Research Article

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Spin(8,C)-Higgs pairs over a compact Riemann surface

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Abstract: Let X be a compact Riemann surface of genus $g \geq 2$, G be a semisimple complex Lie group and $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G. Given a principal G-bundle E over X, a vector bundle $E(V)$ whose typical fiber is a copy of V is induced. A (G, ρ)-Higgs pair is a pair (E, φ), where E is a principal G-bundle over X and φ is a holomorphic global section of $E(V) \otimes L$, L being a fixed line bundle over X. In this work, Higgs pairs of this type are considered for $G = \mathrm{Spin}(8, \mathbb{C})$ and the three irreducible eight-dimensional complex representations which $\mathrm{Spin}(8, \mathbb{C})$ admits. In particular, the reduced notions of stability, semistability, and polystability for these specific Higgs pairs are given, and it is proved that the corresponding moduli spaces are isomorphic, and a precise expression for the stable and not simple Higgs pairs associated with one of the three announced representations of $\mathrm{Spin}(8, \mathbb{C})$ is described.

Keywords: Higgs pair, Higgs bundle, spin, Riemann surface, stability

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1 Introduction

Let X be a compact Riemann surface of genus $g \geq 2$ and G be a semisimple complex Lie group that is equipped with a complex representation $\rho : G \to \mathrm{GL}(V)$. Every principal G-bundle E over X induces a vector bundle $E(V)$ whose typical fiber is a copy of V and which is defined from the direct product $E \times V$ by identifying $(e, v) \sim (eg, \rho(g^{-1})(v))$ for all $e \in E$, $v \in V$ and $g \in G$. A (G, ρ)-Higgs pair (or simply G-Higgs pair, Higgs pair, or pair, when there is not possibility of doubt) over X is defined to be a pair (E, φ), where E is a principal G-bundle over X and φ is a holomorphic global section of the vector bundle $E(V) \otimes L$, the bundle L being a fixed line bundle over X (Definition 1). When the representation ρ is the adjoint one and V coincides with the Lie algebra $\mathfrak{g}$ of G, this concept corresponds to that of G-Higgs bundle over X. Higgs bundles were introduced by Hitchin [1,2] for $G = \mathrm{SL}(2, \mathbb{C})$ and studied in the general case of semisimple (in fact, reductive) Lie groups by Simpson [3,4], who provided notions of stability and polystability aimed at constructing the moduli space of G-Higgs bundles over X. Since these foundational articles were published, moduli spaces of G-Higgs bundles over a compact Riemann surface have been intensely studied from different points of view, including automorphisms and subvarieties of the moduli space [5], stratifications [6,7], representations of the Riemann surface X [8], or Langlands program [9]. The concept of (G, ρ)-Higgs pair was first introduced in Banfield [10] as a natural generalization of that of G-Higgs bundle and has been studied in recent years because they appear in certain geometric contexts, for example as fixed points of certain automorphisms of moduli spaces of Higgs bundles [11]. The latter concept extends, in turn, that of the principal bundle over a curve X, whose

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geometry and topology are also intensively studied, in part along lines analogous to those followed with Higgs pairs and Higgs bundles, such as the study of automorphisms of the corresponding moduli space [12].

In this work, Higgs pairs over X are studied for the structure group $\mathrm{Spin}(8,\mathbb{C})$, the universal cover of $\mathrm{SO}(8,\mathbb{C})$ and $\mathrm{PSO}(8,\mathbb{C})$, and associated with the three eight-dimensional irreducible complex representations that $\mathrm{Spin}(8,\mathbb{C})$ admits, which are defined in (5), (6), and (7) and will be denoted by ρ, ρ_+, and ρ_-. The group $\mathrm{Spin}(8,\mathbb{C})$ is the only simple complex structure group that admits an order 3 outer automorphism, called triality. This singular fact makes $\mathrm{Spin}(8,\mathbb{C})$ a group with interesting geometric peculiarities, which has been the subject of great interest in the literature. For example, fixed point subvarieties have been specifically studied for automorphisms of the moduli space of principal $\mathrm{Spin}(8,\mathbb{C})$-bundles induced by the action of outer automorphismsm of $\mathrm{Spin}(8,\mathbb{C})$ [13]. After that, it was proved that the fixed points for the action of the triality automorphism on the moduli space of $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundles can be described through certain Higgs pairs with structure group isomorphic to G_2 and $\mathrm{PSL}(3,\mathbb{C})$ [5]. In addition, further objects are related to $\mathrm{Spin}(8,\mathbb{C})$-bundles, such as Galois $\mathrm{Spin}(8,\mathbb{C})$-bundles [14], which are essentially fixed points of certain S_3-action defined in the moduli space of principal $\mathrm{Spin}(8,\mathbb{C})$-bundles over a curve and whose moduli space was constructed by Oxbury and Ramanan [14]. The interest in $\mathrm{Spin}(8,\mathbb{C})$-Higgs pairs responds, therefore, to the interest in G-Higgs pairs in general (because they appear in many situations, such as in descriptions of fixed point of automorphisms) and in the interest in $\mathrm{Spin}(8,\mathbb{C})$-bundles over curves in particular.

The three representations ρ, ρ_+ and ρ_- of $\mathrm{Spin}(8,\mathbb{C})$ do not descend to induce representations of either $\mathrm{SO}(8,\mathbb{C})$ or $\mathrm{PSO}(8,\mathbb{C})$, which is why this research is focused on the group $\mathrm{Spin}(8,\mathbb{C})$. Following the general theory on (G,ρ)-Higgs pairs developed in the studies by Garcia-Prada et al. [15,16], in Propositions 4.1, 4.2, and 4.3, the reduced notions of stability, semistability, and polystability for that three types of $\mathrm{Spin}(8,\mathbb{C})$-Higgs pairs are described. The triality automorphism of $\mathrm{Spin}(8,\mathbb{C})$ gives isomorphisms between the three considered representations of $\mathrm{Spin}(8,\mathbb{C})$, which will be proved in Corollary 4.1 that induce isomorphisms between the moduli spaces of polystable $(\mathrm{Spin}(8,\mathbb{C}),\rho)$, $(\mathrm{Spin}(8,\mathbb{C}),\rho_+)$, and $(\mathrm{Spin}(8,\mathbb{C}),\rho_-)$-Higgs pairs over X. After that, an application of this whole study is provided. Specifically, a description of the stable but not simple $(\mathrm{Spin}(8,\mathbb{C}),\rho)$-Higgs pairs over X is provided, in the spirit of previous works devoted to principal or vector bundles [17]. Given any semisimple complex Lie group G, the deformation theory of G-Higgs bundles makes it possible to describe the tangent space of the moduli space of these objects at smooth elements in terms of certain hypercohomology groups. In particular, the smooth points of the moduli space of G-Higgs bundles over X can be identified as the stable and simple Higgs bundles. Taking advantage of this theory, in the study by Garcia-Prada et al. [16], a description of the singular points of the moduli space of G-Higgs bundles is given. That deformation theory is not directly adaptable, as far it has been studied, to the general case of Higgs pairs. However, in Theorem 5.1, it is proved the following description of the stable and not simple $(\mathrm{Spin}(8,\mathbb{C}),\rho)$-Higgs pairs over X.

Theorem. *Let* (E,φ) *be a stable and non-simple* $(\mathrm{Spin}(8,\mathbb{C}),\rho)$*-Higgs pair over X, where* $\rho:\mathrm{Spin}(8,\mathbb{C})\to\mathrm{SO}(8,\mathbb{C})\hookrightarrow\mathrm{GL}(8,\mathbb{C})$ *is given by the double cover* $\mathrm{Spin}(8,\mathbb{C})\to\mathrm{SO}(8,\mathbb{C})$. *Let* E_{SO} *be the principal* $\mathrm{SO}(8,\mathbb{C})$*-bundle over X associated with E through ρ. Then the underlying vector bundle of* E_{SO} *is isomorphic to one of the following vector bundles*:
(1) $L_k\oplus L_{8-k}$ *for* $k=0,1,2,3,4$;
(2) $F_{2r}\oplus L_k\oplus L_{8-2r-k}$ for $r=1,2,3$ *and* $k=0,1,\ldots,4-r$;
(3) $F_{2r}\oplus F_{2s}\oplus L_k\oplus L_{8-2r-2s-k}$ *for* $r=1,2,3$, $s=r,\ldots,4-r$, *and* $k=0,\ldots,4-r-s$;
(4) $F_2\oplus F_2\oplus F_{2r}\oplus L_k\oplus L_{4-2r-k}$ *for* $r=1,2$ *and* $k=2-r,\ldots,4-2r$,

where F_j *is an* $\mathrm{SL}(j,\mathbb{C})$*-bundle and* L_j *is an* $\mathrm{SO}(j,\mathbb{C})$*-bundle for all* $j\geq 1$, $F_0=0$, $L_0=0$, *and* $L_1=O$.

This article is organized as follows. In Section 2, the concept of (G,ρ)-Higgs pair over a compact Riemann surface X associated with a semisimple complex Lie group G and a complex representation ρ of it is defined, and the notions of stability, semistability, and polystability for Higgs pairs are presented to establish the precise formulation of the Hitchin-Kobayashi correspondence for Higgs pairs. Section 3 is devoted to presenting the main properties of the groups $\mathrm{Spin}(2n,\mathbb{C})$ for $n\geq 2$, focusing on $\mathrm{Spin}(8,\mathbb{C})$. The interest in Spin groups with even rank other than 8 is that they will naturally appear in the description of stable and not simple $(\mathrm{Spin}(8,\mathbb{C}),\rho)$-Higgs pairs made in Theorem 5.1. The reduced notions of stability and polystability for Higgs pairs with structure group $\mathrm{Spin}(8,\mathbb{C})$ and associated with the representations ρ, ρ_+, and ρ_- introduced earlier are stated and proved in Section 4, where it is also proved that the Higgs pairs corresponding to the three representations mentioned earlier are in bijective correspondence through a correspondence that preserves the polystability condition. Finally, in Section 5, a precise description of the stable and not simple $(\mathrm{Spin}(8,\mathbb{C}),\rho)$-Higgs pairs over X is given.

2 Stability and polystability notions for Higgs pairs

Let X be a compact Riemann surface of genus $g \geq 2$, G be a semisimple complex Lie group, and $\rho: G \to \mathrm{GL}(V)$ be a complex representation of G. In this section, the concept of (G,ρ)-Higgs pair over X is introduced and reduced notions of stability, semistability, and polystability are provided for such pairs. The survey material presented in this section has been adapted from the study by Garcia-Prada et al. [16].

Definition 1. Let G be a semisimple complex Lie group and $\rho: G \to \mathrm{GL}(V)$ be a complex representation of G. A (G,ρ)-Higgs pair over X is a pair (E,φ), where E is a principal G-bundle over X and $\varphi \in H^0(X, E(V)\otimes L)$, $E(V)$ being the vector bundle obtained by making the quotient of $E \times V$ where the identification $(e,v) \sim (eg, \rho(g^{-1})(v))$ is made for all $g \in G$ and all $(e,v) \in E \times V$, and L being a fixed line bundle over X.

Observe that the notion of (G,ρ)-Higgs pair extends that of G-Higgs bundle, for which the representation ρ is the adjoint one and V is the underlying vector bundle of the Lie algebra $\mathfrak{g}$ of G.

