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Abstract

A cross-shaped matrix $X \in \mathbb{C}^{n\times n}$ has nonzero elements located on the main diagonal and the anti-diagonal, so that the sparsity pattern has the shape of a cross. It is shown that X can be factorized into products of identity-plus-rank-two matrices and can be symmetrically permuted to block diagonal form with 2×2 diagonal blocks and, if n is odd, a 1×1 diagonal block. Exploiting these properties we derive explicit formulae for its determinant, inverse, and characteristic polynomial.

1 Introduction

In this note we study some properties of a class of cross-shaped matrices, which can be viewed as a generalization of a type of compound Jacobi rotation matrices used in the parallel Jacobi algorithm for symmetric eigenvalue problems [2, sect. 5.5], [3]. Any matrix $X \in \mathbb{C}^{n\times n}$ in the class has the form

$$X = \begin{bmatrix} x_{11} & & & & & & x_{1,2k+1} \\ & x_{22} & & & & x_{2,2k} & \\ & & \ddots & & \iddots & & \\ & & & x_{k+1,k+1} & & & \\ & & \iddots & & \ddots & & \\ & x_{2k,2} & & & & x_{2k,2k} & \\ x_{2k+1,1} & & & & & & x_{2k+1,2k+1} \end{bmatrix}, \tag{1.1}$$

for $n = 2k + 1$, $k \in \mathbb{N}^+$, and, if $n = 2k$,

$$X = \begin{bmatrix} x_{11} & & & & & & & x_{1,2k} \\ & x_{22} & & & & & x_{2,2k-1} & \\ & & \ddots & & & \iddots & & \\ & & & x_{kk} & x_{k,k+1} & & & \\ & & & x_{k+1,k} & x_{k+1,k+1} & & & \\ & & \iddots & & & \ddots & & \\ & x_{2k-1,2} & & & & & x_{2k-1,2k-1} & \\ x_{2k,1} & & & & & & & x_{2k,2k} \end{bmatrix}. \tag{1.2}$$

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To the best of our knowledge, this class of matrices were first exploited in [1, sect. 5.8] for studying eigenvalues of real symmetric matrices. Any real symmetric matrix A can be transformed to a symmetric cross-shaped matrix through orthogonal similarity transformations that preserve the eigenvalues of A, which then can be calculated exploiting the cross-shaped form by solving uncoupled quadratic characteristic equations.

2 Factorization into products of identity-plus-rank-two matrices

Let us first consider the case where X in (1.1) has order $n = 2k+1$. The matrix can be factorized as

$$X = \begin{bmatrix} x_{11} & & x_{1,2k+1} \\ & I_{2k-1} & \\ x_{2k+1,1} & & x_{2k+1,2k+1} \end{bmatrix} \begin{bmatrix} 1 & & 0 \\ & X(2\colon 2k, 2\colon 2k) & \\ 0 & & 1 \end{bmatrix}, \qquad (2.1)$$

where I_{2k-1} denotes the identity matrix of order $2k-1$, and the submatrix $X(2\colon 2k, 2\colon 2k)$ of order $2k-1$ has the same cross-shape form so can be factorized in the same way as

$$X(2\colon 2k, 2\colon 2k) = \begin{bmatrix} x_{22} & & x_{2,2k} \\ & I_{2k-3} & \\ x_{2k,2} & & x_{2k,2k} \end{bmatrix} \begin{bmatrix} 1 & & 0 \\ & X(3\colon 2k-1, 3\colon 2k-1) & \\ 0 & & 1 \end{bmatrix},$$

where the submatrix $X(3\colon 2k-1, 3\colon 2k-1)$ of order $2k-3$ has the same cross-shape form. This process can be continued k times, until we obtain in the $(2,2)$ block of the right-hand factor the matrix

$$\begin{aligned} \begin{bmatrix} x_{kk} & & x_{k,k+2} \\ & x_{k+1,k+1} & \\ x_{k+2,k} & & x_{k+2,k+2} \end{bmatrix} &\equiv X(k\colon k+2, k\colon k+2) \\ &= \begin{bmatrix} x_{kk} & & x_{k,k+2} \\ & 1 & \\ x_{k+2,k} & & x_{k+2,k+2} \end{bmatrix} \begin{bmatrix} 1 & & 0 \\ & x_{k+1,k+1} & \\ 0 & & 1 \end{bmatrix}. \end{aligned}$$

