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# The Hooley-Huxley contour method for problems in number fields I: Arithmetic functions

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## 1. Introduction

Let  $K$  be a number field of degree  $n = r_1 + 2r_2$ . For an integral ideal  $\mathfrak{q}$  let  $I_{\mathfrak{q}}$  denote the group of fractional ideals of  $K$  whose prime decomposition contains no prime factors of  $\mathfrak{q}$ . Let

$$P_{\mathfrak{q}\infty} = \{(a) \in I_{\mathfrak{q}} : \alpha \in K^*, \alpha \equiv 1 \pmod{\mathfrak{q}}, \alpha \succ 0\}.$$

Denoting by  $\chi$  a narrow ideal class character mod  $\mathfrak{q}$ , that is, a character on  $I_{\mathfrak{q}}/P_{\mathfrak{q}\infty}$ , we can follow Landau in defining the  $L$ -function

$$\zeta_K(s, \chi) = \sum \chi(\mathfrak{a}) N\mathfrak{a}^{-s} \tag{1}$$

for  $\text{Re } s > 1$ . The sum here is over integral ideals coprime to  $\mathfrak{q}$ . The series has a meromorphic continuation to  $s \in \mathbb{C}$ ; the continuation is entire unless  $\chi \equiv 1$  on  $I_{\mathfrak{q}}$  when it has a simple pole at  $s = 1$  and no other singularity.

Define  $\log \zeta_K(s, \chi)$  by the series

$$\sum_{m \geq 1} \sum_{\mathfrak{p} \nmid \mathfrak{q}} \frac{\chi(\mathfrak{p}^m)}{m N\mathfrak{p}^{ms}}$$

for  $\text{Re } s > 1$ , where the inner sum is over prime ideals of  $K$ . Then by analytic continuation we can define  $\log \zeta_K(s, \chi)$  in any simply connected domain containing  $\text{Re } s > 1$  and not containing any zero or pole of  $\zeta_K(s, \chi)$ . Given  $z \in \mathbb{C}$  define  $(\zeta_K(s, \chi))^z$  as  $\exp(z \log \zeta_K(s, \chi))$ . Now set

$$P(s) = \prod_{\mathfrak{q}} \prod_{\chi \pmod{\mathfrak{q}}} \zeta_K(s, \chi)^{z_{\chi}} \left( \zeta_K^{(n_{\chi})}(s, \chi) \right)^{m_{\chi}} (\log \zeta_K(s, \chi))^{r_{\chi}} \tag{2}$$

where  $z_{\chi} \in \mathbb{C}$ ,  $n_{\chi} \in \mathbb{N}$  and  $m_{\chi}, r_{\chi}$  are non-negative integers that, for all but a finite number of  $\mathfrak{q}$  and  $\chi$ , satisfy  $z_{\chi} = 0$ , and  $m_{\chi} = r_{\chi} = 0$ .

Let  $b(\mathfrak{a}) \in \mathbb{C}$  be such that  $F_0(s) = \sum_{\mathfrak{a}} b(\mathfrak{a}) N\mathfrak{a}^{-s}$  is absolutely convergent in  $\text{Re } s \geq \frac{1}{2} + \delta$  for any  $\delta > 0$ . Putting

$$F(s) = P(s) F_0(s) = \sum_{\mathfrak{a}} \frac{a(\mathfrak{a})}{N\mathfrak{a}^s}, \tag{3}$$

say, for  $\text{Re } s > 1$ , we assume that, for all  $\varepsilon > 0$ ,  $a(\mathfrak{a}) \ll_{\varepsilon} (N\mathfrak{a})^{\varepsilon}$ . If we construct  $F(s)$  from Dirichlet series in the manner of (3) the verification on  $a(\mathfrak{a})$  will be straightforward. If, alternatively, we start with  $F(s)$  and check that it has a decomposition of type (3) then the verification will be easier if we note that the required bound on  $a(\mathfrak{a})$  necessarily follows from  $b(\mathfrak{a}) \ll_{\varepsilon} (N\mathfrak{a})^{\varepsilon}$ .

In this paper we study the distribution of the coefficients  $a(\mathfrak{a})$  when the ideals  $\mathfrak{a}$  are restricted geometrically. Following Hecke [7] let  $(\lambda_1, \dots, \lambda_{n-1})$  be a basis for the torsion-free characters on  $P_{(1)\infty}$  that satisfy

$$\lambda_i(\varepsilon) = 1, \quad 1 \leq i \leq n-1,$$

for all units  $\varepsilon \succ 0$  in  $\mathfrak{O}_K$ . Fixing an extension of each  $\lambda_i$  to a character on  $I = I_{(1)}$  then  $\lambda_i(\mathfrak{a}), 1 \leq i \leq n-1$  are defined for all ideals  $\mathfrak{a}$  and we can define  $\psi(\mathfrak{a}) = (\psi_i(\mathfrak{a})) \in \mathbb{R}^{n-1}/\mathbb{Z}^{n-1} = \mathbb{T}^{n-1}$  by  $\lambda_j(\mathfrak{a}) = e^{2\pi i \psi_j(\mathfrak{a})}$ . As in [3] set

$$\mathcal{S}(x, \psi_0, \ell) = \{\mathfrak{a} \in I, x(1-\ell) \leq N\mathfrak{a} \leq x(1+\ell), |\psi_j(\mathfrak{a}) - \psi_{0j}|_{\mathbb{T}} < \ell, 1 \leq j \leq n-1\}$$

where  $\psi_0 \in \mathbb{T}^{n-1}, 0 \leq \ell < \frac{1}{2}$  and  $|\alpha|_{\mathbb{T}} = \beta$  where  $-\frac{1}{2} < \beta < \frac{1}{2}$  and  $\beta \equiv \alpha \pmod{1}$ . We are interested in

$$A(x, \psi_0, \ell) = \sum_{\mathfrak{a} \in \mathcal{S}(x, \psi_0, \ell)} a(\mathfrak{a}).$$

When  $a(\mathfrak{a}) = \Lambda(\mathfrak{a})$ , von-Mangoldt's function, this sum has been studied in [3] while, if  $a(\mathfrak{a})$  is the characteristic function for relative norms of prime ideals from some number field extension of  $K$ , it has been studied in [4].

The main result of this paper is

**Theorem 1** *Let  $\varepsilon > 0$  be given and  $x, X$  be sufficiently large. Define*

$$\mathcal{C}_o = \{s \in \mathbb{C}; |s-1| = c_o, s \neq 1 - c_o\}$$

*traversed in the anti-clockwise direction. Here  $c_o$  is chosen so that  $F(s)$  has no singularities on the boundary or in the interior of the circle, radius  $c_o$ , centre 1. Set*

$$I(x, \ell) = \frac{(2\ell)^{n-1}}{2\pi i} \int_{x(1-\ell)}^{x(1+\ell)} \int_{\mathcal{C}_o} y^{s-1} F(s) ds dy.$$

*Then*

$$A(x, \psi, \ell) - I(x, \ell) \ll_{\varepsilon} x \ell^n \exp(-R(x)) \quad (4)$$

*for  $\ell > x^{-5/12n+10\varepsilon}$ , and*

$$\int_{\mathbb{T}^{n-1}} \int_X^{2X} |A(x, \psi, \ell) - I(x, \ell)|^2 dx d\psi \ll_{\varepsilon} X^3 \ell^{2n} \exp(-R(X)) \quad (5)$$

*for  $\ell = \ell(X) > X^{-5/6n+20\varepsilon}$ . Here  $R(x) = c(\log x)^{1/3}(\log_2 x)^{-1/3}$  where  $c$  is a constant that need not be the same at each occurrence and  $\log_2 x = \log \log x$ .*

The method of proof follows that given by Ramachandra [9] in the rational case. The results of that paper have been extended (and the misprints corrected) in [12] and [10]. It may be possible to follow the latter paper and, at the cost of stronger bounds on the coefficients  $a(\mathfrak{a})$ , remove the dependency on  $\varepsilon$  of the implied constants in (4) and (5). But the interest of Theorem 1 lies in the range of  $\ell$  and the  $\varepsilon$ 's that occur here come from our zero density results in Theorem 15.

## 2. Applications

Let  $f : I \rightarrow \mathbb{C}$ ,  $F : I \rightarrow \mathbb{N} \cup \{0\}$  denote multiplicative and additive arithmetic functions respectively. Given  $f$  and  $F_i$ ,  $1 \leq i \leq N$ , define, formally,

$$G(s, \mathbf{z}) = \sum_{\mathbf{a}} \frac{f(\mathbf{a}) \mathbf{z}^{\mathbf{F}(\mathbf{a})}}{N \mathbf{a}^s}$$

where  $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^N$  and  $\mathbf{z}^{\mathbf{F}(\mathbf{a})} = z_1^{F_1(\mathbf{a})} \dots z_N^{F_N(\mathbf{a})}$ . We are interested in the examples when  $G(s, \mathbf{z})$  can be expressed in the form (3) for all  $|\mathbf{z}| \leq 1$  where  $|\mathbf{z}| = \max_{1 \leq i \leq N} |z_i|$  and  $s$  in some half-plane.

To this end let  $\mathfrak{q} \in I$  be given. let  $\mathcal{C}(\mathfrak{q})$  denote the ideal class group  $\text{mod } \mathfrak{q}$  and  $\mathcal{C}^+(\mathfrak{q})$  the narrow ideal class group  $\text{mod } \mathfrak{q}$ . Set  $h = |\mathcal{C}(\mathfrak{q})|$  and  $h^+ = |\mathcal{C}^+(\mathfrak{q})|$ . We will assume

- (i) for  $\mathfrak{p} \nmid \mathfrak{q}$ ,  $f(\mathfrak{p})$  and  $F(\mathfrak{p})$  depend only on the class  $C \in \mathcal{C}_q^+$  containing  $\mathfrak{p}$ ,
- (ii)  $f(\mathfrak{p}^r) \ll c^r$  as  $r \rightarrow \infty$ , for some constant  $c < q_o^{1/2}$  where  $q_o$  is the smallest norm of the prime ideals of  $K$ , and
- (iii) given  $\varepsilon > 0$ ,  $f(\mathbf{a}) \ll_\varepsilon (N \mathbf{a})^\varepsilon$  for all  $\mathbf{a}$ .

