B_2(A - \sigma B)^{-1}B(A + \sigma B)^{-1}B(A - \bar{\sigma} B)^{-1}B(A + \bar{\sigma} B)^{-1}B_1$$

if σ is complex. In both cases, it is shown in [101] that S is real and skew-Hamiltonian. Each eigenvalue has multiplicity 2, and if x is a Schur vector, so is Jx. The Krylov subspace $\mathcal{K}_k(S, v)$ has the property that it is *isotropic*, i.e., $y^*Jx = 0$ for all $x, y \in \mathcal{K}_k(S, v)$, which means that $J\mathcal{K}_k(S, v)$ is orthogonal to $\mathcal{K}_k(S, v)$.

The isotropic Arnoldi process [101] produces a Hessenberg matrix H_k and a basis V_k such that $V_k^*V_k = I$ and $V_k^*JV_k = 0$. It differs from the classical Arnoldi process in that the process orthogonalizes the columns of V_k as well as the columns of JV_k, and in exact arithmetic the process terminates after $n - 1$ steps if the size of S is $2n$. The matrices H_k and V_k satisfy the recurrence relation (6.2). Each eigenvalue of S has multiplicity 2, and if x is a Schur vector, so is Jx. The isotropic Arnoldi method removes all Schur vectors of the form Jx, so only one version of a multiple eigenvalue is computed. Note that this method is valid for real Hamiltonian/skew-Hamiltonian pencils only.

If the matrices of the gyroscopic system are complex, one can use the generalized Cayley transform on the Hamiltonian $B_1^{-1}AB_2^{-1}$. The resulting operator is

$$S = B_2(A - \sigma B)^{-1}(A + \bar{\sigma} B)(A - \bar{\sigma} B)^{-1}(A + \sigma B)B_2^{-1},$$

which is symplectic and real if (A, B) is real. Its eigenvalues occur in quadruples $\mu, \bar{\mu}, \mu^{-1}, \bar{\mu}^{-1}$, so that shifts σ close to the imaginary axis must be avoided. Structure-preserving Lanczos-like methods that use both S and S^{-1} in a symmetric manner can be applied (see Benner and Fassbender [15] for details of the method).

6.2. Projection Methods Applied Directly to the QEP. This class of methods builds orthonormal bases V_k for the subspace $\mathcal{K}_k$ and solves the projected problem $V_k^*Q(\lambda)V_k z = 0$ of smaller size using one of the methods described in section 5. As an orthogonal projection preserves symmetry and skew-symmetry, the numerical methods of sections 5.2 and 5.3 can be used to solve the projected problem.

In this section, we propose techniques for building the columns of V_k. We assume that $V_k = [v_1, \ldots, v_k]$ is given and we have computed the Ritz pair $(\widetilde{\lambda}, \widetilde{x} = V_k z)$ so that $V_k^*Q(\widetilde{\lambda})V_k z = 0$ and $(\widetilde{\lambda}, \widetilde{x})$ is close to a target (or shift) σ. The goal is to add a new vector v_{k+1} so that the Ritz pair becomes more accurate.

6.2.1. Residual Iteration Method. The subspace $\mathcal{K}_k$ can be expanded with the direction

$$v = (\sigma^2 M + \sigma C + K)^{-1}(\widetilde{\lambda}^2 M + \widetilde{\lambda} C + K)\widetilde{x} = Q(\sigma)^{-1} r, \tag{6.5}$$

where $r = Q(\widetilde{\lambda})\widetilde{x}$ is the residual. Then v is orthonormalized against $v_1, \ldots, v_k$ into v_{k+1}. The subspace is extended by a quadratic Cayley transform applied to the Ritz vector. The method is studied in [100], [108] and a variation is proposed in [77]. If the matrices are real, then $(\widetilde{\lambda}, \widetilde{x})$ and $(\bar{\widetilde{\lambda}}, \bar{\widetilde{x}})$ are both Ritz pairs and the two directions v and $\bar{v}$ or, equivalently, $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$ can be added to the subspace.

A problem arises when $\sigma = \widetilde{\lambda}$ in the quadratic Cayley transform since $v = Q(\sigma)^{-1}Q(\widetilde{\lambda})\widetilde{x} = \widetilde{x}$ and no new direction is added to the subspace. One possibility is to replace (6.5) by

$$v = Q(\sigma)^{-1} Q'(\sigma)\widetilde{x},$$

where $Q'(\sigma) = 2\sigma M + C$, which can be regarded as a generalization of the shift-and-invert transformation. Another alternative is to use the Jacobi–Davidson method as discussed in the next section.