On completion of the process, we obtain a factorization

$$X = \begin{bmatrix} x_{11} & & x_{1,2k+1} \\ & I_{2k-1} & \\ x_{2k+1,1} & & x_{2k+1,2k+1} \end{bmatrix} \begin{bmatrix} 1 & & & & 0 \\ & x_{22} & & x_{2,2k} & \\ & & I_{2k-3} & & \\ & x_{2k,2} & & x_{2k,2k} & \\ 0 & & & & 1 \end{bmatrix} \cdots$$

$$\begin{bmatrix} I_{k-1} & & & & 0 \\ & x_{kk} & & x_{k,k+2} & \\ & & 1 & & \\ & x_{k+2,k} & & x_{k+2,k+2} & \\ 0 & & & & I_{k-1} \end{bmatrix} \begin{bmatrix} I_k & & \\ & x_{k+1,k+1} & \\ & & I_k \end{bmatrix}, \tag{2.2}$$

and each of the first k factors is a rank-2 perturbation of the identity matrix and the last one is a rank-1 perturbation of the identity matrix.

For the case where X in (1.2) has order $n = 2k$, it is easy to show by using the technique above that the factorization has a similar form to (2.2) except that the last factor is a rank-2 perturbation of the identity matrix:

$$X = \begin{bmatrix} x_{11} & & x_{1,2k} \\ & I_{2k-2} & \\ x_{2k,1} & & x_{2k,2k} \end{bmatrix} \begin{bmatrix} 1 & & & & 0 \\ & x_{22} & & x_{2,2k-1} & \\ & & I_{2k-4} & & \\ & x_{2k-1,2} & & x_{2k-1,2k-1} & \\ 0 & & & & 1 \end{bmatrix} \cdots$$

$$\begin{bmatrix} I_{k-2} & & & & 0 \\ & x_{k-1,k-1} & & x_{k-1,k+2} & \\ & & I_2 & & \\ & x_{k+2,k-1} & & x_{k+2,k+2} & \\ 0 & & & & I_{k-2} \end{bmatrix} \begin{bmatrix} I_{k-1} & & & \\ & x_{kk} & x_{k,k+1} & \\ & x_{k+1,k} & x_{k+1,k+1} & \\ & & & I_{k-1} \end{bmatrix}, \tag{2.3}$$

where the ith factor is the identity matrix with the intersection of its ith and $(2k+1-i)$st rows and columns replaced by that of X.

The factorizations (2.2) and (2.3) can be exploited to derive many important properties of the cross-shaped matrices, as we will show in the next sections. Before that we now derive a formula for the determinant and inverse of the matrix factors in these factorizations.

Consider first the $2k \times 2k$ matrix

$$Y_i := \begin{bmatrix} I_{i-1} & & & & 0 \\ & x_{ii} & & x_{i,2k+1-i} & \\ & & I_{2k-2i} & & \\ & x_{2k+1-i,i} & & x_{2k+1-i,2k+1-i} & \\ 0 & & & & I_{i-1} \end{bmatrix}, \tag{2.4}$$

which is the identity matrix I_{2k} with the intersection of its ith and $(2k+1-i)$st rows and columns replaced by that of X. We can form

$$P_iY_iP_i = \begin{bmatrix} I_{i-1} & & & & 0 \\ & x_{ii} & x_{i,2k+1-i} & & \\ & x_{2k+1-i,i} & x_{2k+1-i,2k+1-i} & & \\ & & & I_{2k-2i} & \\ 0 & & & & I_{i-1} \end{bmatrix} =: \widetilde{Y}_i, \qquad (2.5)$$

where P_i is the *elementary* permutation matrix (which is symmetric) formed by swapping the $(i+1)$st and $(2k+1-i)$st columns of the identity matrix I_{2k}, and $\widetilde{Y}_i$ is in block diagonal form so its determinant and inverse can be easily computed.

