Research Article

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F_4 and $\mathrm{PSp}(8, \mathbb{C})$-Higgs pairs understood as fixed points of the moduli space of E_6-Higgs bundles over a compact Riemann surface

https://doi.org/10.1515/math-2022-0543
received March 6, 2022; accepted December 5, 2022

Abstract: Let X be a compact Riemann surface of genus $g \geq 2$ and $\mathcal{M}(E_6)$ be the moduli space of E_6-Higgs bundles over X. We consider the automorphisms σ_+ of $\mathcal{M}(E_6)$ defined by $\sigma_+(E, \varphi) = (E^*, -\varphi^t)$, induced by the action of the outer involution of E_6 in $\mathcal{M}(E_6)$, and σ_- defined by $\sigma_-(E, \varphi) = (E^*, \varphi^t)$, which results from the combination of σ_+ with the involution of $\mathcal{M}(E_6)$, which consists on a change of sign in the Higgs field. In this work, we describe the fixed points of σ_+ and σ_-, as F_4-Higgs bundles, F_4-Higgs pairs associated with the fundamental irreducible representation of F_4, and $\mathrm{PSp}(8, \mathbb{C})$-Higgs pairs associated with the second symmetric power or the second wedge power of the fundamental representation of $\mathrm{Sp}(8, \mathbb{C})$. Finally, we describe the reduced notions of semistability and polystability for these objects.

Keywords: Higgs pairs, Lie group E_6, automorphism, stability, fixed points

MSC 2020: 14D20, 14H10, 14H60

1 Introduction

In this article, we are interested in a family of Higgs pairs over a compact Riemann surface X whose structure group is the exceptional simple complex Lie group F_4 or the classical complex Lie group $\mathrm{PSp}(8, \mathbb{C})$, and the centerless group whose universal cover is $\mathrm{Sp}(8, \mathbb{C})$. Given a complex reductive Lie group G together with a complex representation $\rho : G \to \mathrm{GL}(V)$ of G, a G-Higgs pair over X is a pair (E, φ), where E is a principal G-bundle over X and φ is a holomorphic global section of the vector bundle $E(V) \otimes K$, where $E(V)$ is the vector bundle associated with E by ρ with typical fiber V and K is the canonical line bundle over X (Definition 1). The notion of G-Higgs bundle can be reconstructed from that of G-Higgs pair by considering the adjoint representation of G. The geometry of the moduli spaces associated with G-Higgs pairs has been intensively studied and admits a great amount of applications to very diverse areas of Mathematics and Theoretical Physics such as non-abelian Hodge theory, integrable systems, string theory, and the theory of branes among others (for details, see [1]). In this work, we study F_4-Higgs bundles and F_4-Higgs pairs whose associated complex representation is the fundamental irreducible representation of F_4, which is 26 dimensional, and $\mathrm{PSp}(8, \mathbb{C})$-Higgs pairs associated with the 36-dimensional representation $\mathrm{Sym}^2\iota$ or the 28-dimensional representation $\wedge^2\iota$, where ι is the inclusion $\mathrm{Sp}(8, \mathbb{C}) \to \mathrm{GL}(8, \mathbb{C})$. These bundles and pairs appear as fixed points of certain automorphisms of the moduli space of E_6-Higgs bundles over X, as we will see in Proposition 3.2. We will also give in Propositions 4.1–4.3 a concrete description of the reduced notions of semistability and polystability for these objects.

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Consider the exceptional simple complex Lie group E_6. Details about this group can be found in [2, Chapter 5, Sections 1.3 and 1.5] and also in [3,4]. The group E_6 is the only exceptional simple complex Lie group that admits outer automorphisms. In particular, it admits only one outer automorphism σ, which is an involution. This automorphism acts in the moduli space $\mathcal{M}(E_6)$ of E_6-Higgs bundles over X according to an action of the group of outer automorphisms of G in the moduli space of G-Higgs bundles, which is described for a general complex Lie group G in [5]. We will denote by σ_+ the automorphism of $\mathcal{M}(E_6)$ induced by the outer involution σ of E_6. This automorphism acts as follows: if (E, φ) is an E_6-Higgs bundle over X, then $\sigma_+(E, \varphi) = (E^*, -\varphi^t)$. In this work, we study the subvariety of $\mathcal{M}(E_6)$ of fixed points of σ_+. These fixed points are described in Proposition 3.2 as F_4-Higgs bundles over X or $\mathrm{PSp}(8, \mathbb{C})$-Higgs pairs associated with the representation $\mathrm{Sym}^2\iota$. We also consider the automorphism σ_- of $\mathcal{M}(E_6)$ that results from combining σ_+ with the involution of $\mathcal{M}(E_6)$ defined by a change of sign in the Higgs field, that is, $\sigma_-(E, \varphi) = (E^*, \varphi^t)$. In Proposition 3.2, it is also proved that the fixed points of σ_- can be described as F_4-Higgs pairs with an associated complex representation of the fundamental irreducible representation of F_4 or $\mathrm{PSp}(8, \mathbb{C})$-Higgs pairs with the representation $\wedge^2\iota$. In Propositions 4.1–4.3, the stability and polystability conditions for all these fixed points are discussed.

The study of automorphisms of moduli spaces of principal bundles and Higgs bundles over compact Riemann surfaces and the corresponding subvarieties of fixed points is a topic of great interest in Geometry, which is being worked on intensively. In [6], for example, we study fixed points of certain automorphism of the moduli space of principal $\mathrm{Spin}(8, \mathbb{C})$-bundles over a curve induced by the action in it of the triality automorphism, and in [5], we extend the study to the case of $\mathrm{Spin}(8, \mathbb{C})$-Higgs bundles, and we obtain that the fixed points of certain automorphisms of the moduli space of $\mathrm{Spin}(8, \mathbb{C})$-Higgs bundles over the Riemann surface are described as certain G_2-Higgs pairs or $\mathrm{PSL}(3, \mathbb{C})$-Higgs pairs whose notions of stability, semistability, and polystability are described following the general concepts explained in [7]. In [8,9], the group of automorphisms of the moduli space of symplectic principal bundles over a compact Riemann surface is computed, and in [4], the same is computed for the moduli space of principal E_6-bundles. Finally, in [10], the groups of automorphisms of vector Higgs bundles moduli spaces are determined. In this work, we study the case of $\mathcal{M}(E_6)$ in the spirit of [5] and the following [7]. In fact, in [7,11], the authors study fixed points of automorphisms of the moduli space of Higgs bundles in the general case where the gauge group G is semisimple, but here an explicit description of these fixed points is given for the group E_6, using specific techniques adjusted to the particular groups we are working with, and in addition, we provide reduced versions of the notions of stability of the fixed points obtained.

