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OPTIMAL PRECONDITIONING FOR RAVIART-THOMAS MIXED FORMULATION OF SECOND-ORDER ELLIPTIC PROBLEMS

CATHERINE E. POWELL* AND DAVID J. SILVESTER†

Abstract. We evaluate two preconditioning strategies for the indefinite linear system obtained from Raviart-Thomas mixed finite element formulation of a second-order elliptic problem with variable coefficients. In contrast to other approaches, our emphasis is on linear algebra; we establish the optimality of the preconditioning using basic properties of the finite element matrices.

The underlying saddle-point problem is well-posed in two function spaces, $H(div) \times L^2$ and $L^2 \times H^1$, leading to the possibility of two distinct types of preconditioner. For homogeneous Dirichlet boundary conditions, the discrete problems are identical. This motivates the use of Raviart-Thomas approximation in both frameworks yielding a non-conforming method in the second case.

We construct two block-diagonal preconditioners and establish a new uniform eigenvalue bound. Trials of preconditioned MINRES illustrate that both preconditioners are robust with respect to the mesh parameter, and with respect to jumps and anisotropies in the diffusion coefficients.

Key words. Second-order elliptic problems, saddle-point problems, variable coefficients, mixed finite elements, Raviart-Thomas, MINRES, preconditioning.

1. Introduction. Let Ω be a bounded domain in $I\!R^2$. We consider scalar, second-order elliptic problems of the form,

$$\begin{aligned} -\nabla \cdot \mathcal{A}\nabla p &= f \quad \text{in } \Omega, \\ p &= g \quad \text{on } \partial\Omega_D, \\ \mathcal{A}\nabla p \cdot \vec{n} &= 0 \quad \text{on } \partial\Omega_N. \end{aligned} \tag{1.1}$$

where $\partial\Omega_D \neq \emptyset$ and $\mathcal{A} = \mathcal{A}(\vec{x})$ is a 2×2 bounded, symmetric and uniformly positive definite matrix-valued function. This implies that there exist positive constants γ and Γ with $0 < \gamma \leq \Gamma$ such that,

$$\gamma(\vec{v},\vec{v}) \leq (\mathcal{A}^{-1}\vec{v},\vec{v}) \leq \Gamma(\vec{v},\vec{v}), \tag{1.2}$$

for every $\vec{v} : \Omega \to I\!R^2$.

Boundary-value problems of this type occur in mathematical models of important physical processes such as fluid flow through porous media. To fix ideas, we call p and $\vec{u} = \mathcal{A}\nabla p$ the pressure and velocity solutions respectively. Mixed finite element methods are favoured when $\vec{u}$ is the variable of interest since post-processing primal pressure solutions leads to loss of accuracy. Mixed velocity solutions are insensitive to the variation in the coefficients $\mathcal{A}$, see [14, pp.240–241]. In addition, mixed methods conserve mass locally, a crucial feature in the modelling of groundwater flow. Other advantages of a mixed approximation are discussed in [7].

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1.1. Preconditioning strategies. It is known (see [2], [6], [7], [15], [17], [21]) that mixed discretisation of (1.1) yields an indefinite linear system of the form,

$$\underbrace{\begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix}}_{C} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{g} \\ -\underline{f} \end{pmatrix}, \tag{1.3}$$

where $M \in I\!R^{n\times n}$ is symmetric and positive definite, $B \in I\!R^{m\times n}$ (with $m \leq n$) has rank m and $\underline{u}$ and $\underline{p}$ are the discrete velocity and pressure solutions. The coefficient matrix C becomes ill-conditioned with mesh refinement (see section 1.5). As a result, Krylov solution methods are not robust with respect to the mesh parameter and preconditioners are required. A variety of preconditioning strategies have already been suggested, the majority of which require the transformation of the underlying saddle-point problem to a *positive-definite* one. Indeed, there is a rich literature concerning domain-decomposition and multigrid techniques for these reduced problems (see, for example, [9], [10], [11], [20], [22]). This is not our approach. We follow [2], [17] and [23] and solve the full *indefinite* system (1.3) using preconditioned MINRES (see [13]).

In [17], Rusten and Winther propose a preconditioner of the form,

$$\mathcal{P} = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{S} \end{pmatrix}, \tag{1.4}$$

where $\mathcal{S}$ is an approximation to a Schur complement operator on the pressure space and $\mathcal{I}$ is the identity. Vassilevski and Lazarov, [23], construct a preconditioner for a transformed indefinite problem. Convergence rates are shown to be independent of the mesh parameter but, critically, iteration counts are affected by two artificial parameters and it is not clear what the optimal choices are. Arnold, Falk and Winther observe in [2] that C also has the same mapping properties as the matrix operator,

$$\mathcal{P} = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & \mathcal{I} \end{pmatrix}, \tag{1.5}$$

where $\mathcal{H} : H(div) \times H(div) \to I\!R$ is the $H(div)$ operator defined by $(\mathcal{H}\vec{u}, \vec{v}) = (\vec{u}, \vec{v}) + (\nabla\cdot\vec{u}, \nabla\cdot\vec{v})$. They approximate $\mathcal{H}$ using domain decomposition and multigrid. Their discussion is not matrix based.

1.2. Overview. In this paper, a preconditioner is *optimal* if the MINRES iteration count is bounded as the mesh parameter tends to zero. The aim here is to illustrate the optimality of two preconditioning strategies using simple linear algebra arguments and further to discuss their performance when the ratio $\frac{\Gamma}{\gamma}$ of constants in (1.2) is large.

In the remainder of this section, we review Raviart-Thomas approximation and convergence properties of MINRES. In section 2 we consider a solution to (1.1) in the space $H(div) \times L^2$, derive the discrete problem using Raviart-Thomas elements, and construct an optimal preconditioner associated with this function space. A new uniform eigenvalue bound is established and shown to be tight. In section 3, the solution is sought in the alternative space $L^2 \times H^1$. A second optimal preconditioner is described. Performance of both preconditioners is assessed numerically.

1.3. Notation. Let $\Omega \subseteq I\!R^2$ be a convex polygon with boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$. $L^2(\Omega)$ is the usual space of square-integrable scalar functions with inner-product $(\cdot,\cdot)$. For the space of vector functions $L^2(\Omega)^2$, the definition is understood to hold componentwise. We write $\|\cdot\|_0$ to denote the Lebesque measure for both spaces. The subspace,

$$H(div;\Omega) = \left\{\vec{v} \in L^2(\Omega)^2 \mid \nabla\cdot\vec{v} \in L^2(\Omega)\right\},$$

contains vectors with square-integrable divergence. The associated inner-product is $(\vec{u},\vec{v})_{div} = (\vec{u},\vec{v}) + (\nabla\cdot\vec{u},\nabla\cdot\vec{v})$ and we denote the induced norm $\|\cdot\|_{div}$. The Sobelev space,

$$H^1(\Omega) = \left\{w \in L^2(\Omega) \mid \nabla w \in L^2(\Omega)^2\right\},$$

is equipped with the usual inner-product $(w,v)_1 = (w,v) + (\nabla w, \nabla v)$ and the induced norm $\|\cdot\|_1$.