6.2.2. Jacobi–Davidson Method for the QEP. In the Jacobi–Davidson method the subspace $\mathcal{K}_k$ is not expanded by a vector obtained from inverse iteration but by Newton's method applied to the equations $Q(\lambda)x = 0$ and $x^*x = 1$, where $x = \widetilde{x} + v$ and $\lambda = \widetilde{\lambda} + \eta$. This amounts to the linear system

$$\begin{bmatrix} Q(\widetilde{\lambda}) & Q'(\widetilde{\lambda})\widetilde{x} \\ 2\widetilde{x}^* & 0 \end{bmatrix} \begin{bmatrix} v \\ \eta \end{bmatrix} = - \begin{bmatrix} r \\ 0 \end{bmatrix},$$

where $r = Q(\widetilde{\lambda})\widetilde{x}$ is the residual and $Q'(\widetilde{\lambda}) = 2\widetilde{\lambda}M + C$. Since $\widetilde{\lambda}$ and $\widetilde{x}$ are computed by a projection method, we have that $\widetilde{x}^* Q(\widetilde{\lambda})\widetilde{x} = 0$. Further manipulations lead to the correction equation [131], [133], [132]

$$\left(I - \frac{Q'(\widetilde{\lambda})\widetilde{x}\widetilde{x}^*}{\widetilde{x}^* Q'(\widetilde{\lambda})\widetilde{x}}\right) Q(\widetilde{\lambda}) \left(I - \frac{\widetilde{x}\widetilde{x}^*}{\widetilde{x}^*\widetilde{x}}\right) v = r. \tag{6.6}$$

The new basis vector v_{k+1} is obtained by orthonormalizing v against the previous columns of V_k.

The Jacobi–Davidson method has been successfully used to compute the most unstable eigenvalues related to the incompressible attachment-line flow problem that arises near the leading edge of a swept wing; see [71], where problems with more than 10,000 unknowns were solved. In [131] an acoustic problem with n up to 250,000 was solved with this method.

6.2.3. Comments. A comparison between inverse residual iteration, Jacobi–Davidson, and shift-and-invert Arnoldi was carried out in [100] for an application from acoustics. No clear winner was identified, but some preference was given to shift-and-invert Arnoldi when several eigenvalues are sought.

Both the quadratic residual iteration and the Jacobi–Davidson methods target one eigenvalue at a time, which may lead to fast local convergence but slow global convergence. This contrasts with Krylov subspace methods, in which a group of eigenvalues is computed simultaneously.

One major advantage of the residual iteration and Jacobi–Davidson methods over Krylov subspace methods is that when the size n of the matrices is too large, $Q(\sigma)$ in (6.5) and the correction equation (6.6) can be solved by an iterative solver such as GMRES. Therefore, these methods require only matrix-vector products. On the other hand, the convergence of the iterative solver might be slow. In practice a good preconditioner is necessary to make these methods work well.

Since iterative linear solvers can be used, the residual iteration method and the Jacobi–Davidson method are well suited for parallelization [131], [145].

6.3. Restarting, Locking, and Purging. Projection methods have high storage and computational requirements as the dimension k of the subspace $\mathcal{K}_k$ increases. Restarting schemes and deflation procedures can be used to improve the efficiency.

For Krylov subspace methods, *restarting* means that the starting vector v defining the Krylov subspace $\mathcal{K}_k(S, v)$ is replaced by a new, improved starting vector. The new vector is generally a linear combination of the eigenvectors of interest and can be defined either explicitly [112], [125] or implicitly [94].

Already converged eigenvalues can be deflated. If $(\widetilde{\lambda}, \widetilde{x})$ is a converged Ritz eigenpair that belongs to the set of desired eigenvalues, one may want to lock it and then continue to compute the remaining eigenvalues without altering $(\widetilde{\lambda}, \widetilde{x})$. If the converged $(\widetilde{\lambda}, \widetilde{x})$ does not belong to the set of wanted eigenvalues, one may want to remove it from the current subspace $\mathcal{K}_k$. These two types of *deflation* are called *locking* and *purging*.

One important advantage of working with a linearization of $Q(\lambda)$ and a Krylov subspace method is that one can get at the same time the partial Schur decomposition of the single matrix S that is used to define the Krylov subspaces. The orthogonal Schur basis is useful for building a stable method for purging and locking [50], [93], [123], [136], as we now briefly describe. We recall that the matrix S contains the desired eigenvalues of $Q(\lambda)$ in the exterior of its spectrum (see section 6.1). Let $W_k^* S V_k = S_k$ be the projected matrix with Schur form $U^* S_k U = T_k$, where U is a $k \times k$ unitary matrix and T_k is $k \times k$ upper triangular. Let $V_k U = Q_k$ and $W_k U = Z_k$. Then

