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Quadratic Realizability of Palindromic Matrix Polynomials: The Real Case

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ABSTRACT
Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ be a list consisting of structural data for a matrix polynomial; here $\mathcal{L}_1$ is a sublist consisting of *powers of irreducible* (*monic*) *scalar polynomials* over the field $\mathbb{R}$, and $\mathcal{L}_2$ is a sublist of *nonnegative integers*. For an arbitrary such $\mathcal{L}$, we give easy-to-check necessary and sufficient conditions for $\mathcal{L}$ to be the list of elementary divisors and minimal indices of some *real T-palindromic quadratic* matrix polynomial. For a list $\mathcal{L}$ satisfying these conditions, we show how to explicitly build a real T-palindromic quadratic matrix polynomial having $\mathcal{L}$ as its structural data; that is, we provide a *T-palindromic quadratic realization of* $\mathcal{L}$ over $\mathbb{R}$. A significant feature of our construction differentiates it from related work in the literature; the realizations constructed here are direct sums of blocks with low bandwidth, that transparently display the spectral and singular structural data in the original list $\mathcal{L}$.

KEYWORDS
matrix polynomials, real quadratic realizability, elementary divisors, minimal indices, T-palindromic, inverse problem

1. Introduction

Matrix polynomials, i.e., matrices whose entries are scalar polynomials, arise in a variety of applications and have been intensively studied over the last several decades. Classical references on matrix polynomials and their applications include [16,21], while more modern treatments can be found in [4,17,38]. The most relevant structural data related to matrix polynomials are finite and infinite elementary divisors, together with left and/or right minimal indices, which only exist if the matrix polynomial is rank deficient over the field of rational functions. Matrix polynomials arising in applications often have a special structure, e.g., (*skew*) *symmetric*, (*skew*) *Hermitian*, *T-alternating* and *T-palindromic* [2,5,12,23,26,32,38,40], which typically translates to polynomials having structural data that satisfy additional constraints [2,6,26–29,32]. Here we are mainly concerned with the class of *T-palindromic matrix polynomials over* $\mathbb{R}$, that is, polynomials of the form $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ such that $A_{k-i}^T = A_i \in \mathbb{R}^{n\times n}$, for $i = 0, \dots, k$.

The primary goal of this paper is to further contribute to the theory of inverse eigenproblems of *structured* matrix polynomials. These kinds of polynomial eigenvalue problems have been investigated for at least 40 years [31, Thm. 5.2], including in the classical reference [16]. In recent years, these topics have been actively studied not only from the theoretical standpoint [10], but also in relation to other problems such as the

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stratification of orbits of matrix polynomials [19] — for more details about inverse polynomial eigenproblems see [9, Sec. 1] and the references therein.

In this paper, we consider a particular variation of the inverse polynomial eigenvalue problem that we refer to as the *Quadratic Realizability Problem* (QRP), consisting of the following two subproblems (SPs) [9]:

(SP-1) *Characterization* of those lists $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$, where $\mathcal{L}_1$ comprises the desired spectral structure (elementary divisors) and $\mathcal{L}_2$ the desired singular structure (minimal indices), that can appear as the complete structural data of some *quadratic* matrix polynomial in a given structure class $\mathscr{C}$.

(SP-2) For each such realizable list $\mathcal{L}$, show how to *concretely construct* a quadratic matrix polynomial in the class $\mathscr{C}$ whose structural data is exactly the list $\mathcal{L}$. It is also desirable for this concrete realization to display the given structural data as *simply* and *transparently* as possible.

In [9], the authors provided a complete and transparent solution to the QRP for the class $\mathscr{C}$ of T-palindromic matrix polynomials over an *algebraically closed field*. This was achieved by developing a Kronecker-like *quasi-canonical form* for quadratic T-palindromic matrix polynomials. Here we extend those results and provide a complete solution to the QRP for the class of T-palindromic matrix polynomials over *the field of real numbers*. What makes this extension nontrivial is the fact that elementary divisors of real matrix polynomials can be (powers of) *irreducible quadratics*, in stark contrast to matrix polynomials over an algebraically closed field, whose elementary divisors are all (powers of) linear factors. In order to accommodate this additional type of elementary divisor, new tools and ideas are necessary to construct transparent T-palindromic quadratic realizations over $\mathbb{R}$, as described in Sections 3-5.