Then (i) implies

$$\mathcal{G}(s, \mathbf{z}) = \mathcal{G}_q(s, \mathbf{z}) \prod_{C \in \mathcal{C}^+(\mathfrak{q})} \prod_{\mathfrak{p} \in C} \left( 1 + \frac{f(C) \mathbf{z}^{F(C)}}{N \mathfrak{p}^s} + \sum_{\mathfrak{p}, r \geq 2} \frac{f(\mathfrak{p}^r) \mathbf{z}^{F(\mathfrak{p}^r)}}{N \mathfrak{p}^{rs}} \right)$$

with the obvious notation  $f(C)$  and  $F(C)$ , and where  $\mathcal{G}_q(s, \mathbf{z})$  is a finite Euler product over the prime ideals dividing  $\mathfrak{q}$ . Because

$$\sum_{\mathfrak{p} \in C} \frac{1}{N \mathfrak{p}^s} = \frac{1}{h^+} \sum_{\chi} \bar{\chi}(C) \sum_{\mathfrak{p}} \frac{\chi(\mathfrak{p})}{N \mathfrak{p}^s}, \quad \text{Re } s > 1,$$

where  $\chi$  runs over the character group of  $\mathcal{C}^+(\mathfrak{q})$ , we write

$$\mathcal{G}(s, \mathbf{z}) = \mathcal{G}_q(s, \mathbf{z}) \left( \prod_{\chi} \zeta_K(s, \chi)^{z_\chi} \right) F_0(s, \mathbf{z}) \tag{6}$$

with

$$z_\chi = \frac{1}{h^+} \sum_C \bar{\chi}(C) f(C) \mathbf{z}^{\mathbf{F}(C)}.$$

For  $F_0(s, \mathbf{z})$  we apply the following rewriting of a result due to M. Delange [5].

**Lemma 2** *Assume that  $\{U_{\mathfrak{p}}(s, \mathbf{z})\}_{\mathfrak{p}}$  and  $\{V_{\mathfrak{p}}(s, \mathbf{z})\}_{\mathfrak{p}}$  are sequences of complex valued function defined on  $\mathbb{C} \times \mathbb{C}^N$ . Assume that on some domain  $B \subseteq \mathbb{C} \times \mathbb{C}^N$  there exist positive constants  $U_{\mathfrak{p}}, V_{\mathfrak{p}}$  for all  $\mathfrak{p}$ , satisfying*

$$|U_{\mathfrak{p}}(s, \mathbf{z})| \leq U_{\mathfrak{p}}, \quad |U_{\mathfrak{p}}(s, \mathbf{z}) - V_{\mathfrak{p}}(s, \mathbf{z})| \leq V_{\mathfrak{p}}$$

along with

$$\sum_{\mathfrak{p}} U_{\mathfrak{p}}^2 < +\infty \quad \text{and} \quad \sum_{\mathfrak{p}} V_{\mathfrak{p}} < +\infty.$$

Then the infinite product

$$\prod_{\mathfrak{p}} (1 + U_{\mathfrak{p}}(s, \mathbf{z})) \exp(-V_{\mathfrak{p}}(s, \mathbf{z}))$$

is uniformly convergent on  $B$  and is bounded on  $B$ .

We apply this to

$$\begin{aligned} & \prod_{\mathfrak{p} \nmid \mathfrak{q}} \left( 1 + \sum_{r \geq 1} \frac{f(\mathfrak{p}^r) \mathbf{z}^{\mathbf{F}(\mathfrak{p}^r)}}{N \mathfrak{p}^{rs}} \right) \left( \prod_{\chi} \zeta_K(s, \chi)^{-z_{\chi}} \right) \\ &= \prod_{\mathfrak{p} \nmid \mathfrak{q}} \left( 1 + \sum_{r \geq 1} \frac{f(\mathfrak{p}^r) \mathbf{z}^{\mathbf{F}(\mathfrak{p}^r)}}{N \mathfrak{p}^{rs}} \right) \exp \left( - \sum_{m \geq 1} \frac{f(C_{m, \mathfrak{p}}) \mathbf{z}^{\mathbf{F}(C_{m, \mathfrak{p}})}}{m N \mathfrak{p}^{ms}} \right) \end{aligned} \quad (7)$$

where  $C_{m, \mathfrak{p}}$  is the ideal class containing  $\mathfrak{p}^m$ . Let  $\sigma_1 > 1/2$  be given. Taking

$$U_{\mathfrak{p}}(s, \mathbf{z}) = \sum_{r \geq 1} \frac{f(\mathfrak{p}^r) \mathbf{z}^{\mathbf{F}(\mathfrak{p}^r)}}{N \mathfrak{p}^{rs}}$$

we have

$$|U_{\mathfrak{p}}(s, \mathbf{z})| \leq \sum_{r \geq 1} \frac{|f(\mathfrak{p}^r)|}{N \mathfrak{p}^{r\sigma_1}}$$

(for  $|\mathbf{z}| \leq 1$ ,  $\operatorname{Re} s \geq \sigma_1$ )

$$= \frac{|f(\mathfrak{p})|}{N \mathfrak{p}^{\sigma_1}} + B_{\mathfrak{p}}$$

say. Then

$$\sum_{\mathfrak{p}} |U_{\mathfrak{p}}(s, \mathbf{z})|^2 \ll \sum_{\mathfrak{p}} \frac{1}{N \mathfrak{p}^{2\sigma_1}} + \sum_{\mathfrak{p}} B_{\mathfrak{p}}^2.$$

But assumption (ii) implies  $B_{\mathfrak{p}} < \infty$ . So, choosing  $0 < \tau < (\sigma_1 - \frac{1}{2})/2$ , there exists  $P_0$  such that  $B_{\mathfrak{p}} N \mathfrak{p}^{-\tau} \leq 1$  for all  $N \mathfrak{p} > P_0$ . And then

$$\sum_{N \mathfrak{p} > P_0} B_{\mathfrak{p}}^2 \leq \sum_{N \mathfrak{p} > P_0} B_{\mathfrak{p}} N \mathfrak{p}^{\tau} \leq \sum_{\mathfrak{p}, r \geq 2} \frac{|f(\mathfrak{p}^r)|}{N \mathfrak{p}^{r\sigma_1}} N \mathfrak{p}^{\tau}$$

which converges, again by (ii). Hence

$$\sum_{\mathfrak{p}} |U_{\mathfrak{p}}(s, \mathbf{z})|^2 < +\infty.$$

Since  $f(C_{1,\mathfrak{p}}) = f(C) = f(\mathfrak{p})$  and similarly for  $\mathbf{F}(C_{1,\mathfrak{p}})$  we have, with

$$V_{\mathfrak{p}}(s, \mathbf{z}) = \sum_{m \geq 1} \frac{f(C_{m,\mathfrak{p}}) \mathbf{z}^{\mathbf{F}(C_{m,\mathfrak{p}})}}{mN\mathfrak{p}^{ms}},$$

that

$$U_{\mathfrak{p}}(s, \mathbf{z}) - V_{\mathfrak{p}}(s, \mathbf{z}) = \sum_{r=2}^{\infty} \frac{f(\mathfrak{p}^r) \mathbf{z}^{\mathbf{F}(\mathfrak{p}^r)}}{N\mathfrak{p}^{rs}} - \sum_{m=2}^{\infty} \frac{f(C_{m,\mathfrak{p}}) \mathbf{z}^{\mathbf{F}(C_{m,\mathfrak{p}})}}{mN\mathfrak{p}^{ms}}$$

Then, again by (ii), we can deduce

$$\sum_{\mathfrak{p}} |U_{\mathfrak{p}}(s, \mathbf{z}) - V_{\mathfrak{p}}(s, \mathbf{z})| < +\infty$$

for  $|\mathbf{z}| \leq 1$  and  $\operatorname{Re} s \geq \sigma_1$ .

Therefore, by Lemma 2, the infinite product (7) and hence, by (6),  $F_0(s, \mathbf{z})$  converges uniformly for  $\operatorname{Re} s \geq \sigma_1, |\mathbf{z}| \leq 1$ . Further, each  $U_{\mathfrak{p}}(s, \mathbf{z})$  is a holomorphic function of  $s$  for  $\operatorname{Re} s > \frac{1}{2}, |\mathbf{z}| \leq 1$  and so by uniform convergence, (7) and  $F_0(s, \mathbf{z})$  are holomorphic for  $\operatorname{Re} s > \sigma_1, |\mathbf{z}| \leq 1$ .

So we have in (6) a decomposition of the form (3) and (2). Assumption (iii) implies we can apply Theorem 1. In the following examples  $f$  and  $F_i$  will always satisfy the assumptions (i), (ii), and (iii) above.

### 2.1. Example 1

Assume  $\mathbf{z} = z \in \mathbb{C}$  and  $f(\mathfrak{p}) = F(\mathfrak{p}) = 1$  for all prime ideals. This case has been studied by Grytczuk [6] and Wu [14]. Let

$$\mathcal{G}_0(s, z) = (s-1)^z \frac{\mathcal{G}(s, z)}{s} \frac{\sin \pi z}{\pi}$$

and

$$\mathcal{J}(x, \ell, z) = \int_0^{c_0} x^{-r} k(\ell, r) \mathcal{G}_0(1-r, z) r^{-z} dr$$

where  $c_0$  is as in Theorem 1 and

$$k(\ell, r) = \frac{(1+\ell)^{1-r} - (1-\ell)^{1-r}}{2\ell}.$$

Then Theorem 1 gives

**Theorem 3** *Assume  $|z| \leq 1, z \neq 1$ . Then*

$$\sum_{\mathfrak{a} \in \mathcal{S}(x, \psi, \ell)} f(\mathfrak{a}) z^{\mathbf{F}(\mathfrak{a})} = (2\ell)^n x \mathcal{J}(x, \ell, z) + O_{\varepsilon}(x \ell^n \exp(-R(x))) \quad (8)$$

for  $\frac{1}{2} \geq \ell > x^{-5/12n+\varepsilon}$ . Also

$$\sum_{\mathfrak{a} \in \mathcal{S}(x, \psi, \ell)} f(\mathfrak{a}) = (2\ell)^n x M(f) + O_{\varepsilon}(x \ell^n \exp(-R(x)))$$

for  $\frac{1}{2} \geq \ell > x^{-5/12n+\varepsilon}$ , where

$$M(f) = \rho_K \prod_{\mathfrak{p}} \left( 1 + \sum_{v=2}^{\infty} \frac{f(\mathfrak{p}^v) - f(\mathfrak{p}^{v-1})}{N\mathfrak{p}^v} \right)$$

and  $\rho_K$  is the residue of  $\zeta_K(s)$  at  $s = 1$ .

