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BLACK-BOX PRECONDITIONING FOR MIXED FORMULATION OF SELF-ADJOINT ELLIPTIC PDES

CATHERINE E. POWELL* AND DAVID J. SILVESTER†

Abstract. Mixed finite element approximation of self-adjoint elliptic PDEs leads to symmetric indefinite linear systems of equations. Preconditioning strategies commonly focus on reduced symmetric positive definite systems and require nested iteration. This defficiency is avoided if preconditioned MINRES is applied to the full indefinite system. We outline such a preconditioning strategy, the key building block for which is a fast solver for a scalar diffusion operator based on black-box algebraic multigrid. Numerical results are presented for the Stokes equations arising in incompressible flow modelling and a variable diffusion equation that arises in modelling potential flow. We prove that the eigenvalues of the preconditioned matrices are contained in intervals that are bounded independently of the discretisation parameter and the PDE coefficients.

1. Introduction. We are interested in efficient and robust solution schemes for the general class of saddle-point systems,

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \end{pmatrix}, \tag{1.1}$$

arising in mixed finite element approximation of second-order, self-adjoint elliptic PDEs. The matrix $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite and we assume that the discretisation is inf-sup stable (see [3],[4]). Without preconditioning, Krylov solution methods are not robust with respect to the discretisation parameter or the PDE coefficients; preconditioners are therefore essential. An *optimal* preconditioner is a matrix operator that accelerates the convergence rate of the chosen iterative solver so that convergence to a fixed tolerance is independent of the problem parameters.

Two distinct types of preconditioning scheme are possible. The first consists of solving the positive definite Schur complement system,

$$S\underline{p} = BA^{-1}\underline{f}_1 - \underline{f}_2, \tag{1.2}$$

with $S = BA^{-1}B^T$, using the preconditioned conjugate gradient method. This requires matrix-vector multiplication with S and thus computation of the action of A^{-1}. For the problems to be discussed, A is ill-conditioned and also requires a preconditioner. The result is nested iteration, the success of which depends on the selection of inner and outer stopping tolerances or other parameters (see [12]). Alternatively, system (1.1) can be solved directly using the minimal residual method

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(MINRES). It is easy to show that an optimal preconditioner for this is,

$$P_* = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}. \tag{1.3}$$

Clearly, an exact implementation is infeasible. For the Stokes problem, and the variable diffusion problem to be discussed in §2 and §3, however, one of the blocks of P_* corresponds to a discrete Laplacian operator or a scalar diffusion operator. We will show that one V-cycle or W-cycle of black-box algebraic multigrid is an optimal and feasible preconditioner for those blocks.

In §2, we review a preconditioning strategy presented by Silvester and Wathen in [13] and [14]. In §3, we extend it to a variable diffusion problem and prove that algebraic multigrid preconditioning for S combined with diagonal scaling for A yields an optimal preconditioning scheme. Eigenvalue bounds are established. Numerical results are presented for both problems, demonstrating the effectiveness and simplicity of our approach.

First, we review convergence properties of MINRES. Let us denote the symmetric and indefinite system (1.1) by $C\underline{x} = \underline{b}$ and suppose that a preconditioner with symmetric and positive definite blocks,

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, \tag{1.4}$$

has been chosen. Given a zero initial guess $\underline{x}_0$, the iterative solver MINRES (see [6]) applied to the symmetrically preconditioned system,

$$\begin{pmatrix} P_1^{-\frac{1}{2}} A P_1^{-\frac{1}{2}} & P_1^{-\frac{1}{2}} B^T P_2^{-\frac{1}{2}} \\ P_2^{-\frac{1}{2}} B P_1^{-\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} \underline{y} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} P_1^{-\frac{1}{2}} \underline{f}_1 \\ P_2^{-\frac{1}{2}} \underline{f}_2 \end{pmatrix}, \quad \begin{pmatrix} \underline{y} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} P_1^{\frac{1}{2}} \underline{u} \\ P_2^{\frac{1}{2}} \underline{p} \end{pmatrix}, \tag{1.5}$$

generates a sequence of iterates $\underline{x}_k$, belonging to the Krylov space,

$$K_k = \underline{x}_0 + \mathtt{span}\left(P^{-1}\underline{r}_0, P^{-1}CP^{-1}\underline{r}_0, \ldots, \left(P^{-1}C\right)^{k-1} P^{-1}\underline{r}_0\right),$$

with the minimisation property,

$$\| \underline{b} - C\underline{x}_k \|_{P^{-1}} = \min_{\underline{x} \in K_k} \| \underline{b} - C\underline{x} \|_{P^{-1}},$$

where $\underline{r}_k = \underline{b} - C\underline{x}_k$ is the kth residual and $\| \underline{v} \|_{P^{-1}}^2 = \underline{v}^T P^{-1} \underline{v}$.

Since P is symmetric, it can be shown that a sharp upper bound for the residual error after k iterations (see [5, p.51]) is given by,

$$\frac{\| \underline{r}^{(k)} \|_{P^{-1}}}{\| \underline{r}^{(0)} \|_{P^{-1}}} \le \min_{p_k} \max_{i=1:(n+m)} | p_k(\lambda_i) |, \tag{1.6}$$

where p_k is a polynomial of degree k satisfying $p_k(0) = 1$, and $\{\lambda_i\}_{i=1}^{n+m}$ denotes the eigenvalues of $P^{-1}C$. In summary, the convergence rate of MINRES is completely determined by the eigenvalue spectrum of $P^{-1}C$. For a robust solution scheme, we seek P such that inclusion intervals for the values $\{\lambda_i\}_{i=1}^{n+m}$ are independent of the problem parameters.