To see the results of this paper in a broader context, let us consider some closely related problems. The simplest analog of the QRP is the corresponding "linear realizability problem (LRP)", that is, the inverse eigenvalue problem for matrix pencils. But this problem has a classical solution (1890) in the Kronecker canonical form (KCF) [15,20]. Specifically, the KCF shows how to build a block diagonal matrix pencil in which each individual block realizes exactly one finite or infinite elementary divisor, or one left or right minimal index. The KCF thus provides a "linear realization" of *any* list $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ of structural data as a direct sum of bidiagonal blocks with *simple* and *transparent* spectral and singular structures. We regard the KCF as the prototype of an ideal solution for an inverse polynomial eigenproblem; to achieve something as close as possible to this for quadratic matrix polynomials is the main motivation behind the formulation of the (SP-2) part of the QRP.

Much is also known about *structured* linear realizability problems, starting with the work of Thompson [37] in developing structured Kronecker-like canonical forms for real and complex symmetric and skew-symmetric matrix pencils; [37] also contains an extensive bibliography on related works. More recently, structured Kronecker-like canonical forms have been found for other structure classes of matrix pencils. Examples include the work of Lancaster and Rodman [22] on Hermitian pencils, and the work of Schröder [33–35] and Horn and Sergeichuk [18] on real and complex T-palindromic pencils, thus solving the structured linear realizability problems for those structure classes in the spirit of (SP-1) and (SP-2) above. One way, then, to view the results of this paper is as an extension of the notion of a structured KCF for structured pencils to something analogous for *quadratic*, *real* T-palindromic matrix polynomials; recall that the corresponding T-palindromic QRP over an algebraically closed field was settled in [9].

Other variations of the quadratic inverse eigenproblem have also been studied. These include the general (unstructured) QRP [7,24] and the Hermitian QRP [30], in which Kronecker-like quasi-canonical forms are developed. Much research has also been done on approaches to quadratic inverse eigenproblems that do not involve the development of Kronecker-like forms. One important example of this is work by Tisseur and Zaballa on triangularizing quadratic matrix polynomials [39]. Another is work of Lancaster and Zaballa on real symmetric quadratics, in which information about desired eigenvectors is also part of the given structural data; see, for example, [23] and the references within.

Significant efforts have been made to extend the known results for the quadratic inverse eigenproblem to higher degree [1,3,10,11,36], but none are completely satisfactory. Some lack transparency in displaying the original structural data, some have restrictions on the underlying field. Indeed, the prospect of finding a Kronecker-like form for general matrix polynomials of arbitrary degree and over arbitrary fields seems quite remote, due to the likely combinatorial explosion in the number of realizable but irreducible combinations of structural data as a function of the degree.

This paper, then, adds to our knowledge of structured Kronecker-like forms for *quadratic* matrix polynomials, a setting where such forms are still feasible and of a manageable complexity. Section 2 begins by establishing notation, introducing new definitions, and recalling key results from [9]. In Sections 3, 4, and 5 we provide complete solutions to (SP-1) and (SP-2) of the QRP for T-palindromic matrix polynomials over $\mathbb{R}$, respectively. Finally, in Section 6 we summarize the main contributions in this paper and discuss some open questions.

2. Preliminaries

In the first part of this section we introduce notation, nomenclature and basic facts about matrix polynomials, most of which can be found in [15,16]. In the second part we briefly review the structural properties of T-palindromic matrix polynomials [28].

The *algebraic closure* of a field $\mathbb{F}$ is denoted by $\overline{\mathbb{F}}$, while the ring of univariate polynomials in the variable λ with coefficients in $\mathbb{F}$ is denoted by $\mathbb{F}[\lambda]$. The field of fractions of $\mathbb{F}[\lambda]$ is denoted by $\mathbb{F}(\lambda)$. Matrices with entries in $\mathbb{F}[\lambda]$ are referred to as matrix polynomials. In general, an $m\times n$ matrix polynomial $P(\lambda)$ over $\mathbb{F}$, often denoted simply by P, can be expressed as

$$P(\lambda) \;=\; \sum_{i=0}^{k} A_i\lambda^i\,, \qquad A_i \in \mathbb{F}^{m\times n}\,, \quad i = 0, 1, \ldots, k\,. \tag{1}$$