### Proof of Theorem 3

Here we only indicate how  $\mathcal{J}(x, \ell, z)$  arises. Deform the contour  $\mathcal{C}_o$  of Theorem 1 into the contour  $\mathcal{C}_\delta$ ,  $0 < \delta < c_o$ , of  $[1 - c_o, 1 - \delta]$  with argument  $-\pi$ , the circle  $|s - 1| = \delta$ ,  $s \neq 1 - \delta$  and  $[1 - \delta, 1 - c_o]$  with argument  $\pi$ . When  $|\mathbf{z}| \leq 1$ ,  $z \neq 1$ , the integral over the circular part of  $\mathcal{C}_\delta$  tends to 0 as  $\delta \rightarrow 0$ . The two horizontal components tend to

$$\begin{aligned} & \frac{1}{2\pi i} \left( \int_{c_0}^0 (re^{-i\pi})^{-z} y^{-r} H(1 - r, z) (-dr) \right. \\ & \quad \left. + \int_0^{c_0} (re^{i\pi})^{-z} y^{-r} H(1 - r, z) (-dr) \right) \\ & = \int_0^{c_0} r^{-z} \sin \pi z y^{-r} \frac{H(1 - r, z)}{\pi} dr \end{aligned}$$

where  $H(s, z) = (s - 1)^z \mathcal{G}(s, z)$  is regular at  $s = 1$ . Integrating over  $y$  gives the required result.  $\blacksquare$

Theorem 1 of Wu [14] can be recovered, though with a weaker error term, by taking  $\ell = 1/2$  and summing over appropriate  $x$ .

When  $z = -1$  there is no main term in (8). We then take either  $f = \mu^2$ ,  $F = \omega$  or  $f \equiv 1$ ,  $F = \omega$ , where we are using the notation for well known arithmetic functions on  $\mathbb{Z}$  for the same functions on the integral ideals. Thus we obtain estimates for sums of the mobius function  $\mu$ , and Liouville's function  $\lambda$ , respectively.

### Corollary 4

$$\sum_{\mathfrak{a} \in \mathcal{S}(x, \psi, \ell)} \mu(\mathfrak{a}) \ll_{\varepsilon} x \ell^n \exp(-R(x))$$

for  $\ell > x^{-5/12n+\varepsilon}$ , and

$$\int_{\mathbb{T}^{n-1}} \int_X^{2X} \left| \sum_{\mathfrak{a} \in \mathcal{S}(x, \psi, \ell)} \mu(\mathfrak{a}) \right|^2 dx d\psi \ll_{\varepsilon} X^3 \ell^{2n} \exp(-R(X))$$

for  $\ell = \ell(X) > X^{-5/6n+\varepsilon}$ .

These results hold for  $\lambda$  replacing  $\mu$ . In this way we generalize the results of Ramachandra [9].

## 2.2. Example 2

For  $k \in \mathbb{N} \cup \{0\}$  let

$$\nu_k(x) = \sum_{\substack{N\mathbf{a} \leq x \\ F(\mathbf{a})=k}} f(\mathbf{a}) \quad \text{and} \quad \nu_k(x, \psi, \ell) = \sum_{\substack{\mathbf{a} \in \mathcal{S}(x, \psi, \ell) \\ F(\mathbf{a})=k}} f(\mathbf{a}).$$

At some stage in the analysis of these we must consider

$$\frac{1}{2\pi i} \oint_{|z|=1-\varepsilon} z^{-k-1} F_{\mathbf{q}}(s, z) \prod_{\chi} \zeta_K(s, \chi)^{z_{\chi}} dz,$$

where  $F_{\mathbf{q}}(s, z) = \mathcal{G}_{\mathbf{q}}(s, z) F_0(s, z)$  in the notation of (6). On evaluating, this is a sum of terms

$$\frac{1}{m!} F_{\mathbf{q}}^{(m)}(s, 0) \prod_{\chi} c_{\chi} (\log \zeta_K(s, \chi))^{a_{\chi}} \zeta_K(s, \chi)^{\alpha_{\chi,0}} \quad (9)$$

for some  $c_{\chi} \in \mathbb{R}$ . Here

$$\alpha_{\chi,0} = \frac{1}{h^+} \sum_{F(C)=0} \bar{\chi}(C) f(C)$$

and  $a_{\chi}, m \in \mathbb{N} \cup \{0\}$  satisfy

$$\begin{aligned} & \text{“If expanded in powers of } z, \prod_{\chi} z_{\chi}^{a_{\chi}} z^m \text{ will have,} \\ & \text{as one of its terms, a non-zero multiple of } z^k \text{.”} \end{aligned} \quad (10)$$

Asymptotic expansions for summatory functions with Dirichlet series of the form (9) are given by many authors, e.g. Scourfield [11] and Kaczorowski [8].

We examine a special case and assume

(A) For all  $C^+(\mathbf{q})$ ,  $F(C) \neq 0$  (so  $\alpha_{\chi,0} = 0$  in (9))

and

(B) there exists  $C \in C^+(\mathbf{q}) : F(C) = 1$  and  $\beta := \frac{1}{h^+} \sum_{F(C)=1} f(C)$  satisfies  $0 < \beta \leq 1$  (so

$z_{\chi_0} = \beta z + \text{higher powers of } z$ ).

With these assumptions (10) is satisfied when  $m = 0$ ,  $a_{\chi_0} = k$  and  $a_{\chi} = 0$  for all  $\chi \neq \chi_0$ . This will, in fact, give the dominant contribution from all the terms of the form (9) that might arise. To calculate this contribution we first note that an analogue of (8) holds for  $\sum f(\mathbf{a}) z^{F(\mathbf{a})}$ ,  $\mathbf{a} \in \mathcal{S}(x, \psi, \ell)$ , when  $|\beta z| \leq 1, \beta z \neq 1$ . The only difference is that

$$\mathcal{G}_0^{(\beta)}(s, z) r^{-\beta z} = (s-1)^{\beta z} \frac{\mathcal{G}(s, z)}{s} \frac{\sin \pi \beta z}{\pi} r^{-\beta z}$$

replaces  $\mathcal{G}_0(s, z) r^{-z}$  in the definition of  $\mathcal{J}(x, \ell, z)$ . Multiply both sides of this analogue by  $z^{-k-1}$ , integrate over  $|z| = 1 - \varepsilon'$  (which is  $< 1/\beta$  and so, allowable) and let  $\varepsilon' \rightarrow 0$  to obtain

$$\nu_k(x, \psi, \ell) = (2\ell)^n x \int_0^{c_0} x^{-r} k(\ell, r) W_k^{(\beta)}(r) dr + O(x \ell^n \exp(-R(x))) \quad (11)$$



for  $\ell > x^{-5/12n+\varepsilon}$ . Here  $W_k^{(\beta)}(r)$  is the coefficient of  $z^k$  in Taylors development of  $\mathcal{G}_0^{(\beta)}(1-r, z)r^{-\beta z}$  at  $z=0$ . With the conditions on  $f$  and  $F$  of example 1, (so  $\beta=1$ ),  $W_k(r)$  has been studied by Wu [14]. Here we indicate, without proof, changes to the results of [14]. So, as in Lemma 6 of [14] we have

$$W_k^{(\beta)}(r) = \sum_{j=0}^J r^j Q_{j,k-1}(-\beta \log r) + O_\beta \left( \left( \frac{r}{2c_o} \right)^{J+1} (-\beta \log r)^{k-1} \right) \quad (12)$$

uniformly for  $0 < r \leq c_o$ ,  $J \geq 0$ ,  $k \geq 1$  and where  $Q_{j,k-1}(X)$  is a polynomial with real coefficients of degree  $k-1$  at most. Because  $k(\ell, r) \ll 1$ , the error from (12) contributes the same to the integral in (11) as does the corresponding term to  $\nu_k(x)$  in Theorem 2 of [14], namely

$$\ll_\beta \frac{(\beta \log_2 x)^{k-1}}{(2c_o \log x)^{J+2}} \left( \frac{(k-1)!}{J+1} + (J+1)! \right) \quad (13)$$

(see equation (4.8) of [14]). Writing

$$Q_{j,k-1}(X) = \sum_{n=0}^{k-1} \frac{\alpha_{j,k-n}}{n!} X^n$$

we have, from (4.5) of [4],  $\alpha_{j,m} \ll_\beta (2c_o)^{-j}$ . Note that  $c_o$  depends only on  $\mathfrak{q}$ , not  $\beta$ . The  $j^{th}$  term of the sum in (12) contributes

$$\sum_{n=0}^{k-1} \frac{\alpha_{j,k-n}}{n!} \int_0^{c_o} x^{-r} k(\ell, r) r^j (-\beta \log r)^n dr \quad (14)$$

to the integral in (11). We complete this integral to  $\infty$ , bounding the tail as

$$\ll \int_{c_o}^{\infty} x^{-r} r^j |\log r|^n dr$$

(since  $k(\ell, r) \ll 1$ ,  $\beta < 1$ )

$$= \left( \int_{c_o}^1 + \int_1^{\infty} \right) x^{-r} r^j |\log r|^n dr.$$

In the first integral,  $r = 1/t$  gives

$$\ll x^{-c_o} \int_1^{1/c_o} \frac{(\log t)^n}{t^{j+2}} dt \ll x^{-c_o} (\log 1/c_o)^n.$$

In the second integral,  $r = u \log x$  gives

$$\begin{aligned} &\ll (\log x)^{-j-1} \int_{\log x}^{\infty} e^{-u} u^j (\log u - \log_2 x)^n du \\ &\ll (\log x)^{-j-1} \left( \int_{\log x}^{(\log_2 x)(\log x)} e^{-u} u^j (\log_3 x)^n du + \int_{(\log_2 x)(\log x)}^{\infty} e^{-u} u^j (\log u)^n du \right) \end{aligned} \quad (15)$$

Assume both  $j$  and  $k$  (and thus  $n$ ) are  $\leq \frac{1}{4} \log 2c_0$ . Then  $\log(u^j(\log u)^n) \leq (\log x)(2 \log u) < u/2$  in the range of the second integral above, so (15) is

$$\begin{aligned} &\ll (\log x)^{-j-1} (x^{-1}(\log x \cdot \log_2 x)^j (\log_3 x)^n + x^{-\log_2 x}) \\ &\ll x^{-1}(\log_2 x)^j (\log_3 x). \end{aligned}$$

Thus the error in (14) is

$$\begin{aligned} &\ll \frac{1}{(2c_0)^j} \sum_{n=0}^{k-1} \frac{1}{n!} (x^{-c_0}(\log 1/c_0)^n + x^{-1}(\log_2 x)^j (\log_3 x)) \\ &\ll \frac{1}{(2c_0)^j} (x^{-c_0}(\log 1/c_0)^{k-1} + x^{-1}(\log_2 x)^j (\log_3 x)) \end{aligned}$$

and hence, in the integral (11),

$$\ll \frac{J}{(2c_0)^J} (x^{-c_0}(\log(1/c_0))^{k-1} + x^{-1}(\log_2 x)^J (\log_3 x)^{k-1}). \quad (16)$$

The completed integral,  $I$  say, in (14) can be written as

$$(2\ell)^{-1} ((1+\ell)I(x(1+\ell)) - (1-\ell)I(x(1-\ell)))$$

where

$$\begin{aligned} I(y) &= \int_0^\infty y^{-r} r^j (-\beta \log r)^n dr \\ &= \frac{\beta^n}{(\log y)^{j+1}} \int_0^\infty e^{-u} u^j \sum_{m=0}^n \binom{n}{m} (\log_2 y)^m (-\log u)^{n-m} du \\ &= \frac{\beta^n}{(\log y)^{j+1}} \sum_{m=0}^n \binom{n}{m} (\log_2 y)^m \Gamma^{(n-m)}(j+1) (-1)^{n-m}. \end{aligned}$$

So it is important to calculate

$$\begin{aligned} &(2\ell)^{-1} \left( \frac{(1+\ell)(\log_2 x(1+\ell))^m}{(\log x(1+\ell))^{j+1}} - \frac{(1-\ell)(\log_2 x(1-\ell))^m}{(\log x(1-\ell))^{j+1}} \right) \\ &= \frac{(\log_2 x)^m}{(\log x)^{j+1}} + \frac{m(\log_2 x)^{m-1} - (j+1)(\log_2 x)^m}{(\log x)^{j+2}} \\ &\quad + O\left((j^2 + m^2)\ell \frac{(\log_2 x(1+\xi))^m}{(\log x(1+\xi))^{j+2}}\right) \end{aligned} \quad (17)$$

for some  $|\xi| < \ell$ , on using a mean value result.