This article is organized as follows: in Section 2, we introduce the concept of the G-Higgs pair over a compact Riemann surface and explain the notions of stability, semistability, and polystability as they are introduced in [7]. The moduli space of E_6-Higgs bundles and the automorphisms of this moduli space that will be the subject of our interest are introduced in Section 3, where we also prove that the fixed points of these automorphisms can be described as certain types of F_4 or $\mathrm{PSp}(8, \mathbb{C})$-Higgs pairs. Finally, in Section 4, we study the reduced notions of stability, semistability, and polystability conditions for the families of Higgs pairs introduced in the previous section.

2 Higgs pairs and stability conditions

In this section, we will introduce the notion of Higgs pair and the corresponding stability conditions needed to define the moduli space of Higgs pairs over a compact Riemann surface X of genus $g \geq 2$ and with a given complex semisimple structure group G as set out in [7]. The whole theory developed in this section can be extended to the case in which G is reductive, but we limit ourselves to the semisimple case for simplicity, given that all the groups we will work with will be semisimple.

Definition 1. Let G be a complex semisimple Lie group and let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G. A G-Higgs pair over X (or simply a Higgs pair, if the structure group is clear) is a pair (E, φ), where E is a principal G-bundle over X and $\varphi \in H^0(X, E(V) \otimes K)$. Here, $E(V) = E \times_\rho V$ is the quotient of $E \times V$ by the equivalence relation defined by $(e, v) \sim (eg, \rho(g^{-1})v)$ for all $g \in G$, and K denotes the canonical line bundle over X.

Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and let H be a compact connected Lie subgroup of G with Lie algebra $\mathfrak{h}$ and such that $H^{\mathbb{C}} = G$ (so $\mathfrak{h}^{\mathbb{C}} = \mathfrak{g}$). Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$. For any $\delta \in \mathfrak{c}^*$, we denote by $\mathfrak{g}_\delta$ the root subspace of $\mathfrak{g}$, which corresponds to δ. Let R be the set of roots of $\mathfrak{g}$ and Δ be the subset of R of simple roots. For any subset $A \subseteq \Delta$, the subset of roots R_A defined by

$$R_A = \left\{ \sum_{\beta \in \Delta} m_\beta \beta : m_\beta \geq 0 \quad \forall \beta \in A \right\}$$

satisfies that the subspace $\mathfrak{p}_A$ of $\mathfrak{g}$ defined by

$$\mathfrak{p}_A = \mathfrak{c} \oplus \bigoplus_{\delta \in R_A} \mathfrak{g}_\delta, \tag{1}$$

which is a parabolic subalgebra of $\mathfrak{g}$. Moreover, if R_A^0 denotes the subset of roots

$$R_A^0 = \left\{ \sum_{\beta \in \Delta} m_\beta \beta : m_\beta = 0 \quad \forall \beta \in A \right\},$$

then the subspace $\mathfrak{l}_A$ of $\mathfrak{g}$ defined by

$$\mathfrak{l}_A = \mathfrak{c} \oplus \bigoplus_{\delta \in R_A^0} \mathfrak{g}_\delta \tag{2}$$

is a Levi subalgebra of $\mathfrak{p}_A$. Let P_A and L_A be the subgroups of G whose Lie algebras are $\mathfrak{p}_A$ and $\mathfrak{l}_A$, respectively. The group P_A thus defined is a parabolic subgroup of G and L_A is a Levi subgroup of P_A. For different subsets A of Δ, the corresponding parabolic subgroups are not isomorphic, and all the parabolic subgroups of G may be constructed in this way (for details, see [12, Chapter VII]).

For any $\delta \in \Delta$, we define

$$\lambda_\delta = \frac{2\delta}{(\delta, \delta)}, \tag{3}$$

where $(,)$ denotes the Killing form defined in R. Given any parabolic subgroup P_A of G with Lie algebra $\mathfrak{p}_A$ defined in (1) for some $A \subseteq \Delta$, an antidominant character χ of P_A is said to be an element of $\mathfrak{c}^*$ of the form

$$\chi = \sum_{\delta \in A} n_\delta \lambda_\delta,$$

where $n_\delta \leq 0$ for any $\delta \in A$ and λ_δ is defined in (3). It is called strictly antidominant if $n_\delta < 0$ for all $\delta \in A$. Each antidominant character χ of P_A induces an element $s_\chi \in \mathfrak{c}$ through the isomorphism $\mathfrak{c} \cong \mathfrak{c}^*$ induced by the Killing form. This s_χ belongs in fact to $i\mathfrak{h}$. So each pair composed of a parabolic subgroup P of G and an antidominant character χ of P induces an element $s \in i\mathfrak{h}$.

Let now (E, φ) be a G-Higgs pair associated with some complex representation $\rho : G \to \mathrm{GL}(V)$ of G. For any $A \subseteq \Delta$ and any antidominant character χ associated with P_A, we define

$$\begin{aligned} V_\chi^- &= \left\{ v \in V : \rho\left(e^{ts_\chi}\right)v \quad \text{is bounded as} \quad t \to \infty \right\} \\ V_\chi^0 &= \left\{ v \in V : \rho\left(e^{ts_\chi}\right)v = v \quad \forall t \right\}. \end{aligned} \tag{4}$$

From [7, Lemmas 2.5 and 2.6], it follows that V_χ^- is invariant under the action of P_A and V_χ^0 is invariant under the action of L_A.

Definition 2. Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G and let (E, φ) be a G-Higgs pair over X associated to the representation ρ of G. The G-Higgs pair (E, φ) is said to be semistable (resp. stable) if for every subset A of Δ, any antidominant character χ of P_A and any reduction of structure group E_A of E to P_A such that $\varphi \in H^0(X, E_A(V_\chi^-) \otimes K)$, where P_A is the parabolic subgroup of G whose Lie algebra is defined in (1) and V_χ^- is defined in (4), we have that $\deg \chi_* E_A \geq 0$ (resp. $\deg \chi_* E_A > 0$).