It follows from (2.5) that

$$\begin{aligned} \det(Y_i) &= \det(P_i)\det(\widetilde{Y}_i)\det(P_i) = \det(\widetilde{Y}_i) \\ &= \det\left(\begin{bmatrix} x_{ii} & x_{i,2k+1-i} \\ x_{2k+1-i,i} & x_{2k+1-i,2k+1-i} \end{bmatrix}\right) \\ &= x_{ii}x_{2k+1-i,2k+1-i} - x_{i,2k+1-i}x_{2k+1-i,i} =: \alpha_i \qquad (2.6) \end{aligned}$$

and, if $\alpha_i \neq 0$,

$$\begin{aligned} Y_i^{-1} = P_i\widetilde{Y}_i^{-1}P_i &= P_i \begin{bmatrix} I_{i-1} & & & & 0 \\ & x_{2k+1-i,2k+1-i}/\alpha_i & -x_{i,2k+1-i}/\alpha_i & & \\ & -x_{2k+1-i,i}/\alpha_i & x_{ii}/\alpha_i & & \\ & & & I_{2k-2i} & \\ 0 & & & & I_{i-1} \end{bmatrix} P_i \\ &= \begin{bmatrix} I_{i-1} & & & & 0 \\ & x_{2k+1-i,2k+1-i}/\alpha_i & & -x_{i,2k+1-i}/\alpha_i & \\ & & I_{2k-2i} & & \\ & -x_{2k+1-i,i}/\alpha_i & & x_{ii}/\alpha_i & \\ 0 & & & & I_{i-1} \end{bmatrix}, \qquad (2.7) \end{aligned}$$

which shows that the computation of the determinant and inverse of Y_i only involves that of a 2×2 principal submatrix $Y_i([i, 2k+1-i], [i, 2k+1-i])$.

Note that the determinant and inverse of the first k factors of size $(2k+1) \times (2k+1)$ in (2.2) are easily seen to have the same form as (2.6) and (2.7) (albeit with slightly different indices) since these factors have exactly the same form as Y_i in (2.4). Specifically, for the $(2k+1) \times (2k+1)$ matrix

$$Z_i := \begin{bmatrix} I_{i-1} & & & & 0 \\ & x_{ii} & & x_{i,2k+2-i} & \\ & & I_{2k+1-2i} & & \\ & x_{2k+2-i,i} & & x_{2k+2-i,2k+2-i} & \\ 0 & & & & I_{i-1} \end{bmatrix} \qquad (2.8)$$

the determinant and inverse are given by

$$\det(Z_i) = \det\left(\begin{bmatrix} x_{ii} & x_{i,2k+2-i} \\ x_{2k+2-i,i} & x_{2k+2-i,2k+2-i} \end{bmatrix}\right)$$
$$= x_{ii}x_{2k+2-i,2k+2-i} - x_{i,2k+2-i}x_{2k+2-i,i} =: \beta_i \tag{2.9}$$

and, if $\beta_i \neq 0$,

$$Z_i^{-1} = \begin{bmatrix} I_{i-1} & & & & 0 \\ & x_{2k+2-i,2k+2-i}/\beta_i & & -x_{i,2k+2-i}/\beta_i & \\ & & I_{2k+1-2i} & & \\ & -x_{2k+2-i,i}/\beta_i & & x_{ii}/\beta_i & \\ 0 & & & & I_{i-1} \end{bmatrix}. \tag{2.10}$$

3 Permutation into block diagonal form

Indeed, more can be said about the cross-shaped matrices X: one can always permute X into block diagonal form PXP^T with 2×2 diagonal blocks and a 1×1 diagonal block if n is odd, where P is a permutation matrix. To illustrate this, we again start from the definitions (1.1) and (1.2) of X. Consider first the case $n = 2k$. We have, using the elementary permutation matrices P_i defined in (2.5),

$$P_1XP_1 = \begin{bmatrix} x_{11} & x_{1,2k} & & & \\ x_{2k,1} & x_{2k,2k} & & & \\ & & X(3:2k-2, 3:2k-2) & & \\ & & & x_{2k-1,2k-1} & x_{2k-1,2} \\ & & & x_{2,2k-1} & x_{22} \end{bmatrix}$$
$$\equiv \begin{bmatrix} B_1 & & \\ & X(3\colon 2k-2, 3\colon 2k-2) & \\ & & C_1 \end{bmatrix},$$

where we have defined the 2×2 matrices

$$B_i := \begin{bmatrix} x_{2i-1,2i-1} & x_{2i-1,2k+2-2i} \\ x_{2k+2-2i,2i-1} & x_{2k+2-2i,2k+2-2i} \end{bmatrix}, \quad C_i := \begin{bmatrix} x_{2k+1-2i,2k+1-2i} & x_{2k+1-2i,2i} \\ x_{2i,2k+1-2i} & x_{2i,2i} \end{bmatrix},$$

and then

$$P_3P_1XP_1P_3 = \begin{bmatrix} B_1 & & & & \\ & B_2 & & & \\ & & X(5\colon 2k-4, 5\colon 2k-4) & & \\ & & & C_2 & \\ & & & & C_1 \end{bmatrix}.$$