1.1. Notation. It is now instructive to recall the definitions of some Sobolev spaces and their associated norms. Let $\Omega \subseteq \mathbb{R}^d$, $(d = 2,3)$, be a polygon with boundary $\partial\Omega$. $L^2(\Omega)$ is the usual space of square-integrable scalar functions with inner-product $(\cdot,\cdot)$. For the space of vector functions $L^2(\Omega)^d$, the definition is understood to hold componentwise. We write $\|\cdot\|_0$ to denote the Lebesque measure for both spaces. The subspace,

$$H(div;\Omega) = \left\{\vec{v} \in L^2(\Omega)^d \mid \nabla\cdot\vec{v} \in L^2(\Omega)\right\},$$

contains vectors with square-integrable divergence. The associated inner-product is $(\vec{u},\vec{v})_{div} = (\vec{u},\vec{v}) + (\nabla\cdot\vec{u},\nabla\cdot\vec{v})$ and we denote the induced norm $\|\cdot\|_{div}$. The Sobolev space,

$$H_0^1(\Omega) = \left\{w \in L^2(\Omega) \mid \nabla w \in L^2(\Omega)^d,\, w = 0 \text{ on } \partial\Omega\right\},$$

is equipped with the inner-product $(w,v)_1 = (w,v) + (\nabla w,\nabla v)$ and the induced norm $\|\cdot\|_1$. The space of vector functions $\left(H_0^1(\Omega)\right)^d$ and the associated norm are analogously defined.

2. The Stokes Problem. Let $\Omega \subset \mathbb{R}^d$, $(d = 2,3)$, be a specified flow domain, with piecewise smooth boundary $\partial\Omega$. For simplicity in our description, we consider a 'no-flow' boundary condition*. The classical Stokes problem is to find a velocity $\vec{u}$ and a pressure p satisfying,

$$\begin{aligned} \nu\nabla^2\vec{u} + \nabla p &= \vec{f}, \\ \nabla\cdot\vec{u} &= 0 \quad \text{in } \Omega, \\ \vec{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

The parameter ν represents viscosity of the fluid, $\vec{f}$ is a given forcing term and the second equation is an incompressibility constraint. A standard, conforming mixed finite element approximation is obtained by constructing finite-dimensional subspaces $\vec{V}_h \subset \left(H_0^1(\Omega)\right)^d$ and $W_h \subset L^2(\Omega)$, consisting of piecewise polynomials defined on a partition of Ω into regular polygons of edge length h. We then seek $\vec{u}_h \in \vec{V}_h$, $p_h \in W_h$ satisfying,

$$\nu\left(\nabla\vec{u}_h,\nabla\vec{v}\right) - (p_h,\nabla\cdot\vec{v}) = \left(\vec{f},\vec{v}\right) \quad \forall\,\vec{v} \in \vec{V}_h \tag{2.2}$$

$$(\nabla\cdot\vec{u}_h, w) = 0 \qquad \forall\, w \in W_h. \tag{2.3}$$

Choosing basis functions,

$$\vec{V}_h = \mathtt{span}\left\{\vec{\psi}_i\right\}_{i=1}^{n}, \quad W_h = \mathtt{span}\left\{\phi_j\right\}_{j=1}^{m},$$

leads to a linear algebra problem of the form (1.1). We obtain,

*The generalisation to a non-zero boundary condition is straightforward.

$$\begin{pmatrix} \nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}, \tag{2.4}$$

where $\vec{u}_h = \sum_{i=1}^{n} u_i \vec{\psi}_i$ and $p_h = \sum_{j=1}^{m} p_j \phi_j$ with matrices,

$$\begin{aligned} A(i,j) &= \left(\nabla \vec{\psi}_i, \nabla \vec{\psi}_j\right) \quad i,j = 1:n, \\ B(k,j) &= -\left(\nabla \cdot \vec{\psi}_j, \phi_k\right) \quad k = 1:m,\, j = 1:n, \end{aligned}$$

and vector $f(i) = \left(\vec{f}, \vec{\psi}_i\right)$ for $i = 1:n$. Here, A is a vector Laplacian matrix with d diagonal blocks corresponding to scalar Laplacian matrices.

To ensure existence and uniqueness of a solution to the problem (2.2)–(2.3), $\vec{V}_h$ and W_h must be compatible. Specifically, the following discrete inf-sup stability condition must hold:

$$\beta \parallel w \parallel_0 \leq \sup_{\vec{v} \in \vec{V}_h} \frac{(w, \nabla \cdot \vec{v})}{\parallel \nabla \vec{v} \parallel_0} \quad \forall w \in W_h,\, w \neq \text{ constant} \tag{2.5}$$

where $\beta \geq \beta_0 > 0$ and β_0 is a constant independent of h. See [4] for a thorough discussion on stability criteria for mixed finite element methods. Defining the pressure mass matrix $Q(k,l) = (\phi_k, \phi_l)$, condition (2.5) can be expressed in matrix form as,

$$\beta^2 \leq \frac{\underline{p}^t B A^{-1} B^T \underline{p}}{\underline{p}^t Q \underline{p}} \quad \forall \underline{p} \in I\!R^m \setminus \{\underline{1}\}. \tag{2.6}$$

That is, β is the square-root of the minimum eigenvalue of $Q^{-1} B A^{-1} B^T$ (see [3], [18]). It can also be shown that for a conforming finite element space,

$$\frac{\underline{p}^t B A^{-1} B^T \underline{p}}{\underline{p}^t Q \underline{p}} \leq 1 \quad \forall \underline{p} \in I\!R^m \setminus \{\underline{0}\}. \tag{2.7}$$