$$SQ_k = Z_k T_k + E_k$$

forms an approximate partial Schur form for S, with E_k an error matrix. The Schur form can be reordered. Locking involves moving the already converged eigenvalues to the top corner of T_k and setting the corresponding first columns of E_k to zero. The error made by explicitly setting the first columns of E_k to zero is small when these columns have small norm. This locking operation hence introduces a small backward error in the remaining eigenvalues to be computed. When eigenvalues are not wanted, they can be purged from $\mathcal{K}_k$ by moving them to the bottom corner of T_k and by removing the corresponding generalized Schur vectors in Q_k and Z_k. Purging can be performed implicitly by implicit restarting with exact shifts. Purging reduces the number of basis vectors by only keeping those that are likely to contribute to the convergence of the wanted eigenpairs [107]. After purging, the subspace can be expanded again by additional Krylov iterations.

It is harder to organize locking, purging, and restarting for methods that tackle the problem in its original form, such as the residual iteration or the Jacobi–Davidson method, since there is no Schur form for QEPs. In [7, sect. 9.2], the Jacobi–Davidson method is restarted by keeping a smaller subspace that is spanned by the best $m < k$ Ritz vectors from the last iteration. Meerbergen [100] proposes locking, purging, and

restarting using a partial Schur form for a linearization $A - \lambda B$ of $Q(\lambda)$. A partial Schur form for $A-\lambda B$ is built via the Schur form of $\mathcal{V}_{2k}^*(A-\lambda B)\mathcal{V}_{2k}$, where $\mathcal{V}_{2k}$ is the $2n \times 2k$ matrix $\mathcal{V}_{2k} = \mathrm{diag}(V_k, V_k)$. Special care is needed so that the new matrix $\widetilde{\mathcal{V}}_{2m}$ with fewer columns $2m < 2k$ obtained after locking and purging keeps the 2×2 block form $\mathrm{diag}(V_m, V_m)$. This is vital to preserving the relationship between the projected QEP and the projected GEP.

7. Software. We list available software that, whenever possible, takes advantage of the structure of the matrices defining the problem as well as the spectral properties of the eigenvalue problem.

7.1. General QEPs, Unsymmetric Linearization.

7.1.1. Direct Methods. Most linear algebra–related software packages include subroutines that implement QZ algorithms and can therefore be used to implement Algorithm 5.1.

In LAPACK [2], the routine `xGGEV` computes generalized eigenvalues and optionally the left and/or right generalized eigenvectors. LAPACK is freely available at http://www.netlib.org/lapack/.

In MATLAB, the `polyeig(K,C,M)` command is an implementation of Algorithm 5.1 that returns the $2n$ eigenvalues and, optionally, the right eigenvectors.

7.1.2. Iterative Methods. A variant of the Arnoldi procedure called the implicitly restarted Arnoldi method (IRAM) is provided by the ARPACK software [94], freely available at ftp://ftp.caam.rice.edu/pub/software/ARPACK. It is designed to solve large-scale GEPs with non-Hermitian (A, B). It contains drivers for several spectral transformations (invert, shift-and-invert, Cayley transforms). The user has only to provide the matrix-vector products involved in the Arnoldi process. A parallel version, PARPACK, is also available from the same ftp address. In MATLAB 6, the `eigs` function is an interface to the ARPACK package.

A FORTRAN implementation of the non-Hermitian Lanczos method with look-ahead to cure breakdowns is available in QMRPACK [54] at http://www.netlib.org/linalg/qmrpack.tgz under the name `DUAL` for double precision real and `ZUAL` for double precision complex.

For the solution of QEPs arising in structural engineering, the MSC/Nastran finite element package [82] uses the first companion linearization (3.6) and a shift-and-invert two-sided block Lanczos algorithm to solve the linearized form.

7.2. Symmetric Linearization—Direct Methods.

Most linear algebra–related software packages also include subroutines for the Hermitian GEP $Ax = \lambda Bx$ with positive definite matrix B. In LAPACK [2], the corresponding routines are `xSYGV`, which uses the QR algorithm as the underlying algorithm; `xSYGVX`, which computes all or a selected subset of eigenvalues and optionally eigenvectors and uses the QR algorithm or bisection method and inverse iteration as the underlying algorithms, whichever is more efficient; and `xSYGVD`, which uses the divide-and-conquer algorithm as the underlying algorithm. In MATLAB, the command `eig(A,B)` is the analogue of the LAPACK routine `xSYGV`.

There is no software available that takes advantage of the symmetry and possible spectral properties for definite pairs (A, B), where B is indefinite, or indefinite pairs (A, B).

7.3. Hamiltonian/Skew-Hamiltonian Linearization—Direct Methods.

A Fortran 77 implementation by Benner, Byers, and Barth [13] of Van Loan's square-

reduced method for the computation of the eigenvalues of Hamiltonian matrices is available at http://www.netlib.org/toms/800, and a MATLAB version is available at http://www.math.uni-bremen.de/~benner/software.html.