Analogous to scalar polynomials, the *degree* of $P(\lambda)$, denoted by $\deg(P)$, is defined as the largest integer i such that $A_i \neq 0$ in (1); keep in mind that this largest integer need not be k. The matrix polynomial $P(\lambda)$ in (1) is said to have *grade* k, denoted by $\text{grade}(P) = k$, which is greater than or equal to the $\deg(P)$, and a matter of choice. Now with respect to an arbitrary but fixed grade, one can define the notion of j*-reversal*, which among other things enables us to define T-palindromic matrix polynomials. More specifically, for any nonzero P of degree d and any $j \geq d$, the j*-reversal* of P is the matrix polynomial $\text{rev}_j P$ given by $(\text{rev}_j P)(\lambda) := \lambda^j P(1/\lambda)$ [28, Def. 3.3]. In the special case when $j = d$, i.e., when grade is chosen to be equal to degree, the j-reversal of P is referred to as just the *reversal* of P, and denoted by $\text{rev}\,P$.

The *rank* of $P(\lambda)$ (sometimes called "normal rank") is the rank of $P(\lambda)$ considered as a matrix over the field $\mathbb{F}(\lambda)$, and is denoted by $\text{rank}(P)$. $P(\lambda)$ is said to be *regular* if P is square and $\det(P) \not\equiv 0$, or equivalently if P is $n \times n$ and $\operatorname{rank} P = n$; otherwise $P(\lambda)$ is said to be *singular*. Any singular $P(\lambda)$ has non-trivial left and/or right "rational nullspaces", i.e., subspaces of $\mathbb{F}(\lambda)^m$ or $\mathbb{F}(\lambda)^n$ over the field of rational functions $\mathbb{F}(\lambda)$, which we denote by $\mathcal{N}_\ell(P)$ and $\mathcal{N}_r(P)$, respectively. Note that for each of these rational nullspaces one can always find a *polynomial* basis; e.g., take any rational basis and multiply each of the basis vectors by the product of the denominators of all its entries. The degree of a polynomial vector is the maximum degree of its entries; the *order* of a polynomial basis is defined as the sum of the degrees of its vectors. A polynomial basis with the smallest order is called a *minimal basis* [13]. Any rational subspace has many minimal bases, but it is well known that for any given subspace $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$, the ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same [13,25]. These uniquely defined degrees are called the *minimal indices* of $\mathcal{V}$ [13]. The minimal indices of the left and right nullspaces of P are then called the left and right minimal indices of P.

Two matrix polynomials of the same size $P(\lambda)$ and $Q(\lambda)$ are said to be *unimodularly equivalent*, denoted $P(\lambda) \sim Q(\lambda)$, if there are unimodular matrix polynomials (i.e., square matrix polynomials with a nonzero *constant* determinant) $U(\lambda), V(\lambda)$ such that $U(\lambda)P(\lambda)V(\lambda) = Q(\lambda)$.

Theorem 2.1. (Smith Form [14])
Let $P(\lambda)$ be an $m \times n$ matrix polynomial with $r = \operatorname{rank} P$. Then $P(\lambda)$ is unimodularly equivalent to

$$D(\lambda)_{m\times n} \;:=\; \operatorname{diag}\Big(d_1(\lambda)\,,\,\ldots, d_r(\lambda)\,,\,0_{(m-r)\times(n-r)}\Big)\,, \tag{2}$$

where:

(i) $d_1(\lambda), \ldots, d_r(\lambda)$ *are* monic *scalar polynomials (i.e., each leading coefficient is* 1*),*
(ii) $d_1(\lambda), \ldots, d_r(\lambda)$ *form a* divisibility chain*, i.e., $d_j(\lambda)$ is a divisor of $d_{j+1}(\lambda)$, for $j = 1, \ldots, r-1$,*
(iii) *the polynomials $d_1(\lambda), d_2(\lambda), \ldots, d_r(\lambda)$ are uniquely determined by the multiplicative relations*

$$d_1(\lambda)d_2(\lambda)\cdots d_j(\lambda) \;=\; \gcd\big\{\, \textit{all } j\times j \textit{ minors of } P(\lambda)\,\big\}\,, \quad \textit{for}\;\; j=1,\ldots,r\,.$$