Then we apply a similarity transformation with P_5 to the resulting matrix to exchange the second and last columns and rows of the submatrix $X(5\colon 2k-4, 5\colon 2k-4)$. This process can be continued $\lfloor n/4 \rfloor$ times until we obtain (for either $n = 4m$ or $n = 4m+2$, $m \in \mathbb{N}^+$, we need m two-side permutations applied to X in total)

$$PXP^T = \begin{cases} \operatorname{diag}(B_1, \dots, B_m, C_m, \dots, C_1), & n = 4m, \\ \operatorname{diag}(B_1, \dots, B_m, X(k\colon k+1, k\colon k+1), C_m, \dots, C_1), & n = 4m+2, \end{cases}$$

where $P := P_{2m-1} \cdots P_3 P_1$ (note that $P^T = P_1^T P_3^T \cdots P_{2m-1}^T = P_1 P_3 \cdots P_{2m-1}$ since $P_i = P_i^T$) is a permutation matrix.

The case for $n = 2k+1$ is rather similar but can be slightly different in the final step. Starting with the X from (1.1), we have

$$Q_1 X Q_1 = \begin{bmatrix} x_{11} & x_{1,2k+1} & & & \\ x_{2k+1,1} & x_{2k+1,2k+1} & & & \\ & & X(3:2k-1, 3:2k-1) & & \\ & & & x_{2k,2k} & x_{2k,2} \\ & & & x_{2,2k} & x_{22} \end{bmatrix}$$
$$\equiv \begin{bmatrix} E_1 & & \\ & X(3\colon 2k-2, 3\colon 2k-2) & \\ & & G_1 \end{bmatrix},$$

where Q_i is the elementary permutation matrix (which is symmetric) formed by swapping the $(i+1)$st and $(2k+2-i)$st columns of the identity matrix I_{2k+1} and we have defined the 2×2 matrices

$$E_i := \begin{bmatrix} x_{2i-1,2i-1} & x_{2i-1,2k+3-2i} \\ x_{2k+3-2i,2i-1} & x_{2k+3-2i,2k+3-2i} \end{bmatrix}, \quad G_i := \begin{bmatrix} x_{2k+2-2i,2k+2-2i} & x_{2k+2-2i,2i} \\ x_{2i,2k+2-2i} & x_{2i,2i} \end{bmatrix},$$

and then

$$Q_3 Q_1 X Q_1 Q_3 = \begin{bmatrix} E_1 & & & & \\ & E_2 & & & \\ & & X(5\colon 2k-3, 5\colon 2k-3) & & \\ & & & G_2 & \\ & & & & G_1 \end{bmatrix}.$$

Then we apply a two-side permutation with Q_5 to the resulting matrix to exchange the second and last columns and rows of the submatrix $X(5\colon 2k-3, 5\colon 2k-3)$. We can continue this process $\lfloor n/4 \rfloor$ times to arrive at

$$QXQ^T = \begin{cases} \operatorname{diag}(E_1, \dots, E_m, x_{k+1,k+1}, G_m, \dots, G_1), & n = 4m+1, \\ \operatorname{diag}(E_1, \dots, E_m, X(k\colon k+2, k\colon k+2), G_m, \dots, G_1), & n = 4m+3, \end{cases}$$

where $Q := Q_{2m-1} \cdots Q_3 Q_1$. So for the case of $n = 4m+1$ the right-hand side is already in the desired form, while for the other case one more transformation is needed for the central 3×3 block $X(k\colon k+2, k\colon k+2)$:

$$\widetilde{Q} X \widetilde{Q}^T = \operatorname{diag}(E_1, \dots, E_m, E_{m+1}, x_{k+1,k+1}, G_m, \dots, G_1),$$

where $\widetilde{Q} := Q_{2m+1} Q$. To summarize, for $n = 4m+1$ we need m two-side permutations and for $n = 4m+3$ the number of permutations required is $m+1$, and we have

$$\begin{aligned} QXQ^T &= \operatorname{diag}(E_1, \dots, E_m, x_{k+1,k+1}, G_m, \dots, G_1), && n = 4m+1, \\ \widetilde{Q} X \widetilde{Q}^T &= \operatorname{diag}(E_1, \dots, E_m, E_{m+1}, x_{k+1,k+1}, G_m, \dots, G_1), && n = 4m+3, \end{aligned}$$

where $Q = Q_{2m-1} \cdots Q_3 Q_1$ and $\widetilde{Q} = Q_{2m+1} Q$ are permutation matrices.

