One-sided indecomposable pure-injective modules over string algebras

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2004

MIMS EPrint: **2006.121**

ISSN 1749-9097

ONE-SIDED INDECOMPOSABLE PURE INJECTIVE MODULES OVER STRING ALGEBRAS

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Abstract. We classify one-sided indecomposable pure injective modules over (finite dimensional) string algebras.

1. Introduction

Let A be a finite dimensional string algebra over a field $\Bbbk$ (as an example one may consider the Gelfand–Ponomarev algebra $G_{2,3}$ given by generators α, β and relations $\alpha\beta = \beta\alpha = \alpha^2 = \beta^3 = 0$). A classification of indecomposable finite dimensional A-modules has been known since Butler and Ringel [2]: they are exactly the so-called string and band modules.

Although the classification of arbitrary infinite dimensional modules over a string algebra A is hardly possible, some particular classes of such modules are of special interest. For instance Ringel [11] announced a program to classify indecomposable pure injective modules over string algebras. It is known that over a finite dimensional algebra pure injective modules may be characterized as direct summands of direct products of finite dimensional modules.

Every indecomposable finite dimensional module is pure injective, but there are less obvious examples. For every band (see Section 2) C over a string algebra A there is a one-parameterized family of 'Prüfer' modules and a one-parameterized family of 'adic' modules. Also there is one 'generic' module corresponding to C. We will refer to these modules as infinite dimensional band modules.

Moreover, if v is a one-sided almost periodic string or a two-sided biperiodic string over A, then Ringel [10] associated to v a module, $M(v)$ which is, in his terminology, a direct sum, direct product or 'mixed' module and which is pure injective and indecomposable.

Conjecture 1.1. *(Ringel's conjecture — see [10, p. 48, p. 51]) Let A be a finite dimensional domestic string algebra. Then every infinite dimensional indecomposable pure injective A-module is either a band module or is of the form $M(v)$, where v is either a one-sided almost periodic string or a two-sided biperiodic string.*

2000 *Mathematics Subject Classification.* 16G20, 16D50.

Key words and phrases. Pure injective module, string algebra.

This paper was mostly written during the visit of the second author to the University of Manchester supported by EPSRC grant GR/R44942/01. He would like to thank the University for the kind hospitality.

There is a natural construction which assigns to every element m of a pure injective module M over a string algebra A an (infinite) word $w(m)$. We will say that M is one-sided, if for some $m \in M$, $w(m)$ is a one-sided word. Otherwise M is two-sided. For instance every finite dimensional string module is one-sided.

In this paper we classify one-sided indecomposable pure injective modules over a string algebra A. We prove that if M is an indecomposable pure injective A-module, and $0 \neq m \in M$ is such that $w(m)$ is a one-sided word, then the isomorphism type of M is determined by $w(m)$. Moreover, for every one-sided word w there is an indecomposable pure injective A-module M and $m \in M$ such that $w(m) = w$ and we show that this correspondence is bijective for infinite words.

Thus one-sided indecomposable pure injective modules over a string algebra A are classified by one-sided words over A. Using this we show that over a non-domestic string algebra A there are precisely 2^{ω} non-isomorphic one-sided indecomposable pure injective modules.

However the methods used in the proofs do not give much information about the structure of such modules. For domestic string algebras, using Ringel results, we are able to give a completely satisfactory description of one-sided indecomposable pure injective modules. Precisely, every such module has the form $M(v)$ from Ringel's list, and $M(v) \cong M(w)$ iff $v = w$ or $v = w^{-1}$.

Given a domestic string algebra A, we calculate the Cantor–Bendixson rank of the open set in the Ziegler spectrum formed by the one-sided indecomposable pure injective modules. We prove that this rank is equal to $n+1$, where n is the length of a maximal path in the bridge quiver of A. Note that conjecturally the Cantor–Bendixson rank of the Ziegler spectrum of a domestic string algebra A is equal to $n+2$ (Schröer's conjecture — see [12, p. 84]): we prove that the rank is at least $n+2$.

The paper consists of two parts. In the first part we show how to analyze the open subset of the Ziegler spectrum given by a chain in the lattice of pp-formulae. This part works for modules over an arbitrary ring. In the second part we apply these results to the family of uniserial functors constructed by Prest and Schröer [7] and combine them with Ringel's results.

Note that the problem of classifying indecomposable pure injective modules over a non-domestic string algebra appears to be extremely difficult. Some examples were collected in Baratella and Prest [1], and we use them in this paper to illustrate results. Recently Puninski [9] proved that in the case of a countable field every non-domestic string algebra has a pure injective module without indecomposable direct summands.

So, it may be instructive to see that a general model-theoretic approach combined with relatively unsophisticated algebraic methods (what Ringel [11, p. 48] refers to as 'bare hands') clarifies the situation without exhaustive calculations.

2. String algebras

Almost everywhere in this paper modules will be left modules over a finite dimensional algebra A. Upper case letters such as C, D and E will always denote finite strings.

Given a finite quiver (i.e. an oriented graph) Q we may construct a possibly infinite dimensional algebra $A = \Bbbk Q$ with a $\Bbbk$-basis given by the paths in Q and with multiplication given by the composition of paths. For instance for every vertex $i \in Q$ there is the path of length 0 which is an indecomposable idempotent $e_i \in A$. Given an arrow α in Q its starting point will be denoted by $s(\alpha)$ and its end point will be denoted by $e(\alpha)$. Thus $\alpha\beta$ (β then α) is a path in Q if $s(\alpha) = e(\beta)$ (this fits with our convention that we consider left modules).

We impose some monomial relations (i.e. relations of the form $\alpha_1 \dots \alpha_n = 0$, where α_i are arrows in Q forming a path) on A to make A finite dimensional. Then A is a *string algebra*, if the following holds:

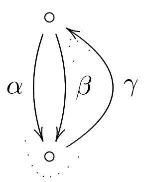
1) every vertex is the starting point of at most two arrows and the end point of at most two arrows;

2) if α, β, γ are arrows such that $e(\alpha) = s(\beta) = s(\gamma)$ (i.e. $\beta\alpha$ and $\gamma\alpha$ are paths in Q), then either $\beta\alpha = 0$ or $\gamma\alpha = 0$ is a relation on A;

3) if α, β, γ are arrows such that $s(\alpha) = e(\beta) = e(\gamma)$ (i.e. $\alpha\beta$ and $\alpha\gamma$ are paths in Q), then either $\alpha\beta = 0$ or $\alpha\gamma = 0$ is a relation on A.

For instance

R_1



with relations $\gamma\alpha = 0$ and $\beta\gamma = 0$ is a string algebra (the relations are indicated by dotted curves).