2.1. Preconditioning Strategy. In [13] and [14], Silvester and Wathen proved that for any stable finite element method, an optimal eigenvalue bound is obtained for the Stokes system matrix if it is preconditioned by a matrix of the form,

$$P = \begin{pmatrix} \nu P_A & 0 \\ 0 & \frac{1}{\nu} P_Q \end{pmatrix}, \tag{2.8}$$

for which there exist constants, λ, Λ, θ and Θ, independent of h such that,

$$\lambda^2 \leq \frac{\underline{u}^t A \underline{u}}{\underline{u}^t P_A \underline{u}} \leq \Lambda^2 \quad \forall \underline{u} \in I\!R^n \setminus \{\underline{0}\}, \tag{2.9}$$

$$\theta^2 \leq \frac{\underline{p}^t Q \underline{p}}{\underline{p}^t P_Q \underline{p}} \leq \Theta^2 \quad \forall \underline{p} \in I\!R^m \setminus \{\underline{0}\}. \tag{2.10}$$

Specifically, they proved the following theorem.

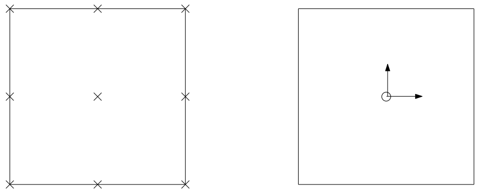

FIG. 2.1. *$Q_2 - P_{-1}$ element; velocity components (left), pressure $\left(p, \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right)$ components (right)*

THEOREM 2.1. *Assume a stable discretisation satisfying (2.6) and (2.7) and a preconditioner of the form (2.8) satisfying (2.9) and (2.10), then the eigenvalues of the preconditioned Stokes matrix lie in the union of the intervals,*

$$\begin{gathered}\left[\tfrac{1}{2}\left(\lambda^2 - \sqrt{\lambda^4 + 4\Theta^2\Lambda^2}\right), \tfrac{1}{2}\left(\lambda^2 - \sqrt{\lambda^4 + 4\beta^2\theta^2\lambda^2}\right)\right], \\ \cup \left[\lambda^2, \tfrac{1}{2}\left(\Lambda^2 + \sqrt{\Lambda^4 + 4\Theta^2\Lambda^2}\right)\right].\end{gathered} \tag{2.11}$$

Proof. See [14]. □

Using a result of Wathen [17] it is possible to show that (2.10) holds with $P_Q = \mathtt{diag}\,(Q)$. In other words, diagonal scaling is an optimal preconditioner for the Stokes pressure mass matrix. One of the best Stokes approximations in $\mathbb{R}^2$ is the $Q_2 - P_{-1}$ finite element (see Fig. 2.1), consisting of piecewise biquadratic velocity and discontinuous piecewise linear pressure. In this case, Q is a diagonal matrix. Thus, $P_Q = Q$ is an optimal choice with $\theta^2 = \Theta^2 = 1$.
The task of defining an efficient preconditioner for (2.4) is thus reduced to finding P_A satisfying (2.9). Since A is a discrete representation of the Laplacian operator ∇^2, any black-box solver for Poisson problems is a potential candidate. Silvester and Wathen demonstrate numerically in [14] that one V-cycle of *geometric* multigrid (see [2]) is an optimal choice. Recently, there has been a resurgence of interest in *algebraic* multigrid (AMG) schemes. Introduced by Ruge and Stüben (see [10], [11]) in the 1980s, AMG represents the new generation of multigrid (see [15] for a review of current trends and related literature). No geometric information is required; coarsening and interpolation are performed by exploiting only information in the coefficient matrix. In this way, AMG is more generally applicable than traditional multigrid with respect to finite element mesh geometry and PDE coefficients. We choose P_A in (2.8) to be one V-cycle or W-cycle of the AMG algorithm `amg1r5` (see [10]) applied to the vector Laplacian matrix A. We implement it as a black-box; no parameters are estimated a-priori.

2.2. Numerical Example. Consider the Stokes equations (2.1), discretised on $[-1,1] \times [-1,1]$ using stable $Q_2 - P_{-1}$ approximation, with $\nu = 1$, $\vec{f} = \vec{0}$ and velocity boundary condition,

$$(u_x, u_y) = \left\{\left(1 - x^4, 0\right) \mid y = 1, -1 \leq x \leq 1\right\}. \tag{2.12}$$

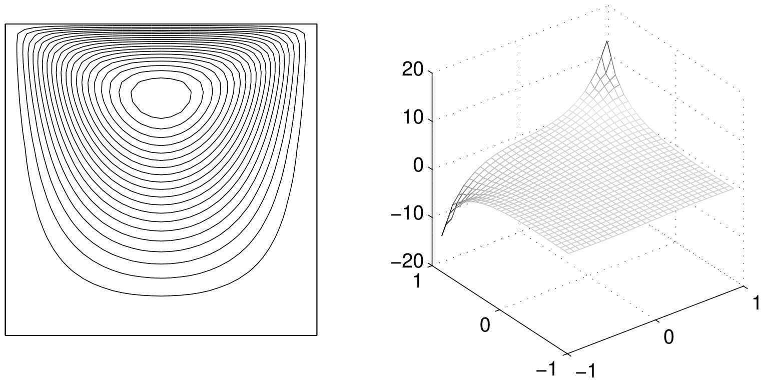

FIG. 2.2. *Contour plot of constant streamlines (left) and pressure solution (right)*

This is the so-called 'regularised driven-cavity' test problem. We apply preconditioned MINRES to the assembled system with a stopping tolerance of 10^{-6} on the relative residual error in the $\|\cdot\|_{P^{-1}}$ norm. The Laplacian blocks of A are preconditioned separately using either one V-cycle ($P_A = A_V$) or one W-cycle ($P_A = A_W$) of `amg1r5` (implemented with symmetric smoothing).