8. Discussion and Related Problems. In this survey, we have treated only quadratic matrix polynomials, but most of the theory and numerical methods extend naturally to matrix polynomials of degree greater than 2. Higher degree polynomial eigenvalue problems arise in a variety of applications in science and engineering. For example, a third degree eigenvalue problem arising from aeroacoustics analysis was studied in [5]. In fluid mechanics the study of the spatial stability of the Orr–Sommerfeld equation [22] yielded a quartic eigenvalue problem. For nonlinear eigenvalue problems $A(\lambda)x = 0$ for which the coefficients in $A(\lambda)$ are not polynomials in λ, we refer to [3], [108], and [122].

For dense problems, there is currently no numerical method that both tackles the problem directly (that is, without using a linearization of $Q(\lambda)$) and computes all the eigenpairs. The QEP is usually linearized and solved by GEP techniques. The GEP techniques applied to the linearized form do not yield backward stable methods: the computed solution is usually the exact solution of a nearby GEP but may not be the exact solution of a nearby QEP because GEP eigensolvers do not respect the structure in the $2n \times 2n$ matrices of the linearization. There is therefore a need to improve the stability of GEP approaches to the solution of the QEP.

In contrast with general QEPs [82], [94], there is currently no software available for the methods described in sections 5 and 6 for symmetric QEPs and QEPs derived from gyroscopic systems, nor for the methods discussed in section 6.2.

In section 6 we omitted inverse iteration and subspace iteration as they are not competitive [66], [125] with the projection methods described in section 6. Some references on inverse iteration and subspace iteration methods for the solution of the QEP are [68], [95], [96], [111], [120], [137], [153].

In the frequency response analysis of vibrating systems [31], [47], [78], [151], and in electromagnetic wave propagation problems [86], one has to solve linear systems of the form

$$(\omega^2 M + \omega C + K)x(\omega) = b(\omega), \qquad b(\omega) \in \mathbb{C}^n \text{ given},$$

over a wide band of frequencies $\omega \in \mathcal{I} = [\omega_l, \omega_h]$. One can use a linearization $A - \omega B$ of the quadratic matrix polynomial $Q(\omega) = \omega^2 M + \omega C + K$. When the size of the problem is not too large, one can compute the generalized Schur decomposition of the $2n \times 2n$ linearized form

$$W^* A Z = T, \quad W^* B Z = S,$$

where W and Z are unitary and T and S are upper triangular. Then, if $A - \omega B$ is the first companion form (3.6) of $Q(\omega)$ and ω is not an eigenvalue,

$$x(\omega) = -[I\ 0]Z[T - \omega S]^{-1} W^* \begin{bmatrix} 0 \\ I \end{bmatrix} b(\omega) = Q^{-1}(\omega) b(\omega).$$

Hence once the generalized Schur decomposition has been computed, we can compute $Q(\omega)^{-1}x$ at a cost of $O((2n)^2)$ flops, since $T - \omega S$ is triangular of dimension $2n$. This approach is not feasible when the matrices are large. An alternative approach is to consider $\omega \in \mathcal{I}$ as a continuous variable and to use a power series–Padé approximation of $x(\omega)$ about a center frequency $\omega_0 \in \mathcal{I}$ [86]. If the interval $\mathcal{I}$ is large, this may require

several power series expansions. Different iterative schemes based on Krylov subspace methods for the solution of the augmented $2n \times 2n$ shifted system

$$(A - \omega B)y(\omega) = d(\omega)$$

have also been proposed [31], [47], [52], [55], [78], [127], [151]. These iterative shifted linear solvers may suffer from slow convergence. For these highly structured linear systems, good preconditioners need to be investigated.

A numerical range for matrix polynomials [97] and self-adjoint quadratic matrix polynomials [90] as well as generalizations of the inertia theorem for quadratic matrix polynomials [19] have been investigated. Several distance quantities derived in the linear case have already been extended to the nonlinear case. For instance, the notion of stability radius, widely used in control theory to measure the size of the smallest perturbation that moves a stable system to an unstable system, has been extended to matrix polynomials for both complex perturbations [60], [113] and real perturbations [59]. Recently, several nearness questions such as, "Where is the nearest definite pair for the Hermitian generalized eigenvalue problem?" [27] or "Where is the nearest nonregular pencil?" [25] have been investigated. New questions arise, including, "Where is the nearest nondefinite or nonoverdamped system?" [75] and "Where is the nearest nonregular quadratic matrix polynomial?"

As we have stressed throughout this survey, QEPs are of growing importance and many open questions are associated with them. We hope that our unified overview of applications, theory, and numerical methods for QEPs will stimulate further research in this area.

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