Definition 3. Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G and let (E, φ) be a G-Higgs pair over X associated to the representation ρ of G. The G-Higgs pair (E, φ) is said to be polystable if it is semistable and for every subset A of Δ, any antidominant character χ of P_A and any reduction of structure group E_A of E to P_A such that $\varphi \in H^0(X, E_A(V_\chi^-) \otimes K)$ and $\deg \chi_* E_A = 0$, where P_A is the parabolic subgroup of G whose Lie algebra is defined in (1) and V_χ^- is defined in (4), there exists a reduction of structure group E_A' of E_A to L_A such that $\varphi \in H^0(X, E_A'(V_\chi^0) \otimes K)$, where L_A is the Levi subgroup of P_A whose Lie algebra is defined in (2) and V_χ^0 is defined in (4).

We will use a formulation of the stability and polystability conditions in terms of filtrations of certain vector bundle associated with the corresponding principal bundle of the Higgs pair. This vector bundle is defined as in the next result, by making use of a fixed representation ρ_G defined in the case in which the representation ρ that defines the principal bundles is faithful. We could have presented the formulation in terms of filtrations as the definition, but we have opted for this presentation of the theory to make this section survey material on this topic. This formulation in terms of filtrations is constructed in detail in [7, Lemma 2.12], which we adapt to formulate the following.

Proposition 2.1. *Let G be a semisimple complex Lie group, $\rho : G \to \mathrm{GL}(V)$ be a faithful complex representation of G, and (E, φ) be a (G, ρ)-Higgs pair over X. Suppose that there exists a representation $\rho_G : G \to \mathrm{GL}(W)$, with $W \cong \mathbb{C}^n$ for some $n \in \mathbb{N}$, such that for any $a, b \in (\mathrm{Ker} \quad d\rho_G)^\perp$ we have that $\langle a, b \rangle = \mathrm{Tr}\, d\rho_G(a) d\rho_G(b)$, where the product is the Euclidean product of W. Denote $E = E(W)$. Then*

(1) *The (G, ρ)-Higgs pair (E, φ) is semistable if for every parabolic subgroup P of G, every antidominant character χ of P and every filtration $E_0 = 0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_k = E$ induced by a reduction of structure group of E to P and such that φ takes values, in each fiber over X, in the space V_χ^- defined in (4), we have that the degree of the filtration, defined by*
$$\lambda_k \deg E + \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) \deg E_j, \tag{5}$$
is greater than or equal to 0, where $\lambda_1 < \cdots < \lambda_k$ are the eigenvalues of $d\rho_G(s_\chi)$.

(2) *It is stable if for every P and χ as before and any filtration $E_0 = 0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_k = E$ induced by a reduction of structure group of E to P and such that φ takes values in V_χ^-, the degree of the filtration defined in (5) is greater than 0.*

(3) *The (G, ρ)-Higgs pair (E, φ) is polystable if it is semistable, and there exists a parabolic subgroup P of G and an antidominant character χ of P such that E admits a decomposition of the form $E = \oplus_{j=1}^k E_j / E_{j-1}$ into vector subbundles, where $E_0 = 0$ and E_j / E_{j-1} is the eigenspace of the eigenvalue λ_j of $d\rho_G(s_\chi)$ for all $j = 1, \ldots, k$, the degree defined in (5) equals 0 and φ takes values, in each fiber over X, in the space V_χ^0 defined in (4).*

The degree defined in (5) coincides with the degree $\deg \chi_* E_A$ considered in the definitions of stable and semistable Higgs pairs considered here, as shown in [7, Lemma 2.12].

3 E_6-Higgs bundles

Let X be a compact Riemann surface of genus $g \geq 2$. A principal E_6-bundle over X can be understood through the fundamental 27-dimensional representation of E_6, as a complex vector bundle E of rank 27 and trivial determinant bundle equipped with a global holomorphic non degenerate symmetric trilinear form Ω (see [4, Section 3], where principal E_6-bundles are studied).

Definition 4. An E_6-Higgs bundle over X is a pair (E, φ), where E is a principal E_6-bundle over X and $\varphi \in H^0(X, E(\mathfrak{e}_6) \otimes K)$, so φ can be seen as a homomorphism from E to $E \otimes K$, which preserves the symmetric trilinear form of E (here, $\mathfrak{e}_6$ denotes the Lie algebra of E_6 and $E(\mathfrak{e}_6)$ is the adjoint vector bundle of E).

We have studied principal E_6-bundles in [4], so most of the details about what we present here concerning these bundles can be found in that reference. In particular, a reduction of the structure group of a principal E_6-bundle E to a parabolic subgroup of E_6 comes with a filtration $0 \subset E_0 \subset \cdots \subset E_r \subset E$ of E into isotropic subbundles for Ω, since every maximal parabolic subgroup of E_6 comes with a filtration of this form (recall that, given a vector subbundle F of a principal E_6-bundle E with holomorphic symmetric trilinear form Ω, we say that F is isotropic for Ω if $\Omega(F, F, F) = 0$) and a reduction of structure group to a Levi subgroup of a parabolic subgroup of E_6 can be seen as a decomposition of E of the form $E = E_0 \oplus E_1/E_0 \oplus \cdots \oplus E/E_r$.

In this section, we first give the reduced stability and polystability conditions for E_6-Higgs bundles, which are compatible with the perspective explained in the last part of the previous section, which follows [7]. For the moment, we do not present the proof, because it is absolutely analogous to that of Proposition 4.1 and will be covered by it.

Proposition 3.1. *Let (E, φ) be an E_6-Higgs bundle with associated symmetric trilinear form Ω. Then (E, φ) is semistable if for every subbundle F of E isotropic for Ω and preserved by φ we have that* $\deg F \leq 0$.

The Higgs bundle (E, φ) is stable if for every filtration $0 \subset E_0 \subset \cdots \subset E_r \subset E$ of E composed by isotropic subbundles for Ω preserved by φ, we have that $\deg E_j \leq 0$ *for all j, and there exists some k such that* $\deg E_k < 0$.