1.4. Raviart-Thomas finite elements. Let T_h denote a partition of Ω into a mesh of triangles or rectangles $\{K_1,\ldots,K_s\}$ with maximum edge size h. The mesh is refined such that $\{T_{h_1}, T_{h_2},\ldots\}$ is a family of shape-regular, affine partitions of Ω. Our numerical experiments are performed on rectangular grids and so, for given h and T_h, the Raviart-Thomas spaces $V_h \subset H(div;\Omega)$ and $W_h \subset L^2(\Omega)$ of integer index k (see [15]) are,

$$V_h = \{\vec{v_h} \in H(div;\Omega) \mid \vec{v_h}\,|_K \in Q_{k+1,k}(K) \times Q_{k,k+1}(K) \quad \forall K \in T_h\}, \quad (1.6)$$

and

$$W_h = \{w_h \in L^2(\Omega) \mid w_h\,|_K \in P_k(K) \quad \forall K \in T_h\}. \quad (1.7)$$

$P_k(K)$ is the space of polynomials of degree less than or equal to k on rectangle K and $Q_{r,s}(K)$ is the set of polynomials of degree less than or equal to r in x and degree less than or equal to s in y. We use the lowest order*$(k=0)$ spaces. Velocities and pressures then have the special piecewise forms,

$$\vec{v_h}\,|_K = \begin{pmatrix} v_1 + v_2 x \\ v_3 + v_4 y \end{pmatrix}, \quad w_h\,|_K = w_1, \quad (1.8)$$

respectively, where v_1, v_2, v_3, v_4 and w_1 are constants. The degrees of freedom for this element are illustrated in figure 1.1.
These are normal components at edges for velocities and the pressure solution is sampled at element centroids. The triangular version is analgously defined.

*For highly discontinuous and anisotropic diffusion coefficients, solution regularity is low.

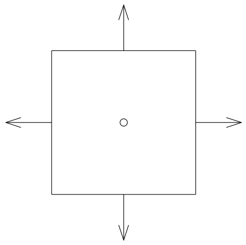

FIG. 1.1. *Degrees of freedom for the lowest order, rectangular Raviart-Thomas element.*

1.5. MINRES. The iterative solver MINRES can be applied to sparse, symmetric and indefinite systems of the form (1.3). (A template is given in [3].) It is known (see [12]) that a sharp upper bound for the kth residual error $\underline{r}^{(k)}$ is given by,

$$\frac{\| \underline{r}^{(k)} \|_2}{\| \underline{r}^{(0)} \|_2} \leq \min_{p_k} \max_{i=1:n+m} | p_k(\lambda_i) |, \tag{1.9}$$

where $\{\lambda_1, \ldots, \lambda_{n+m}\}$ denotes the set of eigenvalues of the coefficient matrix and p_k is a polynomial of degree k satisfying $p_k(0) = 1$. Theoretical eigenvalue bounds for (1.3) are derived by Rusten and Winther in [17] and by Silvester and Wathen in [21]. For Raviart-Thomas discretisations of (1.1) in $I\!R^2$ and mildly varying diffusion coefficients $\mathcal{A}$, $M \approx h^2 I$ while the maximum and minimum singular values of B behave like h and h^2 respectively. The theory of [17] leads to an h-dependent eigenvalue bound of the form $[-ah, -bh^2] \cup [ch^2, dh]$ for positive constants a, b, c, d. For the unpreconditioned problem, the MINRES iteration count then typically increases like h^{-1}. When $\mathcal{A}$ is ill-conditioned, convergence deteriorates further. The solution process becomes prohibitively expensive as the mesh is refined and this is a serious flaw in practical applications.

In sections 2 and 3 we discuss two optimal block-diagonal preconditioners of the form,

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \tag{1.10}$$

where the matrices P_1 and P_2 are discrete representations of suitable norms on the velocity and pressure finite element spaces. We demonstrate that the eigenvalues of the generalised problem,

$$\begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \lambda \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix}, \tag{1.11}$$

for the chosen P_1 and P_2 do not depend on the mesh parameter.

2. H_{div} preconditioning.

2.1. Variational problem. Substituting $\vec{u} = \mathcal{A}\nabla p$ in (1.1) yields the first-order system,

$$\begin{aligned} \mathcal{A}^{-1}\vec{u} - \nabla p &= 0, \\ \nabla \cdot \vec{u} &= -f \quad \text{in } \Omega, \\ p &= g \quad \text{on } \partial\Omega_D, \\ \vec{u}\cdot\vec{n} &= 0 \quad \text{on } \partial\Omega_N. \end{aligned} \tag{2.1}$$

The data requirements are $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega_D)$, the set of traces of $H^1(\Omega)$ functions on $\partial\Omega_D$. Now, (2.1) may be formulated as a saddle-point problem in two function spaces, as discussed in [4]. First, we choose test spaces,

$$V = H_{0,N}(div;\Omega) = \{\vec{v} \in H(div;\Omega) \mid \vec{v}\cdot\vec{n} = 0 \text{ on } \partial\Omega_N\}$$

and $W = L^2(\Omega)$. Multiplying by arbitrary $\vec{v}$ and w in (2.1), integrating the *first* equation by parts and imposing essential boundary conditions yields the variational problem,

find $(\vec{u}, p) \in V \times W$ satisfying,

$$\begin{aligned} (\mathcal{A}^{-1}\vec{u}, \vec{v}) + (p, \nabla\cdot\vec{v}) &= < g, \vec{v}\cdot\vec{n} > \quad \forall \vec{v} \in V, \\ (w, \nabla\cdot\vec{u}) &= -(f, w) \quad \forall w \in W, \end{aligned} \tag{2.2}$$

where $\langle g, \vec{v}\cdot\vec{n}\rangle = \int_{\partial\Omega_D} g\,\vec{v}\cdot\vec{n}\,ds$. Condition (1.2) ensures that $\left(\mathcal{A}^{-1}\vec{u}, \vec{v}\right)$ is bounded.