For every arrow α we introduce a formal inverse α^{-1} for α with $s(\alpha^{-1}) = e(\alpha)$ and $e(\alpha^{-1}) = s(\alpha)$. A *string* (of length k) over A is a sequence of letters (that is, arrows or inverses of arrows) $C = c_1 c_2 \dots c_k$ such that

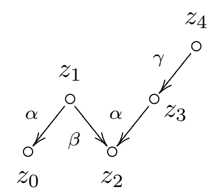
1) $c_i c_{i+1}$ is neither of the form $\alpha\alpha^{-1}$ nor of the form $\beta^{-1}\beta$ for any arrows α or β, for $1 \leq i \leq k-1$;

2) $c_{i+1} \dots c_{i+t}$, $1 \leq i+1 < i+t \leq k$ is neither of the form $\alpha_1 \dots \alpha_t$ nor of the form $\alpha_t^{-1} \dots \alpha_1^{-1}$, where $\alpha_1 \dots \alpha_t = 0$ is any relation on A.

Roughly, a string represents a reduced, non-zero "walk" in Q where arrows may be traversed in either direction.

For instance $\alpha\beta^{-1}\alpha\gamma$ is a string over R_1 (interpreted as 'go along γ then along α, then lift through β and go along α again'). Every string $C = c_1 \dots c_k$ over A defines a *string module* $M(C)$ as follows. $M(C)$ is a $(k+1)$-dimensional vector space with basis $z_0, z_1, \dots, z_k$. Informally c_i will be between z_{i-1} and z_i in $M(C)$ and the action of c_i will be to map z_i to z_{i-1} or vice versa. If c_i is a *direct arrow* (that is, an arrow), say $c_i = \alpha$, then put $z_{i-1} = \alpha z_i$. If c_i is an inverse arrow, say $c_i = \beta^{-1}$, then set $\beta z_{i-1} = z_i$. For each such relation $\alpha z_i = z_j$, say $s(\alpha) = k$ and $e(\alpha) = l$, set $e_k z_i = z_i$ and $e_l z_j = z_j$. All the remaining actions of generators of $\Bbbk Q$ on these basis elements z_i are defined to be zero. It is easy to check that $M(C)$ is a left A-module.

In the sequel we will draw direct arrows from the upper right to the lower left and inverse arrows from the upper left to the lower right. Thus the string module $M(\alpha\beta^{-1}\alpha\gamma)$ over R_1 has the following diagram:



It is known (see [2]) that any string module is indecomposable, and $M(C) \cong M(D)$ iff either $C = D$ or $C^{-1} = D$.

An infinite sequence of letters $v = c_0 c_1 c_2 \dots$ is called a *one-sided string* if $c_0 \dots c_k$ is a string for every k. Similarly we can define a one-sided string $v = \dots c_{-1} c_0$ directed to the left. For instance ${}^\infty(\beta^{-1}\alpha)$, meaning $\dots \beta^{-1}\alpha\beta^{-1}\alpha$, is a one-sided string over R_1, and $(\beta^{-1}\alpha)^\infty$, meaning $\beta^{-1}\alpha\beta^{-1}\alpha \dots$.

For every one-sided string we may define a *direct sum module* with basis $z_0, z_1, \dots$ such that c_i acts between z_i and z_{i-1} as in a finite dimensional string module. If we admit arbitrary (not necessarily finite) tuples and use the same action "pointwise", we obtain a *direct product module*.

A one-sided string v is called *almost periodic* if v is not *periodic* (that is, of the form D^∞ or ${}^\infty D$ for some string D) and $v = CD^\infty$ or $v = {}^\infty DC$ for finite strings C and D. According to Ringel [10] every almost periodic string is either "*expanding*" or "*contracting*" (depending on whether the last letters of C and D are direct or inverse): we will not need the definitions of these terms here.

Fact 2.1. [10, p. 424], [11, p. 50] *Let v be a one-sided almost periodic string over a string algebra A. If v is expanding then the direct product module, which we denote $M(v)$, is pure injective and indecomposable. If v is contracting, then the direct sum module, denoted M(v), is pure injective and indecomposable.*

Note that we use $M(v)$ to denote either the direct sum or direct product module depending on whether v is contracting or expanding (in the above references, $\overline{M}(v)$ is used for the direct product module).

A *band* over A is a string $C = c_1 \dots c_k$ such that following holds:

1) every power C^m is defined;

2) C is not a power of a proper substring;

3) c_1 is a direct arrow and c_k is an inverse arrow.

Thus every band C over A is of the form $\alpha \dots \beta^{-1}$, and clearly $\alpha \neq \beta$. Note that then $C^{-1} = \beta \dots \alpha^{-1}$ is also a band. For instance over R_1 we have the following bands: $C = \alpha\beta^{-1}$ and $C^{-1} = \beta\alpha^{-1}$.

Let C, D be finite strings with the same first letter c. One defines $C < D$ if one of the following holds:

1) $D = C\beta D'$ or

2) $C = D\gamma^{-1}C'$ or

3) $C = E\gamma^{-1}C'$ and $D = E\delta D'$ for some strings D', C', E and arrows β, γ, δ.

Clearly $<$ is a linear order. Also every string C (except the maximal one) has an immediate successor C^+ with respect to this order. For instance, if $C\beta$ is a string for some arrow β, then $C^+ = C\beta\gamma^{-1}\dots$ (as many inverse arrows as possible).

This order obviously can be extended to infinite one-sided strings $v = c\dots$. Then v defines a *cut* on the set of finite strings with first letter c: the 'lower part' of this cut is $\{C \mid C < v\}$ and the 'upper part' is $\{D \mid D > v\}$.

Similarly we may define the 'left order' $<'$ on the set of one-sided strings ending with the same letter c by setting $C <' D$ if $C^{-1} < D^{-1}$. The immediate successor of C with respect to this order will be denoted by ${}^+C$.

3. Some model theory

We recall some basic notions from the model theory of modules. For more on this the reader is referred to [4].

A *pp-formula* $\varphi(x)$ (in one free variable x) is a formula of the form $\exists\, \bar{y}\, B\bar{y} = \bar{b}x$, where $\bar{y} = (y_1, \dots, y_n)$, B is an $m \times n$ matrix over A and $\bar{b}$ is a column over A with m rows. This pp-formula is interpreted as 'B divides $\bar{b}x$'. For instance a *divisibility formula* is a pp-formula of the form $\exists\, y\, (ry = x)$ where $r \in A$; we write $r \mid x$ for short.

If φ is a pp-formula as above and m is an element of a module M we say that φ is *satisfied* by m in M, written $M \models \varphi(m)$, if there is $\overline{m} = (m_1, \dots, m_n) \in M^n$ such that $B\overline{m} = \bar{b}m$. Then $\varphi(M) = \{m \in M \mid M \models \varphi(m)\}$ is a *pp-definable subgroup* of M. Note that $\varphi(M)$ is a (right) S-submodule of M where $S = \mathrm{End}(M)$. For instance, for a divisibility formula we have $(r \mid x)(M) = rM$.

Let φ and ψ be pp-formulae. We write $\psi \to \varphi$ (ψ implies φ) if $\psi(M) \subseteq \varphi(M)$ holds for every module M. The implication relation is reflexive and transitive, therefore defines a (quasi-) order on the set of all pp-formulae. Thus we will often write $\psi \leq \varphi$ instead of $\psi \to \varphi$. We say that pp-formulae φ and ψ are *equivalent* if $\psi \leq \varphi \leq \psi$, i.e. $\psi(M) = \varphi(M)$ for every module M.