TABLE 2.1
MINRES *iterations and (time), TOL* $= 10^{-6}$

h	$P = I$	$P_A = A$		$P_A = A_V$		$P_A = A_W$	
$\frac{1}{8}$	64	22	(0.2)	25	(0.2)	25	(0.2)
$\frac{1}{16}$	147	24	(1.3)	26	(0.5)	26	(0.6)
$\frac{1}{32}$	274	24	(11.5)	28	(2.2)	28	(2.3)
$\frac{1}{64}$	513	24	(96.8)	29	(9.2)	28	(9.2)

TABLE 2.2
Eigenvalues of indefinite preconditioned system; $P_A = A_V$

h	λ^2	Λ^2	β^2	Γ^2	Observed eigenvalues
$\frac{1}{8}$	0.951	1	0.256	0.998	$[-0.615, -0.211] \cup [0] \cup [0.960, 1.613]$
$\frac{1}{16}$	0.917	1	0.235	0.999	$[-0.618, -0.196] \cup [0] \cup [0.939, 1.617]$
$\frac{1}{32}$	0.884	1	0.222	1.000	$[-0.618, -0.186] \cup [0] \cup [0.924, 1.617]$

MINRES iteration counts for the preconditioned system are reported in Table 2.1. The second column contains unpreconditioned counts; the third column lists counts for the exact ($P_A = A$) version of the preconditioner. The time units reported in parentheses are elapsed time in seconds for the total solve using a mex fortran interface in Matlab 6.0 on a SUN ultraSPARC workstation.

Iteration counts do not grow with mesh refinement showing that the AMG pre-

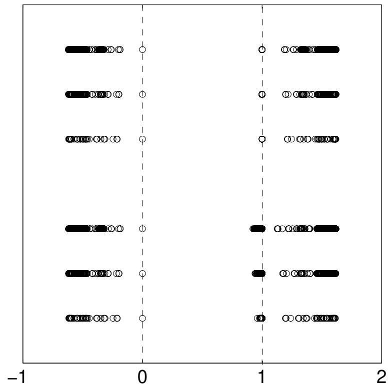

FIG. 2.3. *Eigenvalues of indefinite preconditioned matrix;* $P_A = A$ *(top),* $P_A = A_V$ *(bottom),* $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$

conditioner is optimal. Further, the solve times in the AMG trials grow only linearly with respect to the problem size. The eigenvalues of the preconditioned indefinite system, for the V-cycle case, are reported in Table 2.2. Observe that λ^2 and Λ^2 are independent of h. Substituting $\theta^2 = 1$, $\Theta^2 = 1$, $\lambda^2 = 0.884$, $\Lambda^2 = 1$ and $\beta^2 = 0.222$ in (2.11) yields the theoretical bound, $[-0.651, -0.184] \cup [0.884, 1.618]$. Fig. 2.3 shows the spectral equivalence of the AMG and the exact versions of the preconditioner. Note that in this example the preconditioned system is singular. The nullspace is spanned by the hydrostatic constant pressure solution. This accounts for the zero eigenvalue.

The experiment shows that we have a simple, optimal preconditioner for the Stokes equations. We now consider a second important class of elliptic PDEs.

3. A Variable Diffusion Problem.

Let Ω be a bounded domain in $\mathbb{R}^2$. We consider scalar, second-order elliptic problems of the form,

$$\begin{aligned} -\nabla \cdot \mathcal{A}\nabla p = f \quad & \text{in } \Omega, \\ p = g \quad & \text{on } \partial\Omega_D, \\ \mathcal{A}\nabla p \cdot \vec{n} = 0 \quad & \text{on } \partial\Omega_N, \end{aligned} \tag{3.1}$$

where $\partial\Omega_D \neq \emptyset$, $\vec{n}$ denotes the unit outward normal vector and f and g are given functions. $\mathcal{A} = \sigma(\vec{x})\mathcal{I}$ is a 2×2 bounded, symmetric and uniformly positive definite matrix-valued function with,

$$0 < \sigma_{min} \leq \sigma(\vec{x}) \leq \sigma_{max} < \infty, \quad \forall \vec{x} \in \Omega. \tag{3.2}$$

This is a simple model of fluid flow in porous media; the diffusion coefficient $\sigma(\vec{x})$ is commonly discontinuous or anisotropic. Substituting $\vec{u} = \mathcal{A}\nabla p$ yields the first-order system,

$$
\begin{aligned}
\mathcal{A}^{-1}\vec{u} - \nabla p &= 0, \\
\nabla \cdot \vec{u} &= -f \quad \text{in } \Omega, \\
p &= g \quad \text{on } \partial\Omega_D, \\
\vec{u} \cdot \vec{n} &= 0 \quad \text{on } \partial\Omega_N.
\end{aligned}
\tag{3.3}
$$

This formulation is favoured when the velocity $\vec{u}$ is the variable of interest. It is known that mixed velocity solutions are more robust than standard finite element methods with respect to any variation in $\sigma(\vec{x})$ (see [8]). Another crucial advantage in the modelling of groundwater flow ($f = 0$) is local conservation of mass.