The summation of the  $(j^2 + m^2) (\log_2 x (1 + \xi))^m$  over  $m$  and  $n$  is

$$\begin{aligned} &\ll \frac{(j^2 + k^2)}{(2c_0)^j} \sum_{m=0}^{k-1} \frac{(\log_2 x (1 + \xi))^m \beta^m}{m!} \left( \sum_{m \leq n \leq k-1} \frac{\Gamma^{(n-m)}(j+1)}{(n-m)!} \right) \\ &\ll \frac{j!(j^2 + k^2)}{(2c_0)^j} \beta \log x. \end{aligned}$$

So the contribution of the error term of (17) to the integral of (14) is, on summing over  $0 \leq j \leq J$ , dominated by (13) if  $\ell < (\log x)^{-J}$  as we now assume. But further, if we demand  $J(x) \log_2 x \sim R(x)$  then both the error terms (13) and (16) will be dominated by  $\exp(-R(x))$ , as long as  $k \ll J(x)$ .

Finally substituting the main terms from (17) into (14) and summing over  $j$  gives the main terms for  $\nu_k(x, \psi, \ell)$  in

**Theorem 5** *Put  $J(x) := (\log x)^{1/3} (\log_2 x)^{-4/3}$ . Then for  $1 \leq k \leq C_1 J(x)$ , and  $C_2 \exp(-R(x)) > \ell > x^{-5/12n+\varepsilon}$  we have*

$$\nu_k(x, \psi, \ell) = \frac{(2\ell)^n x}{\log x} \sum_{0 \leq j \leq C_3 J(x)} \frac{P_{j,k-1}(\log_2 x)}{(\log x)^j} + O(x \ell^n \exp(-R(x))),$$

where  $P_{j,k-1}(X)$  is a polynomial of degree at most  $k-1$ . The main term is

$$\frac{(2\ell)^n x \beta^k (\log_2 x)^{k-1}}{(k-1)! \log x} \prod_{\mathfrak{p}} \left( 1 + \sum_{\substack{r \geq 1 \\ F(\mathfrak{p}^r) = 0}} \frac{f(\mathfrak{p}^r)}{N \mathfrak{p}^r} \right).$$

The result on the main term follows from

$$\alpha_{0,1} = \beta \prod_{\mathfrak{p} | \mathfrak{q}} \left( 1 + \sum_{\substack{r \geq 1 \\ F(\mathfrak{p}^r) = 0}} \frac{f(\mathfrak{p}^r)}{N \mathfrak{p}^r} \right) \prod_{\mathfrak{p} \nmid \mathfrak{q}} \left( 1 + \sum_{\substack{r \geq 2 \\ F(\mathfrak{p}^r) = 0}} \frac{f(\mathfrak{p}^r)}{N \mathfrak{p}^r} \right)$$

and  $F(\mathfrak{p}) = 0$  for all  $\mathfrak{p} \nmid \mathfrak{q}$ .

As a special case consider  $K = \mathbb{Q}$ ,  $C_1 = \{n : n \equiv 1 \pmod{4}\}$  and  $C_2 = \{n : n \equiv 3 \pmod{4}\}$  with

$$\begin{aligned} f(p^r) &= 1 \text{ for all } r \geq 1 & \text{if } p \in C_1 \text{ or } p = 2, \\ f(p^r) &= \begin{cases} 1 & \text{if } 2|r \\ 0 & \text{if } 2 \nmid r \end{cases} & \text{if } p \in C_2. \end{aligned}$$

So  $f(n) = 1$  if and only if  $n$  is the sum of two squares (which we write as  $n = 2\Box$ ). It is the question of counting such  $n$  in small intervals that led originally to the Hooley-Huxley contour. Let  $F(p^r) = 1$  for all primes  $p$  and  $r \geq 1$ . Then following the above proof with  $J = 0$  we obtain

**Corollary 6** *For fixed  $k$*

$$|\{x < n < x + h, n = 2\Box, w(n) = k\}| = \frac{h(\log_2 x)^{k-1}}{2^k(k-1)!\log x} (1 + o_\varepsilon(1))$$

for  $x > h > x^{7/12+\varepsilon}$ .

Further deductions from Theorem 1 with  $F(s)$  of the form (6) with  $\mathbf{z} \in \mathbb{C}^N$ ,  $N > 1$ , are given in a paper in preparation.

### 2.3. Example 3

For a fixed  $\alpha > 0$  define

$$\mathcal{B}_\alpha(\mathfrak{a}) = \sum_{\mathfrak{p}|\mathfrak{a}} N^\alpha(\mathfrak{p}).$$

When  $K = \mathbb{Q}(i)$ , Zarzycki [15] has studied the local distribution of  $\mathcal{B}_\alpha(\mathfrak{a})$ . Though the Hooley-Huxley method is used, it is only applied to the norm of the ideals  $\mathfrak{a}$ . There is a far weaker restriction on the argument of the  $\mathfrak{a}$ .

To apply Theorem 1 note that

$$\begin{aligned} \sum_{\mathfrak{a}} \frac{\mathcal{B}_\alpha(\mathfrak{a})/N\mathfrak{a}^\alpha}{N\mathfrak{a}^s} &= \zeta_K(s + \alpha) \sum_{\mathfrak{p}} \frac{1}{N\mathfrak{p}^s} \\ &= \zeta_K(s + \alpha) (\log \zeta_K(s) + G(s)) \end{aligned}$$

for  $\text{Re } s > 1$  where  $G(s)$  is a regular function for  $\text{Re } s > \frac{1}{2}$ . The integral over  $\mathcal{C}_0$  in Theorem 1 has, in this case, the particularly simple form of

$$\frac{1}{2\pi i} \int_{\mathcal{C}_0} y^{s-1} \zeta_K(s + \alpha) \log \frac{1}{s-1} ds = \frac{\zeta_K(1 + \alpha)}{\log y} + O_\alpha \left( \frac{1}{\log^2 y} \right)$$

for  $0 < c_0 < \alpha/2$ , say. So the error here dominates the contribution to our results of the pole of  $\zeta_K(s + \alpha)$  at  $1 - \alpha$ . Hence

**Theorem 7** *For  $\alpha > 0$  and  $\frac{1}{2} \geq \ell \geq x^{-5/12n+\varepsilon}$  we have*

$$\sum_{\mathfrak{a} \in \mathcal{S}(x, \psi, \ell)} \frac{\mathcal{B}_\alpha(\mathfrak{a})}{N\mathfrak{a}^\alpha} = (2\ell)^{n-1} x \zeta(1 + \alpha) \int_{x(1-\ell)}^{x(1+\ell)} \frac{dy}{\log y} + O \left( \frac{x\ell^n}{\log^2 x} \right).$$

To clear the denominator, we use

$$\sum_{\mathfrak{a} \in \mathcal{S}(x, \psi, \ell)} \mathcal{B}_\alpha(\mathfrak{a}) = x^\alpha (1 + O(\ell)) \sum_{\mathfrak{a} \in \mathcal{S}(x, \psi, \ell)} \frac{\mathcal{B}_\alpha(\mathfrak{a})}{N\mathfrak{a}^\alpha},$$

obtaining

**Corollary 8** *For  $\alpha > 0$ ,  $q \geq 1$  fixed and  $\frac{1}{2} \geq \ell \geq x^{-5/12n+\varepsilon}$*

we have

$$\sum_{\mathfrak{a} \in \mathcal{S}(x, \psi, \ell)} \mathcal{B}_\alpha(\mathfrak{a}) = (2\ell)^n x \left( \frac{\zeta_K(1 + \alpha)}{\log x} + O_\alpha \left( \frac{1}{\log^2 x} \right) \right).$$

When  $K = \mathbb{Q}(i)$  Zarzycki has, in [16], given another application of the Hooley-Huxley method. This time both the norm and argument of the ideals are equally constrained as in our Theorem 1. Unfortunately, [16] lacks references to necessary results such as zero-free regions for Hecke  $L$ -functions which we hope the present paper furnishes. Also, the quality of the final results in [16] depend on zero density results such as (30) and there are too few details in the equivalent result, Lemma 2 of [16], to verify the quoted result. Further, the application in [16] to prime ideals in sectors can be dealt with by more classical methods, as in [3].

### 3. Sums over Gaussian Integers

One of the motivating situations for the present work is when the arithmetic functions are defined on the Gaussian integers. Then, the natural region of localization might be considered to be a disc  $\mathcal{D}(\omega, r) = \{z \in \mathbb{C} : |z - \omega| < r\}$ , rather than  $\mathcal{S}(x, \psi, \ell)$ . Of course, a sum over  $\alpha \in \mathcal{D}(\omega, r)$  can be decomposed into a union of sums over  $\alpha : (\alpha) \in \mathcal{S}(y, \psi, \ell)$  for various  $(y, \psi)$ , along with  $\alpha$  near the boundary of  $\mathcal{D}(\omega, r)$ . With  $\ell$  sufficiently small compared to  $r$  these points near the boundary can be shown to be relatively few in number. On the remaining points we can apply results of the form of the previous section. The restrictions of these results, namely that  $\ell$  cannot be too small lead, in turn, to similar restrictions on the radius  $r$ . To simplify the application of this idea we will, below, replace the union of sums by an integral.