The Higgs bundle (E, φ) is polystable if it can be written as a direct sum of vector subbundles

$$E = E_0 \oplus E_1/E_0 \oplus \cdots \oplus E/E_r,$$

where $E_0, E_1, \ldots, E_r$ are degree 0 proper subbundles of E isotropic for Ω, which form a filtration of E, $E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_r \subsetneq E$, and the Higgs field φ preserves this decomposition.

Definition 5. The moduli space $\mathcal{M}(E_6)$ of E_6-Higgs bundles over X is then the complex algebraic variety that parametrizes isomorphism classes of polystable E_6-Higgs bundles over X.

The group $\mathrm{Out}(E_6)$ of outer automorphisms of E_6 is isomorphic to $\mathbb{Z}_2$, since it coincides with the groups of symmetries of the Dynkin diagram of $\mathfrak{e}_6$. In [5], we proved that there is a natural action of $\mathrm{Out}(G)$ on the moduli space of G-Higgs bundles for any complex reductive Lie group G: if $f \in \mathrm{Out}(G)$ and $A \in \mathrm{Aut}(G)$ represents f, then $f \cdot (E, \varphi) = (A(E), \mathrm{d}A(\varphi))$, where $A(E)$ is the principal G-bundle whose total space is that of E and the action of G in $A(E)$ is defined by $e \cdot g = eA^{-1}(g)$ for $e \in A(E)$ and $g \in G$. This construction depends only on the equivalence class of the automorphism A by the following equivalence relation defined on the group $\mathrm{Aut}(G)$: two automorphisms of G are equivalent if they are conjugate by an inner automorphism of G [6]. We will explicitly describe this action for the case of $\mathcal{M}(E_6)$. Let σ be the outer involution of E_6. The involution σ induces the following automorphism of $\mathcal{M}(E_6)$:

$$\sigma_+(E, \varphi) = (E^*, -\varphi^t), \tag{6}$$

since the involution σ acts in E_6 by transposing the automorphisms (we are considering the elements of E_6 through its 27-dimensional fundamental representation), $d\sigma(A) = -A^t$ for any $A \in \mathfrak{e}_6$ and the transformation on the Higgs field coincides with the action of $d\sigma$ in it, that is, $-\varphi^t = d\sigma(\varphi)$. We are also interested in the automorphism that results from composing σ_+ with the involution ι of $\mathcal{M}(E_6)$ whose effect is a change of sign on the field: $\iota(E, \varphi) = (E, -\varphi)$. We then define the automorphism σ_- of $\mathcal{M}(E_6)$ as follows:

$$\sigma_-(E, \varphi) = (E^*, \varphi^t). \tag{7}$$

The Lie algebra involution $d\sigma : \mathfrak{e}_6 \to \mathfrak{e}_6$ induces a vector space decomposition of $\mathfrak{e}_6$ of the form $\mathfrak{e}_6 = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_+$ is the Lie subalgebra of fixed points of $d\sigma$ and $\mathfrak{g}_-$ is the (-1)-eigenspace of $d\sigma$. In [13, Theorem 5.10], it is proved that the subalgebra of fixed points of $d\sigma$ is isomorphic to $\mathfrak{f}_4$ or to $\mathfrak{sp}(8,\mathbb{C})$, so $\mathfrak{g}_+$ is isomorphic to $\mathfrak{f}_4$ or to $\mathfrak{sp}(8,\mathbb{C})$. If $\mathfrak{g}_+ \cong \mathfrak{f}_4$, then, since the dimension of $\mathfrak{e}_6$ is 78 and the dimension of $\mathfrak{f}_4$ is 52, the dimension of $\mathfrak{g}_-$ is 26. The restriction of the adjoint representation of $\mathfrak{e}_6$ to $\mathfrak{g}_+$ induces the adjoint representation of $\mathfrak{f}_4$ when it is restricted in the image to $\mathfrak{gl}(\mathfrak{g}_+)$, and also a representation $d\rho : \mathfrak{g}_+ \to \mathfrak{gl}(\mathfrak{g}_-)$ of $\mathfrak{f}_4$ is induced, since $d\sigma$ is a Lie algebra automorphism. The first representation gives the adjoint representation of F_4 and the second induces its fundamental irreducible representation,

$$\rho : F_4 \to \mathrm{GL}(\mathfrak{g}_-) \tag{8}$$

(observe that $\dim \mathfrak{g}_- = 26$), as we will notice in Proposition 3.2.

We consider now the case in which $\mathfrak{g}_+ \cong \mathfrak{sp}(8,\mathbb{C})$. The corresponding subgroup of E_6 is $\mathrm{PSp}(8,\mathbb{C})$ (observe that the center of $\mathrm{Sp}(8,\mathbb{C})$ is isomorphic to $\mathbb{Z}_2$, while the center of E_6 is isomorphic to $\mathbb{Z}_3$), which admits the symplectic group $\mathrm{Sp}(8,\mathbb{C})$ as its universal cover through a 2 to 1 covering map $\pi_{\mathrm{Sp}} : \mathrm{Sp}(8,\mathbb{C}) \to \mathrm{PSp}(8,\mathbb{C})$. It is also relevant to consider the general symplectic group $\mathrm{GSp}(8,\mathbb{C})$, defined as the group of invertible 8×8 complex matrices that leave invariant certain symplectic form modulo scalars (a classical reference on this group where details about its definition and properties can be found is [14]). We would like to point out that the notions of isotropic vector subspace and symplectic complement make sense when a general symplectic structure is considered in a complex vector space (and also the corresponding notions on principal GSp-bundles). The general symplectic group admits a projection

$$\pi_{\mathrm{GSp}} : \mathrm{GSp}(8,\mathbb{C}) \to \mathrm{PSp}(8,\mathbb{C}) \tag{9}$$

over $\mathrm{PSp}(8,\mathbb{C})$. This is a 4:1 covering map that factors through the 2:1 covering map $\mathrm{GSp}(8,\mathbb{C}) \to \mathrm{Sp}(8,\mathbb{C})$ defined by the universal covering of $\mathrm{Sp}(8,\mathbb{C})$, with the standard projection $\mathrm{Sp}(8,\mathbb{C}) \to \mathrm{PSp}(8,\mathbb{C})$, defined by the quotient by the center of $\mathrm{Sp}(8,\mathbb{C})$. Given a principal $\mathrm{PSp}(8,\mathbb{C})$-bundle E_0 over X, it always admits a lift to a $\mathrm{GSp}(8,\mathbb{C})$-bundle E through π_{GSp}, two of such lifts differing by a line bundle with trivial second power.