Let G be a semisimple complex Lie group with Lie algebra $\mathfrak{g}$. Having fixed a maximal compact connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$ and such that $\mathfrak{h}^{\mathbb{C}} = \mathfrak{g}$, and denoting by Δ the set of simple roots of $\mathfrak{g}$, and in the stud by Garcia-Prada et al. [16, Section 2.5], it is proved that the proper subsets of Δ and the parabolic subalgebras of $\mathfrak{g}$ (hence the parabolic subgroups of G) are in bijective correspondence. Given any parabolic subgroup P of G and any antidominant character χ of G, which belongs to the dual $\mathfrak{c}^*$ of the Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$, the Killing form induces an element $s_\chi \in \mathfrak{c}$, which is in fact an element of $i\mathfrak{h}$. Denote by P_{s_χ} the maximal parabolic subgroup induced by s_χ and by L_{s_χ} a choice of a Levi subgroup of P_{s_χ}.

Let now $\rho: G \to \mathrm{GL}(V)$ be a complex representation of G. Given a parabolic subgroup P of G and an antidominant character χ of P, the following subspaces of V are defined [11]:

$$\begin{aligned} V_\chi^- &= \{v \in V : \rho(e^{ts_\chi})v \quad \text{is bounded as } t \to \infty\}, \\ V_\chi^0 &= \{v \in V : \rho(e^{ts_\chi})v = v \quad \forall t\}. \end{aligned} \tag{1}$$

The subspaces V_χ^- and V_χ^0 thus defined are invariant under the action of P_{s_χ} and L_{s_χ}, respectively, on them.

Definition 2. Let G be a semisimple complex Lie group, $\rho: G \to \mathrm{GL}(V)$ be a complex representation of G, and (E,φ) be a (G,ρ)-Higgs pair over X. Then (E,φ) is stable (resp. semistable) if for every parabolic subgroup P of G, every antidominant character χ of P, and every reduction of structure group E_P of E to P such that φ takes values in $E_P(V_\chi^-)\otimes L$, where V_χ^- is defined in (1), and it is satisfied that $\deg \chi_* E_P > 0$ (resp. $\deg \chi_* E_P \geq 0$).

The (G,ρ)-Higgs pair (E,φ) is polystable if it is semistable, and for every parabolic subgroup P of G, every antidominant character χ of P, and every reduction of structure group E_P of E to P such that φ takes values in $E_P(V_\chi^-)\otimes L$, where V_χ^- is defined in (1), and such that $\deg \chi_* E_P = 0$, there exists a reduction of structure group E_L of E_P to a Levi subgroup L of P such that φ takes values in $E_L(V_\chi^0)\otimes L$, where V_χ^0 is also defined in (1).

The precise notions of stability and polystability of Higgs pairs, which extend that of Higgs bundles, were given by García-Prada et al. [16] to obtain a bijective correspondence between polystable Higgs pairs and solutions to the Hermite-Einstein equations.

The condition of polystability of a (G, ρ)-Higgs pair (E, φ) when applied to a faithful representation ρ can be expressed in terms of filtrations of certain vector bundle associated with E through other fixed representation ρ_G of G satisfying the hypothesis stated in the following result, which is derived from the study by Garcia-Prada et al. [16, Lemma 2.12]. The idea is applying this in the cases in which G is naturally embedded in some $\mathrm{GL}(n, \mathbb{C})$, for example, $\mathrm{SL}(n, \mathbb{C})$, where ρ_G is the natural embedding.

Proposition 2.1. *Let G be a semisimple complex Lie group, $\rho : G \to \mathrm{GL}(V)$ be a faithful complex representation of G, and (E, φ) be a (G, ρ)-Higgs pair over X. Suppose that there exists a representation $\rho_G : G \to \mathrm{GL}(W)$, with $W \cong \mathbb{C}^n$ for some $n \in \mathbb{N}$, such that for any $a, b \in (\mathrm{Ker}\,\mathrm{d}\rho_G)^{\perp}$ it is satisfied that $\langle a, b\rangle = \mathrm{Tr}\,\mathrm{d}\rho_G(a)\mathrm{d}\rho_G(b)$, where the product is the Euclidean product of W. Denote $E = E(W)$. Then*

(1) *The (G, ρ)-Higgs pair (E, φ) is semistable if for every parabolic subgroup P of G, any antidominant character χ of P, and any filtration $E_0 = 0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_k = E$ induced by a reduction of structure group of E to P and such that φ takes values in the space V_χ^- defined in* (1) *in each fiber over X, it is satisfied that the degree of the filtration, defined by*

$$\lambda_k \deg E + \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) \deg E_j, \tag{2}$$

is greater than or equal to 0, *where $\lambda_1 < \cdots < \lambda_k$ are the eigenvalues of $\mathrm{d}\rho(s_\chi)$.*

(2) *The (G, ρ)-Higgs pair (E, φ) is polystable if it is semistable, and there exists a parabolic subgroup P of G and an antidominant character χ of P such that E admits a decomposition of the form $E = \oplus_{j=1}^{k} E_j/E_{j-1}$ into vector subbundles, where $E_0 = 0$ and E_j/E_{j-1} is the λ_j-eigenspace of $\mathrm{d}\rho(s_\chi)$ for all $j = 1, \dots, k$, the degree defined in* (2) *equals* 0, *and φ takes values, in each fiber over X, in the space V_χ^0 defined in* (1).

The Hitchin-Kobayashi correspondence for Higgs pairs will be now introduced. This was first formulated and proved by Hitchin [2] for the case of rank 2 Higgs bundles and was generalized by Simpson [3,4] for Higgs bundles whose structure group is any semisimple complex Lie group. The version presented in this work, developed in the study by Garcia-Prada et al. [16], covers the case of Higgs pairs, which are the objects of interest. Given a semisimple complex Lie group G and any complex representation $\rho : G \to \mathrm{GL}(V)$, and having fixed a maximal compact subgroup H of G, a Hermitian structure h on V, and a Hermitian metric h_L on the line bundle L over X, whose curvature will be denoted by F_L, let $\rho_H : H \to \mathrm{U}(V)$ be the unitary representation of H obtained by restriction of ρ to H. Given also a (G, ρ)-Higgs pair (E, φ) as in Definition 1, the vector bundle $E(V)$ admits a Hermitian metric induced by that of V and the same is true for the vector bundle $E_H(V)$, which is canonically isomorphic to $E(V)$, where E_H is any reduction of structure group of E to H. Let F_H be the curvature on $E_H(V)$, which corresponds to the Chern connection. From the fact that $H^0((X, \mathrm{End}(V) \otimes L)^*) = H^0(X, \mathrm{End}(V)^*) = H^0(X, E(\mathfrak{u}(V))^*)$, the existence of a skew-symmetric element $\varphi \otimes \varphi^{*h,h_L}$ of $H^0(X, E(\mathfrak{u}(V))^*)$ is deduced. Define

$$\mu(\varphi) = \rho_H^*\left(-\frac{i}{2}\varphi \otimes \varphi^{*h,h_L}\right), \tag{3}$$

which may be understood as an element of $H^0(X, E_H(\mathfrak{h}))$, since $\mathfrak{h} \cong \mathfrak{h}^*$ and $\mathrm{d}\rho_H^*$ induces an isomorphism $E(\mathfrak{u}(V))^* \cong E_H(\mathfrak{h})^*$. Notice that, throughout this explanation, the same symbols have been used to denote the C^∞-objects and their holomorphic structures, by a slight abuse of notation.

Theorem 2.1. *Let G be a semisimple complex Lie group, $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G, and (E, φ) be a (G, ρ)-Higgs pair over X. Then (E, φ) is polystable if and only if E admits a reduction of structure group E_H to a maximal compact subgroup H of G such that*

$$\wedge(F_H + F_L) + \mu(\varphi) = 0,$$

where $\mu(\varphi)$ is defined in (3) *and $\wedge$ denotes the adjoint wedging with the volume form on X.*

The following result, which is where the interest in the Hitchin-Kobayashi correspondence lies, derives directly from Theorem 2.1.

Corollary 2.1. *Let G be a semisimple complex Lie group, $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G, G' be a subgroup of G, and $\rho_{G'}$ be the restriction of ρ to G'. Let (E, φ) be a polystable (G, ρ)-Higgs pair over X and let $E_{G'}$ be a reduction of structure group of E to G' such that φ takes values in $E_{G'}(V) \otimes L$. Then the $(G', \rho_{G'})$-Higgs pair $(E_{G'}, \varphi)$ over X is polystable.*

3 The groups Spin(8, $\mathbb{C}$) and Spin(2n, $\mathbb{C}$)

It will now be considered the simple complex Lie group $\mathrm{Spin}(8,\mathbb{C})$, whose Lie algebra is $\mathfrak{so}(8,\mathbb{C})$, of type D_4. The group $\mathrm{Spin}(8,\mathbb{C})$ is the simply connected complex group with Lie algebra $\mathfrak{so}(8,\mathbb{C})$, and it is a double cover of $\mathrm{SO}(8,\mathbb{C})$, and a cover of order 4 of the projective group $\mathrm{PSO}(8,\mathbb{C})$, the centerless group with Lie algebra $\mathfrak{so}(8,\mathbb{C})$. Let Z be the center of $\mathrm{Spin}(8,\mathbb{C})$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The group $\mathrm{Out}(\mathrm{Spin}(8,\mathbb{C}))$ of outer automorphisms of $\mathrm{Spin}(8,\mathbb{C})$ acts faithfully on Z, from which follows the existence of a nontrivial injective homomorphism $\mathrm{Out}(\mathrm{Spin}(8,\mathbb{C})) \to S(Z\backslash\{1\})$. Since the last group is isomorphic to the group S_3 of permutations of three elements and $\mathrm{Out}(\mathrm{Spin}(8,\mathbb{C}))$ is isomorphic to the group of symmetries of the Dynkin diagram of D_4, which is also a copy of S_3, it is deduced that the homomorphism $\mathrm{Out}(\mathrm{Spin}(8,\mathbb{C})) \to S(Z\backslash\{1\})$ is actually an isomorphism of groups [18, Section 1].

The three elements of $Z\backslash\{1\}$ correspond, in the following sense, to the three irreducible complex representations of dimension 8 that $\mathrm{Spin}(8,\mathbb{C})$ admits: each one of these representations leaves invariant exactly one element of $Z\backslash\{1\}$ and permutes the other two. The set of nontrivial outer involutions of $\mathrm{Spin}(8,\mathbb{C})$, the set of eight-dimensional irreducible complex representations of G, and the set $Z\backslash\{1\}$ are in bijective correspondence in a way that each $z \in Z\backslash\{1\}$ admits exactly one outer involution σ of $\mathrm{Spin}(8,\mathbb{C})$ such that every representative of order 2 of σ in $\mathrm{Aut}(\mathrm{Spin}(8,\mathbb{C}))$ leaves z invariant, and exactly one eight-dimensional irreducible complex representation ρ of $\mathrm{Spin}(8,\mathbb{C})$ such that $\rho(z) = 1$. This representation ρ thus defined actually descends to a representation of $\mathrm{Spin}(8,\mathbb{C})/\langle 1, z\rangle \cong \mathrm{SO}(8,\mathbb{C})$. The triality automorphism τ of $\mathrm{Spin}(8,\mathbb{C})$ is a choice of a nontrivial outer automorphism of $\mathrm{Spin}(8,\mathbb{C})$ of order 3 whose effect on Z turns out to be to permute the three nontrivial elements of Z without leaving fixed points. The triality automorphism τ interchanges then the three eight-dimensional irreducible complex representations of $\mathrm{Spin}(8,\mathbb{C})$, in the sense that a choice of an order 3 representative T of τ in $\mathrm{Aut}(\mathrm{Spin}(8,\mathbb{C}))$ acts as an order 3 permutation on the set of the aforementioned three representations [18].