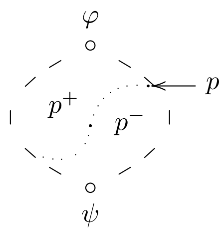
Factorizing the set of all pp-formulae by the equivalence relation, we obtain a partial order $L(A)$. In fact $L(A)$ is a modular lattice, where the meet operation $\wedge$

("and") is conjunction of pp-formulae and the join operation $+$ is given by the rule $(\varphi+\psi)(x) = \exists\, y\, (\varphi(y) \wedge \psi(x-y))$. If $\psi < \varphi$ are pp-formulae then (φ/ψ) will, in this paper, denote the interval $[\psi; \varphi]$ in $L(A)$.

A *pp-type* $p(x)$ is a collection of pp-formulae which is closed with respect to implication and (finite) conjunction. For instance, if m is an element of a module M, then the set of all pp-formulae satisfied by m in M is a pp-type, denoted $pp_M(m)$. By [4, Ch. 4] for every pp-type p there is a 'minimal' pure injective module $M = N(p)$ containing an element $m \in M$ such that $p = pp_M(m)$. This module is unique (up to isomorphism over m) and will be called a *pure injective envelope* of p.

We say that a pp-type p is *indecomposable*, if $N(p)$ is an indecomposable module. The *positive part*, p^+, of a pp-type p consists of all pp-formulae $\varphi \in p$ (i.e. $p^+ = p$) and its *negative part* p^- consists of those pp-formulae ψ with $\psi \notin p$.

We say that an interval (φ/ψ) is *open in a pp-type* p, writing $p \in (\varphi/\psi)$, if $\varphi \in p^+$ and $\psi \in p^-$. In this case p defines a cut on (φ/ψ), whose 'upper' part consists of pp-formulae in p^+ (and below φ) and whose 'lower' part consists of pp-formulae in p^- (and above ψ):



The following useful result says that the pure injective envelope of an indecomposable pp-type p is uniquely determined by any (local) cut of p.

cut **Fact 3.1.** *Let $\psi < \varphi$ be pp-formulae and let $p, q \in (\varphi/\psi)$ be indecomposable pp-types which define the same cut on the interval (φ/ψ). Then $N(p) \cong N(q)$.*

Proof. We have $\varphi \in p, q$ and $\psi \in p^-, q^-$. If $N(p)$ and $N(q)$ were non-isomorphic then, by a result of Ziegler, [15], see [4, Lemma 9.2], there would exist a pp-formula θ such that $\psi < \theta < \varphi$ and either $\theta \in p \setminus q$ or $\theta \in q \setminus p$. Thus p and q would define different cuts on the interval (φ/ψ), a contradiction. $\square$

In general not every cut on an interval (φ/ψ) leads to an indecomposable pp-type. But this is the case if (φ/ψ) is a chain.

chain **Lemma 3.2.** *Given any cut on a chain (φ/ψ) there is an indecomposable pp-type q which defines this cut on (φ/ψ). Moreover the (indecomposable pure injective) module $N(q)$ is uniquely (up to an isomorphism) determined by the original cut.*

Proof. Since (φ/ψ) is a chain, the upper part of the cut, denote it p^+, is closed with respect to conjunctions and the lower part of the cut, p^-, is closed with respect to sums. Also the set of formulas $p^+ \cup \neg\, p^-$ is consistent.

Let us extend $p^+ \cup \neg p^-$ to a maximal pp-type q (i.e. such that $q^+ \supseteq p^+$ and is maximal with respect to $q^+ \cap p^- = \emptyset$). From [4, Thm. 4.32] it follows that q is indecomposable. By the construction, q defines the original cut on the chain (φ/ψ). Suppose that q' is another indecomposable pp-type that defines the same cut on (φ/ψ). Then $N(q) \cong N(q')$ by Fact 3.1. $\square$

We say that an indecomposable pure injective module M *opens an interval* (φ/ψ), written $M \in (\varphi/\psi)$, if there is $m \in M$ such that $m \in \varphi(M) \setminus \psi(M)$, i.e. $p \in (\varphi/\psi)$ where $p = pp_M(m)$.

Thus we obtain the following 'rough' classification of indecomposable pure injective modules living on the chain.

chclas **Theorem 3.3.** *Let (φ/ψ) be a chain in the lattice of all pp-formulae over A. Then there is a natural surjection from the set of cuts on (φ/ψ) to the set of (isomorphism types of) indecomposable pure injective A-modules opening this interval.*

Proof. This follows from Fact 3.1 and Lemma 3.2. $\square$

In general this map is not monic: different cuts may lead to isomorphic indecomposable pure injective modules.

4. Preliminary results

We say that a pair (M, m) is a *free realization* of a pp-formula $\varphi(x)$ if M is a finitely presented (=finite dimensional in the context of modules over finite-dimensional algebras) module, $M \models \varphi(m)$, and $\varphi \to \psi$ for every pp-formula ψ such that $M \models \psi(m)$. In particular $pp_M(m)$ is generated as a pp-type by φ. By [4, Ch. 8] every pp-formula has a free realization. For instance the pair (A, r) is a free realization of the formula $r \mid x$.

The following example of a free realization will be of special importance. Let $M = M(CD)$ be a string module over a string algebra A (we allow C or D to be empty) and let z be the element of a canonical basis of M lying between C and D (in the sense of the construction of string modules). Let $(C.)$ be a pp-formula describing the part of M to the left of z. For instance, continuing the notation of an earlier example, if $M = M(\alpha\beta^{-1}\alpha\gamma)$ over R_1 and $C = \alpha\beta^{-1}$, $D = \alpha\gamma$, then $z = z_2$ and $(C.)$ will say 'there exist z_0, z_1 such that $\alpha z_1 = z_0 \wedge \beta z_1 = z$', and is equivalent to $\beta \mid x$.

If $C = \emptyset$ then we take $C.$ to be the conjunction of any formulas necessary to specify the annihilator of z.

Similarly let $(.D)$ be a pp-formula describing the part of M to the right of z (in this example $\alpha\gamma \mid x$ will do but in general one will need more complicated formulas). By $(C.D)$ we denote the conjunction of the formulae $(C.)$ and $(.D)$. Note that if $e_j \in A$ is the unique basic idempotent such that $e_j z = z$, then $(C.D) \to e_j \mid x$.

CD **Remark 4.1.** *Let $M = M(CD)$ be a string module and let z be an element of the canonical basis of M between C and D. Then (M, z) is a free realization of $(C.D)$.*

Proof. By definition $(C.D) \in p = pp_M(z)$. Let (N, n) be a free realization of $(C.D)$. From the description of $(C.D)$ it it easy to construct a morphism $f : M \to N$ such that $f(z) = n$. Now, if $\psi \in p$, then $M \models \psi(z)$ so $N \models \psi(n)$, therefore $(C.D) \to \psi$ by the definition of a free realization. $\square$

If M is a module and D is a string then $(.D)M$ is a pp-subgroup of M. For instance, in our example, if $D = \alpha$ then for $(.D)$ we may take $\exists z_1\, (\alpha z_1 = x \wedge \beta z_1 = 0)$, so $(.\alpha)M = \alpha\, \mathrm{ann}_M(\beta)$, and if $D = \beta^{-1}$ then for $(.D)$ we may take $\gamma\beta x = 0$ so $(.\beta^{-1})M = \mathrm{ann}_M(\gamma\beta)$.

