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Hamilton's Discovery of the Quaternions and Creativity in Mathematics

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Abstract

Creativity can be defined as the process of forming new patterns from pre-existing component parts. As an illustration, we explain how Hamilton's discovery of the quaternions, and Cayley and Graves's subsequent discoveries of the octonions, could have resulted from considering the properties of complex numbers and asking for each one, "how might this be different?". We give some general suggestions on differences to look for and explain why creativity can be much richer when people work in small groups rather than individually.

Sir William Rowan Hamilton's discovery of the quaternions is a famous example of a flash of inspiration. Hamilton had for many years been trying to extend the notion of complex numbers $a+\mathrm{i}b$ to triplets $a+\mathrm{i}b+\mathrm{j}c$, while retaining the property that the modulus of a product is the product of the moduli. As he was walking with his wife along the Royal Canal in Dublin one day in 1843 "an electric circuit seemed to close; and a spark flashed forth" [6], and he realized that he needed to use not triplets but quadruples $a+b\mathrm{i}+c\mathrm{j}+d\mathrm{k}$, where

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1.$$

So excited was he that he carved these formulas on a stone of Brougham Bridge.

Hamilton has written about the long process that led to his discovery, for example in [5]. He realized that he needed to give up commutativity (which, he admitted, "must, at first sight, seem strange and almost unallowable" [4]), before he eventually saw the need to go to four, rather than three, dimensions.

In modern terminology, Hamilton was trying to construct a normed division algebra [3], and we now know that such algebras exist only for dimensions 1, 2, 4 and 8.

Hamilton's discovery is a wonderful example of creativity in mathematics. Can we learn any lessons from it about how to stimulate creativity?

To explore that question, let us first identify the key features of the complex numbers that Hamilton was trying to generalize.

(1) A complex number $a+\mathrm{i}b$ has two parts: a real part, a, and an imaginary part, b.

(2) Multiplication of complex numbers is commutative.

(3) Multiplication of complex numbers is associative.

(4) $|z_1 z_2| = |z_1||z_2|$ for any complex numbers z_1 and z_2 (the law of moduli).

(5) $\mathrm{i}^2 = -1$.

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Figure 1: Sir William Rowan Hamilton. Etching after J. Kirkwood. Credit: Wellcome Library, London. Wellcome Images. Source: https://commons.wikimedia.org/wiki/File:Sir_William_Rowan_Hamilton._Etching_after_J._Kirkwood_after_Wellcome_V0002552.jpg.

Here, by comparison, are the corresponding features of quaternions.

(1) A quaternion $a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ has four parts: a real part a and three imaginary parts b, c, and d.

(2) Multiplication of quaternions is non-commutative.

(3) Multiplication of quaternions is associative.

(4) $|q_1 q_2| = |q_1||q_2|$ for any quaternions q_1 and q_2.

(5) $\mathrm{i}^2 = -1$, $\mathrm{j}^2 = -1$, $\mathrm{k}^2 = -1$, and $\mathrm{ijk} = -1$.

As can be seen, for quaternions, items (1) and (5) are extensions of their complex number equivalents, whilst items (3) and (4) are in essence identical to them. The lack of commutativity stated in property (2) is perhaps surprising, as commutativity is a property that we usually take for granted. Hamilton's discovery came before the development of matrix algebra, by Cayley in 1858 [2], which is probably the best-known example of a non-commutative algebra, but he would have been familiar with the non-commutativity of rotations in three or more dimensions.

It is, of course, very easy to make this comparison with hindsight. But the comparison gives an important clue as to how to be creative. The features of quaternions can all be discovered by asking "how might this be different?" of each known feature of complex numbers, as exemplified by these possibilities (we keep the law of moduli, which ensures that the product of two nonzero numbers is nonzero).

(1) A number might have two, three, four, ... imaginary parts.

(2) Multiplication might be commutative or non-commutative.

(3) Multiplication might be associative or non-associative.

(4) $i^2 = -1$ (since Hamilton wanted a complex number to be a special case of the new number) and other imaginary units might also square to -1. The products of different imaginary units need to be determined.

This list defines the possibilities to be explored, and some careful thinking along with well-focused trial and error will identify a combination that works.

Later in 1843, John T. Graves discovered the 8-dimensional octonions, and Cayley independently discovered them in 1845. The octonions are both non-commutative and non-associative under multiplication, so they arise from taking a different choice of possibilities from the list.

It is unlikely that Hamilton, Graves and Cayley actually discovered quaternions and octonions in this way. But they might have done so. Asking "how might this be different?" of all the features of something that exists now is a hugely powerful way of discovering ideas, for the question gets to the heart of what creativity is: something different from, and hopefully better than, something that exists now.

Hamilton's discovery of the quaternions is just one of many examples of creativity in mathematics that fit the pattern of combining pre-existing components in different ways. Others include Andrew Wiles's proof of Fermat's Last Theorem, Henri Poincaré's discovery of automorphic forms, and Olga Taussky's work on determinantal conditions for matrix nonsingularity [7]. Furthermore, seventy years of research on iterative refinement for the numerical solution of linear systems can be interpreted as arising from changing different features of the basic algorithm programmed by Wilkinson in 1948 for the Pilot ACE computer at the National Physical Laboratory [8].

In considering "How might this be different?" in any context, we can think about the following aspects. *Size*: can some quantity be bigger or smaller? In the quaternion example, we need 4 rather than 3 components. Often, thinking *much* bigger or *much* smaller (a quantity tending to infinity or to zero) will trigger useful ideas. *Sequence*: in a process involving a sequence of steps, do those steps have to be done in a certain order or can they be reordered or some steps even merged? *Established practice*: if a property or condition is conventionally assumed, can it nevertheless be modified or dropped?

The process for generating ideas that we have outlined is the basis of a six-step process that one of us (Dennis) has developed over the last twenty years [7]. It can be applied to any focus of attention for which one wants to generate ideas and is well suited to tackling mathematical problems.

While the process can be carried out by individuals, it is at its most powerful when used in small groups. Creativity is much richer when people speak to each other, ask each other questions, and share knowledge. A group can spot more problem features and can produce a wider selection of ways in which those features could be different than any individual is likely to do.

The authors met in 2011 at a creativity workshop sponsored by the Creativity@home initiative[1] of the Engineering and Physical Sciences Research Council. We have since collaborated on several creativity workshops, most recently one for the Manchester Numerical Linear Algebra group. Our creativity workshops consist of up to 20 people working in groups of up to 8, usually meeting for two days at an off-site location. They invariably generate a large number of ideas for later evaluation. These workshops are an excellent way to train people in creativity and help build a team environment in which creativity flourishes [7].

We have run some creativity workshops virtually in the last couple of years, and while they were successful we think that face to face events work better for all concerned. Our experience is consistent with recent research that

[1] `https://www.ukri.org/councils/epsrc/guidance-for-applicants/types-of-funding-we-offer/transformative-research/creativity-at-home/`

reports that "videoconferencing hampers idea generation because it focuses communicators on a screen, which prompts a narrower cognitive focus" [1].

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