The stability theory of Brezzi and Babuška (see [7], [16]) tells us that a unique solution $(\vec{u}, p)$ exists if and only if there are constants α, β, bounded away from zero, such that,

$$\sup_{\vec{v}\in V} \frac{(w, \nabla\cdot\vec{v})}{\| \vec{v} \|_{div}} \geq \beta \| w \|_0 \qquad \forall v \in W, \tag{2.3}$$

$$(\mathcal{A}^{-1}\vec{v}, \vec{v}) \geq \alpha \| \vec{v} \|_{div}^2 \qquad \forall \vec{v} \in Z, \tag{2.4}$$

where $Z = \{\vec{v} \in V \mid (w, \nabla\cdot\vec{v}) = 0 \quad \forall w \in W\}$. We refer to these conditions as inf-sup stability and coercivity respectively. Since $\nabla \cdot V \subset W$, the constraint space Z contains only divergence-free velocities and $\|\cdot\|_{div}$ is equivalent to $\|\cdot\|_0$ on this space. Thus, $\alpha = \gamma$ in (1.2). The constant β depends only on the shape of the domain Ω.

To implement the conforming Raviart-Thomas discretisation, we choose $V_h \subset V$ and $W_h \subset W$ defined in section 1.4. Solving (2.2) over these spaces leads to the discrete problem,

find $(\vec{u_h}, p_h) \in V_h \times W_h$ satisfying,

$$\begin{aligned} (\mathcal{A}^{-1}\vec{u_h}, \vec{v_h}) + (p_h, \nabla\cdot\vec{v_h}) &= \langle g, \vec{v_h}\cdot\vec{n}\rangle \quad \forall \vec{v_h} \in V_h, \\ (w_h, \nabla\cdot\vec{u_h}) &= -(f, w_h) \quad \forall w_h \in W_h. \end{aligned} \tag{2.5}$$

Stability of the approximation is established in [15]. A crucial property of this scheme is that $\nabla \cdot V_h \subset W_h$ and hence the discrete version of condition (2.4) is trivially satisfied if (1.2) holds.

2.2. Finite element matrices. To express (2.5) as a linear algebra problem we choose bases such that $V_h = \text{span}\{\vec{\varphi}_i\}_{i=1}^n$ and[†] $W_h = \text{span}\{\phi_j\}_{j=1}^m$. Defining $M \in I\!R^{n\times n}$ by $M_{i,j} = \left(\mathcal{A}^{-1}\vec{\varphi}_i, \vec{\varphi}_j\right)$ for $1 \le i,j \le n$, and $B \in I\!R^{m\times n}$ by $B_{k,j} = (\phi_k, \nabla \cdot \vec{\varphi}_j)$ for $1 \le k \le m$, we have,

$$(\vec{u_h}, p_h) = \left(\sum_{i=1}^n u_i \vec{\varphi}_i \,,\, \sum_{j=1}^m p_j \phi_j \right),$$

where $\underline{u} = [u_1, \ldots, u_n]$ and $\underline{p} = [p_1, \ldots, p_m]$ are the vectors of coefficients satisfying,

$$\underbrace{\begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix}}_{C} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{g} \\ \underline{f} \end{pmatrix}, \tag{2.6}$$

with $g_i = \langle g, \vec{\varphi}_i \cdot \vec{n} \rangle$ and $f_k = -(f, \phi_k)$ for $1 \le i \le n$ and $1 \le k \le m$.

Next, we define the velocity mass matrix $M_{\mathcal{I}} \in I\!R^{n\times n}$, the velocity divergence matrix $D \in I\!R^{n\times n}$ and the pressure mass matrix, $N \in I\!R^{m\times m}$ by,

$$\begin{aligned} M_{\mathcal{I}\,i,j} &= (\vec{\varphi}_i, \vec{\varphi}_j) & 1 \le i,j \le n, \\ D_{i,j} &= (\nabla \cdot \vec{\varphi}_i, \nabla \cdot \vec{\varphi}_j) & 1 \le i,j \le n, \\ N_{r,s} &= (\phi_r, \phi_s) & 1 \le r,s \le m. \end{aligned}$$

These give useful discrete representations of the L^2 and $H(div)$ velocity norms and the L^2 pressure norm since for any $\vec{v_h}$ in V_h and w_h in W_h we can write,

$$\| \vec{v_h} \|_0^2 = \underline{v}^T M_{\mathcal{I}} \underline{v}, \quad \| \vec{v_h} \|_{div}^2 = \underline{v}^T (M_{\mathcal{I}} + D) \underline{v}, \quad \| w_h \|_0^2 = \underline{w}^T N \underline{w}, \tag{2.7}$$

where $\underline{v}$ and $\underline{w}$ are vectors of coefficients corresponding to the expansions of $\vec{v_h}$ and w_h in the chosen basis sets.

It is clear that for the lowest-order Raviart-Thomas spaces, D and N are cheap to assemble. The entries are constants and no numerical integration is required. N is a diagonal matrix with entries $\{\Box_1, \ldots, \Box_m\}$, where $\Box_i$ denotes the area of the ith rectangle. Notice also that for unit diffusion coefficients $\mathcal{A} = \mathcal{I}$, (the identity matrix), the mass matrix $M_{\mathcal{I}}$ is identical to M. The algebraic relationship between B, D and N is nicely summarised in the following lemma.

LEMMA 2.1. *If $\nabla \cdot V_h \subset W_h$, then,*

$$D = B^T N^{-1} B.$$

[†] n = # edges - # edges on $\partial\Omega_N$, m = # elements.

Proof. Consider the matrices B, N and D in operator form. That is,

$$
\begin{aligned}
(\mathcal{B}\vec{x_h}, z_h) &= (z_h, \nabla\cdot\vec{x_h}) \;=\; \left(\mathcal{B}^T z_h,\, \vec{x_h}\right) && \forall \vec{x_h} \in V_h, z_h \in W_h, \\
(\mathcal{D}\vec{x_h}, \vec{y_h}) &= (\nabla\cdot\vec{x_h}, \nabla\cdot\vec{y_h}) && \forall \vec{x_h}, \vec{y_h} \in V_h, \\
(\mathcal{N} z_h, w_h) &= (z_h, w_h) \;=\; \left(z_h,\, \mathcal{N}^{-1} w_h\right) && \forall z_h, w_h \in W_h,
\end{aligned}
$$

with $\mathcal{N}$ acting as the identity operator on W_h. If by construction we have $\nabla\cdot V_h \subset W_h$ then,

$$
\begin{aligned}
(\mathcal{D}\vec{x_h}, \vec{x_h}) &= (\nabla\cdot\vec{x_h}, \nabla\cdot\vec{x_h}) \\
&= (\nabla\cdot\vec{x_h},\, \mathcal{N}\nabla\cdot\vec{x_h}) \\
&= (\mathcal{B}\vec{x_h},\, \mathcal{N}\nabla\cdot\vec{x_h}) \\
&= \left(\mathcal{N}^{-1}\mathcal{B}\vec{x_h}, \nabla\cdot\vec{x_h}\right) \\
&= \left(\mathcal{B}^T\mathcal{N}^{-1}\mathcal{B}\vec{x_h},\, \vec{x_h}\right),
\end{aligned}
$$

which proves the result. $\square$

By writing the inf-sup condition (2.3) in discrete form we see that the inf-sup constant β satisfies,

$$
\beta^2 \leq \frac{\underline{w}^T B\,(M_{\mathcal{I}} + D)^{-1}\, B^T \underline{w}}{\underline{w}^T N \underline{w}} \qquad \forall\, \underline{w} \in I\!R^m,
$$

and so β is $\sqrt{\lambda_{min}}$, where λ_{min} is the minimum eigenvalue of

$$
B\,(M_{\mathcal{I}} + D)\, B^T \underline{w} = \lambda N \underline{w}. \tag{2.8}
$$