It is quite straightforward to check that if E, F are finite strings such that $E \leq F$ then $(.F) \to (.E)$. Similarly, if C, D are finite strings such that $C \leq' D$, then $(D.) \to (C.)$.

The following lemma says that $(.D)$ defines a homogeneous subspace in every direct sum or direct product module (we allow the sum below to be infinite in a direct product module).

hom **Lemma 4.2.** *Let $M = M(v)$ be either the direct sum or direct product module corresponding to a one-sided string v and let D be a finite string. Then $(.D)(M)$ is a homogeneous subspace of M, i.e. $\sum_i \lambda_i z_i \in (.D)(M)$ iff $z_i \in (.D)(M)$ for every i such that $\lambda_i \neq 0$.*

Proof. Similar to [1, Lemma 3.4]. B-P $\square$

The following (almost obvious lemma) will be useful in what follows.

aux **Lemma 4.3.** *Let (M, m) be a free realization of a pp-formula φ. Suppose that $m = n + k$, that (M, n) is a free realization of φ_1, and $M \models \varphi_2(k)$. If ψ is a pp-formula such that $\varphi_2 \to \psi$, then $\varphi + \psi$ is equivalent to $\varphi_1 + \psi$.*

Proof. Since (M, m) is a free realization of φ, and $M \models (\varphi_1 + \varphi_2)(m)$, we obtain $\varphi \to \varphi_1 + \varphi_2 \to \varphi_1 + \psi$. Therefore $\varphi + \psi \to \varphi_1 + \psi$.

So it remains to prove that $\varphi_1 + \psi \to \varphi + \psi$. Since $(M, n = m - k)$ is a free realization of φ_1, and $M \models (\varphi + \varphi_2)(n)$, we conclude that $\varphi_1 \to \varphi + \varphi_2$. Then $\varphi_1 + \psi \to \varphi + \varphi_2 + \psi = \varphi + \psi$. $\square$

Note that in the above lemma it may happen that $\varphi < \varphi_1 + \varphi_2$ even if k is also a free realization of φ_2. For instance, let $A = G_{2,2}$, and let M be the following string module:

$$\begin{tikzcd} & z_1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ z_0 & & z_2 \end{tikzcd}$$

Set $m = z_0 + z_2$, $n = z_0$, and $k = z_2$. Then φ may be taken to be the formula $(\alpha + \beta) \mid x$ which is stronger than $\alpha \mid x + \beta \mid x$ (the sum of the pp-types of n and k in M).

The next result is a key one in what follows.

Lemma 4.4. *Let CD be a string over a string algebra A. Then every formula in the interval $(C.D)/({}^+C.D)$, apart from $({}^+C.D)$, is equivalent to a formula $({}^+C.D)+(C.D_i)$ for some $D_i \geq D$ such that CD_i is a string. In particular the interval $(C.D)/({}^+C.D)$ is a chain.*

Proof. All this follows from [7, Thm. 3.2]. We just add some explanations.

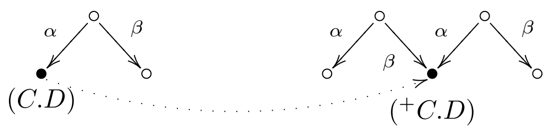
It is clear that the formulae $(C.D_i)$ with $D_i \geq D$ are linearly ordered, therefore the same is true for the formulae $({}^+C.D) + (C.D_i)$. Thus it suffices to prove that every pp-formula strictly between $(C.D)$ and $({}^+C.D)$ is of the required form. Note that such a formula can be obtained in the following way: take any pp-formula φ below $(C.D)$ and add $({}^+C.D)$.

Let z be the element of a canonical basis of $M = M(CD)$ between C and D. Let (N, m) be a free realization of φ. Since $(C.D) \geq \varphi$, there is a morphism $f : M \to N$ taking z to m. Since any sum of pp-formulas of the form $({}^+C.D) + (C.D_i)$ is equivalent to a single one of them we may assume that N is indecomposable, therefore is either a string or a band module.

If N is a band module, then from the proof of Theorem 3.2 in [7] it follows that $\varphi \to ({}^+C.D)$, therefore φ is taken to $({}^+C.D)$ by summation.

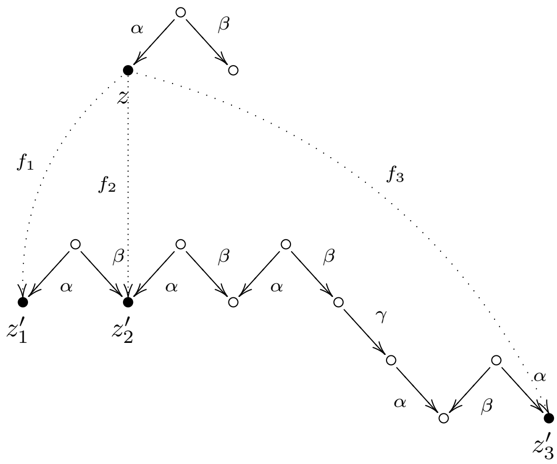
Otherwise N is a string module, therefore (by Crawley-Boevey [3]) f is a linear combination of graph maps $f_i : M \to N = M_i = M(C_iD_i)$, $i = 1, \ldots, n$, where $f_i(z) = z'_i$ with z'_i lying between C_i and D_i.

To understand the situation better let us look at the following example of pp-formulas over R_1, where $(C.D)$ is the formula $\exists\, z_1\, (\alpha z_1 = z \wedge \gamma\beta z_1 = 0)$ (that is stronger than $\alpha \mid x$).



Thus C is empty in this case, and $({}^+C.)$ is $\beta \mid x$.

Let $f_i : M \to N$, $i = 1, 2, 3$ extend the map $z \mapsto z'_i$ as the following diagram shows:



Let $f = f_1 + f_2 + f_3 : M \to N$, in particular $m = f(z) = z_1' + z_2' + z_3'$.

Note that f_1, f_2 preserve the orientation of M, but f_3 flips it over. By Remark 4.1, (C_i, D_i) generates the pp-type p_i of z_i' in N.

Clearly there is an endomorphism h of N which send z_1' to z_3'. Since N is indecomposable, and h strictly increases pp-types (since there is more divisibility on z_3' than z_1' does), by [4, Thm. 4.27], h is in the Jacobson radical of $\mathrm{End}(N)$. Since $1 + h$ is an automorphism of N, the pp-type of $(1+h)(z_1') = z_1' + z_3' = n$ in N is $p_1 = pp(z_1')$.

