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# *Acceleration bundles on Banach and Fréchet manifolds*

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# Acceleration bundles on Banach and Fréchet manifolds

*JGP Editorial Board Scientific Meeting*
*In commemoration of André Lichnerowicz*
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**Abstract** The second order tangent bundle $T^2M$ of a smooth manifold $M$ consists of the equivalence classes of curves on $M$ that agree up to their acceleration. Dodson and Radivoiovici [6] showed that in the case of a finite $n$-dimensional manifold $M$, $T^2M$ becomes a vector bundle over $M$ if and only if $M$ is endowed with a linear connection.

We have extended this result to $M$ modeled on an arbitrary Banach space and more generally to those Fréchet manifolds which can be obtained as projective limits of Banach manifolds. Various structural properties have been deduced.

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## Introduction

It is an honour to be able to contribute to the memory of André Lichnerowicz through this meeting. The point of contact with his legacy is our attempt to follow his tradition of developing global differential geometric structures that help to model real phenomena. This characteristic of Lichnerowicz's work Bourguignon has highlighted in the 1999 memorial article [1] and it is evident also through the 1976 volume in honour of Lichnerowicz's $60^{th}$ birthday [2].

Lichnerowicz himself firmly placed in the context of quantum and statistical mechanics his work on deformations of algebras of smooth functions on a smooth Banach manifold [12].

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Our constructions have provided in the Fréchet manifold case a suitable principal bundle of frames $F^2M$ for the second tangent bundle $T^2M$, which is a vector bundle in the presence of a linear connection. Then $T^2M$ is associated with $F^2M$ and a one to one correspondence between their connections is provided.

Fréchet spaces of sections arise naturally as configurations of a physical field and evolution equations naturally involve second order operators.

We mention first some areas of potential application for our results.

The moduli space of inequivalent configurations is the quotient of the infinite-dimensional configuration space $\mathcal{X}$ by the appropriate symmetry gauge group.

Typically, $\mathcal{X}$ is modelled on a Frechét space of smooth sections of a vector bundle over a closed manifold and is a Hilbert Lie group.

Inverse limit Hilbert manifolds and inverse limit Hilbert groups, introduced by Omori [16, 17], provide an appropriate setting for the study of the Yang-Mills and Seiberg-Witten field equations.

Let $M$ be a finite-dimensional path-connected Riemannian manifold. The free loop space of all smooth maps from the circle group $S^1$ to $M$ is a Fréchet manifold $\Lambda M$, cf. Manoharan [13, 14].

A string structure is defined as a lifting of the structure group to an $S^1$-central extension of the loop group. Suppose that $\widetilde{G} \to \widetilde{P} \to X$ is a lifting of a principal Fréchet bundle $G \to P \to X$ over a Fréchet manifold $X$ and further that $S^1 \to \widetilde{G} \to G$ is an $S^1$-central extension of $G$. Manoharan showed that every connection on the principal bundle $G \to P \to X$ together with a $\widetilde{G}$-invariant connection on $S^1 \to \widetilde{P} \to P$ defines a connection on $\widetilde{G} \to \widetilde{P} \to X$.

The group $\mathcal{D}$ of orientation preserving smooth diffeomorphisms of a compact manifold $M$ is homeomorphic to the product of the group of volume preserving diffeomorphisms $\mathcal{D}_\mu$, of a volume element $\mu$ on $M$, times the set $\mathcal{V}$ of all volumes $v > 0$ with $\int v = \int \mu$. In this case, $\mathcal{D}_\mu$ can be realized as a projective limit of Hilbert-modelled manifolds (see Omori [16, 17]) and forms the appropriate framework for the study of hydrodynamics of an incompressible fluid.

Moreover, there is a close relationship between geodesics on $\mathcal{D}_\mu$ and the classical Euler equations for a perfect fluid. Namely, if $\eta_t \in \mathcal{D}_\mu$ is a geodesic of $\mathcal{D}_\mu$ as above and $v_t = d\eta_t/dt$ the velocity, then the vector field $u_t = v_t \circ \eta_t^{-1}$ of $M$ is a solution to the classical Euler equations.

The space $J^\infty E$ of infinite jets of the sections of a Banach modelled vector bundle $E$ can be realized as the projective limit of the finite corresponding jets $\{J^k E\}_{k\in\mathbb{N}}$.

This approach makes possible the definition of a Fréchet modelled vector bundle on $J^\infty E$ and thus the use of the latter for the description of Lagrangians and source equations as certain types of differential forms, cf. Galanis [7] and Takens [18].