Let now $\iota : \mathrm{Sp}(8,\mathbb{C}) \to \mathrm{GL}(8,\mathbb{C})$ be the representation of $\mathrm{Sp}(8,\mathbb{C})$ induced by the natural inclusion of groups. Then the representations

$$\mathrm{Sym}^2\iota \tag{10}$$

and

$$\wedge^2\iota \tag{11}$$

of $\mathrm{Sp}(8,\mathbb{C})$ clearly descend to define representations of $\mathrm{PSp}(8,\mathbb{C})$. If (E_0,φ) is a $(\mathrm{PSp}(8,\mathbb{C}),\mathrm{Sym}^2\iota)$-Higgs pair over X and E is a principal $\mathrm{GSp}(8,\mathbb{C})$-bundle that lifts E_0, then φ can be seen as a holomorphic global section of the vector bundle $\mathrm{Sym}^2E\otimes K$. Analogously, a $(\mathrm{PSp}(8,\mathbb{C}),\wedge^2\iota)$-Higgs pair (E_0,φ) induces the existence of a principal $\mathrm{GSp}(8,\mathbb{C})$-bundle E, which lifts E_0 such that φ takes values in $\wedge^2E\otimes K$. As it will be proved in Proposition 3.2, the representation $\mathfrak{g}_+$ of $\mathrm{PSp}(8,\mathbb{C})$ should be isomorphic to $\mathrm{Sym}^2\iota$ and $\mathfrak{g}_-$ to $\wedge^2\iota$.

Proposition 3.2.

(1) *Let (E,φ) be an E_6-Higgs bundle fixed by σ_+ defined in* (6). *Then E admits a reduction of structure group E' to F_4 or to* $\mathrm{PSp}(8,\mathbb{C})$ *such that $\varphi(E') \subseteq E' \otimes K$. In the case of F_4, φ defines an endomorphism of E' tensored by K and, in the case of* $\mathrm{PSp}(8,\mathbb{C})$, *$\varphi$ defines a holomorphic global section of* $\mathrm{Sym}^2E'\otimes K$.

(2) *Let (E,φ) be an E_6-Higgs bundle fixed by σ_- defined in* (7). *Then E admits a reduction of structure group E' to F_4 or to* $\mathrm{PSp}(8,\mathbb{C})$ *and φ takes values in $E'(\mathfrak{g}_-)\otimes K$. In the case of F_4, $\varphi = 0$ and in the case of* $\mathrm{PSp}(8,\mathbb{C})$, *$\varphi$ defines a holomorphic global section of $\wedge^2E'\otimes K$.*

Proof. Under the conditions of the statement of the proposition, since $(E,\varphi)\cong(E^*,\pm\varphi^t)$ (where the + or $^-$ sign depends on whether (E,φ) is a fixed point of σ_- or σ_+, respectively), there exists an isomorphism

$f : (E, \varphi) \to (E^*, \pm\varphi^t)$. Then $f : E \to E^*$ is an isomorphism of vector bundles. Let $g = (f^t)^{-1} \circ f$, which is an automorphism of E. Exactly one of the following possibilities is satisfied:

- The automorphism g is represented by a central element of E_6. Then there exists $\lambda \in \mathbb{C}$ with $\lambda^3 = 1$ such that $g = \lambda I$. This implies that $f = \lambda f^t$. By transposing, we have that $f^t = \lambda f$, so $f = \lambda^2 f^t$. Finally, $\lambda^2 = \lambda$, so $\lambda = 1$. Therefore, $f = f^t$ and E admits a reduction of the structure group to F_4.
- The automorphism g is not represented by a central element of E_6. Let $\theta \neq 1$ be an eigenvalue of g. Let v be an eigenvector of g of eigenvalue θ. Then $f(v) = \theta f^t(v)$. Since $v \in \operatorname{Ker}(f - \theta f^t)$, it is also true that $v \in \operatorname{Ker}(f^t - \theta f)$, so $f(v) = \theta^2 f(v)$. This implies that $\theta^2 = 1$, so $\theta = -1$. Then E admits a reduction of structure group to some group of type C_n. Therefore, E admits a reduction of the structure group to a copy of $\mathrm{PSp}(8, \mathbb{C})$, since this group is the maximal subgroup of E_6 of type C_n, as it is proved in [15], and the center of $\mathrm{Sp}(8, \mathbb{C})$ is isomorphic to $\mathbb{Z}_2$, while the center of E_6 is isomorphic to $\mathbb{Z}_3$.

All this proves that E admits a reduction of structure group E' to F_4 or $\mathrm{PSp}(8, \mathbb{C})$ defined by f and such that φ takes values in $\mathfrak{g}_+$ or $\mathfrak{g}_-$ depending on the considered automorphism, σ_+ or σ_-, respectively.

In the F_4 case, we have the following:

(1) For the automorphism σ_+, the Higgs field φ always satisfies $f \circ \varphi + \varphi^t \circ f = 0$, that is, f defines a quadratic form in E', which φ respects, so φ is an endomorphism of E' as a principal F_4-bundle.

(2) For the automorphism σ_-, φ satisfies $f \circ \varphi = \varphi^t \circ f$. Since the dimension of $\mathfrak{g}_-$ is 26 in this case, the only possibility for φ is to define a holomorphic global section of $\operatorname{End}(E') \otimes K$, in view of the possibilities for the representations of F_4. The condition $f \circ \varphi = \varphi^t \circ f$ tells us that this global section should satisfy

$$\langle\langle\varphi, a\rangle\varphi, b\rangle - \langle a, \langle\varphi, b\rangle\varphi\rangle = \langle\varphi, a\rangle\langle\varphi, b\rangle - \langle\varphi, b\rangle\langle a, \varphi\rangle = 0$$

for any $a, b \in E'$ (we are omitting the reference to the element in the canonical bundle K), so (E', φ) defines an F_4-Higgs pair for the representation ρ defined in (8).