Note that p_2 includes $\beta \mid x$, hence $\varphi_2 = (C_2.D_2) \to \psi = ({}^+C.D)$. If $k = z_2'$, then $m = n + k$. By Lemma 4.3, $\psi + \varphi$ is equivalent to $\psi + (C_1.D_1)$, and $C_1 = C$ is empty.

Now we consider the general case.

Comparing pp-types $\lambda_1 z_1'$ and $\lambda_n z_n'$ as above (and using that $E \neq E^{-1}$ for every finite string E), we may drop one of these elements. Now, using Lemma 4.3, we may dispose of all z_i' in the 'middle' of N. □

Note that in the above lemma, if ${}^+CD$ is not a string or if ${}^+C$ is undefined, then $({}^+C.D)$ degenerates to $x = 0$, hence the interval $(C.D)/x = 0$ in the lattice of all pp-formulae is a chain.

5. One-sided pure injective modules

Let M be an indecomposable pure injective A-module, and let $m \in Me_i$ for some i. Using the standard ε-σ formalism (see [2, p. 158]) we may separate strings going in and out of the vertex i into two classes, such that the notions of a 'right hand' string and a 'left hand string' make sense. If w is a right hand string, then $m \in wM$ is defined in an obvious manner.

For every n let u_n be a maximal (with respect to $<$) string of length $\leq n$ such that $m \in u_n M$. Since M is pure injective, there is a (usually infinite) string u

such that $u|n = u_n$ for every n and $m \in uM$. Similarly m determines a left hand (infinite) string v. Then $w(m) = vu$ is a (two-sided) string constructed using m.

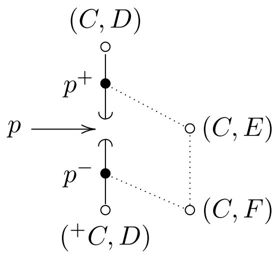
def **Definition 5.1.** *An indecomposable pure injective module M is said to be* one-sided, *if M opens an interval $(C.D)/(^+C.D)$ for some string CD (then say that M ends with C on the left) or M opens an interval $(C.D)/(C.D^+)$ (then say that M ends with D on the right).*

Otherwise we say that M is two-sided.

Clearly this is the same as to say that there exists $m \in M$ such that the string $w(m)$ is one-sided.

For instance every finite dimensional string module $M(CD)$ opens both pairs $(C.D)/(^+C.D)$ and $(C.D)/(C.D^+)$, hence $M(CD)$ is one-sided. Also, if v is a one-sided almost periodic string, then (the direct sum or direct product) module $M(v)$ from Ringel's list is one-sided.

Let M be a one-sided indecomposable pure injective module and, with notations as in the definition, choose $m \in (C.D)(M) \setminus (^+C.D)(M)$ (and such that $m \in e_i M$ for a basic idempotent that corresponds to the vertex between C and D). Then the pp-type $p = pp_M(m)$ defines a cut, by intersection with p^+ and p^-, in the chain $(C.D)/(^+C.D)$:



Moreover, by Fact cut 3.1, M is determined up to isomorphism by this cut. However this cut (therefore this pp-type) may be 'non-homogeneous'. For instance, it is (at least conjecturally) possible to have $(C.E) \in p^-$ but $(^+C.D) + (C.E) \in p^+$. To avoid this possibility we will improve p slightly. We say that a pp-type $p \in (C.D)/(^+C.D)$ is *homogeneous* (with respect to this chain) if $(^+C.D) + (C.E) \in p^+$ implies $(C.E) \in p^+$ for every $E \geq D$ such that CE is a string.

homo **Lemma 5.2.** *Let $p \in (C.D) \setminus (^+C.D)$ be an indecomposable pp-type. Then there is a homogeneous pp-type q such that $N(p) \cong N(q)$.*

Proof. First include in q^+ all pp-formulae $(C.E)$ such that $(^+C.D) + (C.E) \in p^+$. Since C is fixed, these formulae in q^+ form a chain. Now include in q^- all pp-formulae $(^+C.D) + (C.F) \in p^-$. These also form a chain.

We prove that $q^+ \cup \neg q^-$ is consistent. Indeed otherwise we obtain that $(C.E) \to (^+C.D) + (C.F)$, for some E and F such that $(^+C.D) + (C.E) \in p^+$ and $(^+C.D) + (C.F) \in p^-$. Note that $(C.E) > (C.F)$. If $M = M(CE)$ and z is between C and

E in the canonical basis of M, then, by Remark 4.1, (M, z) is a free realization of $(C.E)$.

Since $z \in (C.E)(M)$ and $(C.E) \to (^{+}C.D) + (C.F)$, therefore $z \in (^{+}C.D)(M) + (C.F)(M)$. By Lemma 4.2 we deduce that either $z \in (^{+}C.D)(M)$ or $z \in (C.F)(M)$, therefore either $(C.E) \to (^{+}C.D)$ or $(C.E) \to (C.F)$.

Thus we obtain either a morphism $f : M(^{+}CD) \to M(CE)$, with $f(z') = z$ where z' is between ^{+}C and D; or a morphism $g : M(CF) \to M(CE)$, with $g(z'') = z$ where z'' is between C and F. From the description of morphisms between string modules in [3] we may assume that f, respectively g, is a graph map in either case, which clearly leads to a contradiction.

Thus $q^{+} \cup \neg\, q^{-}$ is consistent. Now we extent this type to a maximal pp-type containing q^{+} and omitting q^{-}. The result (denote it also by q) will be indecomposable by [4, Thm. 4.32] and $N(p) \cong N(q)$ by Fact 3.1 and Lemma 4.4. $\square$

Recall that the Ziegler spectrum of A, Zg_A, is a topological space whose points are isomorphism types of indecomposable pure injective A-modules (e.g. see [6]). The topology on Zg_A is given by basic open sets $(\varphi/\psi) = \{M \in \mathrm{Zg}_A \mid \psi(M) < \varphi(M)\}$, where $\psi < \varphi$ are pp-formulae. It is known that Zg_A is quasi-compact.

Lemma 5.3. *Let q be a homogeneous pp-type as in Lemma 5.2. Then the pairs $(C.E)/((^{+}C.D) + (C.F))$, where $D \leq E < F$ such that CE and CF are strings, $(C.E) \in q^{+}$ and $(^{+}C.D) + (C.F) \in q^{-}$, form a neighbourhood basis of open sets for $N(q)$.*

Proof. Since p opens the interval $(C.D)/(^{+}C.D)$, by Ziegler [15, Thm. 4.9], a neighbourhood basis of $N(q)$ can be taken to be those pairs (φ/ψ) such that $(^{+}C.D) \leq \psi < \varphi \leq (C.D)$. It remains to apply Lemma 4.4 and homogeneity of q. $\square$

Now we are in a position to prove the main theorem of the paper.

Theorem 5.4. *Let A be a finite dimensional string algebra. Then there is a natural one-to-one correspondence between the set of pairs $\{v, v^{-1}\}$ of one-sided strings over A and the set of isomorphism types of one-sided indecomposable pure injective A-modules.*

Proof. Let M be a one-sided indecomposable pure injective A-module. First we assign to M a one-sided string $w = w(M)$.