Notice that the triality automorphism τ defines an outer automorphism of order 3 of $\mathrm{PSO}(8,\mathbb{C})$, but it does not define an outer automorphism of $\mathrm{SO}(8,\mathbb{C})$, because for that to happen, a representative of τ in $\mathrm{Aut}(\mathrm{Spin}(8,\mathbb{C}))$ should leave invariant the center of $\mathrm{SO}(8,\mathbb{C})$, which is not possible. On the other hand, the three announced eight-dimensional complex representations of $\mathrm{Spin}(8,\mathbb{C})$ do not descend to give rise to representations of $\mathrm{PSO}(8,\mathbb{C})$ (specifically, they descend to projective representations of $\mathrm{PSO}(8,\mathbb{C})$). For these reasons, throughout the article, only Higgs pairs over a compact Riemann surface whose structure group is $\mathrm{Spin}(8,\mathbb{C})$ are considered.

The construction of the three eight-dimensional irreducible complex representations of $\mathrm{Spin}(8,\mathbb{C})$, which have been considered in the previous paragraphs will now be sketched following the study by Fulton and Harris[19, Chapter 20]. Let V be an eight-dimensional complex vector space equipped with a nondegenerate quadratic form q. Then $\mathrm{SO}(8,\mathbb{C})$ is isomorphic to the group $\mathrm{SL}(V, q)$ of determinant 1 complex automorphisms of the vector space V, which preserves the quadratic form q. Let

$$\pi : \mathrm{Spin}(8,\mathbb{C}) \to \mathrm{SO}(8,\mathbb{C}) \tag{4}$$

be the double covering. The representation ρ of $\mathrm{Spin}(8,\mathbb{C})$, which will be faithful, is then defined to be the representation

$$\rho : \mathrm{Spin}(8, \mathbb{C}) \xrightarrow{\pi} \mathrm{SO}(8, \mathbb{C}) \cong \mathrm{SL}(V, q) \hookrightarrow \mathrm{GL}(V), \tag{5}$$

where the last map $\mathrm{SL}(V, q) \hookrightarrow \mathrm{GL}(V)$ is the natural inclusion of $\mathrm{SL}(V, q)$ in the general linear group associated with V. The different choices of the isomorphism $\mathrm{SL}(V, q) \cong \mathrm{SO}(8, \mathbb{C})$ induce of course equivalent representations of $\mathrm{Spin}(8, \mathbb{C})$.

Consider now the Clifford algebra $C(V, q)$ associated with V. It can be understood as a quotient of the tensor algebra of V where the identification $v \otimes v = q(v) \cdot 1$ is made for all $v \in V$. Let now W be a maximal isotropic complex subspace of V (recall that a subspace of V is isotropic if $q(v) = 0$ for all v in that subspace). In the study by Fulton and Harris [19, Lemma 20.9], it is proved that $C(V, q)$ is isomorphic to $\mathfrak{gl}(\wedge W)$, where $\wedge W = \oplus_{k=0}^{8} \wedge^k W$. Let $\wedge W = \wedge^+ W \oplus \wedge^- W$ be the decomposition of $\wedge W$ into the direct sum of even and odd exterior powers, respectively, and let $C(V, q)^+$ be the subalgebra of $C(V, q)$ of even tensor powers. In [19, Lemma 20.7], it is also proved that the Lie algebra $\mathfrak{gl}(V, q)$ is contained in $C(V, q)^+$, which is, by [19, Lemma 20.9], isomorphic to $\mathfrak{gl}(\wedge^+ W) \oplus \mathfrak{gl}(\wedge^- W)$, so the Lie algebra $\mathfrak{so}(8, \mathbb{C})$ comes with two representations: $\mathfrak{so}(8, \mathbb{C}) \to \mathfrak{gl}(\wedge^+ W)$ and $\mathfrak{so}(8, \mathbb{C}) \to \mathfrak{gl}(\wedge^- W)$, thus constructed. Since W has complex dimension 4, it is easy to check that $\dim \wedge^+ W = \dim \wedge^- W = 8$. This defines two eight-dimensional faithful complex representations of $\mathrm{Spin}(8, \mathbb{C})$:

$$\rho_+ : \mathrm{Spin}(8, \mathbb{C}) \to \mathrm{GL}(\wedge^+ W), \tag{6}$$

$$\rho_- : \mathrm{Spin}(8, \mathbb{C}) \to \mathrm{GL}(\wedge^- W). \tag{7}$$

These representations are irreducible [19, Proposition 20.15]. The triality automorphism τ interchanges the three representations ρ, ρ_+, and ρ_- of $\mathrm{Spin}(8, \mathbb{C})$. Specifically, in the study by Fulton and Harris [19, Section 20.3], it is constructed a complex linear automorphism of vector spaces

$$J : V \oplus \wedge^+ W \oplus \wedge^- W \to V \oplus \wedge^+ W \oplus \wedge^- W, \tag{8}$$

such that $J(V) = \wedge^+ W$, $J(\wedge^+ W) = \wedge^- W$, and $J(\wedge^- W) = V$, which satisfies

$$J \circ \rho = (\rho_+ \circ T) \circ J, \tag{9}$$

$$J \circ \rho_+ = (\rho_- \circ T) \circ J, \tag{10}$$

$$J \circ \rho_- = (\rho \circ T) \circ J, \tag{11}$$

in the sense that $J(\rho(g)(v)) = \rho_+(T(g))(J(v))$ for all $g \in \mathrm{Spin}(8, \mathbb{C})$ and all $v \in V$, where T is some order 3 representative of τ in $\mathrm{Aut}(\mathrm{Spin}(8, \mathbb{C}))$ (and analogous expressions for the other two identities).

For the study of $(\mathrm{Spin}(8, \mathbb{C}), \rho)$-Higgs pairs made in Theorem 5.1, it will be necessary to consider Higgs pairs whose structure group is $\mathrm{Spin}(2n, \mathbb{C})$ for $n = 2, 3$. For any integer number $n \geq 2$, $\mathrm{Spin}(2n, \mathbb{C})$ is the simply connected complex Lie group with Lie algebra $\mathfrak{so}(2n, \mathbb{C})$, and it is the universal cover of the group $\mathrm{SO}(2n, \mathbb{C})$ through the double covering

$$\pi_{2n} : \mathrm{Spin}(2n, \mathbb{C}) \to \mathrm{SO}(2n, \mathbb{C}). \tag{12}$$

Let V_{2n} be a complex vector space of dimension $2n$ equipped with a holomorphic nondegenerate quadratic form q_{2n}. Then the representation $\mathfrak{so}(2n, \mathbb{C}) \to \mathfrak{gl}(2n, \mathbb{C})$ given by the natural inclusion lifts to a faithful complex irreducible representation

$$\rho_{2n} : \mathrm{Spin}(2n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{C}), \tag{13}$$

which factors through $\mathrm{SO}(2n, \mathbb{C})$, so it induces a representation

$$\rho_{2n}^{\mathrm{SO}} : \mathrm{SO}(2n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{C}) \tag{14}$$

given by the inclusion of groups. Observe that, with this notation, $\pi_8 = \pi$ and $\rho_8 = \rho$, where π and ρ were defined in (4) and (5), respectively.

To conclude, it is useful to establish some facts about the parabolic subgroups of $\mathrm{Spin}(2n, \mathbb{C})$ for $n \geq 2$, which will be done following the study by Procesi [20, Chapters 10 and 11]. Parabolic subgroups of $\mathrm{Spin}(2n, \mathbb{C})$

are in bijective correspondence, through the covering map π_{2n} defined in (12), with the parabolic subgroups of $\mathrm{SO}(2n,\mathbb{C})$. For its part, a parabolic subgroup of $\mathrm{SO}(2n,\mathbb{C})$ corresponds to a filtration of V of the form

$$0 \subset U_1 \subset \cdots \subset U_k \subseteq U_k^{\perp} \subset \cdots \subset U_1^{\perp} \subset V_{2n}, \tag{15}$$

where $U_1,\dots,U_k$ are complex vector subspaces of V_{2n} isotropic for q_{2n} and $\perp$ denotes the orthogonality with respect to the nondegenerate symmetric bilinear form induced by q_{2n} in V_{2n}. The conjugacy class of the parabolic subgroup is univocally determined by the number k and the ranks of the subbundles. The parabolic subgroup is maximal exactly when $k = 1$ in the preceding filtration, that is, when the induced filtration of V_{2n} is of the form

$$0 \subset U_1 \subseteq U_1^{\perp} \subset V_{2n} \tag{16}$$

for some isotropic subbundle U_1 of V_{2n}.

4 Stability conditions for Higgs pairs with structure group Spin(8, ℂ)

Let X be a compact Riemann surface of genus $g \geq 2$. In this section, the reduced stability, semistability, and polystability conditions for Higgs pairs over X with structure group $\mathrm{Spin}(8,\mathbb{C})$ and associated with the representations ρ, ρ_+, and ρ_- of it defined, respectively, in (5), (6), and (7), will be given.

Given any principal $\mathrm{Spin}(8,\mathbb{C})$-bundle E over X, the covering map π defined in (4) induces a principal $\mathrm{SO}(8,\mathbb{C})$-bundle E_{SO} given by the image of E by the map

$$H^1(X,\mathrm{Spin}(8,\mathbb{C})) \to H^1(X,\mathrm{SO}(8,\mathbb{C})), \quad E \mapsto E_{\mathrm{SO}}, \tag{17}$$

which comes from the exact sequence of groups

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(8,\mathbb{C}) \to \mathrm{SO}(8,\mathbb{C}) \to 1.$$

The principal $\mathrm{SO}(8,\mathbb{C})$-bundle E_{SO} can be understood as a holomorphic complex vector bundle of rank 8 over X equipped with a global nondegenerate holomorphic symmetric bilinear form ω. For any $n \geq 2$, the covering map π_{2n} defined in (12) also defines a map

$$H^1(X,\mathrm{Spin}(2n,\mathbb{C})) \to H^1(X,\mathrm{SO}(2n,\mathbb{C})), \quad E \mapsto E_{\mathrm{SO}}, \tag{18}$$

where E_{SO} is a holomorphic complex vector bundle of rank $2n$ over X equipped with a global nondegenerate holomorphic symmetric bilinear form ω_{2n} (of course, $\omega_8 = \omega$).

Let V be an eight-dimensional complex vector space equipped with a nondegenerate quadratic form q and let W be a maximal isotropic vector subspace of V. Let ρ, ρ_+, and ρ_- be the representations of $\mathrm{Spin}(8,\mathbb{C})$ defined in (5), (6), and (7), whose associated vector spaces are V, $\wedge^+W$, and $\wedge^-W$, respectively. The rank 8 holomorphic vector bundle $E(V)$ is also a special orthogonal vector bundle, and the vector bundles $E(\wedge^+W)$ and $E(\wedge^-W)$ are also rank 8 holomorphic vector bundles, which are subbundles of $\wedge^+E_{\mathrm{SO}}$ and $\wedge^-E_{\mathrm{SO}}$, the even and odd exterior powers, respectively, of E_{SO}.

