Bistatic two dimensional synthetic aperture radar as a tensor tomography problem

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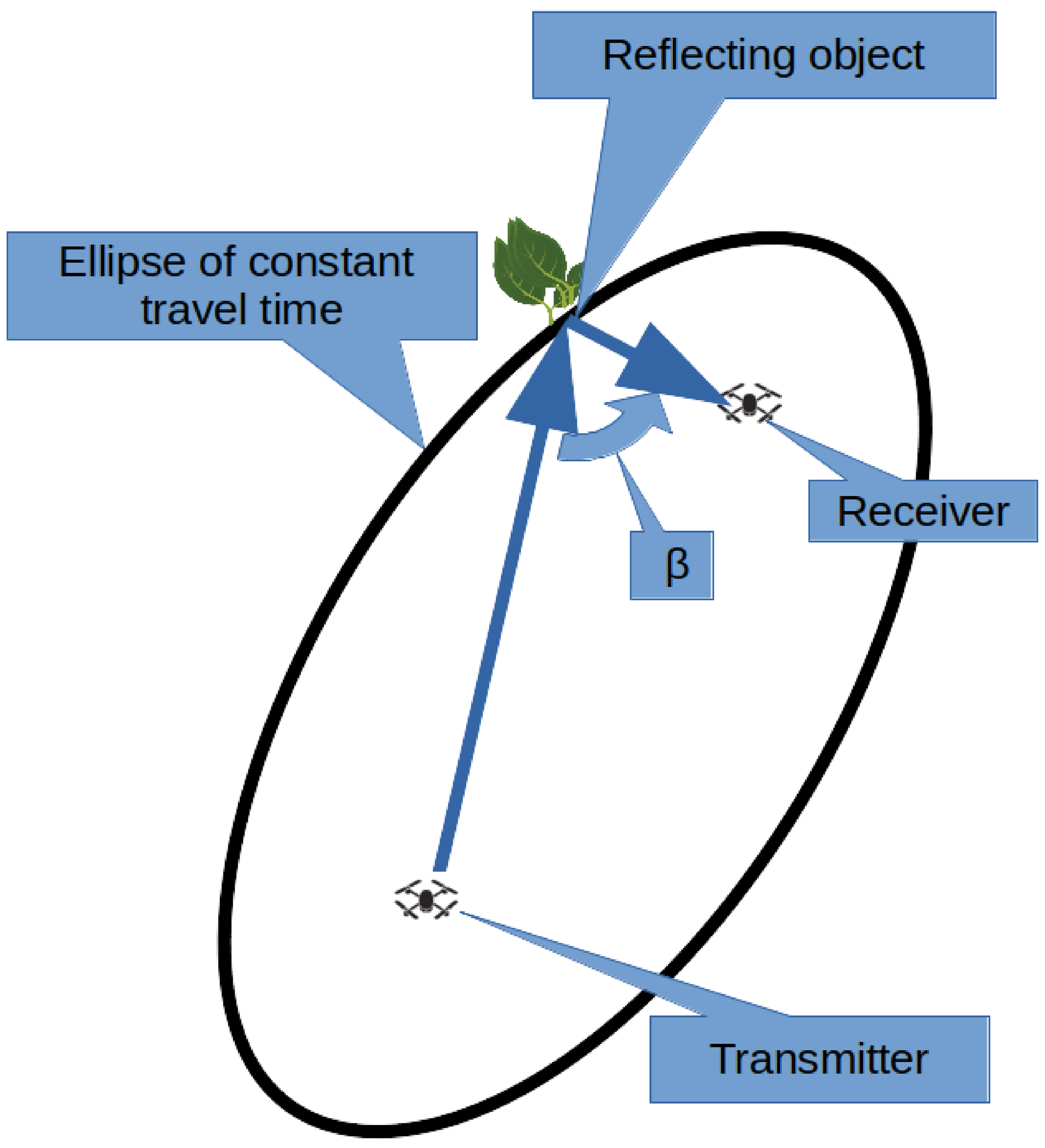
Bistatic radar uses a separate location for the transmitter and receiver, for example on two aircraft or UAVs. The transmitter sends out a pulse and the receiver records its magnitude, for each travel time (we consider the incoherent case, most systems in practice record also the phase).
For each transmitter and receiver position a measurement at a given travel time is the sum of reflections on a prolate spheroid (ellipse in two dimensional case) with the transmitter and receiver as its foci. If the transmitter and receiver are distant from reflectors supported in a small ball the spheroids can be approximated by planes.
Although reflectors are commonly assumed to be isotropic, in practice the relative amplitude of reflection depends on the incoming and outgoing direction, described by the (bistatic) radar scattering cross-section.
In this poster we relate this problem to a tensor Radon transform, which in two dimensions has an explicit singular value decomposition.