In  $\mathbb{Q}(i)$ , a principal ideal domain, the basis for the group of groessencharaktere consists simply of  $\lambda((\alpha)) = (\alpha/|\alpha|)^4$  and so  $\psi((\alpha))$  is the fractional part of  $2(\arg \alpha)/\pi$ . Our arithmetic functions will be assumed to be functions of ideals only. To simplify matters we will only take generators of ideals that lie in the first quadrant. Because of this we modify the definition of  $\mathcal{D}(\omega, r)$  to read

$$\mathcal{D}(\omega, r) = \{z \in \mathbb{C} : 0 \leq \arg z < \pi/2 \text{ and there exists a unit } \varepsilon \text{ of } \mathbb{Z}[i] \text{ such that } |\varepsilon z - \omega| < r\}.$$

And we note that  $(\alpha) \in \mathcal{S}(x, \psi, \ell)$  then implies  $||\alpha|^2 - y| < \ell y$  and  $|2(\arg \alpha)/\pi - \psi| < \ell$ . For  $\alpha \in \mathbb{Z}[i]$  from the first quadrant and  $\ell$  fixed we introduce a weight function

$$w(\alpha) = \iint_{(\alpha) \in \mathcal{S}(y, \psi, \ell)} \frac{dy}{y} d\psi = (2\ell) \log \left( \frac{1 + \ell}{1 - \ell} \right) = c_\ell$$

say, independent of  $\alpha$ . Then, for our arithmetic function  $f$ ,

$$\begin{aligned} \sum_{\alpha \in \mathcal{D}(\omega, r)} f(\alpha) &= c_\ell^{-1} \sum_{\alpha \in \mathcal{D}(\omega, r)} f(\alpha) \iint_{(\alpha) \in \mathcal{S}(y, \psi, \ell)} \frac{dy}{y} d\psi \\ &= c_\ell^{-1} \iint_{\mathcal{D}_1(\omega, r)} \sum_{\substack{(\alpha) \in \mathcal{S}(y, \psi, \ell) \\ \alpha \in \mathcal{D}(\omega, r)}} f(\alpha) \frac{dy}{y} d\psi \end{aligned} \tag{18}$$

where

$$\mathcal{D}_1(\omega, r) = \{(y, \psi) : \text{there exists } \alpha \in \mathcal{D}(\omega, r) \text{ with } (\alpha) \in \mathcal{S}(y, \psi, \ell)\}.$$

The main contribution to this integral will come from the region

$$\mathcal{D}_2(\omega, r) = \{(y, \psi) : \text{If } \mathfrak{a} \in \mathcal{S}(y, \psi, \ell) \text{ then } \mathfrak{a} = (\alpha) \text{ with } \alpha \in \mathcal{D}(\omega, r)\}.$$

The final result will be given as an integral over

$$\mathcal{D}_0(\omega, r) = \left\{ (y, \psi) : y^{1/2} e^{i\frac{\pi}{2}\psi} \in \mathcal{D}(\omega, r) \right\}.$$

**Lemma 9** *There exists a constant  $c > 0$  such that*

$$(i) \quad \mathcal{D}_1(\omega, r) \subseteq \mathcal{D}_0(\omega, r + c|\omega|\ell),$$

$$(ii) \quad \mathcal{D}_0(\omega, r - c|\omega|\ell) \subseteq \mathcal{D}_2(\omega, r).$$

**Proof**

(i)  $(y, \psi) \in \mathcal{D}_1(\omega, r)$  implies that there exists  $\alpha \in \mathbb{Z}[i]$  from the first quadrant with  $||\alpha|^2 - y| < \ell y$  and  $|2(\arg \alpha)/\pi - \psi| < \ell$  and a unit  $\varepsilon$  such that  $|\varepsilon\alpha - \omega| < r$ . But then

$$\begin{aligned} \left| \varepsilon y^{1/2} e^{i(\pi/2)\psi} - \omega \right| &= \left| \varepsilon y^{1/2} e^{i(\pi/2)\psi} - \varepsilon|\alpha| e^{i(\pi/2)\psi} + \varepsilon|\alpha| e^{i(\pi/2)\psi} - \varepsilon\alpha + \varepsilon\alpha - \omega \right| \\ &\leq \left| y^{1/2} e^{i(\pi/2)\psi} - |\alpha| e^{i(\pi/2)\psi} \right| + |\alpha| \left| e^{i(\pi/2)\psi} - e^{i\arg \alpha} \right| + |\varepsilon\alpha - \omega| \\ &\leq |y^{1/2} - |\alpha|| + |\alpha| |\pi\psi/2 - \arg \alpha| + r \end{aligned}$$

(using  $|e^{i\zeta_1} - e^{i\zeta_2}| \leq |\zeta_1 - \zeta_2|$ )

$$\leq y^{1/2}\ell + |\alpha|\frac{\pi}{2}\ell + r \leq c|\omega|\ell + r$$

for some  $c$ , as required.

(ii) Assume  $(y, \psi) \in \mathcal{D}_0(\omega, r - c|\omega|\ell)$ , with  $c$  as above. So there exists a unit  $\varepsilon$  such that

$$\left| \varepsilon y^{1/2} e^{i(\pi/2)\psi} - \omega \right| \leq r - c|\omega|\ell.$$

Assume  $\mathfrak{a} \in \mathcal{S}(x, \psi, \ell)$  has been chosen and  $\mathfrak{a} = (\alpha)$  with  $\alpha$  from the first quadrant. So, with the unit above,

$$\begin{aligned} |\varepsilon\alpha - \omega| &\leq \left| \alpha - y^{1/2} e^{i(\pi/2)\psi} \right| + r - c|\omega|\ell \\ &\leq c|\omega|\ell + r - c|\omega|\ell \end{aligned}$$

by the argument in part (i). Thus  $\alpha \in \mathcal{D}(\omega, r)$  and hence  $(y, \psi) \in \mathcal{D}_2(\omega, r)$ . ■

We can now state our main result as

**Proposition 10** *Let  $f$  be an arithmetic function defined on the ideals of  $\mathbb{Q}(i)$ . For  $\omega \in \mathbb{C}$  assume that  $\ell = \ell(\omega)$  satisfies  $\ell(\omega) \rightarrow 0$  and  $|\omega|\ell(\omega) \rightarrow \infty$  as  $|\omega| \rightarrow \infty$ . Then for  $0 < r < |\omega|$ ,*

$$\sum_{\alpha \in \mathcal{D}(\omega, r)} f(\alpha) = (2\ell)^{-2} \iint_{\mathcal{D}_0(\omega, r)} \sum_{(\alpha) \in \mathcal{S}(y, \psi, \ell)} f(\alpha) \frac{dy}{y} d\psi + E, \quad (19)$$

where, in all cases,

$$E \ll r|\omega|\ell \max_{\alpha \in D(\omega, r)} |f(\alpha)|.$$

If we know further that  $f$  is of constant sign and

$$\sum_{(\alpha) \in \mathcal{S}(y, \psi, \ell)} f(\alpha) \ll y\ell^2(\log y)^a$$

for some  $a \in \mathbb{Z}$ , then  $E \ll r|\omega|\ell(\log |\omega|)^a$ .

**Proof** Continuing from (18)

$$\begin{aligned} \sum_{\alpha \in \mathcal{D}(\omega, r)} f(\alpha) &= c_\ell^{-1} \iint_{\mathcal{D}_2(\omega, r)} \sum_{(\alpha) \in \mathcal{S}(y, \psi, \ell)} f(\alpha) \frac{dy}{y} d\psi + E_1, \\ &= c_\ell^{-1} \iint_{\mathcal{D}_0(\omega, r)} \sum_{(\alpha) \in \mathcal{S}(y, \psi, \ell)} f(\alpha) \frac{dy}{y} d\psi + E_1 + E_2. \end{aligned}$$

Here

$$E_1 = c_\ell^{-1} \iint_{\mathcal{D}_1 \setminus \mathcal{D}_2} \sum_{\substack{(\alpha) \in \mathcal{S}(y, \psi, \ell) \\ \alpha \in \mathcal{D}(\omega, r)}} f(\alpha) \frac{dy}{y} d\psi.$$

Letting  $M = \max_{\alpha \in D(\omega, r)} |f(\alpha)|$  we see that the inner sum here is  $\leq M |\mathcal{S}(y, \psi, \ell)|$ . It is implicit in the proof of Lemma 1 in [4] that  $\mathcal{S}(y, \psi, \ell) \ll (y^{1/2}\ell + 1)^2$  which is  $\ll y\ell^2$  by our assumptions on  $\ell$ . So

$$E_1 \ll M \iint_{\mathcal{D}_1 \setminus \mathcal{D}_2} dy d\psi \ll M \iint_{\substack{\mathcal{D}_0(\omega, r+c|\omega|\ell) \\ \setminus \mathcal{D}_0(\omega, r-c|\omega|\ell)}} dy d\psi$$

by Lemma 9. On changing the variable to  $t = y^{1/2}$  this double integral is seen to be the area (expressed in polar coordinates) of  $\mathcal{D}_0(\omega, r+c|\omega|\ell) \setminus \mathcal{D}_0(\omega, r-c|\omega|\ell)$  which is  $\ll r|\omega|\ell$ . Hence  $E_1 \ll Mr|\omega|\ell$ .

Assuming that the additional properties described in the proposition hold for our  $f$  we enlarge  $E_1$  by dropping the  $\alpha \in \mathcal{D}(\omega, r)$  condition. And then we have

$$E_1 \ll \iint_{\mathcal{D}_1 \setminus \mathcal{D}_2} (\log y)^a dy d\psi \ll (\log |\omega|)^a \iint_{\mathcal{D}_1 \setminus \mathcal{D}_2} dy d\psi \ll (\log |\omega|)^a r|\omega|\ell.$$

Finally,

$$c_\ell^{-1} = \frac{1}{(2\ell)^2} (1 + O(\ell))$$

while the double integral in (19) is  $\ll M\ell^2 r^2$  in general and  $\ll \ell^2 r^2 (\log |\omega|)^a$  with the stronger assumptions. Hence  $c_\ell^{-1}$  can be replaced by  $(2\ell)^{-2}$  with errors  $\ll M\ell^3 r^2$  or  $\ll \ell^3 r^2 (\log |\omega|)^a$  which, because  $r < |\omega|, \ell < 1$ , are less than the errors appearing in the statement of the proposition. ■

The following results for  $\mathbb{Q}(i)$  are now immediate from §2

**Corollary 11** *Given  $\omega \in \mathbb{C}$  with  $|\omega| > 1$ , the following hold for  $|\omega| > r > |\omega|^{7/12+\varepsilon}$ .*

(i) *For an arithmetic function  $f : \mathbb{Q}(i) \rightarrow \mathbb{C}$  satisfying the conditions of Theorem 3,*

$$\sum_{\alpha \in \mathcal{D}(\omega, r)} f(\alpha) = 4r^2 M(f) + O_\varepsilon(r^2 \exp(-R(|\omega|))).$$

(ii) *For the Mobius function  $\mu$  we have*

$$\sum_{\alpha \in \mathcal{D}(\omega, r)} \mu(\alpha) \ll r^2 \exp(-R(|\omega|)).$$

(iii) *For fixed  $k \geq 1$*

$$|\{\alpha \in \mathcal{D}(\omega, r), \omega(\alpha) = k\}| = 4r^2 \frac{(\log_2 |\omega|^2)^{k-1}}{(k-1)! \log(|\omega|^2)} (1 + o(1)).$$

When  $k = 1$  this last result, (iii), shows that  $\mathcal{D}(\omega, r)$  contains the expected proportion of Gaussian primes as long as  $r > |\omega|^{7/12+\varepsilon}$ . We might remark that assuming the Riemann Hypothesis for all Hecke  $L$ -functions on  $\mathbb{Q}(i)$  then

$$\Psi(x, \psi, \ell) = (2\ell)^2 x + O\left(x^{2/3} \ell^{2/3} \log^{4/3} x\right)$$

where  $\Psi(x, \psi, \ell) = \sum \Lambda(\mathfrak{a}), \mathfrak{a} \in \mathcal{S}(x, \psi, \ell)$ . (See [3].) It is then a straightforward deduction from Proposition 10 that, subject to the extended Riemann Hypothesis,  $\Psi(\omega, r) = 4r^2 (1 + o(1))$  (with the obvious notation) as long as  $r (|\omega|^{1/2} \log |\omega|)^{-1} \rightarrow \infty$  as  $|\omega| \rightarrow \infty$ .