Since M is one-sided, M opens a pair, say $(C.D)/(^{+}C.D)$, on a (nonzero) $m \in M$ (such that $m \in Me_i$ for some i).

Shifting along C we may further assume that C is empty. Thus M opens the interval $((.D)/(^{+}.D))$, and this interval is a chain by Lemma 4.4.

Moreover, by Lemma 5.2, we may assume that m is such that $p = pp_M(m)$ is a homogeneous pp-type, so $M = N(p)$. Then the isomorphism type of M is determined by the cut of p on the above interval (see Lemma 3.2). If E is a string, then $(.E)+(^+.D) \in p^+ \cap (.D)/(^+.D)$ iff E is an initial part of the one-sided string w determined by m (as before Definition 5.1). Thus the cut and the string determine each other and we assign this string to M.

Conversely, let w be a one-sided infinite (right hand) string. Take any finite string D such that $D \leq w$. Then the interval $(.D)/(^+.D)$ is a chain and w defines a cut on it as above. By Lemma 5.2 and Lemma 3.2 there is an indecomposable (homogeneous) pp-type p such that p defines on this interval the same cut as w.

Then we assign to w the (one-sided) indecomposable pure injective module $N(p)$. Since $N(p)$ is determined by w, we may use notation $N(w)$.

It remains to prove that for different one-sided strings $v \neq w$, both infinite to the right, the corresponding modules $M = N(v)$ and $N = N(w)$ are not isomorphic. Assume first that v and w start at the same vertex (so we may compare v and w with respect to the ordering $<$ on strings).

Looking for a contradiction, we may assume that $v < w$ and $M \cong N$. By Corollary 5.3, a basis for N in Zg_A can be chosen to consist of pairs of the form $(.G)/((.H) + (^+.))$, where $G \leq w < H$ are finite strings. Choose G, H such that $G \mid n = H \mid n$ for some n large enough that the initial segments of v and w of length n are different. In particular, $M \cong N \in (.G)/((.H) + (^+.))$.

Similarly, a basis of open sets for M can be chosen to consist of pairs of the form $(.E)/((.F)+(^+.))$, where $E \leq v < F$ are finite strings. We have two neighbourhood bases of M so we may choose E, F such that $(.E)/((.F)+(^+.)) \subseteq (.G)/((.H)+(^+.))$. We prove that this leads to a contradiction.

Indeed, let v_k be an initial part of v of length k. If k is large enough, $E \leq v_k < F$, and also $v_k < G$. Let $M_k = M(v_k)$ be the corresponding indecomposable string module with the basis $z_1, \ldots, z_k$. Clearly $M_k \in (.E)/((.F)+(^+.))$, where z_1 realizes the corresponding pp-type. By choice of E and F, $M_k \in (.G)/((.H) + (^+.))$, therefore there is $z \in M_k$ which opens this pair.

By homogeneity, 4.2, we may assume that z is one of the basis elements z_i. Since $(^+.)$ is in the pp-type of z_i for $2 \leq i \leq k-1$, we conclude that $z = z_1$ or $z = z_k$. From $v_k < G$ it follows that the pp-type of z_1 does not contain $(.G)$, contradiction. Thus we must have $z = z_k$.

Thus for every (large enough) k, the pp-type of z_k in M_k would open the pair $(.G)/(.H)$, in particular the n-initial part of the string defined from z_k in M_k would be equal to $w \mid n$, which is clearly not possible (as k varies).

A similar argument applies if v and w start at different vertices. □

6. Applications

Corollary 6.1. *Let A be a non-domestic string algebra. Then there are 2^ω one-sided indecomposable pure injective modules over A.*

Proof. Since A is non-domestic, by Ringel [10, Prop. 2], A has 2^{ω} one-sided (non-periodic) strings. So we can apply Theorem 5.4. □

If the field $\Bbbk$ is countable, the existence of 2^{ω} points in the Ziegler spectrum of A was already known (see [5, p.450] and [13, Prop. 2]) and can be proved as follows. Using Shröer [14] it is not difficult to show that some interval $(.D)/(^{+}.D) + (.E)$ (where $E > D$ are strings) contains a dense subchain. Then apply a result of Ziegler, see [4, Thm. 10.15].

str **Question 6.2.** *What is the algebraic structure of one-sided indecomposable pure injective modules over a non-domestic string algebra?*

We have defined a one-sided pure injective module M to be a module with an element m such that the string $w(m)$ is one-sided. A negative answer to the following question would allow us to separate one-sided and two-sided pure injective modules completely.

two **Question 6.3.** *Let M be a one-sided indecomposable pure injective module. Is it possible to have $m \in M$ such that the string $w(m)$ is two-sided?*

It will follow from what we show below that the answer to this question is negative for domestic string algebras.

Given a string algebra A, $n_d(A)$ will denote the number of 1-parameterized families required to cover all but finitely many indecomposable A-modules of dimension d. We say that A is *domestic* if there is N such that for every d, $n_d(A) \leq N$.

The following characterization of domestic string algebras is contained in [10, Prop. 8.2].

alm **Fact 6.4.** *A string algebra A is domestic if and only if every one-sided string over A is almost periodic.*

For instance the following string algebra

$$R_2 \qquad \begin{tikzcd} \circ \arrow[d, bend right, "\alpha_2"'] \arrow[d, bend left, "\alpha_1"] & \circ \arrow[l, "\delta"'] \\ \circ \arrow[r, "\beta"'] & \circ \arrow[u, bend left, "\gamma_2"] \arrow[u, bend right, "\gamma_1"'] \end{tikzcd}$$

(all zero-relations have length 2 and are shown by dotted curves) is domestic. Indeed up to inversion every (one-sided or two-sided) string v over R_2 is a substring of the following string:

$$^{\infty}(\alpha_1^{-1}\alpha_2)\delta\gamma_1(\gamma_2^{-1}\gamma_1)^n\beta(\alpha_1\alpha_2^{-1})^{\infty}, \quad n \in \mathbb{Z},$$

therefore v is almost periodic.

2clas **Theorem 6.5.** *Let M be a one-sided indecomposable pure injective module over a domestic string algebra A. Then M is isomorphic to a module $M(v)$ from Ringel's list, where v is a one-sided string. Moreover $M(v) \cong M(w)$ iff $v = w$ or $v = w^{-1}$.*

Proof. Choose $m \in M$, $m \neq 0$ and set $v = w(m)$. Choose a finite string $D \leq v$. By Theorem 5.4 we may assume that $M = N(v) = N(p)$, where p is a homogeneous pp-type in the interval $(.D)/(^{+}.D)$ determined by D.

Since A is domestic, Fact 6.4 yields that v is almost periodic. Let $M = M(v)$ be the direct sum or direct product module determined by v. Let z_1 be the first element of a standard basis of M, and let $q = pp_M(z_1)$. By Lemma 4.2, q is homogeneous.