The diagonal matrix $D(\lambda)$ in (2) is thus unique, and is known as the Smith form *of P.*

The scalar polynomials $d_j(\lambda)$, for $j = 1, 2, \ldots, r$, in the Smith form of P are called the *invariant polynomials* of P. The roots $\lambda_0 \in \overline{\mathbb{F}}$ of the product $d_1(\lambda)\cdots d_r(\lambda)$ in (2) are the (finite) *eigenvalues* of P. Furthermore, ∞ is said to be an eigenvalue of P whenever 0 is an eigenvalue of $\operatorname{rev}_j P$, where $j = \operatorname{grade} P$. Given that the definition of the reversal depends on the choice for the grade of P, one has to explicitly specify it when considering an eigenvalue at ∞. It is well known that if $\operatorname{grade}(P) > \deg(P)$, then P definitely has an eigenvalue at ∞ [8, Lem. 2.17]. On the other hand, if $\operatorname{grade}(P) = \deg(P)$, then P has an eigenvalue at ∞ if and only if the rank of the leading coefficient of P is strictly less than $\operatorname{rank}(P)$ [8, Rem. 2.14].

Definition 2.2. Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ be of grade k and have $\operatorname{rank}(P) = r$. Then:

(i) For any nonzero $\pi(\lambda) \in \mathbb{F}[\lambda]$ and $i = 1, \ldots, r$, there exist unique nonnegative integers α_i and scalar polynomials q_i such that

$$d_i(\lambda) \,=\, \pi(\lambda)^{\alpha_i} q_i(\lambda)\,, \qquad \text{with} \quad \gcd(\pi(\lambda), q_i(\lambda)) \,\equiv\, 1\,,$$

where $d_i(\lambda)$'s are invariant polynomials of P. By the divisibility chain property of the Smith form, the sequence of exponents $(\alpha_1, \ldots, \alpha_r)$ satisfies the condition $0 \leq \alpha_1 \leq \ldots \leq \alpha_r$.

(ii) Let $\pi(\lambda)$ be a nonconstant monic *irreducible* scalar polynomial over $\mathbb{F}$ with a nonzero sequence of exponents $(\alpha_1, \ldots, \alpha_r)$, and let $1 \leq g \leq r$ be the index such that $\alpha_1 = \ldots = \alpha_{g-1} = 0$ and $\alpha_j > 0$ for all $g \leq j \leq r$. Then each $\pi(\lambda)^{\alpha_j}$ is said to be a *finite elementary divisor* of P, and $\alpha_g, \alpha_{g+1}, \ldots, \alpha_r$ are the associated *partial multiplicities of* $\pi(\lambda)$.

(iii) The *infinite elementary divisors* of P correspond to the elementary divisors at λ of $\text{rev}_k P$. More specifically, if $\lambda^{\beta_1}, \ldots, \lambda^{\beta_\ell}$ with $0 < \beta_1 \leq \cdots \leq \beta_\ell$ are the elementary divisors for $\text{rev}_k P$, then P has ℓ corresponding elementary divisors at ∞, denoted by $\omega^{\beta_1}, \ldots, \omega^{\beta_\ell}$. Here we use the notation ω^β to denote the elementary divisors at ∞ as a way to prevent possible confusion with elementary divisors at λ.

We conclude this section by introducing key concepts that play a role in the solution of the T-palindromic QRP over $\mathbb{R}$. The subsequent definition closely follows [10, Def. 2.17].

Definition 2.3. (Structural data of a matrix polynomial).
Let $P(\lambda)$ be an $m \times n$ matrix polynomial with grade k over a field $\mathbb{F}$.

(i) The collection of *all* finite and infinite elementary divisors, including repetitions, is said to comprise the *spectral structure* of P.
(ii) The *left* and *right minimal indices* of P are the minimal indices of $\mathcal{N}_\ell(P)$ and $\mathcal{N}_r(P)$, respectively, and together comprise the *singular structure of* P.
(iii) The *structural data* of P consists of the elementary divisors (spectral structure) of P, together with the left and right minimal indices (singular structure) of P.