## Preliminaries

In the case of a finite $n$-dimensional manifold $M$, if and only if $M$ is endowed with a linear connection, $T^2M$ becomes a vector bundle over $M$ with structure group the general linear group $GL(2n;\mathbb{R})$ and, therefore, a $3n$-dimensional manifold [6].

## Banach case

Consider a manifold $M$ modeled on an arbitrarily chosen Banach space $\mathbb{E}$. Using the Vilms [20] point of view for connections on infinite dimensional vector bundles and a new formalism, we prove that $T^2M$ can be thought of as a Banach vector bundle over $M$ with structure group $GL(\mathbb{E}\times\mathbb{E})$ if and only if $M$ admits a linear connection.

Let $M$ be a $C^\infty-$manifold modeled on a Banach space $\mathbb{E}$ and atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$. This gives atlas $\{(\pi_M^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha\in I}$ for the tangent bundle $TM$ of $M$ with

$$\Psi_\alpha : \pi_M^{-1}(U_\alpha) \longrightarrow \psi_\alpha(U_\alpha) \times \mathbb{E} : [c, x] \longmapsto (\psi_\alpha(x), (\psi_\alpha \circ c)'(0)),$$

where $[c, x]$ is the equivalence class of smooth curves $c$ of $M$ with $c(0) = x$ and $(\psi_\alpha \circ c)'(0) = [d(\psi_\alpha \circ c)(0)](1)$. The trivializing system of $T(TM)$ is denoted by

$$\{(\pi_{TM}^{-1}(\pi_M^{-1}(U_\alpha)), \widetilde{\Psi}_\alpha)\}_{\alpha\in I}.$$

A connection on $M$ is a vector bundle morphism:

$$D : T(TM) \longrightarrow TM$$

with smooth mappings $\omega_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \to \mathcal{L}(\mathbb{E}, \mathbb{E})$ defined by the local forms of D:

$$D_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \psi_\alpha(U_\alpha) \times \mathbb{E}$$

with $D_\alpha := \Psi_\alpha \circ D \circ (\widetilde{\Psi}_\alpha)^{-1}$, $\alpha \in I$, via the relation

$$D_\alpha(y, u, v, w) = (y, w + \omega_\alpha(y, u) \cdot v).$$

$D$ is linear if and only if $\{\omega_\alpha\}_{\alpha \in I}$ are linear in the second variable.

Such a connection $D$ is characterized by the Christoffel symbols $\{\Gamma_\alpha\}_{\alpha\in I}$ , smooth mappings

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \longrightarrow \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E}, \mathbb{E}))$$

defined by $\Gamma_\alpha(y)[u] = \omega_\alpha(y, u)$, $(y, u) \in \psi_\alpha(U_\alpha) \times \mathbb{E}$. On chart overlaps:

$$\Gamma_\alpha(\sigma_{\alpha\beta}(y))(d\sigma_{\alpha\beta}(y)(u))[d(\sigma_{\alpha\beta}(y))(v)] + (d^2\sigma_{\alpha\beta}(y)(v))(u) \\ = d\sigma_{\alpha\beta}(y)((\Gamma_\beta(y)(u))(v)),$$

for all $(y, u, v) \in \psi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{E} \times \mathbb{E}$. Here $d$, $d^2$ stand for the first and the second differential and by $\sigma_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1}$ of $\mathbb{E}$.

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## Projective system of Banach manifolds

Let $\{M^i; \varphi^{ji}\}_{i,j\in\mathbb{N}}$ be a projective system of Banach manifolds modeled on the Banach spaces $\{\mathbb{E}^i\}$ respectively. We assume that

**(i)** the models form also a projective limit $\mathbb{F} = \varprojlim \mathbb{E}^i$,

**(ii)** for each $x = (x^i) \in M$ there exists a projective system of local charts $\{(U^i, \psi^i)\}_{i\in\mathbb{N}}$ such that $x^i \in U^i$ and the corresponding limit $\varprojlim U^i$ is open in $M$.

Then the projective limit $M = \varprojlim M^i$ can be endowed with a Fréchet manifold structure modeled on $\mathbb{F}$ via the charts $\{(\varprojlim U^i, \varprojlim \psi^i)\}$. Moreover, the tangent bundle $TM$ of $M$ is also endowed with a Fréchet manifold structure of the same type modeled on $\mathbb{F} \times \mathbb{F}$.

The local structure now is defined by the projective limits of the differentials of $\{\psi^i\}$ and $TM$ turns out to be an isomorph of $\varprojlim TM^i$.

Here we adopt the definition of Leslie [9], [10] for the differentiability of mappings between Fréchet spaces. However, the differentiability proposed by Kriegl and Michor [8] is also suited to our study.