Calculating in $M(v)$ we see that q and p coincide on formulas $(.E)$, $E \geq D$ (since realizations of q and p define the same string v). Since p and q are homogeneous, they define the same cut on the chain $(.D)/(^{+}.D)$. Then $N(v) \cong M(v)$ by Fact 3.1.

The last assertion can be checked directly. □

To highlight that some new effects may occur in a non-domestic case, let us consider some examples.

ex1 **Example 6.6.** [1] *Let $A = G_{2,2}$, where the characteristic of $\Bbbk$ is not equal to 2, and let M be the following direct sum module:*

$$\begin{tikzcd}[column sep=small]
 & z_1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & z_3 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & z_5 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & \\
z_0 & & z_2 & & z_4 & & z_6 & \cdots
\end{tikzcd}$$

Let $p = pp_M(z_0)$. Then p is homogeneous but decomposable. Moreover the embedding of M into the corresponding direct product module is not pure.

Proof. The pp-type p is homogeneous by Lemma 4.2. Also $z_0 = 1/2 \cdot (z_0 + z_2) + 1/2 \cdot (z_0 - z_2)$. But clearly $(\alpha \pm \beta) \mid x \in pp_M(z_0 \pm z_2)$. Calculating strings on $z_0 \pm z_2$ as in [1, p. 26] we obtain that $p \subset pp_M(z_0 \pm z_1)$. That p is decomposable then follows from [4, Cor. 4.30].

Since $(\alpha - \beta)(z_1 + z_3 + \dots) = z_0$, $\alpha - \beta$ divides z_0 in the corresponding direct product module $\overline{M}$. Also clearly $\alpha - \beta$ does not divide z_0 in M. Thus M is not pure in $\overline{M}$. □

Note that in this example the defining string $v = (\alpha\beta^{-1})^{\infty}$ for M is expanding. Therefore in the direct product module $M(v)$ the pp-type of z_0 is indecomposable.

But in general there are indecomposable pp-types of a completely different shape. If $v = v_1 v_2 \dots$ is a one-sided string, then $v(i)$ will denote the string $v_i v_{i+1} \dots$ and $v \mid i$ will denote the string $v_1 \dots v_i$.

W **Lemma 6.7.** *Let v be a one-sided string over a string algebra A and let $M = M(v)$ be a direct sum module with the standard basis $z_1, z_2, \dots$. Suppose that there is i*

such that for some n, $v(i) \mid n \neq v(j) \mid n$ holds for every $j \neq i$. Then the pp-type $pp_M(z_i)$ is indecomposable.

Proof. Similar to [1, Prop. 6.2]. □

ex2 **Example 6.8.** *Let $A = G_{2,3}$, $a = \alpha\beta^{-1}$, $b = \alpha\beta^{-2}$ and let $v = aba^2ba^3b\dots$. Let $M = M(v)$ be the corresponding direct sum module, and let z_i be an element of the canonical basis of M. Then the pp-type $p_i = pp_M(z_i)$ is indecomposable.*

Proof. Using Lemma 6.7 it is easy to check that p_i is indecomposable. For instance, for z_1 we may take $n = 5$, i.e. ab, as the required part of v. □

Nevertheless, even in this case we do not know if the pure injective envelope of M is indecomposable.

Note also that the $N(p_i)$ are not isomorphic to any module of the kind on Ringel's list for the domestic case.

7. The Cantor–Bendixson rank

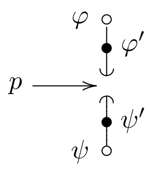
The Cantor–Bendixson analysis on Zg_A runs as follows. At the first step we delete from Zg_A the isolated points, i.e. by [4, Cor. 13.4] exactly the indecomposable finite dimensional A-modules. What remains is a closed subset, Zg'_A, the first derivative of Zg_A. Removing isolated points from this space we obtain the second derivative Zg''_A and so on. At limit stages we put $\mathrm{Zg}_A^{(\lambda)} = \bigcap_{\mu<\lambda} \mathrm{Zg}_A^{(\mu)}$.

If this process reaches the empty set at stage $\lambda + 1$, then the CB-rank of Zg_A is defined to be λ. In this case for every point $M \in \mathrm{Zg}_A$ we may define the CB-rank of M to be the least μ such that $M \in \mathrm{Zg}_A^{(\mu)} \setminus \mathrm{Zg}_A^{(\mu+1)}$.

Note that if V is an open subset in Zg_A, then the CB-rank of every point in V can be calculated inside V. We define $\mathrm{CB}(V)$ as the supremum of CB-ranks of points in V.

The notion of m-dimension of a lattice L, $\mathrm{mdim}(L)$, can be found in [4, Ch. 10]. For instance the m-dimension of a finite lattice is zero and $\mathrm{mdim}(\omega + 1) = 1$.

Let (φ/ψ) be a chain in the lattice of pp-formulae over A and let $p \in (\varphi/\psi)$ be an indecomposable pp-type. We define the m-dimension of p, $\mathrm{mdim}(p)$, as the infimum of m-dimensions of intervals (φ'/ψ') such that $\psi \leq \psi' < \varphi' \leq \varphi$ and $p \in (\varphi'/\psi')$.



sv **Proposition 7.1.** *Let A be any ring. Let (φ/ψ) be a chain in the lattice of all pp-formulae over A and let $p \in (\varphi/\psi)$ be an indecomposable pp-type. Then* $\mathrm{CB}(p) =$

mdim(p). *Also* mdim(φ/ψ) *is the supremum of m-dimensions of indecomposable pp-types* $p \in (\varphi/\psi)$.

Proof. The proof in [8, Thm. 3.1] can be applied in this situation to show that the isolation property (see e.g. [6, p. 382]) holds true for the open set (φ/ψ): for every theory T of A-modules every isolated point in $T \cap (\varphi/\psi)$ is isolated by a minimal pair.

Now the result is easily proved by induction, similarly to [4, Prop. 10.19]. $\square$

It follows from [14, Prop. 6.1] that for every non-domestic string algebra A, the CB-rank of Zg_A is undefined. For a domestic string algebra Schröer conjectured (see [12, p. 84]) that $\mathrm{CB}(\mathrm{Zg}_A)$ is finite and can be calculated from the bridge quiver of A.

The precise definition of the bridge quiver of a domestic string algebra A can be found in [14]. We hope that from the following example it will be clear how to calculate the bridge quiver for a particular string algebra.

Let A be the domestic string algebra R_2 (see after Fact 6.4). The bands over A are the following: $C = \alpha_1\alpha_2^{-1}$, $C^{-1} = \alpha_2\alpha_1^{-1}$, and $D = \gamma_1\gamma_2^{-1}$, $D^{-1} = \gamma_2\gamma_1^{-1}$.