It follows from Definition 2.3(i) that if $\mathbb{F}$ is an algebraically closed field (i.e., $\mathbb{F} = \overline{\mathbb{F}}$), then all the finite elementary divisors of P are of the form $\pi(\lambda)^\alpha$, where $\alpha > 0$ and $\pi(\lambda) = (\lambda - \lambda_0)$ for some $\lambda_0 \in \mathbb{F}$. On the other hand, if $\mathbb{F} = \mathbb{R}$, then finite elementary divisors are $\pi(\lambda)^\alpha$ with $\alpha > 0$, where either $\pi(\lambda) = (\lambda - \lambda_0)$ for some $\lambda_0 \in \mathbb{R}$, or $\pi(\lambda) = \lambda^2 + b\lambda + c$ is an irreducible quadratic polynomial over $\mathbb{R}$ with $b, c \in \mathbb{R}$.

The following well-known property of the structural data of direct sums of matrix polynomials plays a crucial background role for our solution of the real T-palindromic QRP.

Lemma 2.4. (Spectral and singular structures of a direct sum).
Let $P(\lambda)$ and $Q(\lambda)$ be two grade k matrix polynomials over an arbitrary field $\mathbb{F}$, with $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ denoting the lists of elementary divisors and minimal indices of P and Q, respectively. Then the list of elementary divisors and minimal indices of the grade k matrix polynomial $\text{diag}(\,P\,,\,Q\,)$ *is simply the concatenation of the lists $\mathcal{L}(P)$ and $\mathcal{L}(Q)$, i.e., $c(\mathcal{L}(P), \mathcal{L}(Q))$ as in (5).*

Proof. A proof of the concatenation property for finite elementary divisors can be found in [15]. The same result in [16, Prop. S1.5] is given only for algebraically closed fields, but the argument is easily adapted to arbitrary fields. For infinite elementary divisors, just apply the result for the finite case to the elementary divisors associated with zero in $\text{rev}_k(\text{diag}(P,Q)) = \text{diag}(\text{rev}_k P, \text{rev}_k Q)$. The concatenation property for minimal indices of direct sums can be found in [25]. $\square$

This direct sum property of elementary divisors and minimal indices will form the basis for our transparent quadratic realization of structural data, analogous to how the KCF achieves a direct sum realization of structural data for matrix pencils. Here, then, is the "direct sum strategy" for solving the QRP. Given a list of structural data for a real T-palindromic quadratic matrix polynomial, break up that list into the simplest, most primitive sublists possible that can each still be realized by a real T-palindromic quadratic matrix polynomial in its own right. Then show how to construct a canonical block realizing each of these finitely many primitive sublists. The final realization of the complete structural data list is then just the direct sum of these canonical blocks. Although we will try to make each direct summand as simple as possible, it will turn out that for quadratic realizations this is *necessarily* significantly more complicated than it is for pencils. In particular, it will no longer be possible for each quadratic canonical block to contain just a single elementary divisor or a single minimal index, as is the case in the KCF for matrix pencils.

2.1. Spectral and singular structure of T-palindromic matrix polynomials

In this section we briefly recall some well-known facts about T-palindromic matrix polynomials that will be used in the rest of the paper.

Definition 2.5. [26, Table 2.1] (T-palindromic).
A nonzero $n \times n$ matrix polynomial P of degree $d \geq 1$ is said to be T-palindromic if $(\text{rev}_j P)(\lambda) = P^T(\lambda)$, for some $j \geq d$.

It is important to highlight the fact that the notion of T-palindromicity in Definition 2.5 is defined "with respect to some grade j." For example, consider the degree-one scalar polynomial $p(\lambda) = \lambda$. Then $\text{rev}_1 p(\lambda) = \lambda \cdot (1/\lambda) = 1 \neq \lambda = p(\lambda)^T$, that is, $p(\lambda)$ is *not* T-palindromic with respect to its degree. On the other hand, $\text{rev}_2 p(\lambda) = \lambda^2 \cdot (1/\lambda) = p(\lambda)^T$, and so $p(\lambda)$ *is* T-palindromic with respect to grade two. In [28, Prop. 4.3], the authors proved that if a degree d polynomial P is T-palindromic, then there is a unique $j \geq d$ such that $\text{rev}_j P = P^T$. This j is known as the *grade of palindromicity of* P. Consequently, for the rest of the paper whenever we refer to a T-palindromic matrix polynomial P with grade k, we are assuming that k is its unique grade of palindromicity.