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## $T^2M$ for infinite dimensional Banach manifolds

Let $M$ be a smooth manifold modeled on the infinite dimensional Banach space $\mathbb{E}$ and $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ a corresponding atlas.

For each $x \in M$ we define an equivalence relation $\approx_x$ on

$$C_x = \{f : (-\varepsilon, \varepsilon) \to M \mid f \text{smooth and } f(0) = x, \varepsilon > 0\} :$$

$$f \approx_x g \Leftrightarrow f'(0) = g'(0) \text{ and } f''(0) = g''(0).$$

**Definition**

We define the *tangent space of order two* of $M$ at the point $x$ to be the quotient $T^2_xM = C_x/\approx_x$ and the *tangent bundle of order two* of $M$ the union of all tangent spaces of order 2: $T^2M := \bigcup_{x\in M} T^2_xM$.

Of course, $T^2_xM$ is a topological vector space isomorphic to $\mathbb{E}\times\mathbb{E}$ via the bijection

$$T^2_xM \xleftrightarrow{\simeq} \mathbb{E}\times\mathbb{E} : [f,x]_2 \longmapsto ((\psi_\alpha\circ f)'(0),(\psi_\alpha\circ f)''(0)),$$

where $[f,x]_2$ stands for the equivalence class of $f$ with respect to $\approx_x$. However, this structure depends on the choice of the chart $(U_\alpha,\psi_\alpha)$, hence a definition of a vector bundle structure on $T^2M$ cannot be achieved. With a linear connection we solve this problem.

**Theorem**

Given a linear connection $D$ on $M$, then $T^2M$ becomes a Banach vector bundle with structure group the general linear group $GL(\mathbb{E} \times \mathbb{E})$. Moreover, $T^2M$ is isomorphic to $TM \times TM$ since both bundles are characterized by the same cocycle $\{(d\sigma_{\alpha\beta} \circ \psi_\beta) \times (d\sigma_{\alpha\beta} \circ \psi_\beta)\}_{\alpha,\beta \in I}$ of transition functions.

We have also a converse

**Theorem**

Let $M$ be a smooth manifold modeled on the Banach space $\mathbb{E}$. If the second order tangent bundle $T^2M$ of $M$ admits a vector bundle structure, with fibres of type $\mathbb{E} \times \mathbb{E}$, isomorphic to the product of vector bundles $TM \times TM$, then a linear connection can be defined on $M$.

## Conjugacy classes for Banach bundles $T^2M$

Using results of Vassiliou [19], Dodson, Galanis and Vassiliou [5] investigated the classification of the vector bundle structures induced on $T^2M$ by a linear connection on $M$ via the conjugacy classes of second order differentials.

Given smooth $g : M \to N$ between Banach manifolds with linear connections, $(M, \nabla_M)$ and $(N, \nabla_N)$, the second order differential is well-defined by

$$T^2g : T^2M \longrightarrow T^2N : [(c, x)]_2 \mapsto [(g \circ c, g(x))]_2.$$

$\nabla_M$ and $\nabla_N$ are called *g-conjugate* if they commute with the differentials of $g : (M, \nabla_M) \to (N, \nabla_N)$ :

$$Tg \circ \nabla_M = \nabla_N \circ T(Tg).$$

The local expression of the latter is the following:

$$DG(\phi_\alpha(x))(\Gamma^M_\alpha(\phi_\alpha(x))(u)(u)) =$$
$$\Gamma^N_\beta(G(\phi_\alpha(x)))(DG(\phi_\alpha(x))(u))(DG(\phi_\alpha(x))(u)) +$$
$$+D(DG)((\phi_\alpha(x))(u,u),$$

for every $(x, u) \in U_\alpha \times \mathbb{E}$.

## Examples

**1.** In the case of a constant map $g$, the condition collapses to a trivial identification of zero quantities, since the local expression $G$ is constant. As a result, all linear connections are conjugate through constant maps.

**2.** If we consider the identity map $g = id_M$, then

$$D\phi_{\beta\alpha}(\phi_\alpha(x))(\Gamma^M_\alpha(\phi_\alpha(x))(u)(u)) =$$

$$\Gamma^N_\beta(\phi_\beta(x))(D\phi_{\beta\alpha}(\phi_\alpha(x))(u))(D\phi_{\beta\alpha}(\phi_\alpha(x))(u)) + D^2\phi_{\beta\alpha}(\phi_\alpha(x))(u,u).$$

The latter is equivalent to the chart overlap compatibility condition satisfied by the Christoffel symbols of a connection on $M$. Any $id_M$-conjugate connections have to be equal and the conjugation relationship in this case reduces to equality.