We now analyze the case of $\mathrm{PSp}(8, \mathbb{C})$.

(1) For the automorphism σ_+. By the description of the representations of $\mathrm{PSp}(8, \mathbb{C})$ and the dimensional restrictions (see the discussion before the statement), φ should be a holomorphic global section of $\mathrm{Sym}^2 E' \otimes K$ or $\wedge^2 E' \otimes K$. In any case, if E_0 is a lift of E' by the projection map defined in (9), φ can be understood as a global section of $\otimes^2 E_0 \otimes K$. Without loss of generality, we can assume that $\varphi = v \otimes w$, by making a slight abuse of notation, for certain $v, w \in E_0$, instead of a linear combination of summands of this kind. If $\langle , \rangle$ denotes the symplectic form defined in E_0 and $a, b \in E'$, we have that

$$\langle\varphi(a), b\rangle + \langle a, \varphi(b)\rangle = \langle w, a\rangle\langle v, b\rangle - \langle v, a\rangle\langle w, b\rangle,$$

because $\varphi(a) = \langle w, a\rangle v$ and $\varphi(b) = \langle w, b\rangle v$, the field φ understood as a vector endomorphism of E' (again, we omitted the reference to the K element). Now, since φ satisfies $f \circ \varphi + \varphi^t \circ f = 0$, one has that the aforementioned expression should be equal 0, which is equivalent to the state that $\varphi \in \mathrm{Sym}^2 E' \otimes K$.

(2) For the automorphism σ_-, the same argument shows that φ defines a global section of $\wedge^2 E' \otimes K$ in this case, only by observing that $f \circ \varphi = \varphi^t \circ f$. □

Remark. Proposition 3.2 can be seen as a consequence of [11, Proposition 3.9], which addresses the general case of a complex semisimple Lie group G. The novelty here is that we have made a specific proof for the particular case of $G = E_6$.

4 Stability conditions for the F_4 and PSp(8, $\mathbb{C}$)-Higgs pairs

Proposition 3.2 tells that, among the fixed points of the automorphism σ_+ of $\mathcal{M}(E_6)$ defined in (6), we find F_4-Higgs bundles over X, and among the fixed points of σ_- defined in (7), F_4-Higgs pairs associated to the fundamental representation of F_4 are found. In the first part of this section, we will give the reduced notions of stability for the bundles and pairs with structure group F_4, which will be done by using the theory set out in Section 2.

The group F_4 is the subgroup of E_6 which consists of automorphisms of a complex vector space V of dimension 26 that preserve a certain holomorphic nondegenerate symmetric trilinear form Ω and a certain holomorphic nondegenerate symmetric bilinear form ω. A principal F_4-bundle over the compact Riemann surface X is then a rank 26 holomorphic complex vector bundle E equipped with a global holomorphic nondegenerate symmetric trilinear form Ω and with a global holomorphic symmetric bilinear form ω (for details, see [4, Section 3]).

In [4], we described the parabolic subgroups of F_4. We will recall here the form of the filtrations induced on a principal F_4-bundle by a reduction of the structure group of the bundle to a parabolic subgroup of F_4.

Let E be a principal F_4-bundle over X. A reduction of structure group of E to a parabolic subgroup of F_4 induces a filtration of E into vector subbundles of the form

$$0 \subsetneq E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_r \subseteq E_r^{\perp} \subsetneq \cdots \subsetneq E_1^{\perp} \subsetneq E_0^{\perp} \subsetneq E, \tag{12}$$

where the orthogonality $\perp$ is taken with respect to ω and $E_0, \ldots, E_r$ are isotropic for ω and Ω.

Proposition 4.1. *Let (E, φ) be an F_4-Higgs bundle, Ω be the holomorphic nondegenerate symmetric trilinear form defined in E, and ω be the holomorphic nondegenerate symmetric bilinear form defined in E. Then (E, φ) is semistable if for every proper subbundle F of E isotropic for ω and Ω such that $\varphi(F) \subseteq F \otimes K$, we have that* $\deg F \leq 0$.

The Higgs bundle (E, φ) is stable if for every filtration as in (12), *where $E_0, \ldots, E_r$ are isotropic for ω and Ω and such that $\varphi(E_k) \subseteq E_k \otimes K$ for all k with* $0 \leq k \leq r$, *we have that* $\deg E_j \leq 0$ *for all j and there exists some k such that* $\deg E_k < 0$.

The Higgs bundle (E, φ) is polystable if E admits a filtration into vector subbundles as in (12), *where $E_0, \ldots, E_r$ are isotropic for ω and Ω,* $\deg E_0 = \deg E_1 = \cdots = \deg E_r = 0$, *such that $\varphi(E_k) \subseteq E_k \otimes K$ for all k with* $0 \leq k \leq r$ *and E can be written in the following form*:

$$E = E_0 \oplus E_1/E_0 \oplus E_2/E_1 \oplus \cdots \oplus E_r/E_{r-1} \oplus E_r^{\perp}/E_r \oplus E_{r-1}^{\perp}/E_r^{\perp} \oplus \cdots \oplus E_0^{\perp}/E_1^{\perp} \oplus E/E_0^{\perp}.$$

Proposition 4.2. *Let (E, φ) be an F_4-Higgs pair associated with the representation of F_4 defined in* (8). *Let Ω be the holomorphic nondegenerate symmetric trilinear form defined in E and ω be the holomorphic nondegenerate symmetric bilinear form defined in E. The Higgs pair (E, φ) is semistable if for every proper subbundle F of E isotropic for ω and Ω such that φ takes values in $F^{\perp} \otimes K$, we have that* $\deg F \leq 0$.

The Higgs pair (E, φ) is stable if for every filtration of the form (12), *where $E_0, \ldots, E_r$ are isotropic for ω and Ω and such that φ takes values in $E_r^{\perp} \otimes K$, we have that* $\deg E_j \leq 0$ *for all j and there exists some k such that* $\deg E_k < 0$.