We have shown the cross-shaped matrix X is unitarily similar to a block diagonal matrix with 2×2 blocks and a 1×1 block if n is odd. This actually provides a more straightforward proof to the formulas of the determinant and eigenvalues of X. It is not hard to see from the discussion above that any powers of X has the same shape (including negative powers if X is nonsingular), as we are essentially powering the symmetrically permuted block diagonal form and then recovering the permutation. Furthermore, with a similar argument we can conclude that any matrix functions, for example, the matrix exponential and the matrix logarithm, of X has the same cross-shaped form.

4 The determinant

From the factorizations (2.2), (2.3), using the explicit formulae (2.6), (2.9) for each of the factors, it is straightforward to obtain the formula

$$\det(X) = \begin{cases} \prod_{i=1}^{k} (x_{ii} x_{2k+1-i,2k+1-i} - x_{i,2k+1-i} x_{2k+1-i,i}) & n = 2k, \\ x_{k+1,k+1} \prod_{i=1}^{k} (x_{ii} x_{2k+2-i,2k+2-i} - x_{i,2k+2-i} x_{2k+2-i,i}), & n = 2k+1. \end{cases} \tag{4.1}$$

5 The inverse

We start by showing an important proerty of the factorizations of X discussed in Section 2 which we will need later in this section. Again, let us consider the case where X has order $n = 2k+1$ to illustrate. At the start of the factorization process (2.1), we could alternatively form

$$X = \begin{bmatrix} 1 & & 0 \\ & X(2\colon 2k, 2\colon 2k) & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} x_{11} & & x_{1,2k+1} \\ & I_{2k-1} & \\ x_{2k+1,1} & & x_{2k+1,2k+1} \end{bmatrix},$$

(In fact, it is not hard to see that the two factor matrices on the right-hand side commute) and, similarly, for the submatrix $X(2\colon 2k, 2\colon 2k)$ we can have

$$X(2\colon 2k, 2\colon 2k) = \begin{bmatrix} 1 & & 0 \\ & X(3\colon 2k-1, 3\colon 2k-1) & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} x_{22} & & x_{2,2k} \\ & I_{2k-3} & \\ x_{2k,2} & & x_{2k,2k} \end{bmatrix}.$$

This process can also be continued for k times, until we obtain

$$\begin{aligned} \begin{bmatrix} x_{kk} & & x_{k,k+2} \\ & x_{k+1,k+1} & \\ x_{k+2,k} & & x_{k+2,k+2} \end{bmatrix} &\equiv X(k\colon k+2, k\colon k+2) \\ &= \begin{bmatrix} 1 & & 0 \\ & x_{k+1,k+1} & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} x_{kk} & & x_{k,k+2} \\ & 1 & \\ x_{k+2,k} & & x_{k+2,k+2} \end{bmatrix}. \end{aligned}$$

On completion of this process we obtain a factorization

$$\begin{aligned} X = &\begin{bmatrix} I_k & & \\ & x_{k+1,k+1} & \\ & & I_k \end{bmatrix} \begin{bmatrix} I_{k-1} & & & & 0 \\ & x_{kk} & & x_{k,k+2} & \\ & & 1 & & \\ & x_{k+2,k} & & x_{k+2,k+2} & \\ 0 & & & & I_{k-1} \end{bmatrix} \cdots \\ &\begin{bmatrix} 1 & & & & 0 \\ & x_{22} & & x_{2,2k} & \\ & & I_{2k-3} & & \\ & x_{2k,2} & & x_{2k,2k} & \\ 0 & & & & 1 \end{bmatrix} \begin{bmatrix} x_{11} & & x_{1,2k+1} \\ & I_{2k-1} & \\ x_{2k+1,1} & & x_{2k+1,2k+1} \end{bmatrix}, \end{aligned} \tag{5.1}$$

where the factors are in the reverse order of that in (2.2). Similarly, for the case where X has order $n = 2k$, the alternative form of the factorization is