From the description in terms of filtrations of the parabolic subgroups of $\mathrm{Spin}(2n,\mathbb{C})$ given in (15) and (16), if follows that a reduction of structure group of a principal $\mathrm{Spin}(2n,\mathbb{C})$-bundle E over X to a parabolic subgroup of $\mathrm{Spin}(2n,\mathbb{C})$ gives a filtration of E_{SO} of the form

$$0 \subset F_1 \subset \cdots \subset F_k \subseteq F_k^{\perp} \subset \cdots \subset F_1^{\perp} \subset E_{\mathrm{SO}}, \tag{19}$$

where $F_1,\dots,F_k$ are isotropic holomorphic vector subbundles of E_{SO} for $1 \leq k \leq n$ (isotropy and orthogonality are taken with respect to the holomorphic symmetric bilinear form ω_{2n} of E_{SO}). A reduction of structure group of E to a maximal parabolic subgroup of $\mathrm{Spin}(2n,\mathbb{C})$ gives a filtration of the form

$$0 \subset F \subseteq F^{\perp} \subset E_{\mathrm{SO}}, \tag{20}$$

where F is an isotropic subbundle of E_{SO}.

Proposition 4.1. *Let* (E, φ) *be a* $(\mathrm{Spin}(2n, \mathbb{C}), \rho_{2n})$*-Higgs pair over* X *for the representation* ρ_{2n} *of* $\mathrm{Spin}(2n, \mathbb{C})$ *defined in* (13) *for some* $n \geq 2$. *Let* E_{SO} *be the principal* $\mathrm{SO}(2n, \mathbb{C})$*-bundle defined in* (18), *and let* ω_{2n} *be its associated global nondegenerate holomorphic symmetric bilinear form. The* $(\mathrm{Spin}(2n, \mathbb{C}), \rho_{2n})$*-Higgs pair* (E, φ) *is stable (resp. semistable) if for every proper subbundle* F *of* E_{SO}, *which is isotropic for* ω_{2n} *and such that* φ *takes values in* $F^{\perp} \otimes L$, *it is satisfied that* $\deg F < 0$ (*resp.* $\deg F \leq 0$). *In particular, if* (E, φ) *is a* $(\mathrm{Spin}(8, \mathbb{C}), \rho)$*-Higgs pair over* X *for the representation* ρ of $\mathrm{Spin}(8, \mathbb{C})$ *defined in* (5) *and* ω *is the global nondegenerate holomorphic symmetric bilinear form of* E_{SO}, *then* (E, φ) *is stable* (*resp. semistable*) *if for every proper subbundle* F *of* E_{SO} *isotropic for* ω *and such that* φ *takes values in* $F^{\perp} \otimes L$ *we have that* $\deg F < 0$ (*resp.* $\deg F \leq 0$).

The $(\mathrm{Spin}(8, \mathbb{C}), \rho)$*-Higgs pair* (E, φ) *is polystable if it is semistable and* E_{SO} *admits a filtration of the form*

$$0 \subset F_1 \subset \cdots \subset F_k \subseteq F_k^{\perp} \subset \cdots \subset F_1^{\perp} \subset E_{\mathrm{SO}}$$

described in (19), *where* $F_1, \ldots, F_k$ *are holomorphic vector subbundles of* E_{SO} *isotropic for* ω *with* $\deg F_1 = \cdots = \deg F_k = 0$, *such that* φ *takes values in* $F_k^{\perp}/F_k \otimes L$, *and* E_{SO} *admits the following decomposition into a direct sum of subspaces*:

$$E_{\mathrm{SO}} = F_1 \oplus F_2/F_1 \oplus \cdots \oplus F_k/F_{k-1} \oplus F_k^{\perp}/F_k \oplus F_{k-1}^{\perp}/F_k^{\perp} \oplus \cdots \oplus E_{\mathrm{SO}}/F_1^{\perp}.$$

Proof. Let P be a parabolic subgroup of $\mathrm{Spin}(8, \mathbb{C})$, χ be any antidominant character of P, and s_χ be the associated element of $i\mathfrak{h}$. If the filtration of E_{SO} is as described in (19), then the element s_χ diagonalizes in the form

$$\begin{pmatrix} \lambda_1 I_{F_1} & & & & & & \\ & \ddots & & & & & \\ & & \lambda_k I_{F_k/F_{k-1}} & & & & \\ & & & 0 I_{F_k^{\perp}/F_k} & & & \\ & & & & -\lambda_k I_{F_{k-1}^{\perp}/F_k^{\perp}} & & \\ & & & & & \ddots & \\ & & & & & & -\lambda_1 I_{E_{\mathrm{SO}}/F_1^{\perp}} \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $\lambda_1 < \cdots < \lambda_k < 0$. The degree defined in (2) takes the value

$$\sum_{i=1}^{k} (\lambda_i - \lambda_{i+1})(\deg F_i + \deg F_i^{\perp}),$$

so it is greater than or equal to 0 exactly when $\deg F_i \leq 0$ for all i and it is equal to 0 when $\deg F_i = 0$ for all i. On the other hand, the condition for φ of taking values in the space V_χ^- (resp. in V_χ^0) defined in (1) clearly requires that φ takes values in $F_k^{\perp} \otimes L$ (resp. in $F_k^{\perp}/F_k \otimes L$). Then the semistability condition requires that $\deg F_i \leq 0$ for all i whenever φ takes values in $F_k^{\perp} \otimes L$ and for every filtration as the considered one. Since the satisfaction of this condition for filtrations induced by reductions to a maximal parabolic subgroup as in (20) gives the fulfilment of the condition for every filtration, the first part of the result is proved. For polystability, observe that a reduction of structure group of (E, φ) to a Levi subgroup of P gives a decomposition of E_{SO} into a direct sum of vector subbundles of the form

$$E_{\mathrm{SO}} = F_1 \oplus F_2/F_1 \oplus \cdots \oplus F_k/F_{k-1} \oplus F_k^{\perp}/F_k \oplus F_{k-1}^{\perp}/F_k^{\perp} \oplus \cdots \oplus E_{\mathrm{SO}}/F_1^{\perp}.$$

Therefore, (E, φ) is polystable if it admits a decomposition into direct sum of vector subbundles as those described in the statement such that $\deg F_j = 0$ for all j and φ takes values in $F_k^{\perp}/F_k \otimes L$. □

The reduced notions of stability and polystability for $(\mathrm{Spin}(8, \mathbb{C}), \rho_+)$ and $(\mathrm{Spin}(8, \mathbb{C}), \rho_-)$-Higgs pairs over X, where ρ_+ and ρ_- are the eight-dimensional complex representations of $\mathrm{Spin}(8, \mathbb{C})$ defined in (6) and (7), respectively, will be now described. The proofs of Propositions 4.2 and 4.3 keep many analogous elements with each other and with the proof of Proposition 4.1. However, it has been preferred to keep the details of the proofs, at the risk of being repetitive, to make explicit the differences that exist between the cases considered in the three results.

Proposition 4.2. *Let* (E,φ) *be a* $(\mathrm{Spin}(8,\mathbb{C}),\rho_+)$*-Higgs pair over X for the representation* ρ_+ *of* $\mathrm{Spin}(8,\mathbb{C})$ *defined in* (6). *Let* E_{SO} *be the principal* $\mathrm{SO}(8,\mathbb{C})$*-bundle defined in* (17), *and let* ω *be its global nondegenerate holomorphic symmetric bilinear form. The* $(\mathrm{Spin}(8,\mathbb{C}),\rho_+)$*-Higgs pair* (E,φ) *is stable (resp. semistable) if for every proper isotropic subbundle F of* E_{SO} *such that* φ *takes values in*

$$(F\wedge E_{\mathrm{SO}}+F^{\perp}\wedge F^{\perp})\coprod(F\wedge F\wedge E_{\mathrm{SO}}\wedge E_{\mathrm{SO}}+F\wedge F^{\perp}\wedge F^{\perp}\wedge E_{\mathrm{SO}}+F^{\perp}\wedge F^{\perp}\wedge F^{\perp}\wedge F^{\perp}),$$

where the reference to the line bundle L is omitted for clarity, and it is satisfied that $\deg F<0$ (*resp.* $\deg F\le 0$).

The $(\mathrm{Spin}(8,\mathbb{C}),\rho_+)$*-Higgs pair* (E,φ) *is polystable if one of the following conditions holds* (*again, the reference to L has been omitted in the vector subbundles where, in each case,* φ *takes values, for clarity*):

(1) *There exists a proper isotropic subbundle F of* E_{SO} *with* $\mathrm{rk}F\le 3$ *such that* E_{SO} *admits a decomposition of the form*

$$E_{\mathrm{SO}}=F\oplus F^{\perp}/F\oplus E_{\mathrm{SO}}/F^{\perp}$$

and φ *takes values in*

$$\begin{aligned}&(F\wedge E_{\mathrm{SO}}/F^{\perp}\oplus\ F^{\perp}/F\wedge F^{\perp}/F)\coprod\\&\quad(F\wedge F\wedge E_{\mathrm{SO}}/F^{\perp}\wedge E_{\mathrm{SO}}/F^{\perp}\oplus\ F\wedge F^{\perp}/F\wedge F^{\perp}/F\wedge E_{\mathrm{SO}}/F^{\perp}\oplus\ F^{\perp}/F\wedge F^{\perp}/F\wedge F^{\perp}/F\wedge F^{\perp}/F).\end{aligned}$$

(2) *There exists a rank* 4 *isotropic subbundle F of* E_{SO} *such that* E_{SO} *admits a decomposition of the form*

$$E_{\mathrm{SO}}=F\oplus E_{\mathrm{SO}}/F$$

and φ *takes values in*

$$(F\wedge E_{\mathrm{SO}}/F^{\perp})\coprod(F\wedge F\wedge E_{\mathrm{SO}}/F^{\perp}\wedge E_{\mathrm{SO}}/F^{\perp}).$$

(3) *There exists a filtration* $0\subsetneq F_1\subsetneq F_2$ *of* E_{SO} *into isotropic subbundles with* $2\le\mathrm{rk}F_2\le 3$ *such that* E_{SO} *admits a decomposition of the form*

$$E_{\mathrm{SO}}=F_1\oplus F_2/F_1\oplus F_2^{\perp}/F_2\oplus F_1^{\perp}/F_2^{\perp}\oplus E_{\mathrm{SO}}/F_1^{\perp}$$

and φ *takes values in*

$$\begin{aligned}&(F_1\wedge E_{\mathrm{SO}}/F_1^{\perp}\oplus\ F_2/F_1\wedge F_1^{\perp}/F_2^{\perp}\oplus\ F_2^{\perp}/F_2\wedge F_2^{\perp}/F_2)\coprod\\&\quad(F_1\wedge F_1\wedge E_{\mathrm{SO}}/F_1^{\perp}\wedge E_{\mathrm{SO}}/F_1^{\perp}\oplus\ F_1\wedge F_2^{\perp}/F_2\wedge F_2^{\perp}/F_2\wedge E_{\mathrm{SO}}/F_1^{\perp}\\&\quad\oplus F_1\wedge F_2/F_1\wedge F_1^{\perp}/F_2^{\perp}\wedge E_{\mathrm{SO}}/F_1^{\perp}\oplus\ F_2^{\perp}/F_2\wedge F_2^{\perp}/F_2\wedge F_2^{\perp}/F_2\wedge F_2^{\perp}/F_2\\&\quad\oplus F_2/F_1\wedge F_2^{\perp}/F_2\wedge F_2^{\perp}/F_2\wedge F_1^{\perp}/F_2^{\perp}\oplus\ F_2/F_1\wedge F_2/F_1\wedge F_1^{\perp}/F_2^{\perp}\wedge F_1^{\perp}/F_2^{\perp}).\end{aligned}$$