The Higgs pair (E, φ) is polystable if E admits a filtration into vector subbundles as in (12), *where $E_0, \ldots, E_r$ are isotropic for ω and Ω,* $\deg E_0 = \deg E_1 = \cdots = \deg E_r = 0$ *and such that φ takes values in $E_r^{\perp}/E_r \otimes K$ and E can be written in the following form*:

$$E = E_0 \oplus E_1/E_0 \oplus E_2/E_1 \oplus \cdots \oplus E_r/E_{r-1} \oplus E_r^{\perp}/E_r \oplus E_{r-1}^{\perp}/E_r^{\perp} \oplus \cdots \oplus E_0^{\perp}/E_1^{\perp} \oplus E/E_0^{\perp}.$$

Due to their absolute similarity, we give together the proofs of Propositions 4.1 and 4.2.

Proof. Let (E, φ) be an F_4-Higgs pair for the adjoint representation or for the representation of F_4 defined in (8). Let Ω be the holomorphic nondegenerate symmetric trilinear form defined on E and ω be the holomorphic nondegenerate symmetric bilinear form defined on E. Let P be a parabolic subgroup of F_4, χ be an antidominant character of P and s_{χ} be the corresponding element of $i\mathfrak{h}$. A reduction of structure group of E to P induces a filtration of E of the form

$$0 \subsetneq E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_r \subseteq E_r^{\perp} \subsetneq \cdots \subsetneq E_1^{\perp} \subsetneq E_0^{\perp} \subsetneq E,$$

where $E_0, \ldots, E_r$ are isotropic for ω and Ω. Let d_j be the rank of E_j for each $j = 0, \ldots, r$. The element s then diagonalizes in the following form:

$$\begin{pmatrix} \lambda_0 I_{d_0} & & & & & & \\ & \ddots & & & & & \\ & & \lambda_r I_{d_r - d_{r-1}} & & & & \\ & & & 0 I_{26-2d_r} & & & \\ & & & & -\lambda_r I_{d_r - d_{r-1}} & & \\ & & & & & \ddots & \\ & & & & & & -\lambda_0 I_{d_0} \end{pmatrix}, \tag{13}$$

where $\lambda_k < 0$, I_k denotes the identity matrix of rank k for each k and $\lambda_0 < \lambda_1 < \cdots < \lambda_r$. A few simple computations show that the degree defined in (5) becomes

$$\lambda_r(\deg E_r + \deg E_r^{\perp}) + \sum_{j=0}^{r-1} (\lambda_j - \lambda_{j+1})(\deg E_j + \deg E_j^{\perp}).$$

This shows that the expression (5) is a linear combination of the numbers $\deg E_j + \deg E_j^{\perp}$ with negative coefficients. Then we have the following:

(1) For the automorphism σ_+.
   (a) The Higgs bundle (E, φ) is semistable if $\deg E_j \leq 0$ for all j and for every filtration as mentioned earlier for which $\varphi(E_k) \subseteq E_k \otimes K$ for all k with $0 \leq k \leq r$ (this is the condition for φ expressed in Proposition 2.1). This is clearly equivalent to the assertion of the statement since every isotropic subbundle of E can be placed in a filtration as the considered one.
   (b) Analogously, by Proposition 2.1, the Higgs bundle (E, φ) is stable if for every filtration as mentioned earlier for which $\varphi(E_k) \subseteq E_k \otimes K$ for all k with $0 \leq k \leq r$, we have that $\lambda_r(\deg E_r + \deg E_r^{\perp}) + \sum_{j=0}^{r-1}(\lambda_j - \lambda_{j+1})(\deg E_j + \deg E_j^{\perp}) > 0$, which is equivalent to requiring that $\deg E_j \leq 0$ for all j and that there exists at least one $k \in \{0, 1, \ldots, r\}$ such that $\deg E_k < 0$.
   (c) Finally, the third part of the statement is clearly a re-reading of the third part of Proposition 2.1.

(2) For the automorphism σ_-.
   (a) The Higgs pair (E, φ) is semistable if and only if for each filtration as mentioned earlier for which φ takes values in $E_r^{\perp} \otimes K$, $\deg E_j \leq 0$ for all j, by Proposition 2.1. As in the first case, this is again equivalent to the assertion of the statement.
   (b) Analogously, by Proposition 2.1, (E, φ) is stable if and only if for each filtration as mentioned earlier for which φ takes values in $E_r^{\perp} \otimes K$, it is satisfied that $\deg E_j \leq 0$ for all j and that there exists at least one $k \in \{0, 1, \ldots, r\}$ such that $\deg E_k < 0$.
   (c) As in the case of σ_+, the third part of the result is a re-reading of the third part of Proposition 2.1. □

We will now deal with the case of Higgs pairs with the structure group $\mathrm{PSp}(8, \mathbb{C})$.

Proposition 4.3. *Let* $\mathrm{Sym}^2\iota$ (*resp.* $\wedge^2\iota$) *be the representation of* $\mathrm{PSp}(8, \mathbb{C})$ *defined in* (10) (*resp. in* (11)), *let* (E_0, φ) *be a* $(\mathrm{PSp}(8, \mathbb{C}), \mathrm{Sym}^2\iota)$*-Higgs pair* (*resp.* $(\mathrm{PSp}(8, \mathbb{C}), \wedge^2 \iota)$*-Higgs pair*) *over* X, *and let E be a principal* $\mathrm{GSp}(8, \mathbb{C})$*-bundle over X, which lifts* E_0. *Then* (E_0, φ) *is semistable if* $\frac{\deg F}{\mathrm{rk} F} \leq \frac{\deg E}{\mathrm{rk} E}$ *for every isotropic subbundle F of E satisfying that* φ *takes values in the subbundle*

$$((F \otimes_S E) \oplus (F^{\perp}/F \otimes_S F^{\perp}/F)) \otimes K$$

(*resp.*

$$((F \wedge E) \oplus (F^{\perp}/F \wedge F^{\perp}/F)) \otimes K)$$

of $\mathrm{Sym}^2 E \otimes K$ (*resp. of* $\wedge^2 E \otimes K$).