$$\begin{aligned} X = &\begin{bmatrix} I_{k-1} & & & \\ & x_{kk} & x_{k,k+1} & \\ & x_{k+1,k} & x_{k+1,k+1} & \\ & & & I_{k-1} \end{bmatrix} \begin{bmatrix} I_{k-2} & & & & 0 \\ & x_{k-1,k-1} & & x_{k-1,k+2} & \\ & & I_2 & & \\ & x_{k+2,k-1} & & x_{k+2,k+2} & \\ 0 & & & & I_{k-2} \end{bmatrix} \cdots \\ &\begin{bmatrix} 1 & & & & 0 \\ & x_{22} & & x_{2,2k-1} & \\ & & I_{2k-4} & & \\ & x_{2k-1,2} & & x_{2k-1,2k-1} & \\ 0 & & & & 1 \end{bmatrix} \begin{bmatrix} x_{11} & & x_{1,2k} \\ & I_{2k-2} & \\ x_{2k,1} & & x_{2k,2k} \end{bmatrix}. \end{aligned} \tag{5.2}$$

Now assume that X is nonsingular. For $n = 2k+1$, we invert both sides of (2.2), using the explicit inversion formula (2.10) for Z_i, to get

$$
\begin{aligned}
X^{-1} &= \begin{bmatrix} I_k & & \\ & x_{k+1,k+1} & \\ & & I_k \end{bmatrix}^{-1} Z_k^{-1} Z_{k-1}^{-1} \cdots Z_2^{-1} Z_1^{-1} \\
&= \begin{bmatrix} I_k & & \\ & 1/x_{k+1,k+1} & \\ & & I_k \end{bmatrix}
\begin{bmatrix} I_{k-1} & & & & 0 \\ & x_{k+2,k+2}/\beta_k & & -x_{k,k+2}/\beta_k & \\ & & 1 & & \\ & -x_{k+2,k}/\beta_k & & x_{kk}/\beta_k & \\ 0 & & & & I_{k-1} \end{bmatrix} \cdots \\
&\quad \begin{bmatrix} 1 & & & & 0 \\ & x_{2k,2k}/\beta_2 & & -x_{2,2k}/\beta_2 & \\ & & I_{2k-3} & & \\ & -x_{2k,2}/\beta_2 & & x_{22}/\beta_2 & \\ 0 & & & & 1 \end{bmatrix}
\begin{bmatrix} x_{2k+1,2k+1}/\beta_1 & & -x_{1,2k+1}/\beta_1 \\ & I_{2k-1} & \\ -x_{2k+1,1}/\beta_1 & & x_{11}/\beta_1 \end{bmatrix},
\end{aligned}
$$

whose right-hand side has exactly the same form as that of the factorization (5.1), and so, by comparing them, we have

$$
X^{-1} = \begin{bmatrix}
x_{2k+1,2k+1}/\beta_1 & & & & & & -x_{1,2k+1}/\beta_1 \\
& x_{2k,2k}/\beta_2 & & & & -x_{2,2k}/\beta_2 & \\
& & \ddots & & \iddots & & \\
& & & 1/x_{k+1,k+1} & & & \\
& & \iddots & & \ddots & & \\
& -x_{2k,2}/\beta_2 & & & & x_{22}/\beta_2 & \\
-x_{2k+1,1}/\beta_1 & & & & & & x_{11}/\beta_1
\end{bmatrix},
$$

where the β_i, $i = 1\colon k$, are defined in (2.9).

Similarly, for $n = 2k$, we invert both sides of (2.3), using the explicit inversion formula (2.7) for Y_i, to obtain