(4) *There exists a filtration* $0\subsetneq F_1\subsetneq F_2$ *of* E_{SO} *into isotropic subbundles with* $\mathrm{rk}F_2=4$ *such that* E_{SO} *admits a decomposition of the form*

$$E=F_1\oplus F_2/F_1\oplus F_1^{\perp}/F_2^{\perp}\oplus E_{\mathrm{SO}}/F_1^{\perp}$$

and φ *takes values in*

$$\begin{aligned}&(F_1\wedge E_{\mathrm{SO}}/F_1^{\perp}\oplus\ F_2/F_1\wedge F_1^{\perp}/F_2^{\perp})\coprod\\&\quad(F_1\wedge F_1\wedge E_{\mathrm{SO}}/F_1^{\perp}\wedge E_{\mathrm{SO}}/F_1^{\perp}\oplus\ F_1\wedge F_2/F_1\wedge F_1^{\perp}/F_2^{\perp}\wedge E_{\mathrm{SO}}/F_1^{\perp}\oplus F_2/F_1\wedge F_2/F_1\wedge F_1^{\perp}/F_2^{\perp}\wedge F_1^{\perp}/F_2^{\perp}).\end{aligned}$$

(5) *There exists a filtration* $0\subsetneq F_1\subsetneq F_2\subsetneq F_3$ *of* E_{SO} *into isotropic subbundles with* $\mathrm{rk}F_3=3$ *such that* E_{SO} *admits a decomposition of the form*

$$E_{\mathrm{SO}}=F_1\oplus F_2/F_1\oplus F_3/F_2\oplus F_3^{\perp}/F_3\oplus F_2^{\perp}/F_3^{\perp}\oplus F_1^{\perp}/F_2^{\perp}\oplus E_{\mathrm{SO}}/F_1^{\perp}$$

and φ *takes values in*

$$
\begin{aligned}
&(F_1 \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus \ F_2/F_1 \wedge F_1^{\perp}/F_2^{\perp} \oplus \ F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp} \oplus \ F_3^{\perp}/F_3 \wedge F_3^{\perp}/F_3) \coprod \\
&\quad (F_1 \wedge F_1 \wedge E_{\mathrm{SO}}/F_1^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus \ F_1 \wedge F_2/F_1 \wedge F_1^{\perp}/F_2^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \\
&\quad \oplus F_1 \wedge F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus \ F_1 \wedge F_3^{\perp}/F_3 \wedge F_3^{\perp}/F_3 \wedge E_{\mathrm{SO}}/F_1^{\perp} \\
&\quad \oplus F_2/F_1 \wedge F_2/F_1 \wedge F_1^{\perp}/F_2^{\perp} \wedge F_1^{\perp}/F_2^{\perp} \oplus \ F_2/F_1 \wedge F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp} \wedge F_1^{\perp}/F_2^{\perp} \\
&\quad \oplus F_2/F_1 \wedge F_3^{\perp}/F_3 \wedge F_3^{\perp}/F_3 \wedge F_1^{\perp}/F_2^{\perp} \oplus \ F_3/F_2 \wedge F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp} \wedge F_2^{\perp}/F_3^{\perp} \\
&\quad \oplus F_3/F_2 \wedge F_3^{\perp}/F_3 \wedge F_3^{\perp}/F_3 \wedge F_2^{\perp}/F_3^{\perp} \oplus \ F_3^{\perp}/F_3 \wedge F_3^{\perp}/F_3 \wedge F_3^{\perp}/F_3 \wedge F_3^{\perp}/F_3).
\end{aligned}
$$

(6) *There exists a filtration* $0 \subsetneq F_1 \subsetneq F_2 \subsetneq F_3$ *of* E_{SO} *into isotropic subbundles with* $\mathrm{rk} F_3 = 4$ *such that* E_{SO} *admits a decomposition of the form*

$$E_{\mathrm{SO}} = F_1 \oplus F_2/F_1 \oplus F_3/F_2 \oplus F_2^{\perp}/F_3^{\perp} \oplus F_1^{\perp}/F_2^{\perp} \oplus E_{\mathrm{SO}}/F_1^{\perp}$$

and φ *takes values in*

$$
\begin{aligned}
&(F_1 \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus \ F_2/F_1 \wedge F_1^{\perp}/F_2^{\perp} \oplus \ F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp}) \coprod \\
&\quad (F_1 \wedge F_1 \wedge E_{\mathrm{SO}}/F_1^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus \ F_1 \wedge F_2/F_1 \wedge F_1^{\perp}/F_2^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \\
&\quad \oplus F_1 \wedge F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus \ F_2/F_1 \wedge F_2/F_1 \wedge F_1^{\perp}/F_2^{\perp} \wedge F_1^{\perp}/F_2^{\perp} \\
&\quad \oplus F_2/F_1 \wedge F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp} \wedge F_1^{\perp}/F_2^{\perp} \oplus \ F_3/F_2 \wedge F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp} \wedge F_2^{\perp}/F_3^{\perp}).
\end{aligned}
$$

(7) *There exists a filtration* $0 \subsetneq F_1 \subsetneq F_2 \subsetneq F_3 \subsetneq F_4$ *of* E_{SO} *into isotropic subbundles such that* E_{SO} *admits a decomposition of the form*

$$E_{\mathrm{SO}} = F_1 \oplus F_2/F_1 \oplus F_3/F_2 \oplus \oplus F_4/F_3 \oplus F_3^{\perp}/F_4^{\perp} \oplus F_2^{\perp}/F_3^{\perp} \oplus F_1^{\perp}/F_2^{\perp} \oplus E_{\mathrm{SO}}/F_1^{\perp}$$

and φ *takes values in*

$$
\begin{aligned}
&(F_1 \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus \ F_2/F_1 \wedge F_1^{\perp}/F_2^{\perp} \oplus \ F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp} \oplus \ F_4/F_3 \wedge F_3^{\perp}/F_4^{\perp}) \coprod \\
&\quad (F_1 \wedge F_1 \wedge E_{\mathrm{SO}}/F_1^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus \ F_1 \wedge F_2/F_1 \wedge F_1^{\perp}/F_2^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \\
&\quad \oplus F_1 \wedge F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus \ F_1 \wedge F_4/F_3 \wedge F_3^{\perp}/F_4^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \\
&\quad \oplus F_2/F_1 \wedge F_2/F_1 \wedge F_1^{\perp}/F_2^{\perp} \wedge F_1^{\perp}/F_2^{\perp} \oplus \ F_2/F_1 \wedge F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp} \wedge F_1^{\perp}/F_2^{\perp} \\
&\quad \oplus F_2/F_1 \wedge F_4/F_3 \wedge F_3^{\perp}/F_4^{\perp} \wedge F_1^{\perp}/F_2^{\perp} \oplus \ F_3/F_2 \wedge F_3/F_2 \wedge F_2^{\perp}/F_3^{\perp} \wedge F_2^{\perp}/F_3^{\perp} \\
&\quad \oplus F_3/F_2 \wedge F_4/F_3 \wedge F_3^{\perp}/F_4^{\perp} \wedge F_2^{\perp}/F_3^{\perp} \oplus \ F_4/F_3 \wedge F_4/F_3 \wedge F_3^{\perp}/F_4^{\perp} \wedge F_3^{\perp}/F_4^{\perp}).
\end{aligned}
$$

Proof. Let P be any parabolic subgroup of $\mathrm{Spin}(8, \mathbb{C})$ and χ be any antidominant character of P. With the notation of Section 2, the associated element $s_\chi \in i\mathfrak{h}$ diagonalizes in the form

$$
\begin{pmatrix}
\lambda_1 I_{F_1} & & & & & & \\
& \ddots & & & & & \\
& & \lambda_k I_{F_k/F_{k-1}} & & & & \\
& & & 0 I_{F_k^{\perp}/F_k} & & & \\
& & & & -\lambda_k I_{F_{k-1}^{\perp}/F_k^{\perp}} & & \\
& & & & & \ddots & \\
& & & & & & -\lambda_1 I_{E_{\mathrm{SO}}/F_1^{\perp}}
\end{pmatrix}.
$$

for the filtration

$$0 \subset F_1 \subset \cdots \subset F_k \subseteq F_k^{\perp} \subset \cdots \subset F_1^{\perp} \subset E_{\mathrm{SO}}$$

of E_{SO} induced by a restriction of structure group of E_{SO} to P, where $\lambda_1 < \cdots < \lambda_k < 0$. Since the space V_χ defined in (1) is a subspace of the corresponding space induced by a reduction to a maximal parabolic subgroup (that is, when $k = 1$), it is enough to check the semistability condition on filtrations of the form $0 \subset F \subsetneq F^{\perp} \subset E_{\mathrm{SO}}$. In this case, the space V_{χ}^{-} is clearly

$$(F \wedge E_{\mathrm{SO}} + F^{\perp} \wedge F^{\perp}) \coprod (F \wedge F \wedge E_{\mathrm{SO}} \wedge E_{\mathrm{SO}} + F \wedge F^{\perp} \wedge F^{\perp} \wedge E_{\mathrm{SO}} + F^{\perp} \wedge F^{\perp} \wedge F^{\perp} \wedge F^{\perp}),$$

so the first part of the result is proved. It is easily checked that the kernel of the corresponding endomorphism of $\wedge^{+} E_{\mathrm{SO}}$ is exactly the space announced in each one of the seven cases described in the second part of the statement; the case depends on the value of $k = 1, 2, 3, 4$ and the ranks of the isotropic subspaces involved:
(1) $k = 1$ and $\mathrm{rk} F \leq 3$. In this case, $F \subsetneq F^{\perp}$.
(2) $k = 1$ and $\mathrm{rk} F = 4$. In this case, $F = F^{\perp}$.
(3) $k = 2$ and $\mathrm{rk} F_2 \leq 3$. In this case, $F_2 \subsetneq F_2^{\perp}$.
(4) $k = 2$ and $\mathrm{rk} F_2 = 4$. In this case, $F_2 = F_2^{\perp}$.
(5) $k = 3$ and $\mathrm{rk} F_3 \leq 3$. In this case, $F_3 \subsetneq F_3^{\perp}$.
(6) $k = 3$ and $\mathrm{rk} F_3 = 4$. In this case, $F_3 = F_3^{\perp}$.
(7) $k = 4$. In this case, necessarily $F_4 = F_4^{\perp}$.

This finally shows the result. □

Proposition 4.3. *Let (E, φ) be a $(\mathrm{Spin}(8,\mathbb{C}), \rho_{-})$-Higgs pair over X for the representation ρ_{-} of $\mathrm{Spin}(8,\mathbb{C})$ defined in* (7). *Let E_{SO} be the principal $\mathrm{SO}(8,\mathbb{C})$-bundle defined in* (17), *and let ω be its global nondegenerate holomorphic symmetric bilinear form. The $(\mathrm{Spin}(8,\mathbb{C}), \rho_{-})$-Higgs pair (E, φ) is stable (resp. semistable) if for every proper isotropic subbundle F of E_{SO} such that φ takes values in*

$$\begin{aligned}&(F^{\perp}) \coprod (F \wedge F^{\perp} \wedge E_{\mathrm{SO}} \oplus F^{\perp} \wedge F^{\perp} \wedge F^{\perp})\\ &\quad \coprod (F \wedge F \wedge F^{\perp} \wedge E_{\mathrm{SO}} \wedge E_{\mathrm{SO}} \oplus F \wedge F^{\perp} \wedge F^{\perp} \wedge F^{\perp} \wedge E_{\mathrm{SO}}),\end{aligned}$$

where the reference to the line bundle L is omitted for clarity, it is satisfied that $\deg F < 0$ (*resp.* $\deg F \leq 0$).