## 4. Introduction of smooth weights

As in [3] we introduce smooth weights as follows. Given  $\Delta \leq \ell$ , Vinogradov [13, lemma 12] constructs a continuous function  $f$  satisfying

$$\begin{aligned} f(y) &= 1 & \text{for } |y| \leq \ell - \Delta, \\ 0 \leq f(y) &\leq 1 & \text{for } \ell - \Delta < |y| \leq \ell, \\ f(y) &= 0 & \text{for } \ell \leq |y| \leq \frac{1}{2}, \end{aligned}$$

and defined for all  $y$  by periodicity. Importantly,  $f$  can be replaced by a Fourier series  $\sum a_m e^{2\pi i m y}$  where

$$a_m \ll \begin{cases} a_0 = 2\ell + \Delta \\ 1/|m|, & m \neq 0. \end{cases}$$

From [1] we have a continuous function  $g$  satisfying



$$\begin{aligned}
g(y) &= 1 && \text{for } x(1 - (\ell - \Delta)) \leq y \leq x(1 + (\ell - \Delta)), \\
0 \leq g(y) &\leq 1 && \text{for } x(1 - \ell) \leq y \leq x(1 - (\ell - \Delta)) \\
&&& \text{or } x(1 + (\ell - \Delta)) \leq y \leq x(1 + \ell), \\
g(y) &= 0 && \text{for } y \leq x(1 - \ell) \text{ or } y \geq x(1 + \ell).
\end{aligned}$$

Importantly, the mellin transform,  $\widehat{g}(s)$ , satisfies  $\widehat{g}(1) = 2\ell x(1 + O(\Delta))$  and  $\widehat{g}(\sigma + it) \ll \ell x^\sigma$  for all  $t$ . Then, in place of  $A(x, \psi, \ell)$  we examine

$$\sum_{\mathbf{a}} a(\mathbf{a}) g(N\mathbf{a}) \prod_{j=1}^{n-1} f(\psi_j(\mathbf{a}) - \psi_j) = \sum_{\mathbf{a}} a(\mathbf{a}) \theta_{x, \chi}(\mathbf{a})$$

say, denoted by  $A(\theta_{x, \chi})$ . To recover results for  $A(x, \psi, \ell)$  we will “strip the weights” using

**Lemma 12** For  $\psi \in \mathbb{T}^{n-1}$ ,  $0 < \ell < \frac{1}{2}$  and  $0 \leq \Delta \leq \ell$ ,

$$|\mathcal{S}(x, \psi, \ell) \setminus \mathcal{S}(x, \psi, \ell - \Delta)| \ll x \ell^{n-1} \Delta.$$

**Proof** For  $\mathbf{a} \in \mathcal{S}(x, \psi, \ell) \setminus \mathcal{S}(x, \psi, \ell - \Delta)$  then

$$\begin{aligned}
&\text{either (a) } x(1 - \ell) < N\mathbf{a} < x(1 - (\ell - \Delta)) \\
&\text{or (b) } x(1 + (\ell - \Delta)) < N\mathbf{a} < x(1 + \ell) \\
&\text{or (c) there exists } 1 \leq j \leq n-1 \text{ such that} \\
&\text{either } -\ell < |\psi_j(\mathbf{a}) - \psi_j|_{\mathbb{T}} < -\ell + \Delta \\
&\text{or } \ell - \Delta < |\psi_j(\mathbf{a}) - \psi_j|_{\mathbb{T}} < \ell.
\end{aligned} \tag{20}$$

If  $x(1 - \ell) < N\mathbf{a} < x(1 - (\ell - \Delta))$  then necessarily  $\widetilde{x}(1 - \Delta) < N\mathbf{a} < \widetilde{x}(1 + \Delta)$  with  $\widetilde{x} = x(1 - \ell)/(1 - \Delta)$ . Splitting each of the  $n-1$  conditions  $|\psi_j(\mathbf{a}) - \psi_j|_{\mathbb{T}} < \ell$  into  $\ll (\ell/\Delta + 1)$  conditions of the form  $|\psi_j(\mathbf{a}) - \psi_{ij}|_{\mathbb{T}} < \Delta$ , the ideals satisfying the first condition of (20) lie in at least one of  $\ll (\ell/\Delta + 1)^{n-1}$  sets of the form  $\mathcal{S}(\widetilde{x}, \widetilde{\psi}, \Delta)$ .

Similarly, the same result holds for all the other possibilities in (20).

As noted in the proof of Proposition 10  $\mathcal{S}(\widetilde{x}, \widetilde{\psi}, \Delta) \ll (x^{1/n} \Delta + 1)^n$ . Hence

$$|\mathcal{S}(x, \psi, \ell) \setminus \mathcal{S}(x, \psi, \ell - \Delta)| \ll \left(\frac{\ell}{\Delta} + 1\right)^{n-1} (x^{1/n} \Delta + 1)^n \ll x \ell^{n-1} \Delta$$

as required.  $\blacksquare$

Thus, since  $a(\mathbf{a}) \ll (N\mathbf{a})^\varepsilon$ , we have  $A(x, \psi, \ell) - A(\theta_{x, \chi}) \ll x^{1+\varepsilon} \ell^{n-1} \Delta$  which is sufficiently small if we choose  $\Delta = \ell x^{-2\varepsilon}$ .

Rewriting in terms of the Fourier series and Mellin transform,

$$A(\theta_{x, \chi}) = \frac{1}{2\pi i} \sum_{\mathbf{m}} a_{\mathbf{m}} e^{-2\pi i \mathbf{m} \cdot \psi_0} \int_{c-i\infty}^{c+i\infty} \widehat{g}(s) \sum_{\mathbf{a}} \frac{a(\mathbf{a}) \lambda^{\mathbf{m}}(\mathbf{a})}{N\mathbf{a}^s} ds \tag{21}$$

where  $\mathbf{m} \in \mathbb{Z}^{n-1}$ ,  $c > 1$  and  $a_{\mathbf{m}} = \prod_{j=1}^{n-1} a_{m_j}$ , with  $a_{m_j}$  the coefficient of the Fourier series.

When  $\mathbf{m} = 0$  the inner sum here is  $F(s)$  which has a factorization given by (3) and (2). Because  $\lambda^{\mathbf{m}}$  is totally multiplicative the inner sum in (21)  $F(s, \lambda^{\mathbf{m}})$  say, has a similar factorization with the  $\zeta_K(s, \chi)$  in (2) replaced by

$$L(s, \chi \lambda^{\mathbf{m}}) = \sum_{(\mathbf{a}, \mathbf{q})=1} \frac{\chi(\mathbf{a}) \lambda^{\mathbf{m}}(\mathbf{a})}{N \mathbf{a}^s},$$

$\text{Res} > 1$ ; the Hecke  $L$ -functions with Grossencharakteres. See [7] for properties of these  $L$ -functions. Here we just note that  $L(s, \chi \lambda^{\mathbf{m}})$  has an analytic continuation to the whole plane with the single exception of a pole at  $s=1$  when  $\chi = \chi_0$  and  $\mathbf{m} = \mathbf{0}$ . So the main contribution to (21) can only come from  $\mathbf{m} = \mathbf{0}$ . We now state our weighted form of Theorem 1.

**Theorem 1'** *Let  $g$  and  $f$  be as above, with the associated  $\theta = \theta_{x, \chi}$ . Let  $\varepsilon > 0$  be given. Then, with the notation of Theorem 1,*

$$A(\theta) - \frac{a_0}{2\pi i} \int_{\mathcal{C}_0} \hat{g}_x(s) F(s) ds \ll x \ell^n \exp(-R(x)) \quad (22)$$

for  $\ell > x^{-5/12n+10\varepsilon}$ . If  $\ell = \ell(X)$ ,  $\Delta = \Delta(X)$  are functions only of  $X$ , then

$$\int_{\mathbb{T}^{n-1}} \int_X^{2X} \left| A(\theta_{x, \psi}) - \frac{a_0}{2\pi i} \int_{\mathcal{C}_0} \hat{g}_x(s) F(s) ds \right|^2 dx d\psi \ll_{\varepsilon} X^3 \ell^{2n} \exp(-R(X)), \quad (23)$$

for  $\ell(X) > X^{-5/6n+20\varepsilon}$ .

**Theorem 1' implies Theorem 1**

$$\begin{aligned} & A(x, \psi, \ell) - I(x, \ell) \\ & \ll |A(x, \psi, \ell) - A(\theta)| + x \ell^n \exp(-R(x)) \\ & + \left| \frac{a_0}{2\pi i} \int_{\mathcal{C}_0} \hat{g}_x(s) F(s) ds - \frac{(2\ell)^{n-1}}{2\pi i} \int_{x(1-\ell)}^{x(1+\ell)} \int_{\mathcal{C}_0} y^{s-1} F(s) ds dy \right| \end{aligned}$$

by (22). The first term on the right has been estimated previously. For the third term we note that  $a_0 = (2\ell)^{n-1} + O(\Delta \ell^{n-2})$  and

$$\hat{g}_x(s) = \int_{-\infty}^{\infty} g_x(y) y^{s-1} dy$$

to obtain the bound

$$\ll \Delta \ell^{n-2} \int_{x(1-(\ell-\Delta))}^{x(1+(\ell-\Delta))} |\mathcal{F}(y)| dy + \ell^{n-1} \left( \int_{x(1-\ell)}^{x(1-(\ell-\Delta))} + \int_{x(1+(\ell-\Delta))}^{x(1+\ell)} \right) |\mathcal{F}(y)| dy \quad (24)$$

where

$$\mathcal{F}(y) = \int_{\mathcal{C}_0} y^{s-1} F(s) ds.$$

Deform  $\mathcal{C}_0$  into  $\mathcal{C}_\delta$  of the proof of Theorem 3, with  $\delta = 1/\log x$ . Observe that

$$F(s) = (s-1)^{-z} \left( \log \left( \frac{1}{s-1} \right) \right)^n \eta(s)$$

for some  $z \in \mathbb{C}, n \in \mathbb{N} \cup \{0\}$  and function  $\eta(s)$  regular and bounded in some disc about  $s=1$  containing  $\mathcal{C}_0$ . Then it is easy to show that

$$\mathcal{F}(y) \ll (\log x)^{\operatorname{Re} z - 1} (\log_2 x)^n.$$

Hence (24) is

$$\ll \Delta \ell^{n-1} x (\log x)^{\operatorname{Re} z - 1} (\log_2 x)^n \ll x^{1-\varepsilon/2} \ell^n$$

by our choice of  $\Delta$ . Hence (4) follows.