Note that β does not depend on $\mathcal{A}$ and so the method is inf-sup stable even when $\frac{\Gamma}{\gamma}$ is large. By Cauchy-Schwarz we also have,

$$
(w_h, \nabla\cdot\vec{v_h}) \leq \| w_h \|_0 \| \nabla\cdot\vec{v_h} \|_0 \leq \| w_h \|_0 \| \vec{v_h} \|_{div} \quad \forall \vec{v_h} \in V_h, w_h \in W_h
$$

and so $\lambda_{max} \leq 1$ in (2.8).

Consider now preconditioning (2.6) with the block-diagonal matrix,

$$
P_{div} = \begin{pmatrix} M + D & 0 \\ 0 & N \end{pmatrix}.
$$

We call P_{div} an '$H(div)$' preconditioner since for any $\vec{v_h}$ in V_h,

$$
\underline{v}^T(\frac{1}{\Gamma}M + D)\underline{v} \;\leq \| \vec{v_h} \|^2_{div} \leq\; \underline{v}^T(\frac{1}{\gamma}M + D)\underline{v} \tag{2.9}
$$

where γ and Γ are the constants that describe the boundedness of the diffusion coefficients in (1.2). That is, $M + D$ engenders a discrete representation of a norm that is equivalent to $\| \cdot \|_{div}$ on V_h.

2.3. Eigenvalue bounds. We consider first the trivial case of unit diffusion coefficients.

LEMMA 2.2. *If $\mathcal{A} = \mathcal{I}$, the $n+m$ eigenvalues of the generalised eigenvalue problem,*

$$\begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \lambda \begin{pmatrix} M+D & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} \tag{2.10}$$

lie in the intervals $\left[-1, -\beta^2\right] \cup [1]$ where β is the inf-sup constant.

Proof. The eigenvalues $\{\lambda_i\}_{i=1}^{m+n}$ satisfy,

$$M\underline{u} + B^T\underline{p} = \lambda\,(M+D)\,\underline{u},$$
$$B\underline{u} = \lambda N\underline{p}.$$

Suppose $\lambda = 1$, then $B^T N^{-1} B\underline{u} = D\underline{u}$, which is true, by lemma 2.1, for all $\underline{u} \in I\!R^n$. Since $D \in I\!R^{n\times n}$ is symmetric, there are n *distinct* linearly independent eigenvectors corresponding to $\lambda = 1$ and hence n eigenvalues equal to 1. In addition, $\underline{u}$ and $\underline{p}$ satisfy,

$$B^T\underline{p} = (\lambda - 1)\,(M+D)\,\underline{u} + D\underline{u},$$
$$B\underline{u} = \lambda N\underline{p}.$$

Applying lemma 2.1 yields,

$$\begin{aligned} B\,(M+D)^{-1}\,B^T\underline{p} &= (\lambda-1)\,B\underline{u} + B\,(M+D)^{-1}\,D\underline{u}, \\ &= \lambda\,(\lambda-1)\,N\underline{p} + B\,(M+D)^{-1}\,B^T N^{-1} B\underline{u} \\ &= \lambda\,(\lambda-1)\,N\underline{p} + \lambda B\,(M+D)^{-1}\,B^T\underline{p}. \end{aligned}$$

Thus, if $\lambda \neq 1$,

$$B\,(M+D)^{-1}\,B^T\underline{p} = -\lambda N\underline{p}.$$

Now if $\mathcal{A} = \mathcal{I}_{2\times 2}$, $M = M_{\mathcal{I}}$ and so the result follows from the discussion in section 2.2. □

For general diffusion coefficients, we have the following eigenvalue bound. This seems to be a new result and extends the bound established by Vassilevski and Lazarov[‡] in [23].

LEMMA 2.3. *The $n+m$ eigenvalues of the generalised eigenvalue problem,*

$$\begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \lambda \begin{pmatrix} M+D & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} \tag{2.11}$$

[‡]The analysis here is is matrix form, with a different choice of X and no penalty parameters

lie in the intervals,

$$\left(-1, -\frac{\mu_{min}}{\Box_{max} + \mu_{min}}\right] \cup [1], \tag{2.12}$$

where μ_{min} is the minimum eigenvalue of the Schur complement $BM^{-1}B^T$ and $\Box_{max}$ is the area of the largest rectangle in the mesh.

Proof. $\lambda = 1$ with multiplicity n by lemma 2.1 since $B^T N^{-1} B = D$ is symmetric. For $\lambda \neq 1$ we have,

$$B\left(M + D\right)^{-1} B^T \underline{p} = -\lambda N \underline{p}.$$

Since $D = B^T N^{-1} B$, the negative eigenvalues $\{\lambda_i\}_{i=1}^m$ are the eigenvalues of the matrix,

$$-N^{-\frac{1}{2}} B \left(M + B^T N^{-1} B\right)^{-1} B^T N^{-\frac{1}{2}}.$$

Now,
$$\begin{aligned}
&N^{-\frac{1}{2}} B \left(M + B^T N^{-1} B\right)^{-1} B^T N^{-\frac{1}{2}} \\
&= N^{-\frac{1}{2}} B M^{-\frac{1}{2}} \left(I + M^{-\frac{1}{2}} B^T N^{-\frac{1}{2}} N^{-\frac{1}{2}} B M^{-\frac{1}{2}}\right)^{-1} M^{-\frac{1}{2}} B^T N^{-\frac{1}{2}} \\
&= X\left(I + X^T X\right)^{-1} X^T
\end{aligned}$$

where $X = N^{-\frac{1}{2}} B M^{-\frac{1}{2}}$. Following lemma 3.1 of [23], we apply the Sherman-Morrison-Woodbury formula to obtain,

$$\left(I + X^T X\right)^{-1} = I - X^T \left(I + XX^T\right)^{-1} X.$$

Hence,

$$X\left(I + XX^T\right)^{-1} X^T = X\left(I - X^T\left(I + XX^T\right)^{-1} X\right) X^T.$$