The pair (E_0, φ) *is polystable if* E *admits a decomposition of the form* $E = E_0 \oplus E_1/E_0 \oplus \cdots \oplus E_r/E_{r-1} \oplus E_r^\perp/E_r \oplus \cdots \oplus E/E_0^\perp$ *with* $\frac{\deg E_0}{\operatorname{rk} E_0} = \frac{\deg E}{\operatorname{rk} E}$, $\deg E_j = 0$ *for all* $j \geq 1$ *and such that* φ *takes values in the subbundle*

$$(\oplus_{i=0}^{r-1}(E_i \otimes_S E_{i+1}^\perp/E_i) \oplus (E_r^\perp/E_r \otimes_S E_r^\perp/E_r)) \otimes K$$

(*resp.*

$$(\oplus_{i=0}^{r-1}(E_i \wedge E_{i+1}^\perp/E_i) \oplus (E_r^\perp/E_r \wedge E_r^\perp/E_r)) \otimes K)$$

of $\operatorname{Sym}^2 E \otimes K$ (*resp. of* $\wedge^2 E \otimes K$).

Proof. We will develop the proof for the case of the representation $\operatorname{Sym}^2\iota$ (the case of $\wedge^2\iota$ is absolutely analogous). Given a $(\mathrm{PSp}(8, \mathbb{C}), \operatorname{Sym}^2\iota)$-Higgs pair (E_0, φ) and given E as in the statement, a reduction of structure group of (E_0, φ) to a parabolic subgroup P of $\mathrm{PSp}(8, \mathbb{C})$ comes with a filtration of E into isotropic subbundles for its symplectic form

$$0 \subsetneq E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_r \subseteq E_r^\perp \subsetneq \cdots \subsetneq E_1^\perp \subsetneq E_0^\perp \subsetneq E.$$

If χ is an antidominant character of P and s_χ is the associated element of $i\mathfrak{h}$, then s_χ diagonalizes as in (13). The degree defined in (5) has the following form

$$-\lambda_0 \deg E + \lambda_r(\deg E_r + \deg E_r^\perp) + \sum_{j=0}^{r-1}(\lambda_j - \lambda_{j+1})(\deg E_j + \deg E_j^\perp),$$

so it is greater than or equal to 0 for every family of weights $\{\lambda_i\}$ if and only if $\frac{\deg E_i}{\operatorname{rk} E_i} \leq \frac{\deg E}{\operatorname{rk} E}$ for every $i \geq 0$ and it is equal to 0 for every such family if $\frac{\deg E_0}{\operatorname{rk} E_0} = \frac{\deg E}{\operatorname{rk} E}$ and $\deg E_j = 0$ for all $j \geq 1$. On the other hand, if $v \otimes_S w$ is a generic element that belongs to a summand of the differentiable decomposition $E_0 \oplus E_1/E_0 \oplus \cdots \oplus E_r/E_{r-1} \oplus E_r^\perp/E_r \oplus \cdots \oplus E/E_0^\perp$, then

$$\operatorname{Sym}^2\iota\left(e^{ts_\chi}\right)(v \otimes_S w) = e^{t(\alpha+\beta)} v \otimes_S w,$$

where α is the eigenvalue of v and β is the eigenvalue of w. This expression is bounded as $t \to \infty$ if and only if $\alpha + \beta \leq 0$. If we suppose that $\alpha \leq \beta$, this implies that v belongs to some E_i (i.e., α coincides with some λ_i and w belongs to $E_{i-1}^\perp$, or both vectors v and w belong to $E_r^\perp/E_r$. The semistability condition then demands that $\frac{\deg E_i}{\operatorname{rk} E_i} \leq \frac{\deg E}{\operatorname{rk} E}$ for every i whenever φ takes values in

$$\oplus_{j=1}^{r}(E_j \otimes_S E_{j-1}^\perp) \oplus (E_r^\perp/E_r \otimes_S E_r^\perp/E_r).$$

The semistability condition applied to a filtration of the form $0 \subsetneq F \subsetneq F^\perp \subsetneq E$ induced by a reduction of structure group to a maximal parabolic subgroup then demands the condition exposed in the statement. Since the satisfaction of the semistability condition for these filtrations given by reductions to maximal parabolic subgroups implies the satisfaction of the semistability condition for every other filtration, the result for semistability holds.

For polystability, observe that, with the preceding notation, $\alpha + \beta = 0$ if and only if $\alpha = \beta = 0$ or $\alpha = \lambda_i$ and $\beta = -\lambda_i$ for some i, and so polystability requires that $\frac{\deg E_0}{\operatorname{rk} E_0} = \frac{\deg E}{\operatorname{rk} E}$, $\deg E_j = 0$ for all $j \geq 1$ whenever φ takes values in $(\oplus_{i=0}^{r-1}(E_i \otimes_S E_{i+1}^\perp/E_i) \oplus (E_r^\perp/E_r \otimes_S E_r^\perp/E_r)) \otimes K$. □

5 Conclusion

Let X be a compact Riemann surface of genus $g \geq 2$ and let $\mathcal{M}(E_6)$ be the moduli space of E_6-Higgs bundles over X. It is well known that E_6 is the only exceptional simple complex Lie group that admits a nontrivial outer automorphism. This automorphism, which has been called σ in the paper, is an involution, and it acts on $\mathcal{M}(E_6)$. By combining this with the multiplication by +1 or –1 on the Higgs field, the automorphisms σ_+ and σ_- of $\mathcal{M}(E_6)$, defined in (6) and (7), are obtained. The main result of this article describes the subvariety of fixed points of $\mathcal{M}(E_6)$ for these two automorphisms (Proposition 3.2). The techniques employed make use of the specific properties of group E_6, so the proof is also specific for this case. As a result, it has been obtained that the indicated fixed points can be described as certain Higgs pairs associated with the structure groups F_4 and $\mathrm{PSp}(8,\mathbb{C})$ and to certain representations of these groups, which are defined in (8), (10), and (11). Finally, in Section 4, the stability conditions for these Higgs pairs are obtained from the general theory of G-Higgs pairs developed in Section 2 (Propositions 4.1–4.3). A study of how the group inclusions $F_4 \hookrightarrow E_6$ and $\mathrm{PSp}(8,\mathbb{C}) \hookrightarrow E_6$ influences the natural maps of the moduli spaces of F_4 or $\mathrm{PSp}(8,\mathbb{C})$-Higgs pairs in $\mathcal{M}(E_6)$ is proposed as a line of future research.

Funding information: This research received no external funding.

Conflict of interest: The author states no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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