Similarly (5) follows from (23).  $\blacksquare$

## 5. Proof of Theorem 1'

### 5.1. The Hooley-Huxley Contour

The important results from [13] and [1] are that the sums and integrals in (21) can be truncated at  $W = [\Delta^{-1} \log^3 1/\ell]$  with a negligible error as long as  $x$  is sufficiently large. So we need only examine

$$\frac{1}{2\pi i} \sum_{\|\mathbf{m}\| < W} a_{\mathbf{m}} e^{-2\pi i \mathbf{m} \cdot \psi_0} \int_{c-iW}^{c+iW} \hat{g}(s) F(s, \lambda^{\mathbf{m}}) ds. \quad (25)$$

Let  $\rho_{\mathbf{m}\chi} = \beta_{\mathbf{m}\chi} + i\gamma_{\mathbf{m}\chi}$  denote a zero of the  $L$ -function  $L(s, \chi \lambda^{\mathbf{m}})$ . Define

$$\mathcal{Z}(W) = \left\{ \rho_{\mathbf{m}\chi} : \begin{array}{l} 0 < \beta_{\mathbf{m}\chi} < 1, \ |\gamma_{\mathbf{m}\chi}| < W \text{ and } \rho_{\mathbf{m}\chi} \text{ is a zero of} \\ \text{one of the } L\text{-functions implicit in (25).} \end{array} \right\}$$

and  $\mathcal{Z}_{\mathbf{m}}(W)$  those zeros  $\rho_{\mathbf{m}'\chi}$  in  $\mathcal{Z}(W)$  with  $\mathbf{m}' = \mathbf{m}$ .

We can now move the line of integration in (25) to the left of  $\operatorname{Re} s = 1$  except, when  $\mathbf{m} = \mathbf{0}$ , for a loop about  $s = 1$ . The new contour has to stay within a region free of zero of the  $L$ -functions in (25). Such a region is given in

**Theorem 12** [2]. *There exists  $c > 0$  such that if  $\rho_{\mathbf{m}\chi} \in \mathcal{Z}(W)$  then*

$$\beta_{\mathbf{m}\chi} \leq 1 - c (\log W)^{-2/3} (\log_2 W)^{-1/3}. \quad (26)$$

This follows from the order result.

**Theorem 13** *There exists constants  $c_1$  and  $c_2$ , depending only on  $K$ , such that*

$$L(\sigma + it, \chi \lambda^{\mathbf{m}}) \ll_{\mathbf{q}} V^{c_1(1-\sigma)^{3/2}} \log^{2/3} V \quad (27)$$

for  $2 > \sigma > 1 - c_2$  (and  $|t| > 2$  when  $\chi = \chi_0, \mathbf{m} = \mathbf{0}$ ) where  $V^2 = e + t^2 + \sum_{i=1}^{n-1} m_i^2$ .

The idea of the Hooley-Huxley contour is that the density of zeros with large real part is low. So it should be possible to deform the contour of integration around these few zero and go into the region (26) frequently.

Let  $R(\sigma_1, \sigma_2, T_1, T_2)$  denote the rectangle with the corners

$$\sigma_1 + iT_1, \quad \sigma_1 + iT_2, \quad \sigma_2 + iT_1 \quad \text{and} \quad \sigma_2 + iT_2.$$

Let  $R_r, r \in \mathbb{Z}$ , be the rectangle

$$R\left(\frac{1}{2}, 1 + \frac{1}{\log W}, (100r + 50)(\log W)^2, (100r - 50)(\log W)^2\right)$$

where  $\left|(100r \pm 50)(\log W)^2\right| \leq W + 100(\log W)^2$ . Let  $r_0$  be the largest integer satisfying this last inequality. For each  $\|\mathbf{m}\| < W$  and  $|r| < r_0$ , pick a zero  $\rho_{\mathbf{m},r} \in \mathcal{Z}_{\mathbf{m}}(W)$  lying in  $R_{r-1} \cup R_r \cup R_{r+1}$  with the greatest real part  $\beta_{\mathbf{m},r}$ . To exclude the possibility that no such zero exists we follow [12] in giving to the points  $\frac{1}{2} + im, m \in \mathbb{Z}$ , the same treatment as is given to the zeros in  $\mathcal{Z}(W)$ . On  $R_r$  fix a new right hand side  $V_{\mathbf{m},r} : \sigma = \beta_{\mathbf{m},r}$ . Connecting the  $V_{\mathbf{m},r}$  by horizontal lines gives the edge of the regions  $R_{\mathbf{m}}$  say, into which we can deform the contour in (25) except, as before, for a loop about  $s = 1$  when  $\mathbf{m} = \mathbf{0}$ . The resulting line of integration should lie close to the edge of the region so that  $\hat{g}(s)$  in (25) is small. But then the  $F(s, \lambda^{\mathbf{m}})$  might well be large due to singularities on the edge of the region. We control this latter effect by

**Lemma 14.** *Consider a fixed  $\|\mathbf{m}\| < W$  and  $U = 0, \pm 100(\log W)^2, \pm 200(\log W)^2, \dots$  with  $U + 50(\log W)^2 \leq W + 100(\log W)^2$ . Let a constant  $0 \leq 2a \leq 1$  be given. Suppose that  $\sigma$  is the largest real part of all zeros of  $\mathcal{Z}_{\mathbf{m}}(W)$  in*

$$R\left(\frac{1}{2}, 2, U + 150(\log W)^2, U - 150(\log W)^2\right).$$

Then for

$$s \in R\left(\sigma + d(1 - \sigma), 2, U + 55(\log W)^2, U - 55(\log W)^2\right),$$

with the disc  $|s - 1| \ll (\log_2 W)^{-2}$  excluded when  $\chi = \chi_0, \mathbf{m} = \mathbf{0}$ , we have

$$\log L(s, \chi \lambda^{\mathbf{m}}) \ll (\log W)^{(1-d)/(1-2a)} (\log_2 W)^{(d-2a)/(1-2a)} + (\log_2 W)^4 \quad (28)$$

uniformly for  $2a \leq d \leq 1$ . (The  $(\log_2 W)^4$  occurs only when  $\chi = \chi_0, \mathbf{m} = \mathbf{0}$ .)

**Proof.** This follows the proof of Lemma 5 of [9]. First consider  $\chi = \chi_0$ ,  $\mathbf{m} = \mathbf{0}$  and let  $\sigma_0$  be the largest real part of the zero of  $\zeta_K(s)$  in

$$R\left(\frac{1}{2}, 2, 150(\log W)^2, -150(\log W)^2\right).$$

Here,  $\sigma_0$  is far smaller than if we had looked at all the zeros in  $\mathcal{Z}(W)$ , and, in fact,  $(1 - \sigma_0)^{-1} \ll \log^2 W$ . We can then follow the first part of the proof in [9] to conclude

$$\log \zeta_K(s) \ll (\log_2 W)^4$$

for  $s \in R\left(\sigma_0 + 2a(1 - \sigma_0), 2, 55(\log W)^2, -55(\log W)^2\right)$  with the disc  $|s - 1| \ll (\log_2 W)^{-2}$  excluded. This explains the second term in (28).

For all other cases, that is  $(\chi, \mathbf{m}) \neq (\chi_0, \mathbf{0})$  for all  $U$ , or  $(\chi, \mathbf{m}) = (\chi_0, \mathbf{0})$  for all  $U \neq 0$ , we apply the maximum modulus principal to the function

$$\phi(w) = e^{(w-s)^2} Z^{w-s} \log L(w, \chi\lambda^{\mathbf{m}})$$

where

$$w \in R\left(\sigma + 2a(1 - \sigma), 2, U + 60(\log W)^2, U - 60(\log W)^2\right)$$

and

$$s \in R\left(\sigma + d(1 - \sigma), 2, U + 55(\log W)^2, U - 55(\log W)^2\right).$$

For this we need

$$L(1 + it, \chi\lambda^{\mathbf{m}}) \ll \log \log W$$

$2 < |t| \leq W$  which follows by the same proof of Lemma 6 in [9]. We also need a bound on  $L(\sigma + 2a(1 - \sigma), \chi\lambda^{\mathbf{m}})$ . From the foot of p. 322 of [9] this is

$$\begin{aligned} &\ll \log W + (1 - \sigma)^{-1} \log_2 W \\ &\ll \log W + (\log W)^{2/3} (\log_2 W)^{4/3} \quad \text{by Theorem 12,} \\ &\ll \log W. \end{aligned}$$

The choice of  $Z^{(1-2a)(1-\sigma)} = \log W$  gives the first term in (28). ■

As discussed, we require zero density results, that is, bounds for

$$N_K(\sigma, W) = \sum_{\substack{\rho_{\mathbf{m}, \chi} \in \mathcal{Z}(W) \\ 1 \geq \beta_{\mathbf{m}, \chi} \geq \sigma}} 1.$$

**Theorem 15.** *There exist constants  $D$  and  $E$  such that*

$$N_K(\sigma, W) \ll W^{D(1-\sigma)^{3/2}} (\log W)^E \quad (29)$$

in the range of validity of (26).

Given  $\varepsilon > 0$  there exists  $F = F(\varepsilon)$  such that

$$N_K(\sigma, W) \ll W^{(12n/5+\varepsilon)(1-\sigma)} (\log W)^F \quad (30)$$

uniformly for  $\frac{1}{2} \leq \sigma \leq 1$ .