Let $\underline{x}_i$ be an eigenfunction of XX^T and σ_i denote the corresponding eigenvalue. Then,

$$\begin{aligned}
X\left(I + XX^T\right)^{-1} X^T \underline{x}_i &= XX^T \underline{x}_i - XX^T\left(I + XX^T\right)^{-1} XX^T \underline{x}_i \\
&= \sigma_i \underline{x}_i - \left(\frac{\sigma_i^2}{1 + \sigma_i}\right) \underline{x}_i \\
&= \left(\frac{\sigma_i}{1 + \sigma_i}\right) \underline{x}_i.
\end{aligned}$$

The eigenvalues of $X\left(I + XX^T\right)^{-1} X^T$ are then the set of values $\{\frac{\sigma_i}{1+\sigma_i}\}_{i=1}^m$, where $\{\sigma_i\}_{i=1}^m$ are the eigenvalues of $XX^T = N^{-\frac{1}{2}} B M^{-1} B^T N^{-\frac{1}{2}}$. It follows that the negative eigenvalues of our generalised eigenvalue problem (2.11) lie in the interval,

$$\left[-\max_i \frac{\sigma_i}{1 + \sigma_i}, -\min_i \frac{\sigma_i}{1 + \sigma_i}\right].$$

Since $\sigma_i > 0$ for all i, we have

$$\{\lambda_i\}_{i=1}^m \in \left(-1, -\frac{\sigma_{min}}{1 + \sigma_{min}}\right],$$

where σ_{min} is the minimum eigenvalue of $N^{-1}BM^{-1}B^T$. Finally, since $\frac{\mu_{min}}{\Box_{max}} \leq \sigma_{min} \leq \frac{\mu_{min}}{\Box_{min}}$ for lowest-order Raviart-Thomas elements, we obtain,

$$\{\lambda_i\}_{i=1}^{m} \in \left(-1, \frac{-\frac{\mu_{min}}{\Box_{max}}}{1+\frac{\mu_{min}}{\Box_{max}}}\right] = \left(-1, -\frac{\mu_{min}}{\Box_{max}+\mu_{min}}\right].$$

$\Box$

For mildly varying coefficients, μ_{min} behaves like h^2 and the bound in lemma 2.3 is independent of h. Clearly, the bound deteriorates in cases where the coefficients $\mathcal{A}$ are such that μ_{min} is close to zero.

2.3.1. Numerical eigenvalues. First, we study (1.1) on $\Omega = [0,1] \times [0,1]$ with uniform rectangular meshes, $\mathcal{A} = I$ and $\partial\Omega = \partial\Omega_D$. The eigenvalues of the preconditioned system $\{\lambda_1, \ldots, \lambda_{n+m}\}$ are listed in table 2.1 and indicate that our theoretical bounds are tight.

h	$-\beta^2$	μ_{min}	$-\frac{\mu_{min}}{h^2+\mu_{min}}$	λ_1	λ_m	λ_{m+1}	λ_{m+n}
$\frac{1}{8}$	-0.9524	0.3124	-0.9524	-0.9993	-0.9524	1	1
$\frac{1}{16}$	-0.9519	0.0774	-0.9519	-0.9998	-0.9519	1	1
$\frac{1}{32}$	-0.9518	0.0193	-0.9518	-0.9999	-0.9518	1	1

TABLE 2.1
Eigenvalues of preconditioned matrix; unit coefficients

Next, we consider anisotropic coefficients of the form,

$$\mathcal{A}(\vec{x}) = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \quad \forall\, \vec{x} \in \Omega, \tag{2.13}$$

with $\epsilon \to \infty$ or $\epsilon \to 0$ so that $\mathcal{A}$ is ill-conditioned. Using lemma 2.3 we can easily gauge the performance of the preconditioner P_{div}. On rectangular meshes with fixed h, as $\epsilon \to \infty$, all the eigenvalues of the Schur complement are $O(\epsilon)$. As the mesh is refined, the right-hand bound for the negative eigenvalues converges to -1. MINRES convergence is fast. On the other hand, if $\epsilon \to 0$, μ_{min} is $O(1)$ for fixed h and behaves like h^2 with mesh refinement, also yielding an optimal eigenvalue bound. The eigenvalues of the preconditioned system for varying ϵ and fixed h are plotted in figure 2.1. The associated values of μ_{min} are listed in table 2.2.
Note that for anisotropic coefficients and homogeneous Dirichlet boundary conditions, the analytical pressure solution generally has boundary layers. Anisotropic meshes should be used to resolve them. For the calculations above, uniform meshes were used simply to illustrate the asymptotic properties of the eigenvalue bound.

Now suppose that we have 'jumping' or discontinuous coefficients,

$$\mathcal{A}(\vec{x}) = \begin{cases} \epsilon I & \forall\, \vec{x} \in \Omega_*, \\ I & \forall \vec{x} \in \Omega \backslash \Omega_*, \end{cases} \tag{2.14}$$

$h \quad \epsilon$	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$\frac{1}{8}$	0.1718	0.1578	0.1564	0.1562	0.1562	0.1562
$\frac{1}{16}$	0.0425	0.0391	0.0387	0.0387	0.0387	0.0387
$\frac{1}{32}$	0.0106	0.0097	0.0096	0.0096	0.0096	0.0096
$h \quad \epsilon$	10^{1}	10^{2}	10^{3}	10^{4}	10^{5}	10^{6}
$\frac{1}{8}$	1.7182	15.777	1.564e2	1.562e3	1.562e4	1.562e5
$\frac{1}{16}$	0.4254	3.9064	3.872e1	3.868e2	3.868e3	3.868e4
$\frac{1}{32}$	0.1061	0.9742	9.6557	9.647e1	9.646e2	9.646e3

TABLE 2.2

Minimum eigenvalue of Schur complement, anisotropic coefficients

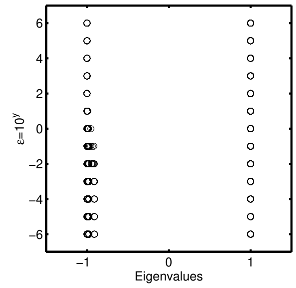

FIG. 2.1. *Eigenvalues of preconditioned system*

where I is the identity matrix and Ω_* is the so-called jump zone. Setting $0 < \epsilon << 1$ corresponds to a zone of low permeability, a feature common to oil reservoir models. We choose $\Omega_* = [0.25, 0.5] \times [0.25, 0.75]$ and values of ϵ between 10^{-6} and 10^{6}. The corresponding values of μ_{min} are listed in table 2.3. If $\epsilon \to \infty$, the right-hand bound for the negative eigenvalues in lemma 2.3 converges rapidly to -1 as $h \to 0$. In the case $\epsilon \to 0$, however, $\mu_{min} = O(\epsilon)$ and so the bound converges to zero as $\epsilon \to 0$.