To formulate the weak problem, we choose finite-dimensional subspaces, $\vec{V}_h \subset H_{0,N}(div;\Omega) = \{\vec{v} \in H(div;\Omega) \mid \vec{v} \cdot \vec{n} = 0 \text{ on } \partial\Omega_N\}$ and $W_h \subset L^2(\Omega)$, consisting of piecewise polynomials defined on a partition of Ω into triangles of edge length h. Then we look for $\vec{u}_h \in \vec{V}_h$, $p_h \in W_h$ satisfying,

$$
\begin{aligned}
(\mathcal{A}^{-1}\vec{u}_h, \vec{v}) + (p_h, \nabla \cdot \vec{v}) &= \; < g, \vec{v} \cdot \vec{n} > \quad \forall \vec{v} \in \vec{V}_h, \\
(w, \nabla \cdot \vec{u}_h) &= -(f, w) \quad \forall w \in W_h,
\end{aligned}
\tag{3.4}
$$

where $\langle g, \vec{v} \cdot \vec{n} \rangle = \int_{\partial\Omega_D} g\, \vec{v} \cdot \vec{n}\, ds$. A unique solution exists (see [4]), if and only if,

$$
\alpha \parallel \vec{v} \parallel_{div}^2 \leq (\mathcal{A}^{-1}\vec{v}, \vec{v}) \qquad \forall\, \vec{v} \in \vec{Z}_h,
\tag{3.5}
$$

with $\vec{Z}_h = \{\vec{v} \in \vec{V}_h \mid (w, \nabla \cdot \vec{v}) = 0 \quad \forall w \in W_h\}$ and,

$$
\beta \parallel w \parallel_0 \leq \sup_{\vec{v} \in V_h} \frac{(w, \nabla \cdot \vec{v})}{\parallel \vec{v} \parallel_{div}} \qquad \forall\, w \in W_h,
\tag{3.6}
$$

where $\alpha \geq \alpha_0 > 0$, $\beta \geq \beta_0 > 0$ and α_0 and β_0 are constants independent of h.

We choose the lowest-order Raviart-Thomas finite element spaces. Stability of the method is established in [9]. For a given triangulation T_h of Ω we have,

$$
\vec{V}_h = \{\vec{v} \in H(div;\Omega) \mid \vec{v}\,|_T \in (P_0^2(T) + [x,y]P_0(T)) \quad \forall T \in T_h\}
\tag{3.7}
$$

and $W_h = \{w \in L^2(\Omega) \mid w\,|_T \in P_0(T) \quad \forall T \in T_h\}$ where $P_0(T)$ denotes the set of polynomials of degree zero defined on triangle T.

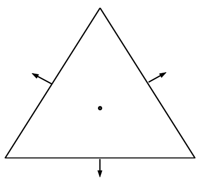

FIG. 3.1. *The lowest order Raviart-Thomas element*

Velocities are linear in each component; the pressure space consists of piecewise constant functions. For an $H(div;\Omega)$ conforming velocity approximation, the degrees of freedom are normal components at edge midsides; the pressure solution is sampled at element centroids (see Fig. 3.1).

$$\left(\begin{array}{cc} A & B^T \\ B & 0 \end{array}\right)\left(\begin{array}{c} \underline{u} \\ \underline{p} \end{array}\right) = \left(\begin{array}{c} \underline{g} \\ \underline{f} \end{array}\right), \tag{3.8}$$

Choosing bases, $\vec{V}_h = \mathtt{span}\left\{\vec{\psi}_i\right\}_{i=1}^{n}$, $W_h = \mathtt{span}\left\{\phi_j\right\}_{j=1}^{m}$, we obtain the linear algebra problem, with matrices A and B defined by,

$$A(i,j) = \left(\mathcal{A}^{-1}\vec{\varphi}_i, \vec{\varphi}_j\right) \quad i,j = 1:n,$$
$$B(k,j) = \left(\nabla\cdot\vec{\varphi}_j, \phi_k\right) \quad k = 1:m,\ j = 1:n,$$

and vectors $g(i) = \langle g, \vec{\varphi}_i \cdot \vec{n}\rangle$, $f(k) = -(f, \phi_k)$ for $i = 1:n$ and $k = 1:m$.

3.1. Preconditioning Strategy. Preconditioning strategies for the indefinite problem (3.8) are proposed in [1] and [16]. The former discusses the case $\sigma(\vec{x}) = 1$; the latter requires the computation of acceleration parameters. A theoretically optimal, parameter-free preconditioner for general coefficients is discussed in [7]. Here, we propose the block-diagonal preconditioner,

$$P = \left(\begin{array}{cc} D_A & 0 \\ 0 & P_S \end{array}\right), \tag{3.9}$$

where $D_A = \mathtt{diag}\,(A)$ and P_S is black-box AMG preconditioning for the approximate Schur-complement $S_D = BD_A^{-1}B^T$. Our choice is motivated by the observation that for $\mathcal{A}$ of the form,

$$\mathcal{A} = \left(\begin{array}{cc} \sigma(\vec{x}) & 0 \\ 0 & \sigma(\vec{x}) \end{array}\right), \tag{3.10}$$

diagonal scaling for A is an optimal choice, independent of $\sigma(\vec{x})$ (see Lemma 3.4 below). The Schur-complement represents the generalised diffusion operator $\nabla\cdot\mathcal{A}\nabla$ and AMG is known to be an efficient *solver* for linear systems arising from discretisations of such operators (see [10]).

We now prove that the preconditioner (3.9) is optimal. Our starting point is the eigenvalue bound established by Rusten and Winther in [12].