The $(\mathrm{Spin}(8,\mathbb{C}), \rho_{-})$-Higgs pair (E, φ) is polystable if one of the following conditions holds (again, the reference to L has been omitted in the vector subbundles where, in each case, φ takes values, for clarity):
(1) *There exists a proper isotropic subbundle F of E_{SO} with $\mathrm{rk} F \leq 3$ such that E_{SO} admits a decomposition of the form*

$$E_{\mathrm{SO}} = F \oplus F^{\perp}/F \oplus E_{\mathrm{SO}}/F^{\perp}$$

and φ takes values in

$$\begin{aligned}&(F^{\perp}/F) \coprod (F \wedge F^{\perp}/F \wedge E_{\mathrm{SO}}/F^{\perp} \oplus \textstyle\bigwedge^{3} F^{\perp}/F)\\ &\quad \coprod (\textstyle\bigwedge^{2} F \wedge F^{\perp}/F \wedge \textstyle\bigwedge^{2} E_{\mathrm{SO}}/F^{\perp} \oplus F \wedge \textstyle\bigwedge^{3} F^{\perp}/F \wedge E_{\mathrm{SO}}/F^{\perp}).\end{aligned}$$

(2) *There exists a filtration $0 \subsetneq F_1 \subsetneq F_2$ of E_{SO} into isotropic subbundles of E_{SO} with $2 \leq \mathrm{rk} F_2 \leq 3$ such that E_{SO} admits a decomposition of the form*

$$E_{\mathrm{SO}} = F_1 \oplus F_2/F_1 \oplus F_2^{\perp}/F_2 \oplus F_1^{\perp}/F_2^{\perp} \oplus E_{\mathrm{SO}}/F_1^{\perp}$$

and φ takes values in

$$\begin{aligned}&(F_2^{\perp}/F_2) \coprod (F_1 \wedge F_2^{\perp}/F_2 \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus F_2/F_1 \wedge F_2^{\perp}/F_2 \wedge F_1^{\perp}/F_2^{\perp} \oplus \textstyle\bigwedge^{3} F_2^{\perp}/F_2)\\ &\quad \coprod (\textstyle\bigwedge^{2} F_1 \wedge F_2^{\perp}/F_2 \wedge \textstyle\bigwedge^{2} E_{\mathrm{SO}}/F_1^{\perp} \oplus F_1 \wedge \textstyle\bigwedge^{3} F_2^{\perp}/F_2 \wedge E_{\mathrm{SO}}/F_1^{\perp}\\ &\quad \oplus F_1 \wedge F_2/F_1 \wedge F_2^{\perp}/F_2 \wedge F_1^{\perp}/F_2^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus \textstyle\bigwedge^{5} F_2^{\perp}/F_2\\ &\quad \oplus F_2/F_1 \wedge \textstyle\bigwedge^{3} F_2^{\perp}/F_2 \wedge F_1^{\perp}/F_2^{\perp} \oplus \textstyle\bigwedge^{2} F_2/F_1 \wedge F_2^{\perp}/F_2 \wedge \textstyle\bigwedge^{2} F_1^{\perp}/F_2^{\perp}).\end{aligned}$$

(3) *There exists a filtration $0 \subsetneq F_1 \subsetneq F_2 \subsetneq F_3$ of E_{SO} into isotropic subbundles of E_{SO} with $\mathrm{rk} F_3 = 3$ such that E_{SO} admits a decomposition of the form*

$$E_{\mathrm{SO}} = F_1 \oplus F_2/F_1 \oplus F_3/F_2 \oplus F_3^{\perp}/F_3 \oplus F_2^{\perp}/F_3^{\perp} \oplus F_1^{\perp}/F_2^{\perp} \oplus E_{\mathrm{SO}}/F_1^{\perp}$$

and φ takes values in

$$
\begin{aligned}
&(F_3^{\perp}/F_3)\\
&\quad \bigsqcup (F_1 \wedge F_3^{\perp}/F_3 \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus\ F_2/F_1 \wedge F_3^{\perp}/F_3 \wedge F_1^{\perp}/F_2^{\perp} \oplus\ F_3/F_2 \wedge F_3^{\perp}/F_3 \wedge F_2^{\perp}/F_3^{\perp} \oplus\ \textstyle\bigwedge^3 F_3^{\perp}/F_3)\\
&\quad \bigsqcup (\textstyle\bigwedge^2 F_1 \wedge F_3^{\perp}/F_3 \wedge \bigwedge^2 E_{\mathrm{SO}}/F_1^{\perp} \oplus\ F_1 \wedge F_2/F_1 \wedge F_3^{\perp}/F_3 \wedge F_1^{\perp}/F_2^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp}\\
&\quad \oplus F_1 \wedge \textstyle\bigwedge^3 F_2^{\perp}/F_2 \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus\ \bigwedge^2 F_2/F_1 \wedge F_3^{\perp}/F_3 \wedge \bigwedge^2 F_1^{\perp}/F_2^{\perp}\\
&\quad \oplus F_2/F_1 \wedge F_3/F_2 \wedge F_3^{\perp}/F_3 \wedge F_2^{\perp}/F_3^{\perp} \wedge F_1^{\perp}/F_2^{\perp} \oplus\ F_2/F_1 \wedge \textstyle\bigwedge^3 F_3^{\perp}/F_3 \wedge F_1^{\perp}/F_2^{\perp}\\
&\quad \oplus F_1 \wedge F_3/F_2 \wedge F_3^{\perp}/F_3 \wedge F_2^{\perp}/F_3^{\perp} \wedge E_{\mathrm{SO}}/F_1^{\perp} \oplus\ \textstyle\bigwedge^5 F_3^{\perp}/F_3 \oplus\ F_3/F_2 \wedge \bigwedge^3 F_3^{\perp}/F_3 \wedge F_2^{\perp}/F_3^{\perp}\\
&\quad \oplus \textstyle\bigwedge^2 F_3/F_2 \wedge F_3^{\perp}/F_3 \wedge \bigwedge^2 F_2^{\perp}/F_3^{\perp}).
\end{aligned}
$$

Proof. The same proof of Proposition 4.2, with the necessary formal differences, works here. A few words to the reduced notion of polystability. If P is any parabolic subgroup of $\mathrm{Spin}(8,\mathbb{C})$, χ is any antidominant character of P and

$$0 \subset F_1 \subset \cdots \subset F_k \subseteq F_k^{\perp} \subset \cdots \subset F_1^{\perp} \subset E_{\mathrm{SO}}$$

is the filtration of E_{SO} induced by a restriction of structure group of E to P, then the kernel of the endomorphism of $\wedge^{-} E_{\mathrm{SO}}$ induced by s_χ is 0 when $\mathrm{rk} F_k = 4$, since $F_k^{\perp}/F_k = 0$ in this case and each wedge product has an odd number of factors, so a factor of type $F_k^{\perp}/F_k$ must appear in every element of the considered kernel. Then it must be $k \leq 3$ and $\mathrm{rk} F_k \leq 3$. Therefore, the possible cases are the following:
(1) $k = 1$ and $\mathrm{rk} F \leq 3$.
(2) $k = 2$ and $\mathrm{rk} F \leq 3$.
(3) $k = 3$ and $\mathrm{rk} F = 3$.

These three cases turn to the cases described in the statement. □

Example. Let E be a rank 8 and trivial determinant vector bundle over X, which admits a globally defined nondegenerate symmetric bilinear form, and whose second Stiefel-Whitney class is 0. Suppose that the maximal isotropic subbundle of E is F. Take L to be the trivial line bundle $\mathcal{O}$ over X. Then E can be understood as a principal $\mathrm{SO}(8,\mathbb{C})$-bundle over X and it lifts to a principal $\mathrm{Spin}(8,\mathbb{C})$-bundle over X (because the second Stiefel-Whitney class is 0). A Higgs pair associated with the representation ρ defined in (5) is, in this situation, given by E together with a holomorphic global section of E. Similarly, a Higgs pair for the representation ρ_+ defined in (6) (resp, ρ_- defined in (7)) is given by E together with a holomorphic global section of $\wedge^k F$ for some even number k with $k \leq 8$ (resp. odd number k with $k \leq 7$). In the case of the representation ρ, the stability of E as a Higgs pair depends on the degree of F and where the global section takes values. If $\deg F \geq 0$, then it should be required that the global section does not take values in $F^{\perp}$ for the Higgs pair to be stable, by Proposition 4.1.

In Proposition 4.4, the action of the group $\mathrm{Out}(G)$ of outer automorphisms of G on the set of principal G-bundles over X, introduced in [21, Section 5] for a semisimple complex Lie group G, is considered. Specifically, if $\sigma \in \mathrm{Out}(G)$ and E is a principal G-bundle over X, $\sigma(E)$ is defined to be the principal G-bundle over X whose total space coincides with that of E and such that the action of G on it derives from that of G in E in the following way: if $g \in G$ and $e \in E$, then

$$e \cdot g = eS^{-1}(g), \tag{21}$$

where S is an automorphism of G that represents σ.

Proposition 4.4. *Let G be a semisimple complex Lie group, $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(W)$ be complex representations of G, σ be an outer automorphism of G, S be a representative of σ in $\mathrm{Aut}(G)$, and $F : V \to W$ be* *an isomorphism of vector spaces such that $F \circ \rho = (\rho' \circ S) \circ F$. Then F induces a bijective correspondence between polystable (G, ρ)-Higgs pairs and (G, ρ')-Higgs pairs over X that preserves polystability.*

Proof. Under the conditions and the notation of the statement, the map f is defined in the following way: if (E, φ) is a polystable (G, ρ)-Higgs pair over X, then $f(E, \varphi) = (\sigma(E), F(\varphi))$, where $\sigma(E)$ is defined in (21) and $F(\varphi)$ acts on each fiber by taking the image by F of the image of φ, that is, if $x \in X$ and $\varphi(x) = [e, v] \otimes l$, where $e \in E$, $v \in V$ and $l \in L$, then $F(\varphi(x)) = [e, F(v)] \otimes L$. Notice that this $F(\varphi)$ is well defined, and gives a global section of $\sigma(E)(W) \otimes L$. To prove this, take any $g \in G$. Since $[e, v] = [eg^{-1}, \rho(g)(v)]$ and $F \circ \rho = (\rho' \circ S) \circ F$, it follows that

$$[eg^{-1}, F(\rho(g)(v))] = [eg^{-1}, \rho'(S(g))(F(v))] = [e\sigma^{-1}(S(g))^{-1}, \rho'(S(g))(F(v))],$$

so $F(\varphi)$ is well defined, and it takes values in W. In fact, $F(\varphi) \in H^0(X, \sigma(E)(W))$.