¿From the description of the two-sided strings over A (see above) we read off the following paths in the bridge quiver of A:

$$\alpha_2\alpha_1^{-1} \xrightarrow{\alpha_2\delta} \gamma_1\gamma_2^{-1} \xrightarrow{\gamma_1\beta} \alpha_1\alpha_2^{-1}$$

Inverting this we obtain:

$$\alpha_2\alpha_1^{-1} \xrightarrow{\beta^{-1}\gamma_1^{-1}} \gamma_2\gamma_1^{-1} \xrightarrow{\delta^{-1}\alpha_2^{-1}} \alpha_1\alpha_2^{-1}$$

Gluing these together we get the bridge quiver of A:

$$\begin{tikzcd} & D \arrow[dr] & \\ C^{-1} \arrow[ur] \arrow[dr] & & C \\ & D^{-1} \arrow[ur] & \end{tikzcd}$$

bridge **Fact 7.2.** [14, Lemma 4.2] *Let A be a domestic string algebra. Then the bridge quiver of A is a finite oriented graph without oriented cycles.*

Note that, directly from the definition, for a string algebra A the one-sided indecomposable pure injective A-modules form an open subset in Zg_A. In the following theorem we calculate the CB-rank of this set.

br **Theorem 7.3.** *Let A be a domestic string algebra and let n be the maximal length of a path in the bridge quiver of A. Let U be the open set in* Zg_A *formed by the one-sided indecomposable pure injective A-modules. Then* $\mathrm{CB}(U) = n + 1$.

Proof. Given a string CD, U_{CD} will denote the open set $(C.D)/({}^+C.D)$ in Zg_A. We prove that $\mathrm{CB}(U_{CD}) \leq n+1$. Since U is a union of such sets it will then follow that $\mathrm{CB}(U) \leq n+1$.

Proposition 7.1 yields $\mathrm{CB}(U_{CD}) = \mathrm{mdim}(\varphi/\psi)$, where $\varphi = (C.D)$ and $\psi = ({}^+C.D)$. From Lemma 4.4 it follows that $\mathrm{mdim}(\varphi/\psi)$ is equal to the m-dimension of the chain $\{(C.D_i) \mid D_i \geq D, CD_i \text{ is a string}\}$. Then the result is easily derived from [14, Thm. 4.3].

For the converse let $C_0, \ldots, C_n$ be bands such that $C_0 \ldots C_i \ldots C_n$ is a path of maximal length in the bridge quiver of A. Along this path we obtain one-sided strings $v_i = C_0^{k_0} \ldots C_1^{k_1} \ldots C_i^{\infty}$, where dots also replace bridges between bands. By induction on $i = n, \ldots, 0$ we prove that (the direct sum or direct product) module $M(v_i)$ has CB-rank $\geq n-i+1$.

For $i = n$, the indecomposable pure injective module $M(v_0) = M(C_0^{k_0} \ldots C_n^{\infty})$ is infinite dimensional, therefore its CB-rank is not less than 1.

For $i < n$ note that $M(v_i)$ is in the Ziegler closure of the modules $M_k = M(w_k)$ where $w_k = C_0^{k_0} \ldots C_i^{k} \ldots C_{i+1}^{\infty}$, $k = 1, \ldots$. Indeed take any finite string $D \leq v_i$. By Lemma 4.4 and [15, Thm. 4.9] a basis of open neighborhoods of $M(v_i)$ can be chosen as $\{(.E)/(({}^+.D) + (.F)) \mid D \leq E \leq v_i < F\}$. Clearly for every such pair there exists k such that $E \leq w_k < F$. Taking a homogeneous realization in M_k we obtain that $M_k \in (.E)/(({}^+.D) + (.F))$.

By the induction assumption $\mathrm{CB}(M_k) \geq n - (i+1) + 1 = n - i$ for every k. By the definition of CB-rank we deduce that $\mathrm{CB}(M(v_i)) \geq n - i + 1$.

Finally for $i = 0$ we have $\mathrm{CB}(M(C_0^{\infty})) \geq n+1$, therefore $\mathrm{CB}(U) = n+1$. □

e **Corollary 7.4.** *Let A be a domestic string algebra and let n be the maximal length of a path in the bridge quiver of A. Then* $\mathrm{CB}(\mathrm{Zg}_A) \geq n+2$.

Proof. From the proof of Theorem 7.3 we have $\mathrm{CB}(M) = n+1$ where $M = M(C_0^{\infty})$. Clearly it suffices to prove that the theory T of M contains a non-isolated point.

Indeed otherwise M is the only isolated point of T. Let $C_0 = \alpha \ldots \beta^{-1}$ and let φ be the pp-formula $\alpha \mid x \wedge \beta \mid x$. Then $\varphi(M)$ is a uniserial right S-module, where $S = \mathrm{End}(M)$. As in [8, Thm. 3.1] it follows that M is isolated in T by a minimal pair. Then as in [4, Prop. 10.17] we obtain that the interval $(\varphi/x = 0)$ in T has finite length. Therefore $\varphi(M)$ has finite length as an S-module, a contradiction. □

References

B-P [1] S. Baratella, M. Prest, Pure injective modules over the dihedral algebras, Comm. Algebra, **25**(1) (1997), 11–31.

B-R [2] M.C.R. Butler, C.M. Ringel, Auslander–Reiten sequence with few middle terms and applications to string algebras, Comm. Algebra, **15**(1-2) (1987), 145–179.

CB [3] W.W. Crawley-Boevey, Maps between representations of zero relation algebras, J. Algebra, **126** (1989), 259–263.

Preb [4] M. Prest, Model Theory and Modules. Cambridge University Press, London Math. Soc. Lecture Note Series, **130** (1987).

Pre1 [5] M. Prest, Maps between finitely presented modules and infinite-dimensional representations, pp. 447–455 in Canad. Math. Soc. Conf. Proc., **24** (1998).

Pre [6] M. Prest, Topological and geometric aspects of the Ziegler spectrum, pp. 369–392 in H. Krause and C.M. Ringel (eds.), Infinite Length Modules, Birkhäuser, 2000.

P-S [7] M. Prest, J. Schröer, Serial functors, Jacobson radical and representation type, J. Pure Appl. Algebra, **170**(2-3) (2002), 295–307.

Pun [8] G. Puninski, The Krull–Gabriel dimension of a serial rings, Comm. Algebra, **31**(12) (2003), 5977–5993.

Pun1 [9] G. Puninski, Super-decomposable pure-injective modules exist over some string algebras, Proc. Amer. Math. Soc., to appear

Rin [10] C.M. Ringel, Some algebraically compact modules. I, pp. 419–439 in: Abelian Groups and Modules, eds. A. Facchini, C. Menini, Kluwer, 1995.

Rin1 [11] C.M. Ringel, Infinite length modules. Some examples as introduction, pp. 1–73 in H. Krause and C.M. Ringel (eds.), Infinite Length Modules, Birkhäuser, 2000.

Sch0 [12] J. Schröer, Hammocks for string algebras, Preprint, 1997.

Sch1 [13] J. Schröer, On the Krull-Gabriel dimension of an algebra, Math. Z., 233 (2000), 287-303.

Sch [14] J. Schröer, On the infinite radical of a module category, Proc. London Math. Soc. (3), **81** (2000), 651–674.

Zig [15] M. Ziegler, Model theory of modules, Annals Pure Appl. Math., **26** (1984), 149–213.

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