**Theorem**

Let $T^2M$, $T^2N$ be the second order tangent bundles defined by the pairs $(M, \nabla_M)$, $(N, \nabla_N)$, and let $g : M \to N$ be a smooth map. If the connections $\nabla_M$ and $\nabla_N$ are $g$-conjugate, then the second order differential $T^2g : T^2M \to T^2N$ is a vector bundle morphism [5].

**Theorem**

Let $\nabla$, $\nabla'$ be two linear connections on $M$. If $g$ is a diffeomorphism of $M$ such that $\nabla$ and $\nabla'$ are $g$-conjugate, then the vector bundle structures on $T^2M$, induced by $\nabla$ and $\nabla'$, are isomorphic [5].

**Corollary**

Up to isomorphism, the elements of the $g$-conjugate equivalence class $[(M, \nabla)]_g$ determine the same vector bundle structure on $T^2M$. Consequently, the latter structure depends not only on a pair $(M, \nabla)$ but also on the entire class $[(M, \nabla)]_g$ [5].

## Fréchet manifolds

Lewis [11] gives some background material on Fréchet spaces and in particular on the Fréchet projective limit of Banach spaces and the Fréchet space of infinite jets—ie Taylor series. For $M$ modeled on a Fréchet (non-Banach) space $\mathbb{F}$, there are complications because of the pathological structure of the general linear groups $GL(\mathbb{F})$, $GL(\mathbb{F} \times \mathbb{F})$, which does not even admit non-trivial topological group structures.

Also, the space of continuous linear mappings between Fréchet spaces does not remain in the same category of topological vector spaces, and we lack a general solvability theory of differential equations on $\mathbb{F}$. These problems are discussed in the 2006 Monastir Summer School Lectures of Neeb [15].

## Fréchet projective limits of Banach manifolds

Restricting ourselves to those Fréchet manifolds which can be obtained as projective limits of Banach manifolds, it is possible to endow $T^2M$ with a vector bundle structure over $M$ with structure group a new topological (and in a generalized sense Lie) group which replaces the pathological general linear group of the fibre type.

This construction is equivalent to the existence on $M$ of a specific type of linear connection characterized by a generalized set of Christoffel symbols.

Let $M$ be a smooth manifold modeled on the Fréchet space $\mathbb{F}$. Taking into account that the latter *always* can be realized as a projective limit of Banach spaces $\{\mathbb{E}^i; \rho^{ji}\}_{i,j\in\mathbb{N}}$ (i.e. $\mathbb{F} \cong \varprojlim \mathbb{E}^i$) we assume that the manifold itself is obtained as the limit of a projective system of Banach modeled manifolds $\{M^i; \varphi^{ji}\}_{i,j\in\mathbb{N}}$. Then, we obtain:

**Proposition**
The second order tangent bundles $\{T^2M^i\}_{i\in\mathbb{N}}$ form also a projective system with limit (set-theoretically) isomorphic to $T^2M$.

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Next we define a vector bundle structure on $T^2M$ by means of a certain type of linear connection on $M$. The problems concerning the structure group of this bundle are overcome by the replacement of the pathological $GL(\mathbb{F}\times\mathbb{F})$ by the new topological (and in a generalized sense smooth Lie) group:

$$\mathcal{H}^0(\mathbb{F}\times\mathbb{F}) := \{(l^i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} GL(\mathbb{E}^i\times\mathbb{E}^i) : \varprojlim l^i \text{ exists}\}.$$

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To be more specific, $\mathcal{H}^0(\mathbb{F}\times\mathbb{F})$ is a topological group being isomorphic to the projective limit of the Banach-Lie groups

$$\mathcal{H}^0_i(\mathbb{F}\times\mathbb{F}) := \{(I^1, I^2, ..., I^i)_{i\in\mathbb{N}} \in \prod_{k=1}^{i} GL(\mathbb{E}^k\times\mathbb{E}^k) : \ \rho^{jk}\circ I^j = I^k\circ\rho^{jk}$$

for $k \leq j \leq i$.

On the other hand, it can be considered as a generalized Lie group via its embedding in the topological vector space $\mathcal{L}(\mathbb{F}\times\mathbb{F})$.

**Theorem**

If a Fréchet manifold $M = \varprojlim M^i$ is endowed with a linear connection $D$ that can be also realized as a projective limit of connections $D = \varprojlim D^i$, then $T^2M$ is a Fréchet vector bundle over $M$ with structure group $\mathcal{H}^0(\mathbb{F} \times \mathbb{F})$.

Conversely, if $T^2M$ is an $\mathcal{H}^0(\mathbb{F} \times \mathbb{F})-$Fréchet vector bundle over $M$ isomorphic to $TM \times TM$, then $M$ admits a linear connection which can be realized as a projective limit of connections.

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