$h \quad \epsilon$	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$\frac{1}{8}$	0.2074	0.0308	0.0036	3.648e-4	3.648e-5	3.649e-6
$\frac{1}{16}$	0.0506	0.0078	0.0008	8.051e-5	8.053e-6	8.054e-7
$\frac{1}{32}$	0.0126	0.0019	0.0002	1.943e-5	1.949e-6	1.949e-7
$h \quad \epsilon$	10^{1}	10^{2}	10^{3}	10^{4}	10^{5}	10^{6}
$\frac{1}{8}$	0.3409	0.3452	0.3456	0.3457	0.3547	0.3547
$\frac{1}{16}$	0.0846	0.0857	0.0858	0.0858	0.0858	0.0858
$\frac{1}{32}$	0.0211	0.0214	0.0214	0.0214	0.0214	0.0214

TABLE 2.3

Minimum eigenvalue of Schur complement, jumping coefficients

However, if $\mathcal{A}$ is a constant diagonal matrix, we can avoid this phenomena when

$\epsilon < 1$ by rescaling the source term f and setting,

$$\mathcal{A}(\vec{x}) = \begin{cases} I & \forall \vec{x} \in \Omega_* \\ \frac{1}{\epsilon} I & \forall \vec{x} \in \Omega \backslash \Omega_* \end{cases} \tag{2.15}$$

so that the minimum eigenvalue of $BM^{-1}B^T$ is $O(1)$ and bounded away from zero, yielding an optimal eigenvalue bound. The solution to the original, unscaled problem can be retrieved by simple post-scaling. The eigenvalues of the preconditioned system for fixed h and varying ϵ are shown in figure 2.2. In the second plot, scaling is used for $\epsilon < 1$.

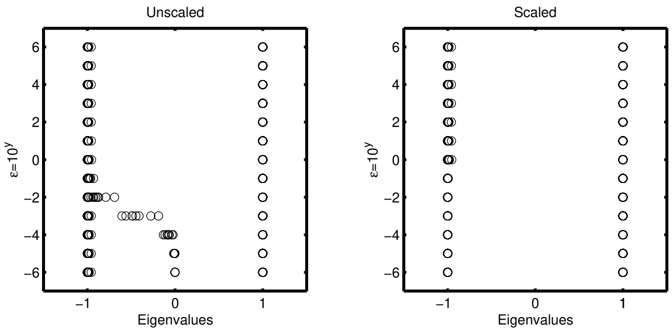

FIG. 2.2. *Eigenvalues of preconditioned system*

2.3.2. Preconditioned MINRES. In this section, we test our H_{div} preconditioner. Problem (1.1) is discretised on $\Omega = [0,1] \times [0,1]$ using rectangular meshes. For a specified *tol*, the stopping criterion is,

$$\frac{\| P_{div}^{-1} \underline{r}^{(i)} \|_0}{\| P_{div}^{-1} \underline{r}^{(0)} \|_0} \leq tol. \tag{2.16}$$

Example 1 We begin with the trivial case of unit coefficients, $\mathcal{A} = I$, $f = 1$ and homogeneous Dirichlet boundary conditions with $tol = 10^{-6}$. Iteration counts for the preconditioned and unpreconditioned problems are given in table 2.4.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
preconditioned	5	5	5	5
unpreconditioned	25	75	165	311

TABLE 2.4
Iteration counts; homogeneous Dirichlet

Alternatively, we can choose mixed boundary conditions, $p = 1$ on $\{0\} \times [0,1]$, $p = 0$ on $\{1\} \times [0,1]$ and $\vec{u} \cdot \vec{n} = 0$ on $(0,1) \times \{0,1\}$. Iteration counts for this modified problem are recorded in table 2.5.

Example 2 The eigenvalues of our preconditioned system do not depend on the structure of $\mathcal{A}$. To demonstrate this fact we run MINRES with coefficients,

$$\mathcal{A}(\vec{x}) = \begin{pmatrix} 1 + 4\left(x^2 + y^2\right) & 3xy \\ 3xy & 1 + 11\left(x^2 + y^2\right) \end{pmatrix}. \tag{2.17}$$

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
preconditioned	6	6	6	6
unpreconditioned	18	34	66	130

TABLE 2.5

Iteration counts; mixed boundary conditions

Iteration counts are reported in table 2.6. Next we set $\mathcal{A}$ to be the diagonal matrix of the eigenvalues of $\mathcal{A}$ in (2.17). (They satisfy $1 \leq \lambda(\mathcal{A}) \leq 25$). Iteration counts are given in table 2.7. The preconditioner performs equally well in both experiments. Convergence does not deteriorate in the non-diagonal case.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
preconditioned	5	5	5	5
unpreconditioned	191	426	849	1615

TABLE 2.6

Iteration counts; non-diagonal coefficients

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
preconditioned	5	5	5	5
unpreconditioned	146	343	706	1381

TABLE 2.7

Iteration counts; diagonal coefficients

Example 3 We now test the performance of our preconditioner when discontinuous permeability coefficients are present. The jump parameter is $\epsilon \in \left(10^{-6}, 10^{+6}\right)$ in the sense that,

$$\mathcal{A}(\vec{x}) = \begin{cases} \epsilon I & \forall\, \vec{x} \in \Omega_* \\ I & \forall\, \vec{x} \in \Omega \backslash \Omega_*. \end{cases}$$

We choose $\Omega_* = \left(\frac{1}{3}, \frac{2}{3}\right) \times \left(\frac{1}{3}, 1\right)$, $f = 0$ and mixed boundary conditions, $\vec{u} \cdot \vec{n} = 0$ on $\partial\Omega_N$, $p = 1 - x$ on $\partial\Omega_D$ where,

$$\partial\Omega_N = [0,1] \times 0 \cup \{0,1\} \times [0, 0.5], \quad \partial\Omega_D = \Omega \backslash \Omega_N.$$

For $\epsilon < 1$, we solve the rescaled problem as discussed in section 2.3.1. The stopping tolerance used is 10^{-9}. This fine tolerance was needed to ensure that the solution of the rescaled problem is the same (to 4 decimal places) as the unscaled problem.

In all of these examples with rectangular meshes, P_{div} performs optimally and is robust with respect to the diffusion coefficients. We can use triangular meshes with equal success except in the case of anisotropic coefficients of the form (2.13). Uniform triangular meshes are unsuitable when $\epsilon \to 0$ since, in that case, all the

h	10^{-6}	10^{-3}	10^{-1}	10^{1}	10^{3}	10^{6}
$\frac{1}{16}$	6	5	6	7	8	7
$\frac{1}{32}$	6	6	6	7	8	7
$\frac{1}{64}$	6	6	6	7	8	7

TABLE 2.8
Iteration counts

eigenvalues of $BM^{-1}B^T$ are $O(\epsilon)$. We now consider an alternative preconditioning strategy.