$$
\begin{aligned}
X^{-1} &= Y_k^{-1} Y_{k-1}^{-1} \cdots Y_2^{-1} Y_1^{-1} \\
&= \begin{bmatrix} I_{k-1} & & & \\ & \frac{x_{k+1,k+1}}{\alpha_k} & \frac{-x_{k,k+1}}{\alpha_k} & \\ & -\frac{x_{k+1,k}}{\alpha_k} & \frac{x_{kk}}{\alpha_k} & \\ & & & I_{k-1} \end{bmatrix}
\begin{bmatrix} I_{k-2} & & & & 0 \\ & \frac{x_{k+2,k+2}}{\alpha_{k-1}} & & -\frac{x_{k-1,k+2}}{\alpha_{k-1}} & \\ & & I_2 & & \\ & -\frac{x_{k+2,k-1}}{\alpha_{k-1}} & & \frac{x_{k-1,k-1}}{\alpha_{k-1}} & \\ 0 & & & & I_{k-2} \end{bmatrix} \cdots \\
&\quad \begin{bmatrix} 1 & & & & 0 \\ & x_{2k-1,2k-1}/\alpha_2 & & -x_{2,2k-1}/\alpha_2 & \\ & & I_{2k-4} & & \\ & -x_{2k-1,2}/\alpha_2 & & x_{22}/\alpha_2 & \\ 0 & & & & 1 \end{bmatrix}
\begin{bmatrix} x_{2k,2k}/\alpha_1 & & -x_{1,2k}/\alpha_1 \\ & I_{2k-2} & \\ -x_{2k,1}/\alpha_1 & & x_{11}/\alpha_1 \end{bmatrix},
\end{aligned}
$$

whose right-hand side has exactly the same form as that of the factorization (5.2), and so, by comparing them, we have

$$X^{-1} = \begin{bmatrix} \frac{x_{2k,2k}}{\alpha_1} & & & & & & -\frac{x_{1,2k}}{\alpha_1} \\ & \frac{x_{2k-1,2k-1}}{\alpha_2} & & & & -\frac{x_{2,2k-1}}{\alpha_2} & \\ & & \ddots & & \iddots & & \\ & & & \frac{x_{k+1,k+1}}{\alpha_k} \quad -\frac{x_{k,k+1}}{\alpha_k} & & & \\ & & & -\frac{x_{k+1,k}}{\alpha_k} \quad \frac{x_{kk}}{\alpha_k} & & & \\ & & \iddots & & \ddots & & \\ & -\frac{x_{2k-1,2}}{\alpha_2} & & & & \frac{x_{22}}{\alpha_2} & \\ -\frac{x_{2k,1}}{\alpha_1} & & & & & & \frac{x_{11}}{\alpha_1} \end{bmatrix},$$

where the $\alpha_i, i = 1\colon k$ are defined as in (2.6).

An alternative approach to get the inverse is via the adjugate of X, which is defined by $\mathrm{adj}(X) = \big((-1)^{1+j}\det(X_{ji})\big)$, where X_{ji} denotes the submatrix of X obtained by deleting row j and column i. Since all X_{ji} are in the cross-shaped form, the right-hand side of the formula $X^{-1} = \mathrm{adj}(X)/\det(X)$ essentially only involves the determinant of cross-shaped matrices, for which we have already obtained explicit formulas.

6 Eigenvalues

Let us first consider the case where $n = 2k$. To study the eigenvalues of X we consider its characteristic polynomial, which, by exploiting the factorization (2.3), is the deteninant of

$$X - \lambda I = \begin{bmatrix} x_{11} - \lambda & & x_{1,2k} \\ & I_{2k-2} & \\ x_{2k,1} & & x_{2k,2k} - \lambda \end{bmatrix} \begin{bmatrix} 1 & & & & 0 \\ & x_{22} - \lambda & & x_{2,2k-1} & \\ & & I_{2k-4} & & \\ & x_{2k-1,2} & & x_{2k-1,2k-1} - \lambda & \\ 0 & & & & 1 \end{bmatrix}$$

$$\cdots \begin{bmatrix} I_{k-1} & & & \\ & x_{kk} - \lambda & x_{k,k+1} & \\ & x_{k+1,k} & x_{k+1,k+1} - \lambda & \\ & & & I_{k-1} \end{bmatrix},$$

and each of the factors on the right-hand side is in the form of (2.4). So using the determinantal formula (2.6), we obtain

$$\det(X - \lambda I) = \prod_{i=1}^{k} \big((x_{ii} - \lambda)(x_{2k+1-i,2k+1-i} - \lambda) - x_{i,2k+1-i}x_{2k+1-i,i}\big).$$

Similarly, for $n = 2k + 1$, using (2.2), (2.8), and (2.9) we have

$$\det(X - \lambda I) = (x_{k+1,k+1} - \lambda) \prod_{i=1}^{k} \big((x_{ii} - \lambda)(x_{2k+2-i,2k+2-i} - \lambda) - x_{i,2k+2-i}x_{2k+2-i,i}\big).$$

Hence the eigenvalues of X are the roots of these $\lfloor n/2 \rfloor$ scalar quadratic equations.

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