Formulation of radar problem

We are considering a transmitter x_T and a receiver x_R at the two focal points of an ellipse where we fix the bistatic angle. The vector equation of the ellipse is $|x - x_T| + |x - x_R| = ct$ for all points x for a given time t (c is the speed of light).



This is the two dimensional approximation for objects on flat ground. The signal received for a given travel time and x_T and x_R is the sum of all the reflectors on the ellipse, hence an elliptical Radon transform [5]. In three dimensions the surface of constant travel time is an ellipsoid, more specifically a prolate spheroid. For the Elliptical Radon transform see our other poster [2].

Approximating by lines (planes).

The gradient vector of the travel time equation at x for fixed t is $-\mathbf{n}$ where

$$\mathbf{n} = \frac{x - x_T}{|x - x_T|} + \frac{x - x_R}{|x - x_R|},$$

and is normal to the ellipse. It is in a direction bisecting the lines from x to x_T and x_R. Provided the transmitter and receiver are distant from the reflecting objects, and those objects are contained in a small disk, we can approximate the ellipse near a point x_0 by the line:

$$0 = \mathbf{n} \cdot (x - x_0)$$

For a scalar distribution of reflectors the elliptical Radon transform is replaced by the standard Radon transform integrating over all lines (planes) [3]. For complete data the $\mathbf{n}$ must cover a complete circle, for example if the transmitter and receiver both fly in a circle, one following a fixed angle behind the other.

Radar Scattering cross-section

The radar scattering cross-section, which describes the magnitude (and phase) of the reflected signal $F(x, \frac{x-x_T}{|x-x_T|}, \frac{x-x_R}{|x-x_R|})$ depends on the incoming and outgoing direction. We could replace the unit vectors by angles θ_T and θ_R resulting in $F(x, \theta_T, \theta_R)$.
Here F might have quite sharp peaks (even distributional) for angular objects, but we assume it is smooth as a function of the angles. It will typically be distributional in x with relatively small support.
The bistatic angle $\beta = |\theta_T - \theta_R|$. The tangent to the ellipse is normal to the $\theta_{AZ} = \frac{\theta_T + \theta_R}{2}$, the average azimuth which is the bisection of two lines to the foci.
Now fixing β, f can then be approximated by a polynomial in angle of some degree as follows:

$$f(x, \theta) = \sum_{i_1 \dots i_m = 1}^{2} a_{i_1 \dots i_m} \Theta^{i_1} \dots \Theta^{i_m},$$

where $\Theta = [\cos\theta, \sin\theta]^T$ is a unit vector and $a_{i_1 \dots i_m}$ are the components of rank m symmetric tensor a in the Cartesian basis (x_1, x_2).

Normal Tensor Radon Transform

For a fixed x_0 our bistatic radar measurements correspond to integrals over lines $L_{\theta_{AZ}, t}$ with normal direction θ_{AZ} and distance from x_0, $c(t - t_0)$. Now

$$\int_{L_{\theta_{AZ}, t}} f(x, \theta_{AZ})\, \mathrm{d}s$$

for arc length s is the Normal Radon transform of the tensor field a, which coincides up to notation with the (longitudinal) ray transform [4].
Kazantsev and Bukhgeim [4] give a singular value decomposition (SVD) of the ray transform of a symmetric tensor field on the unit disk in terms of Zernike disk functions and a Fourier basis (this extends the the SVD of the scalar Radon transform see e.g. [1]).
Specifically we see that there is a **null space**, and only rank m tensor fields that are the symmetric derivative of rank $m - 1$ fields can be recovered. The range of the transform is also characterized, extending the well known Helgason range conditions [1]. This means that a suitable m can be estimated from the data.
It is noted by [6] that in three dimensions and $m = 2$ the normal Radon transform also has a null space, and the only fields that can be recovered are the second symmetric derivatives of a scalar field.

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