If (E, φ) is a semistable (G, ρ)-Higgs pair over X then $F(E, \varphi)$ is also semistable. To show this, take any parabolic subgroup P of G, any antidominant character χ of P, and a representative S of σ in $\mathrm{Aut}(G)$. Then $S(P)$ and $\chi \circ S^{-1}$ are generic parabolic subgroup of G and antidominant character of $S(P)$, so it suffices to check the semistability condition stated in Definition 2, applied to $F(E, \varphi) = (\sigma(E), F(\varphi))$, for $S(P)$ and $\chi \circ S^{-1}$. Let $\sigma(E)_{S(P)}$ be a reduction of structure group of $\sigma(E)$ to $S(P)$ such that $F(\varphi)$ takes values in $W^-_{\chi \circ S^{-1}}$. It is then clear that $\sigma^{-1}(\sigma(E)_{S(P)})$ defines a reduction of structure group E_P of E to P, and this reduction satisfies that φ takes values in V^-_χ because $S(V^-_\chi) = W^-_{\chi \circ S^{-1}}$, as an immediate consequence of the expression $F \circ \rho = (\rho' \circ S) \circ F$ of the hypotheses, the definition of the spaces V^-_χ and $W^-_{\chi \circ \sigma^{-1}}$ given in (1), and the fact that F is linear. Then the reduction E_P satisfies that φ takes values in V^-_χ. Since (E, φ) is semistable, then $\deg \chi_* E_P \geq 0$. This together with the observation that the line bundles $(\chi \circ S^{-1})_* \sigma(E)_{S(P)}$ and $\chi_* E_P$ over X are isomorphic, concludes that $F(E, \varphi)$ is semistable.

Let now (E, φ) be a polystable (G, ρ)-Higgs pair over X. Then $F(E, \varphi)$ is itself semistable. Let P and χ be a parabolic subgroup of G and an antidominant character of P such that $F(E, \varphi) = (\sigma(E), F(\varphi))$ admits a reduction of structure group $\sigma(E)_{S(P)}$ to $S(P)$ with

$$F(\varphi) \in H^0(X, \sigma(E)_{S(P)}(W^-_{\chi \circ S^{-1}}) \otimes L)$$

and $\deg(\chi \circ S^{-1})_* \sigma(E)_{S(P)} = 0$. Then, as mentioned earlier, $E_P = \sigma^{-1}(\sigma(E)_{S(P)})$ is a reduction of structure group of E to P such that φ takes values in V^-_χ and

$$\deg \chi_* E_P = \deg(\chi \circ S^{-1})_* \sigma(E)_{S(P)} = 0.$$

Since (E, φ) is polystable, there exists a reduction of structure group E_L of E_P to a Levi subgroup L of P such that φ takes values in the space V^0_χ defined in (1). Then it is easily checked that $\sigma(E_L)$ defines a reduction of structure group of $\sigma(E)_{S(P)}$ to $S(L)$, which is a Levi subgroup of $S(P)$, and $F(\varphi)$ takes values in $W^0_{\chi \circ S^{-1}}$ since it is satisfied that $S(V^0_\chi) = W^0_{\chi \circ S^{-1}}$, due to the hypothesis relation $F \circ \rho = (\rho' \circ S) \circ F$. This proves that $F(E, \varphi)$ is polystable.

All this proves that the correspondence f is well defined. Observe that the isomorphism of vector spaces F^{-1} defines also a correspondence, which is clearly inverse to f, what proves that f is bijective. □

Corollary 4.1. *The automorphism J defined in* (8) *induces bijective correspondences between polystable* $(\mathrm{Spin}(8, \mathbb{C}), \rho)$*-Higgs pairs,* $(\mathrm{Spin}(8, \mathbb{C}), \rho_+)$*-Higgs pairs, and* $(\mathrm{Spin}(8, \mathbb{C}), \rho_-)$*-Higgs pairs over X that preserves the polystability, where the representations ρ, ρ_+, and ρ_- are defined in* (5), (6), *and* (7), *respectively.*

Proof. Let V be the eight-dimensional vector space on which $\mathrm{Spin}(8, \mathbb{C})$ acts through the representation ρ defined in (5), and let q be its nondegenerate quadratic form. Let W be the maximal isotropic subspace of V such that the representations ρ_+ and ρ_- of $\mathrm{Spin}(8, \mathbb{C})$ defined in (6) and (7), respectively, define actions of $\mathrm{Spin}(8, \mathbb{C})$ on $\wedge^+ W$ and $\wedge^- W$. Consider the automorphism

$$J : V \oplus \wedge^+ W \oplus \wedge^- W \to V \oplus \wedge^+ W \oplus \wedge^- W$$

defined in (8). It takes values in $\wedge^+ W$ when restricted to V, so it defines an isomorphism $J : V \to \wedge^+ W$. From the relation expressed in (9), it is satisfied that $J \circ \rho = (\rho_+ \circ T) \circ J$ for a representative T in Aut(Spin(8, $\mathbb{C}$)) of the triality automorphism τ of Spin(8, $\mathbb{C}$). The hypotheses of Proposition 4.4 are satisfied, so the correspondence defined by $j(E, \varphi) = (\tau(E), J(\varphi))$ given in Proposition 4.4 is bijective. □

5 Stability and simplicity of (Spin(8, $\mathbb{C}$), ρ)-Higgs pairs

Let X be a compact Riemann surface of genus $g \geq 2$. Let (E, φ) be a (Spin($2n$, $\mathbb{C}$), ρ_{2n})-Higgs pair over X. Denote by Aut(E, φ) the group of automorphisms of (E, φ), that is,

$$\mathrm{Aut}(E, \varphi) = \{f \in \mathrm{Aut}(E) : f_{\rho_{2n}}(\varphi) = \varphi\}, \tag{22}$$

where $f_{\rho_{2n}}$ denotes the automorphism of $E(V_{2n})$ induced by f. The space of infinitesimal automorphisms of (E, φ) is also defined to be the space

$$\mathrm{aut}(E, \varphi) = \{f \in \mathrm{End}(E) : f_{\rho_{2n}}(\varphi) = 0\}, \tag{23}$$

Let Z(Spin($2n$, $\mathbb{C}$)) be the center of Spin($2n$, $\mathbb{C}$), which satisfies $Z(\mathrm{Spin}(2n, \mathbb{C})) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $z \in Z(\mathrm{Spin}(2n, \mathbb{C}))$ be a choice of a central element. This choice induces the definition of an automorphism $f^z : E \to E$ of E given by multiplication by z. The corresponding automorphism $f^z_{\rho_{2n}}$ of $E(V_{2n})$ is defined by $f^z_{\rho_{2n}}([e, v]) = [ez, \rho_{2n}(z)(v)]$, where $e \in E$ and $v \in V_{2n}$. This is a good definition since, for any $g \in \mathrm{Spin}(2n, \mathbb{C})$,

$$[egz, \rho_{2n}(z)\rho_{2n}(g^{-1})(v)] = [ezg, \rho_{2n}(g^{-1})\rho_{2n}(z)(v)] = [ez, \rho_{2n}(z)(v)].$$

This fact, together with the additional observation that every central element of Spin($2n$, $\mathbb{C}$) has order 2, proves that Z(Spin($2n$, $\mathbb{C}$)) can be understood as a subgroup of Aut(E, φ).

Definition 3. A (Spin($2n$, $\mathbb{C}$), ρ_{2n})-Higgs pair (E, φ) over X is said to be simple if the group Aut(E, φ) coincides with Z(Spin($2n$, $\mathbb{C}$)).

In previous studies [13,16,22], it is proved that stable and simple Higgs bundles represent smooth points of the moduli space of Higgs bundles for any reductive complex structure group. They use arguments that involve deformation theory and that, as far as it has been studied, are not easily adaptable to the situation in which pairs associated with a representation different from the adjoint one are considered.

Proposition 5.1. *Let (E, φ) be a stable* (Spin($2n$, $\mathbb{C}$), ρ_{2n})*-Higgs pair over X. Then every element of* Aut(E, φ) *is semisimple, where* Aut(E, φ) *is defined in* (22).

Proof. Under the conditions of the statement, the identity component Aut$(E, \varphi)_0$ of Aut(E, φ) is semisimple, since every element of its Lie algebra, the space aut(E, φ) of infinitesimal automorphisms of (E, φ), is semisimple as a consequence of [16, Proposition 2.14]. Let $g \in \mathrm{Aut}(E, \varphi)$, and let g_u be the unipotent part of g. Let $p : \mathrm{Aut}(E, \varphi) \to \pi_0(\mathrm{Aut}(E, \varphi))$ be the projection. Since p is a morphism that preserves the unipotent parts and $\pi_0(\mathrm{Aut}(E, \varphi))$ is a finite group, then every element of $\pi_0(\mathrm{Aut}(E, \varphi))$ is also semisimple, so $p(g_u) = 0$ and $g_u \in \mathrm{Aut}(E, \varphi)_0$; hence, g_u is itself 0 because the elements of Aut$(E, \varphi)_0$ are all of them semisimple. □

Lemma 5.1. *Let G be any semisimple complex Lie group, and let $g \in G$. Then the centralizer $Z_G(g)$ of g in G is defined up to conjugation.*

Proof. Let $g \in V$, $x \in Z_G(g)$, and $h \in G$. It is clear that hxh^{-1} commutes with hgh^{-1} (because g and x commute). This proves that the inner automorphism $i_h : G \to G$ defined by $i_h(y) = hyh^{-1}$ restricts to an isomorphism between Z_G and $Z_G(hgh^{-1})$. □

Lemma 5.2. *Let n be any integer number with* $n \geq 2$, $\pi_{2n} : \mathrm{Spin}(2n, \mathbb{C}) \to \mathrm{SO}(2n, \mathbb{C})$ *be the covering map defined in* (12), *and let* $g \in \mathrm{Spin}(2n, \mathbb{C})$. *Then* $\pi_{2n}(Z_{\mathrm{Spin}(2n,\mathbb{C})}(g)) = Z_{\mathrm{SO}(2n,\mathbb{C})}(\pi_{2n}(g))$.

Proof. Since π_{2n} is a homomorphism of groups, it is obvious that

$$\pi_{2n}(Z_{\mathrm{Spin}(2n,\mathbb{C})}(g)) \subseteq Z_{\mathrm{SO}(2n,\mathbb{C})}(\pi_{2n}(g)).$$

To prove the other contention, take any $x \in Z_{\mathrm{SO}(2n,\mathbb{C})}(\pi_{2n}(g))$ and let $h \in \mathrm{Spin}(2n, \mathbb{C})$ be such that $\pi_{2n}(h) = x$. Then $hgh^{-1}g^{-1} \in \ker \pi_{2n}$. It may be supposed that $hgh^{-1}g^{-1} = 1$ (if this is not the case, then the other element in the fiber $\pi_{2n}^{-1}(g)$ should satisfy this relation), so $h \in Z_{\mathrm{Spin}(2n,\mathbb{C})}(g)$, hence $x \in \pi_{2n}(Z_{\mathrm{Spin}(2n,\mathbb{C})})$. □

Lemma 5.3. *Let n be any integer number with* $n \geq 2$, *and let g be any element of* $\mathrm{SO}(2n, \mathbb{C})$. *Then* $Z_{\mathrm{SO}(2n,\mathbb{C})}(g)$ *is isomorphic to one of the following groups*:
(1) $\mathrm{SO}(2n, \mathbb{C})$ (*if and only if g is a central element*).
(2) $\mathrm{SL}(2r_1, \mathbb{C}) \times \cdots \times \mathrm{SL}(2r_d, \mathbb{C}) \times \mathrm{SO}(k_1, \mathbb{C}) \times \mathrm{SO}(k_2, \mathbb{C})$, *where* $d \geq 0$, $k_1, k_2 \geq 0$, *and* $2r_1 + \cdots + 2r_d + k_1 + k_2 = 2n$.