3. H_0^1 preconditioning.

3.1. Variational problem. The variational problem (2.2) is well-posed in a second pair of norms. To see this, set $V = L^2(\Omega)^2$ and

$$W = H_D^1(\Omega) = \left\{ w \in H^1(\Omega) \mid w = g \text{ on } \partial\Omega_D \right\}. \tag{3.1}$$

Multiplying by arbitrary $\vec{v}$ and w in (2.1), integrating the *second* equation by parts and imposing essential boundary conditions now yields the alternative variational problem,

find $(\vec{u}, p) \in V \times W$ satisfying,

$$\begin{array}{rcll} (\mathcal{A}^{-1}\vec{u}, \vec{v}) - (\vec{v}, \nabla p) & = & 0 & \forall \vec{v} \in V, \\ -(\vec{u}, \nabla w) & = & -(f, w) - \langle g, \vec{u} \cdot \vec{n} \rangle & \forall w \in W. \end{array} \tag{3.2}$$

which is equivalent to (2.2) if $g = 0$ and $\partial\Omega = \partial\Omega_D$. We make this assumption in the sequal and assume, for simplicity, that $\mathcal{A}(\vec{x})$ is a diagonal matrix.

Coercivity certainly holds in $\|\cdot\|_0$ by (1.2) and it can be shown (see [4]) that the problem is inf-sup stable in the new norms with $\beta = \frac{1}{c}$ where c is a constant arising in Friedrich's inequality. Suppose we formulate the discrete problem using the Raviart-Thomas spaces V_h and W_h. Since $W_h \not\subseteq W$ this corresponds to a non-conforming approach and the discrete variational problem is,

find $(\vec{u_h}, p_h) \in V_h \times W_h$ satisfying,

$$\begin{array}{rcll} (\mathcal{A}^{-1}\vec{u_h}, \vec{v_h}) - (\vec{v_h}, \nabla p_h) & = & 0 & \forall \vec{v_h} \in V_h, \\ -(\vec{u_h}, \nabla w_h) & = & -(f, w_h) & \forall w_h \in W_h. \end{array} \tag{3.3}$$

Now, since, $-(\vec{v_h}, \nabla w_h) = (w_h, \nabla \cdot \vec{v_h})$ for $\vec{v_h}$ in H_0^1, the coefficient matrix of the associated linear algebra problem is the same as the $H(div) \times L^2$ formulation in section 2.

3.1.1. Preconditioning the pressure Laplacian. It is known (see [2]) that the matrix operator associated with (3.2) defines an isomorphism from $L^2 \times H_0^1$ to $L^2 \times H^{-1}$. Block-elimination yields,

$$\begin{pmatrix} \mathcal{A}^{-1}\mathcal{I} & \nabla \\ 0 & \nabla \cdot (\mathcal{A}\nabla) \end{pmatrix} \begin{pmatrix} \vec{u_h} \\ p_h \end{pmatrix} = \begin{pmatrix} 0 \\ -f_h \end{pmatrix}. \tag{3.4}$$

It is natural to approximate the coefficient operator in (3.4) by a block-diagonal matrix whose blocks are discrete representations of the operators $\mathcal{A}^{-1}\mathcal{I}$ and $\nabla\cdot(\mathcal{A}\nabla)$ acting on V_h and W_h respectively. The Schur complement $BM^{-1}B^T$ is the obvious choice to approximate $\nabla \cdot (\mathcal{A}\nabla)$. Indeed, it can be shown that

$$P = \begin{pmatrix} M & 0 \\ 0 & BM^{-1}B^T \end{pmatrix}, \tag{3.5}$$

is an optimal preconditioner. To make the approach (3.5) feasible in practice, Rusten and Winther, [17], replace the Schur complement with the Cholesky factors of $BB^T \approx \nabla \cdot (\mathcal{I}\nabla)$. Unfortunately, this scheme cannot be robust with respect to the diffusion coefficients. We want an approximation to $\nabla \cdot (\mathcal{A}\nabla)$ that is simple to construct and cheap to implement. However, $W_h \not\subseteq H^1$ and so it is not clear how to achieve this.

On rectangular meshes, a simple idea is to construct a 5-point finite difference approximation to $\nabla \cdot (\mathcal{A}\nabla)$. Another idea, developed by the authors, is to use the stiffness matrix associated with a standard piecewise linear pressure approximation defined on a Delaunay triangulation of the Raviart-Thomas pressure nodes. However, this is not the focus of this discussion. Given $\mathcal{A}(\vec{x}) = \mathtt{diag}(a_1(\vec{x}), a_2(\vec{x}))$, we use rectangular meshes and assemble a simple 5-point finite difference matrix on the set of element centroids, scaling x and y connections by $a_1(\vec{x})$ and $a_2(\vec{x})$ respectively. The scaling terms should be evaluated at edge-midsides to obtain a symmetric matrix. The boundary condition is imposed by extending the domain by a ficticious layer of rectangles and applying a standard centered difference. We call this new matrix S and define the preconditioner,

$$P_{schur} = \begin{pmatrix} M & 0 \\ 0 & S \end{pmatrix}. \tag{3.6}$$

3.2. Eigenvalue bound. The performance of this second precondtioner is determined by how well S approximates $B\mathrm{M}^{-1}\mathrm{B}^T$. This statement is made more precise in the following result.

LEMMA 3.1. *The eigenvalues of the generalised problem,*

$$\begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \lambda \begin{pmatrix} M & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix}, \tag{3.7}$$

lie in the intervals $[-\hat{b}, -\hat{a}] \cup [1, 1] \cup [1+\hat{a}, 1+\hat{b}]$ *where* $\hat{a}$ *and* $\hat{b}$ *are constants that depend on the minimum and maximum eigenvalues of the 'dual' eigenvalue problem,*

$$BM^{-1}B^T\underline{p} = \sigma S\underline{p}.$$

Proof. If $\lambda = 1$,

$$B^T S^{-1} B\underline{u} = \underline{0} \Leftrightarrow \underline{u} \in null(B). \tag{3.8}$$

Since B has full rank, $dim(null(B)) = n - m$ and there are $n - m$ linearly indepdendent vectors $\underline{u}$ satisfying (3.8). Hence, $\lambda = 1$ with multiplicity $n - m$. If $\lambda \neq 1$,

$$\underline{u} = \frac{1}{(\lambda - 1)} M^{-1} B^T \underline{p} \text{ and } B\underline{u} = \lambda S \underline{p}.$$

Hence,

$$BM^{-1}B^T \underline{p} = \sigma S \underline{p}, \tag{3.9}$$

where $\sigma = \lambda(\lambda - 1)$. The eigenvalue problem (3.9) has m positive eigenvalues $\{\sigma_i\}_{i=1}^m$ which yield m negative and m positive eigenvalues for (3.7), defined by,

$$\lambda_i^1 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\sigma_i}, \quad \lambda_i^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\sigma_i}, \; i = 1 : m.$$

If we now write,

$$\hat{a} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\sigma_{min}}, \quad \hat{b} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\sigma_{max}},$$

then it is clear that these $2m$ eigenvalues of (3.7) lie in $[-\hat{b}, -\hat{a}] \cup [1 + \hat{a}, 1 + \hat{b}]$. $\square$

S is a more robust preconditioner for $BM^{-1}B^T$ with respect to the diffusion coefficients than a block decomposition of BB^T. Intuitively, we expect the eigenvalues of S to exhibit the same h-dependence as those of $BM^{-1}B^T$, leading to an optimal eigenvalue bound.