THEOREM 3.1. *Let $0 < \mu_1 \ldots \le \mu_n$ be the eigenvalues of A and let $0 < \rho_1 \ldots \le \rho_m$ be the singular values of B, then the eigenvalues of the indefinite system matrix (3.8) lie in the union of the intervals,*

$$\begin{array}{c}\left[\frac{1}{2}\left(\mu_1 - \sqrt{\mu_1^2 + 4\rho_m^2}\right),\ \frac{1}{2}\left(\mu_n - \sqrt{\mu_n^2 + 4\rho_1^2}\right)\right] \\ \cup\left[\mu_1,\ \frac{1}{2}\left(\mu_n + \sqrt{\mu_n^2 + 4\rho_m^2}\right)\right].\end{array} \tag{3.11}$$

Proof. See [12]. □

Consider, first, symmetric preconditioning for (3.8) with the exact (no AMG) version of the preconditioner (3.9). We obtain,

$$P^{-\frac{1}{2}}CP^{-\frac{1}{2}} = \left(\begin{array}{cc} D_A^{-\frac{1}{2}}AD_A^{-\frac{1}{2}} & D_A^{-\frac{1}{2}}B^T\left(BD_A^{-1}B^T\right)^{-\frac{1}{2}} \\ \left(BD_A^{-1}B^T\right)^{-\frac{1}{2}}BD_A^{-\frac{1}{2}} & 0 \end{array} \right) = \left(\begin{array}{cc} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{array} \right).$$

Applying Theorem 3.1 to this preconditioned saddle-point system leads, trivially, to the following result.

COROLLARY 3.2. *Let $0 < \tilde{\mu}_1 \ldots \leq \tilde{\mu}_n$ be the eigenvalues of $D_A^{-1}A$, then the eigenvalues of,*

$$P^{-\frac{1}{2}} \left(\begin{array}{cc} A & B^T \\ B & 0 \end{array} \right) P^{-\frac{1}{2}}, \ \textit{with}\ P = \left(\begin{array}{cc} D_A & 0 \\ 0 & BD_A^{-1}B^T \end{array} \right),$$

lie in the union of the intervals,

$$\left[\frac{1}{2}\left(\tilde{\mu}_1 - \sqrt{\tilde{\mu}_1^2+4}\right), \frac{1}{2}\left(\tilde{\mu}_n - \sqrt{\tilde{\mu}_n^2+4}\right)\right] \cup \left[\tilde{\mu}_1, \frac{1}{2}\left(\tilde{\mu}_n + \sqrt{\tilde{\mu}_n^2+4}\right)\right].$$

Proof. Observe that

$$\tilde{B}\tilde{B}^T = \left(BD_A^{-1}B^T\right)^{-\frac{1}{2}} BD_A^{-\frac{1}{2}}D_A^{-\frac{1}{2}}B^T\left(BD_A^{-1}B^T\right)^{-\frac{1}{2}} = I,$$

where I is the identity matrix. The result follows immediately from Theorem 3.1 since the singular values of $\tilde{B}$ satisfy $\tilde{\rho}_1 = \ldots = \tilde{\rho}_m = 1$ and $\tilde{A}$ has the same eigenvalue spectrum as $D_A^{-1}A$. □

Now, given a preconditioner P_S for $BD_A^{-1}B^T$ satisfying,

$$\theta^2 \leq \frac{\underline{p}^t BD_A^{-1}B^T\underline{p}}{\underline{p}^t P_S \underline{p}} \leq \Theta^2 \quad \forall \underline{p} \in I\!\!R^m \setminus \{\underline{0}\}.$$

for some constants θ and Θ, Theorem 3.1 can be extended to obtain the following theoretical eigenvalue bound for the indefinite preconditioned system.

COROLLARY 3.3. *Let $0 < \tilde{\mu}_1 \ldots \leq \tilde{\mu}_n$ be the eigenvalues of $D_A^{-1}A$, then the eigenvalues of,*

$$P^{-\frac{1}{2}} \left(\begin{array}{cc} A & B^T \\ B & 0 \end{array} \right) P^{-\frac{1}{2}}, \ \textit{with}\ P = \left(\begin{array}{cc} D_A & 0 \\ 0 & P_S \end{array} \right),$$

lie in the union of the intervals,

$$\left[\frac{1}{2}\left(\tilde{\mu}_1 - \sqrt{\tilde{\mu}_1^2+4\Theta^2}\right), \frac{1}{2}\left(\tilde{\mu}_n - \sqrt{\tilde{\mu}_n^2+4\theta^2}\right)\right] \cup \left[\tilde{\mu}_1, \frac{1}{2}\left(\tilde{\mu}_n + \sqrt{\tilde{\mu}_n^2+4\Theta^2}\right)\right].$$

Proof. The result follows from Theorem 3.1 and Corollary 3.2 with $\tilde{B} = P_S^{-\frac{1}{2}}BD_A^{-\frac{1}{2}}$. Observe that,

$$\begin{aligned} \underline{p}^T\tilde{B}\tilde{B}^T\underline{p} &= \underline{p}^T P_S^{-\frac{1}{2}}BD_A^{-1}B^TP_S^{-\frac{1}{2}}\underline{p} \\ &\leq \Theta^2\underline{p}^TP_S^{-\frac{1}{2}}P_SP_S^{-\frac{1}{2}}\underline{p} \\ &= \Theta^2\underline{p}^T\underline{p}. \end{aligned}$$

That is, the maximum singular value $\tilde{\rho}_m$ of $\tilde{B}$ satisfies, $\tilde{\rho}_m^2 \leq \Theta^2$. Similarly, it can be shown that $\tilde{\rho}_1^2 \geq \theta^2$. □