Proof. Let $T = \mathrm{SO}(2, \mathbb{C}) \times \overset{n}{\cdots} \times \mathrm{SO}(2, \mathbb{C})$ be a maximal torus of $\mathrm{SO}(2n, \mathbb{C})$. Every element in $\mathrm{SO}(2n, \mathbb{C})$ can be conjugated into T, since $\mathrm{SO}(2n, \mathbb{C})$ is connected. Then every element in $\mathrm{SO}(2n, \mathbb{C})$ is conjugate to an element of the form

$$M = \begin{pmatrix} M_1 & & & & \\ & \overset{d}{\ddots} & & & \\ & & M_d & & \\ & & & I_{k_1} & \\ & & & & -I_{k_2} \end{pmatrix},$$

where

$$M_i = \begin{pmatrix} A_i & & \\ & \overset{r_i}{\ddots} & \\ & & A_i \end{pmatrix},$$

each A_i being an element in $\mathrm{SO}(2, \mathbb{C})$ different from $\pm I$ and such that $A_i \neq A_j$ for $i \neq j$. An element in $\mathrm{SO}(2n, \mathbb{C})$ that commutes with M should preserve the blocks. Since the centralizer of each M_i is isomorphic to $\mathrm{SL}(2r_i, \mathbb{C})$, the result comes. □

Lemma 5.4. *Let n be any integer number with* $n \geq 2$, *let* (E, φ) *be a polystable* $(\mathrm{Spin}(2n, \mathbb{C}), \rho_{2n})$*-Higgs pair over X, and let* $(E_{\mathrm{SO}}, \varphi)$ *the associated* $(\mathrm{SO}(2n, \mathbb{C}), \rho_{2n}^{\mathrm{SO}})$*-Higgs pair over X, where the representations* ρ_{2n} *and* ρ_{2n}^{SO} *are defined in* (13) *and* (14), *respectively. Then* (E, φ) *is simple if and only if* $(E_{\mathrm{SO}}, \varphi)$ *is simple.*

Proof. Observe first that every automorphism of (E, φ) descends to give an automorphism of $(E_{\mathrm{SO}}, \varphi)$ in such a way that two automorphisms of (E, φ) that descend to the same automorphism of $(E_{\mathrm{SO}}, \varphi)$ differ in one central element of $\mathrm{Spin}(2n, \mathbb{C})$. This proves that if $(E_{\mathrm{SO}}, \varphi)$ is simple then (E, φ) is also simple. Reciprocally, suppose that (E, φ) is simple, and take any automorphism f of $(E_{\mathrm{SO}}, \varphi)$. The automorphism f defines an element of the adjoint bundle $\mathrm{Ad}(E_{\mathrm{SO}})$, which is isomorphic to $\mathrm{Ad}(E)$, so f defines an endomorphism of (E, φ), say $\bar{f}$, such that $\bar{f}$ descends to f. For the same reason, there exists an endomorphism $\overline{f^{-1}}$ of (E, φ) that descends to f^{-1}. The endomorphisms $\bar{f}$ and $\overline{f^{-1}}$ clearly differ in one central element of $\mathrm{Spin}(2n, \mathbb{C})$, so they are isomorphisms. Since (E, φ) is simple, $\bar{f}$ must consist of multiplication by a central element of $\mathrm{Spin}(2n, \mathbb{C})$, so f consists also in multiplication by a central element of $\mathrm{SO}(2n, \mathbb{C})$, what proves that $(E_{\mathrm{SO}}, \varphi)$ is itself simple. □

Theorem 5.1. *Let* (E, φ) *be a stable and non-simple* $(\mathrm{Spin}(8, \mathbb{C}), \rho)$*-Higgs pair over X, where the representation* $\rho : \mathrm{Spin}(8, \mathbb{C}) \to \mathrm{GL}(8, \mathbb{C})$ *is defined in* (5). *Let* E_{SO} *be the principal* $\mathrm{SO}(8, \mathbb{C})$*-bundle over X defined in* (17) *associated with E. Then the underlying vector bundle of* E_{SO} *is isomorphic to one of the following vector bundles*:

(1) $L_k \oplus L_{8-k}$ *for* $k = 0, 1, 2, 3, 4$;
(2) $F_{2r} \oplus L_k \oplus L_{8-2r-k}$ *for* $r = 1, 2, 3$ *and* $k = 0, 1, \dots, 4 - r$;
(3) $F_{2r} \oplus F_{2s} \oplus L_k \oplus L_{8-2r-2s-k}$ *for* $r = 1, 2, 3$, $s = r, \dots, 4 - r$, *and* $k = 0, \dots, 4 - r - s$;
(4) $F_2 \oplus F_2 \oplus F_{2r} \oplus L_k \oplus L_{4-2r-k}$ *for* $r = 1, 2$ *and* $k = 2 - r, \dots, 4 - 2r$,

where F_j *is an* $\mathrm{SL}(j, \mathbb{C})$*-bundle and* L_j *is an* $\mathrm{SO}(j, \mathbb{C})$*-bundle for all* $j \geq 1$, $F_0 = 0$, $L_0 = 0$, *and* $L_1 = O$.

Remark. All the Higgs pairs described in the four cases stated in Theorem 5.1 are polystable, as a consequence of Corollary 2.1.

Proof of Theorem 5.1. Let (E, φ) be a stable and non-simple $(\mathrm{Spin}(8, \mathbb{C}), \rho)$-Higgs pair over X. Then there exists an automorphism f of (E, φ) that does not belong to the center Z of $\mathrm{Spin}(8, \mathbb{C})$. Since the group $\mathrm{Spin}(8, \mathbb{C})$ is semisimple and (E, φ) is stable, it is ensured [16, Proposition 2.14] that the space $\mathrm{aut}(E, \varphi)$ defined in (23) is 0. This space is the Lie algebra of the group $\mathrm{Aut}(E, \varphi)$ at the identity, so $\mathrm{Aut}(E, \varphi)_0 = Z_0 = \{1\}$ and, moreover, since $\pi_0(\mathrm{Aut}(E, \varphi))$ is finite because it is an algebraic group, and Z is a normal subgroup of $\mathrm{Aut}(E, \varphi)$, the quotient $\mathrm{Aut}(E, \varphi)/Z$ is a finite group. Let $f_1, \dots, f_k$ be a family of automorphisms of (E, φ) not coming from the center of $\mathrm{Spin}(8, \mathbb{C})$ such that the nontrivial elements of $\mathrm{Aut}(E, \varphi)/Z$ are exactly the set of their classes modulo Z, $\{[f_1], \dots, [f_k]\}$. Each f_i corresponds to an element $g_i \in \mathrm{Spin}(8, \mathbb{C})$. In the study by Garcia-Prada and Oliveira [23, Theorem 3.17], it is shown that, in this situation, E admits a reduction of structure group to the centralizer $Z_{\mathrm{Spin}(8,\mathbb{C})}(g_i)$ of g_i for every $i = 1, \dots, k$. Of course, φ takes values in that reduction, since f_i is an automorphism of (E, φ). Notice that the choice of representatives is well defined except for one element of the center Z, but this does not change the centralizers that are being considered. Let (E^{g_1}, φ^{g_1}) be the reduction of structure group of $(E_{\mathrm{SO}}, \varphi)$ to $Z_{\mathrm{Spin}(8,\mathbb{C})}(\pi(g_1))$, which exists by Lemma 5.2. Then, since g_1 does not belong to Z, from Lemma 5.3 applied to $n = 4$, we deduce the result. □

Remark. The same proof made in Theorem 5.1 works to give a similar description of stable and non-simple $(\mathrm{SO}(8, \mathbb{C}), \iota)$-Higgs pairs, where ι is the representation of $\mathrm{SO}(8, \mathbb{C})$ induced by the natural inclusion of groups $\mathrm{SO}(8, \mathbb{C}) \to \mathrm{GL}(8, \mathbb{C})$. Specifically, if (E, φ) is a stable and non-simple $(\mathrm{SO}(8, \mathbb{C}), \iota)$-Higgs pair over X, then it admits exactly one of the three forms described in Theorem 5.1.

Example. Take a rank 8 and trivial determinant vector bundle E over X, which admits a globally defined nondegenerate symmetric bilinear form, and whose second Stiefel-Whitney class is 0. Then this vector bundle can be understood as a principal $\mathrm{SO}(8, \mathbb{C})$-bundle over X. The bundle E lifts to a principal $\mathrm{Spin}(8, \mathbb{C})$-bundle over X. Suppose that this principal bundle is stable and not simple. Suppose, in addition, that E admits a nonzero holomorphic global section, whose induced line subbundle of E is not isotropic. The pair consisting of E together with this global section is a stable Higgs pair for the representation ρ (5), where the fixed line bundle L considered is the trivial line bundle O over X. In this situation, E satisfies the conditions of Theorem 5.1. Then it is deduced that E admits a decomposition into 2, 3, 4, or 5 vector subbundles. Moreover, if there are more than 2 subbundles, all but perhaps two of them must be of even rank.

6 Conclusion

The group $\mathrm{Spin}(8, \mathbb{C})$ is the only simple complex Lie group that admits an outer automorphism of order 3, called triality automorphism. It also admits three non-isomorphic irreducible eight-dimensional complex representations, so that the triality automorphism acts as an order 3 permutation on the set of these representations. One of them is the representation ρ induced by the double covering $\mathrm{Spin}(8, \mathbb{C}) \to \mathrm{SO}(8, \mathbb{C})$ with which $\mathrm{Spin}(8, \mathbb{C})$ is equipped. If $\alpha : \mathrm{Spin}(8, \mathbb{C}) \to \mathrm{GL}(V)$ is a complex representation of $\mathrm{Spin}(8, \mathbb{C})$ and E is a principal $\mathrm{Spin}(8, \mathbb{C})$-bundle over a compact Riemann surface X, then a complex rank 8 vector bundle $E(V)$ is induced by E and α. A $(\mathrm{Spin}(8, \mathbb{C}), \alpha)$-Higgs pair over X is a pair (E, φ), where E is a principal $\mathrm{Spin}(8, \mathbb{C})$-bundle over X and $\varphi \in H^0(X, E(V) \otimes L)$, L being a fixed line bundle over X. In this work, reduced notions of stability

and polystability for Higgs pairs over X with structure group $\mathrm{Spin}(8,\mathbb{C})$ and associated with the representations cited above are given, and it is proved that the three moduli spaces of Higgs pairs considered are isomorphic. It is also given an explicit expression of the vector bundles associated with the stable and not simple $(\mathrm{Spin}(8,\mathbb{C}),\rho)$-Higgs pairs over X through the representation ρ of $\mathrm{Spin}(8,\mathbb{C})$. Specifically, it has been shown that, if (E,φ) is a stable and non-simple $(\mathrm{Spin}(8,\mathbb{C}),\rho)$-Higgs pair over X, then the vector bundle induced by E and ρ is isomorphic to one of the following:
(1) $L_k \oplus L_{8-k}$ for $k = 0, 1, 2, 3, 4$;
(2) $F_{2r} \oplus L_k \oplus L_{8-2r-k}$ for $r = 1, 2, 3$ and $k = 0, 1, \ldots, 4-r$;
(3) $F_{2r} \oplus F_{2s} \oplus L_k \oplus L_{8-2r-2s-k}$ for $r = 1, 2, 3$, $s = r, \ldots, 4-r$ and $k = 0, \ldots, 4-r-s$;
(4) $F_2 \oplus F_2 \oplus F_{2r} \oplus L_k \oplus L_{4-2r-k}$ for $r = 1, 2$ and $k = 2-r, \ldots, 4-2r$,

where F_j is an $\mathrm{SL}(j,\mathbb{C})$-bundle and L_j is an $\mathrm{SO}(j,\mathbb{C})$-bundle for all $j \geq 1$, $F_0 = 0$, $L_0 = 0$, and $L_1 = \mathcal{O}$.

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