3.2.1. Numerical eigenvalues. We consider (1.1) on $\Omega = [0, 1] \times [0, 1]$ with $\partial\Omega = \partial\Omega_D$ and uniform rectangular meshes. Firstly, we set $\mathcal{A} = \mathcal{I}$ and secondly, we choose mildly varying coefficients,

$$\mathcal{A}(\vec{x}) = \begin{pmatrix} \frac{1}{1+100(x^2+y^2)} & 0 \\ 0 & \frac{1}{1+100(x^2+y^2)} \end{pmatrix}. \tag{3.10}$$

The eigenvalues of the preconditioned matrices $\{\lambda_1, \ldots, \lambda_{n+m}\}$ are recorded in tables 3.1 and 3.2, and plotted in figure 3.1. Observe that on uniform meshes, the eigenvalues are barely affected by the variation in the diffusion coefficients. In fact, on uniform meshes, one observes that $\sigma_{min} \to 1$ and $\sigma_{max} \to 3$ as $h \to 0$ independently of the coefficients.

h	σ_{min}	σ_{max}	λ_1	λ_m	λ_{m+1}	λ_n	λ_{n+1}	λ_{n+m}
$\frac{1}{8}$	1.0260	3	-1.3028	-0.6296	1	1	1.6296	2.3028
$\frac{1}{16}$	1.0064	3	-1.3028	-0.6209	1	1	1.6209	2.3028
$\frac{1}{32}$	1.0016	3	-1.3028	-0.6188	1	1	1.6188	2.3028

TABLE 3.1

Eigenvalues of preconditioned system; unit coefficients

The same behaviour is observed using triangular elements. The only difficulty is to determine the correct scaling when constructing the five-point difference approximation.

h	σ_{min}	σ_{max}	λ_1	λ_m	λ_{m+1}	λ_n	λ_{n+1}	λ_{n+m}
$\frac{1}{8}$	1.0141	3.0243	-1.3095	-0.6243	1	1	1.6243	2.3095
$\frac{1}{16}$	1.0038	3.0063	-1.3045	-0.6220	1	1	1.6220	2.3045
$\frac{1}{32}$	1.0010	3.0016	-1.3032	-0.6185	1	1	1.6185	2.3032

TABLE 3.2

Eigenvalues of preconditioned system; variable coefficients

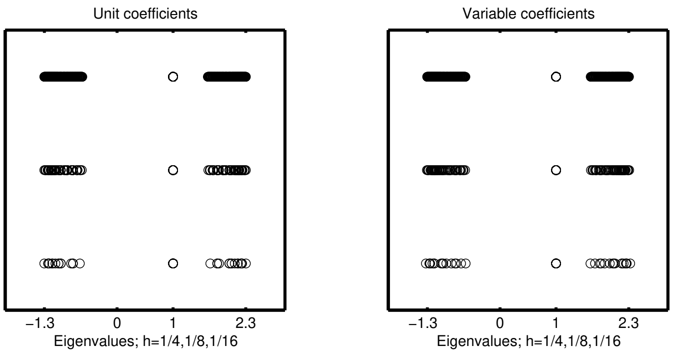

FIG. 3.1. *Eigenvalues of preconditioned systems*

3.2.2. Preconditioned MINRES. Again, we discretise (1.1) on $\Omega = [0,1] \times [0,1]$ using rectangular meshes. For simplicity, we only consider a homogeneous Dirichlet boundary condition in the following numerical examples. However, a Neumann condition can also be discretised very simply.

Example 1 $\mathcal{A} = I$, $f = 1$. The stopping tolerance is 10^{-6}. Iteration counts are given in table 3.3. As predicted, the preconditioner is optimal.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
preconditioned	17	19	19	19
unpreconditioned	25	75	165	311

TABLE 3.3

Iteration counts

Example 2 We choose $\mathcal{A}$ here to be the variable coefficients (3.10). We have seen above that the eigenvalues of the preconditioned system are very close to those of example 1 and this is reflected in the iteration counts shown in table 3.4. Observe that convergence of the unpreconditioned solve has slowed considerably whereas the preconditioner is robust with respect to the variation in the coefficients.

Example 3 Finally, we consider discontinuous coefficients and introduce an interior zone of increasing permeability by choosing $\epsilon \in \left(10^0, 10^6\right)$ and setting,

$$\mathcal{A}(\vec{x}) = \begin{cases} \epsilon I & \forall\, \vec{x} \in \Omega_* \\ I & \forall\, \vec{x} \in \Omega \backslash \Omega_*, \end{cases}$$

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
preconditioned	21	21	21	19
unpreconditioned	142	449	1,178	2,638

TABLE 3.4

Iteration counts

with $\Omega_* = \left(\frac{1}{2}, 1\right) \times \left(0, \frac{1}{2}\right)$. Iteration counts are recorded in table 3.5 below.

h	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$\frac{1}{16}$	19	23	25	23	23	23	23
$\frac{1}{32}$	19	23	25	25	25	25	25
$\frac{1}{64}$	19	23	25	25	25	25	25

TABLE 3.5

Iteration counts

Comparing the numerical results above with those in section 2, we note that both preconditioning strategies are optimal. The P_{div} method however is more widely applicable; modifying the P_{schur} approach in the context of non-diagonal $\mathcal{A}$ is not straight-forward!

4. Concluding remarks. In this paper we have explained–without penalty parameters or reduction techniques—why the preconditioners P_{div} and P_{schur} are optimal and robust with respect to variations in the diffusion coefficients. Our bounds are derived using properties of the finite element matrices. The results apply to the lowest order Raviart-Thomas spaces in $\mathbb{R}^2$ and $\mathbb{R}^3$. To derive practical techniques the issue of efficient implementation of P_{div} and P_{schur} needs to be addressed. We have not focused on this; effective domain decomposition and multigrid methods in the context of the H_{div} operator are described in [2] and specialist multigrid methods, see [5, ch. 6], can be used in the second approach.

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