In section 3.2 we show that black-box AMG is an optimal choice for P_S yielding constants θ and Θ that are independent of h. By Corollary 3.3, an optimal eigenvalue bound is thus obtained if and only if, the constants $\tilde{\mu}_1$ and $\tilde{\mu}_n$ satisfy,

$$\tilde{\mu}_1 \leq \frac{\underline{u}^t A \underline{u}}{\underline{u}^t D_A \underline{u}} \leq \tilde{\mu}_n \quad \forall \underline{u} \in I\!R^n \setminus \{\underline{0}\},$$

and are independent of the discretisation parameter h and the coefficient function $\sigma(\vec{x})$. By a result of Wathen, [17], it is sufficient to consider the *element* matrices. Denoting by λ_{min}^k and λ_{max}^k the minimum and maximum eigenvalues of the diagonally preconditioned element matrix, $\left(D_A^k\right)^{-1} A^k$, associated with triangle k, we have,

$$\min_k \left\{\lambda_{min}^k\right\} \leq \tilde{\mu}_1, \quad \tilde{\mu}_n \leq \max_k \left\{\lambda_{max}^k\right\}.$$

For simple geometries, we can compute explicit bounds for these values. As an illustration, consider uniform triangles as in Fig. 3.2 and fix normal vectors on each edge as shown.

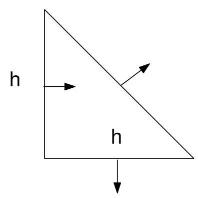

FIG. 3.2. *Uniform triangles with orientated normal vectors*

LEMMA 3.4. *Consider a square domain tiled with uniform triangles of edge length h. Fix normal vectors as shown in Fig. 3.2. Then,*

$$\frac{1}{2} \leq \min_k \left\{\lambda_{min}^k\right\}, \quad \max_k \left\{\lambda_{max}^k\right\} \leq \frac{3}{2}. \tag{3.12}$$

Proof. Labelling the Raviart-Thomas velocity degrees of freedom (x_j, y_j) with $j = 1:3$, the element velocity basis functions are,

$$\vec{\varphi}_1 = \begin{pmatrix} -\frac{x_3}{h} + \frac{x}{h} \\ -1 - \frac{y_1}{h} + \frac{y}{h} \end{pmatrix}, \vec{\varphi}_2 = \begin{pmatrix} -\frac{x_3\sqrt{2}}{h} + \frac{x\sqrt{2}}{h} \\ -\frac{y_1\sqrt{2}}{h} + \frac{y\sqrt{2}}{h} \end{pmatrix}, \vec{\varphi}_3 = \begin{pmatrix} 1 + \frac{x_3}{h} - \frac{x}{h} \\ \frac{y_1}{h} - \frac{y}{h} \end{pmatrix}.$$

For element k, let

$$\mathcal{A}_c^k = \begin{pmatrix} \sigma(x_c^k, y_c^k) & 0 \\ 0 & \sigma(x_c^k, y_c^k) \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

denote $\mathcal{A}$ evaluated at the centroid $\left(x_c^k, y_c^k\right)$ of that element. The element velocity mass matrix is then defined by,

$$A^k(i,j) = \int\int_{\triangle} \left(\mathcal{A}_c^k\right)^{-1} \vec{\varphi}_i \cdot \vec{\varphi}_j \, dxdy \quad i,j = 1:3.$$

Integrating over triangle k yields,

$$A^k = h^2 \begin{pmatrix} \frac{1}{3\sigma} & 0 & \frac{1}{6\sigma} \\ 0 & \frac{1}{3\sigma} & 0 \\ \frac{1}{6\sigma} & 0 & \frac{1}{3\sigma} \end{pmatrix}, \quad \left(D_A^k\right)^{-1} A^k = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}. \tag{3.13}$$

Hence, $\lambda_1^k = \frac{1}{2}$, $\lambda_2^k = 1$, $\lambda_3^k = \frac{3}{2}$. $\square$

Note that for anisotropic coefficients of the form,

$$\mathcal{A}_c = \begin{pmatrix} \sigma_1(x_c, y_c) & 0 \\ 0 & \sigma_2(x_c, y_c) \end{pmatrix}$$

with $\sigma_1(x_c, y_c) \gg \sigma_2(x_c, y_c)$ or vice versa, diagonal preconditioning for A is not efficient on triangles of this type. The diagonally preconditioned element matrix (3.13) has a zero eigenvalue in the limit of the anisotropy. Rectangular Raviart-Thomas finite elements should be used in that case.

The following numerical examples illustrate the optimality of the preconditioner (3.9). We apply one V-cycle ($P_S = S_V$) or one W-cycle ($P_S = S_W$) of `amg1r5` to the sparse matrix $S_D = BD_A^{-1}B^T$.

3.2. Numerical Examples. First, consider (3.3) with $\sigma(\vec{x}) = 1$, (unit coefficients), $f = 1$, $g = 0$ and $\partial\Omega_N = \emptyset$, discretised on $\Omega = [0,1] \times [0,1]$. MINRES iteration counts for the assembled linear system are reported in Table 3.1. The second column corresponds to unpreconditioned MINRES; the symbol * signifies that more than 500 iterations were needed. We use the same stopping criterion as in the Stokes example.