**Proof** Here (29) is part of lemma 1 of [3] while (30) is the first part of Theorem 5 of [4]. ■

We construct the Hooley-Huxley contour by moving the vertical lines,  $V_{\mathbf{m},r}$ , by the rule

$$V_{\mathbf{m},r} \rightarrow V'_{\mathbf{m},r} = \{s' = \sigma' + it \mid \sigma' = \sigma + d(1-\sigma), \sigma + it \in V_{\mathbf{m},r}\}$$

for various  $0 < d < 1$ . Follow [9] in letting  $0 < \theta < 1$  be chosen later. If  $s \in V_{\mathbf{m},r}$  has  $\operatorname{Re} s < \theta$  choose  $d = 3a$ , where  $a$  is a small constant depending on  $\varepsilon$ . If  $s \in V_{\mathbf{m},r}$  has  $\operatorname{Re} s > \theta$  choose  $d = b$  near to 1 to be chosen later. Connect the new vertical lines  $V'_{\mathbf{m},r}$  by horizontal lines. Along with the detour about  $s = 1$  when  $\mathbf{m} = \mathbf{0}$ , this describes the Hooley-Huxley contours,  $H_{\mathbf{m}}$  say.

## 5.2. Completion of the proof.

The line of integration in (25) is, for each  $\|\mathbf{m}\| < W$ , moved back to  $H_{\mathbf{m}}$  (with the horizontal lines  $\operatorname{Im} s = \pm W$ ) along with a loop,  $\mathcal{L}$  say about  $s = 1$  when  $\mathbf{m} = \mathbf{0}$ . Note that  $s \in V_{\mathbf{0},0}$  implies  $\sigma < 1 - c(\log_2 W)^{-2/3}(\log_3 W)^{-1/3}$ . So  $\mathcal{L}$  might have radius as small as  $c(\varepsilon)(\log_2 W)^{-2/3}(\log_3 W)^{-1/3} = r$ , say. Thus the error in replacing this loop by the circle  $\mathcal{C}_0$  of Theorem 1' is

$$E = \frac{1}{2\pi i} a_{\mathbf{0}} \int_{L^\pm} \hat{g}(s) F(s) ds = \frac{1}{2\pi i} a_{\mathbf{0}} \int_{-\infty}^{\infty} g(y) \int_{L^\pm} y^{s-1} F(s) ds$$

with  $s \in L^\pm$  if, and only if,  $s = 1 + \rho e^{\pm i\pi}$ ,  $\rho \in [r, c_0]$ . The inner integral here is

$$\ll \int_r^{c_0} y^{-\rho} \rho^{-\operatorname{Re} z} \left( \log \frac{1}{\rho} \right)^n d\rho$$

which on evaluating has, apart from a number of log terms, a factor of  $\exp(-r \log y) \ll \exp(-(\log y)^\alpha)$  for any  $\alpha < 1$ . Hence

$$E \ll x \ell^n \exp(-(\log x)^\alpha).$$

We now have all the required information to bound the remaining integrals over  $H_{\mathbf{m}} \setminus \mathcal{L}$  as in [9]. To clarify the argument in [9] we present the proof in outline. So (22) will follow if we show

$$\sum_{\|\mathbf{m}\| < W} \int_{H_{\mathbf{m}} \setminus \mathcal{L}} |y^{s-1} F(s, \lambda^{\mathbf{m}})| |ds| \ll \exp(-R(y)) \quad (31)$$

for all  $y$  for which  $g(y) \neq 0$ , (i.e.  $y \asymp x$ ).

Now, if  $s' \in V'_{\mathbf{m},r}$  then either  $\operatorname{Re} s' < \theta + 3a(1 - \theta)$  or  $\operatorname{Re} s' > \theta + b(1 - \theta)$ . In the first region

$$|F(s', \lambda^{\mathbf{m}})| \ll \exp\left((\log W)^\psi\right) \quad (32)$$

for any  $(1 - 3a)/(1 - 2a) < \psi < 1$ , by (28). This holds not only on  $V'_{\mathbf{m},r}$  but also on any connecting horizontal lines to the right of  $V'_{\mathbf{m},r}$ . In the second region

$$|F(s', \lambda^{\mathbf{m}})| \ll \exp\left((\log W)^{\psi'}\right), \quad (33)$$

for any  $1 > \psi' > (1 - b)/(1 - 2a)$ , again not only on  $V'_{\mathbf{m},r}$  but also on connecting lines to the right. In  $H_{\mathbf{m}}$  there are horizontal lines between the two regions above. For  $s'$  on these horizontal lines with  $\operatorname{Re} s' < \theta + b(1 - \theta)$  we have only the weak bound (32). But if  $\operatorname{Re} s' > \theta + b(1 - \theta)$  we are looking at points sufficiently far from the  $V'_{\mathbf{m},r}$  where the horizontal lines originated to enable us to use Lemma 14 to deduce the strong bound (33). These horizontal lines either go to, or from, a  $V'_{\mathbf{m},r}$  with  $\operatorname{Re} s' > \theta + b(1 - \theta)$  which arose from a zero  $\rho_{\mathbf{m},\chi}$  with  $\beta_{\mathbf{m},\chi} > \theta$ . So the number of such lines, when summed over all  $\|\mathbf{m}\| < W$ , is  $\ll N_K(\theta, W)$ . Hence the contribution to (31) from the horizontal lines between the regions  $\operatorname{Re} s' < \theta + 3a(1 - \theta)$  and  $\operatorname{Re} s' > \theta + b(1 - \theta)$  is

$$\begin{aligned} &\ll N_K(\theta, W) \exp\left((\log W)^\psi\right) \int_{\theta+3a(1-\theta)}^{\theta+b(1-\theta)} y^{\sigma'-1} d\sigma' \\ &\ll \exp\left(2(\log W)^\psi\right) \left(\frac{W^{D(1-\theta)^{1/2}(1-b)^{-1}}}{y}\right)^{(1-\theta)(1-b)}. \end{aligned}$$

The two remaining regions are split into vertical strips of width  $1/\log W$ . As in [9] we obtain the bounds

$$\begin{aligned} &\max_{\sigma' > \theta+3a(1-\theta)} N_K(\sigma, W) \exp\left((\log W)^\psi\right) (\log W)^A x^{\sigma'-1} \\ &\ll \exp\left((\log W)^\psi + F(\varepsilon) \log_2 W\right) \left(\frac{W^{(\frac{12n}{5}+\varepsilon)(1-3a)^{-1}}}{y}\right)^{(1-3a)(1-\theta)} \end{aligned}$$

by (30) and

$$\begin{aligned} &\max_{\sigma' > \theta+b(1-\theta)} N_K(\sigma, W) \exp\left((\log_2 W)^{\psi'}\right) (\log W)^A x^{\sigma'-1} \\ &\ll \exp\left(2(\log W)^{\psi'}\right) \left(\frac{W^{D(1-\theta)^{1/2}(1-b)^{-1}}}{y}\right)^{(1-b)c/(\log W)^{2/3}(\log_2 W)^{1/3}} \end{aligned}$$

on using the zero-free region (26). The condition  $\ell > x^{-5/12n+10\varepsilon}$  is sufficient, along with the definition of  $W$  to ensure that

$$W^{(12n/5+\varepsilon)(1-3a)^{-1}} < x^{1-\delta}$$

for some  $\delta = \delta(\varepsilon) > 0$  if  $a = a(\varepsilon)$  is chosen sufficiently small. Then choose  $b$  so that we can take  $\psi' < 1/3$  in (33). Finally choose  $\theta$  such that

$$D(1 - \theta)^{1/2}(1 - b)^{-1} < 1.$$

Then all three bounds above are  $\ll \exp(-R(y))$  as required.

For the proof of (23) the smooth weights  $f$  and  $g$  are defined as before but with  $\ell = \ell(X)$  a function of  $X$ , not  $x$ . In particular  $g(y) = h(y/x)$  where  $h$  is an approximation to the interval  $(1 - \ell, 1 + \ell)$ . Thus  $\hat{g}(s) = x^s \hat{h}(s)$  with  $\hat{h}(s)$  depending only on  $X$ . Hence the left hand side of (23) equals

$$\begin{aligned} & \int_{\mathbb{T}^{n-1}} \int_X^{2X} \left| \frac{1}{2\pi i} \sum_{\|\mathbf{m}\| < W} a_{\mathbf{m}} e^{-2\pi i \mathbf{m} \cdot \psi_0} \int_{H_{\mathbf{m}} \setminus \mathcal{L}} x^s \hat{h}(s) F(s, \lambda^{\mathbf{m}}) ds \right|^2 dx d\psi_0 \\ &= \frac{1}{4\pi^2} \sum_{\|\mathbf{m}\| < W} |a_{\mathbf{m}}|^2 \int_X^{2X} \left| \int_{H_{\mathbf{m}} \setminus \mathcal{L}} x^s \hat{h}(s) F(s, \lambda^{\mathbf{m}}) ds \right|^2 dx \\ &\ll a_{\mathbf{0}}^2 \sum_{\|\mathbf{m}\| < W} \int_X^{2X} \left| \int_{R_{\mathbf{m}}} x^s \hat{h}(s) F(s, \lambda^{\mathbf{m}}) ds \right|^2 dx \end{aligned}$$

(where  $R_{\mathbf{m}}$  represents the parts of  $H_{\mathbf{m}}$  lying in the regions  $R$  described above, i.e. between  $\theta + 3a(1 - \theta)$  and  $\theta + b(1 - \theta)$  or vertical strips of width  $1/\log W$ .)

$$\begin{aligned} &\ll (\log W)^2 \ell^{2(n-1)} \max_R \sum_{\|\mathbf{m}\| < W} \int_X^{2X} \left| \int_{R_{\mathbf{m}}} x^s \hat{h}(s) F(s, \lambda^{\mathbf{m}}) ds \right|^2 dx \\ &\ll (\log W)^2 \ell^{2(n-1)} \max_R \sum_{\|\mathbf{m}\| < W} \int_{R_{\mathbf{m}}} \int_{R_{\mathbf{m}}} \left( \frac{2^{s_1 + \bar{s}_2 + 1} - 1}{s_1 + \bar{s}_2 + 1} \right) X^{s_1 + \bar{s}_2 - 1} \\ &\quad \times \hat{h}(s_1) \overline{\hat{h}(s_2)} F(s_1, \lambda^{\mathbf{m}}) \overline{F(s_2, \lambda^{\mathbf{m}})} ds_1 ds_2 \\ &\ll (\log W)^2 X^3 \ell^{2n} \max_R \sum_{\|\mathbf{m}\| < W} \int_{R_{\mathbf{m}}} \int_{R_{\mathbf{m}}} \left| \frac{2^{s_1 + \bar{s}_2 + 1} - 1}{s_1 + \bar{s}_2 + 1} \right| X^{\sigma_1 + \bar{\sigma}_2 - 2} \\ &\quad \times \left( |F(s_1, \lambda^{\mathbf{m}})|^2 + |F(s_2, \lambda^{\mathbf{m}})|^2 \right) |ds_1 ds_2| \end{aligned}$$

since  $\hat{h}(s) \ll \ell(X)$ . The previous method of proof then gives

$$\ll X^3 \ell^{2n} \exp(-R(X))$$

for  $\ell(X) > (X^2)^{-5/12n+10\varepsilon}$ , as required.  $\blacksquare$



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