TABLE 3.1

MINRES *iterations and (time),* $TOL = 10^{-6}$

h	$P = I$	$P_S = S_D$	$P_S = S_V$	$P_S = S_W$
$\frac{1}{16}$	186	26 (0.5)	26 (0.4)	26 (0.4)
$\frac{1}{32}$	375	26 (2.4)	26 (1.3)	26 (1.6)
$\frac{1}{64}$	*	26 (13.6)	26 (5.3)	26 (6.4)
$\frac{1}{128}$	*	26 (84.0)	26 (23.8)	26 (28.1)

Table 3.2

Eigenvalues of indefinite preconditioned system; $P_S = S_V$

h	$\tilde{\mu}_1$	$\tilde{\mu}_n$	θ^2	Θ^2	Observed eigenvalues
$\frac{1}{8}$	0.5	1.5	0.957	1	$[-0.777,\ -0.532] \cup [0.707,\ 1.896]$
$\frac{1}{16}$	0.5	1.5	0.955	1	$[-0.777,\ -0.508] \cup [0.707,\ 1.939]$
$\frac{1}{32}$	0.5	1.5	0.954	1	$[-0.777,\ -0.496] \cup [0.707,\ 1.964]$

The iteration counts with V-cycle and W-cycle AMG preconditioning are independent of h and match the iteration counts for the exact version of the preconditioner. Again, we observe that solve times grow almost[†] linearly with respect to the problem size. The eigenvalues of the preconditioned matrix with $P_S = S_V$ are shown in Table 3.2. We have $\theta^2 \approx 0.954$ and $\Theta^2 = 1$. Substituting these values in Corollary 3.3 gives the theoretical bound $[-0.781, -0.477] \cup [0.5, 2]$. Note that θ^2 is larger here than in the Stokes example. This can be attributed to the different discretisation method.

We now consider a discontinuous coefficient example. Take

$$\sigma(\vec{x}) = \begin{cases} 10^{-3} & \text{in } \Omega_* = [0.25, 0.75] \times [0.25, 1] \\ 1 & \text{in } \Omega \backslash \Omega_* \end{cases}$$

with $f = 0$ and mixed boundary conditions: $\vec{u} \cdot \vec{n} = 0$ on $\partial\Omega_N$, $p = 1 - x$ on $\partial\Omega_D$, where $\partial\Omega_N = [0,1] \times 0 \cup \{0,1\} \times [0, 0.75]$ and $\partial\Omega_D = \Omega \backslash \Omega_N$. Iteration counts are given in Table 3.3.

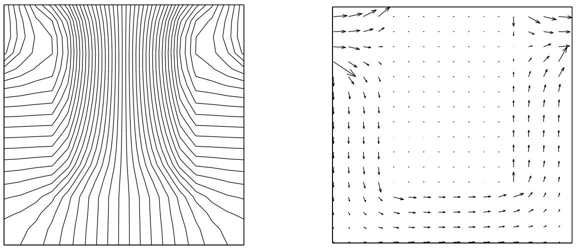

Fig. 3.3. *Pressure contours (left) and velocity field (right)*

The AMG preconditioner is optimal with repect to the mesh parameter and the solve times grow linearly. Eigenvalues for the V-cycle case are reported in Table 3.4. We observe $\theta^2 \approx 0.835$ and $\Theta^2 = 1$, yielding the theoretical bound $[-0.781, -0.433]$ $\cup [0.5, 2]$ for the preconditioned system. Note that θ^2 does change with $\sigma(\vec{x})$ but not significantly. It is bounded below by a constant that is independent of h and not close to zero. Fig. 3.4 illustrates the spectral equivalence of the exact and inexact versions of the preconditioner. Comparing the iteration counts for the two examples, we see that the AMG preconditioner is completely insensitive to the coefficient $\sigma(\vec{x})$.

[†]There is a small overhead associated with the set-up phase of AMG.

Table 3.3
MINRES *iterations and (time),* $TOL = 10^{-6}$

h	$P = I$	$P_S = S_D$		$P_S = S_V$		$P_S = S_W$	
$\frac{1}{16}$	*	25	(0.5)	25	(0.4)	25	(0.5)
$\frac{1}{32}$	*	25	(2.3)	26	(1.3)	28	(1.5)
$\frac{1}{64}$	*	25	(12.7)	27	(5.6)	25	(6.4)
$\frac{1}{128}$	*	25	(81.1)	27	(25.4)	25	(27.6)

Table 3.4
Eigenvalues of indefinite preconditioned system; $P_S = S_V$

h	$\tilde{\mu}_1$	$\tilde{\mu}_n$	θ^2	Θ^2	Observed eigenvalues
$\frac{1}{8}$	0.5	1.5	0.918	1	$[-0.774, -0.480] \cup [0.822, 1.965]$
$\frac{1}{16}$	0.5	1.5	0.845	1	$[-0.775, -0.448] \cup [0.822, 1.965]$
$\frac{1}{32}$	0.5	1.5	0.836	1	$[-0.775, -0.452] \cup [0.822, 1.974]$

4. Concluding Remarks. In this paper we have shown that the symmetric indefinite linear systems arising from mixed finite element formulation of Stokes and variable diffusion problems can be solved efficiently using a generic black-box preconditioning strategy. The preconditioner discussed essentially consists of diagonal scaling for a mass matrix and one V-cycle or W-cycle of algebraic multigrid applied to discrete Laplacian or diffusion blocks. The resulting method is simple, robust with respect to the PDE coefficients, yields optimal rates of convergence for MINRES and there are no parameters to estimate.

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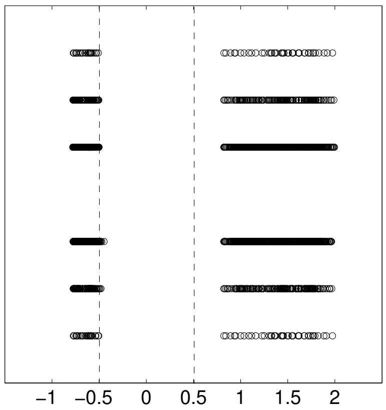

FIG. 3.4. *Eigenvalues of indefinite preconditioned matrix; $P_S = S_D$ (top), $P_S = S_V$ (bottom),* $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$

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