We conclude this section with a remark that collects all of the important facts from [6,28] about the singular and spectral structures of T-palindromic matrix polynomials that are relevant to our present work.

Remark 2.6. Let $Q(\lambda)$ be a matrix polynomial over $\mathbb{R}$ and assume that Q is T-palindromic with grade of palindromicity two. Then the following statements hold:

(i) If $p(\lambda) = (\lambda+1)^\alpha(\lambda-1)^\beta q(\lambda)$, with $q(1) \neq 0 \neq q(-1)$, is any invariant polynomial of $P(\lambda)$, then $q(\lambda)$ is palindromic [28, Thm. 7.6] and monic. Moreover, $q(\lambda)$ can be factored as

$$q(\lambda) = \lambda^\nu \cdot \prod_{i=1}^{\ell} b_i(\lambda)^{m_i} \cdot \prod_{j=1}^{w} \left((c_j d_j(\lambda) \text{rev} d_j(\lambda)\right)^{n_j}, \quad (3)$$

where the m_i's, n_j's, and ν are positive integers, and the irreducible factors $b_i(\lambda)$, $d_j(\lambda)$, and $\text{rev}\, d_j(\lambda)$ are distinct and satisfy the following properties:

(a) Each $b_i(\lambda)$ and each $d_j(\lambda)$ is monic and coprime to λ, $(\lambda+1)$, and $(\lambda-1)$. The nonzero constants $c_j \in \mathbb{R}$ are chosen so that $c_j \text{rev}\, d_j(\lambda)$ is also monic for $j = 1, 2, \ldots, w$.

(b) Each $b_i(\lambda)$ is of degree two and is palindromic with respect to its degree.

(c) Factors $d_j(\lambda)$ and $\text{rev}\, d_j(\lambda)$ are of degree one or two and are *not* palindromic.

(ii) Any odd degree elementary divisor of $Q(\lambda)$ associated with either of the eigenvalues $\lambda_0 = \pm 1$ has even multiplicity [28, Cor. 8.2].

(iii) For any $\beta \geq 1$, the elementary divisors λ^β and ω^β have the same multiplicity (i.e., they appear the same number of times) [28, Cor. 8.1].

(iv) The left and right minimal indices of $Q(\lambda)$ coincide. Namely, if $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_q$ and $\varepsilon_1 \geq \varepsilon_2 \geq \ldots \geq \varepsilon_p$ are the left and right minimal indices of $P(\lambda)$, respectively, then $p = q$ and $\eta_i = \varepsilon_i$, for $i = 1, \ldots, p$ [6, Thm. 3.6].

3. Solution of the first part of the real T-palindromic QRP

In this section we completely solve (SP-1), and lay out the strategy for solving (SP-2) of the real T-palindromic QRP. We adopt the convention that when the field $\mathbb{F}$ is not explicitly mentioned in a definition or a result, then it is to be understood to hold for an arbitrary field. Otherwise, we will specify that $\mathbb{F}$ is algebraically closed with $\text{char}(\mathbb{F}) \neq 2$, or that $\mathbb{F} = \mathbb{R}$.

We start by introducing some basic concepts about lists of elementary divisors and minimal indices, analogous to [9, Def. 3.1].

Definition 3.1. (Lists of elementary divisors and minimal indices).

(i) A *list of finite elementary divisors* is a list of the form

$$\mathcal{L}_{fin} = \left\{ \pi_1(\lambda)^{\alpha_{1,1}}, \ldots, \pi_1(\lambda)^{\alpha_{1,g_1}}, \ldots, \pi_s(\lambda)^{\alpha_{s,1}}, \ldots, \pi_s(\lambda)^{\alpha_{s,g_s}} \right\},$$

where $\pi_1(\lambda), \ldots, \pi_s(\lambda) \in \mathbb{F}[\lambda]$ are nonconstant monic polynomials, irreducible over $\mathbb{F}$, such that $\gcd(\pi_i(\lambda), \pi_j(\lambda)) \equiv 1$ for $i \neq j$, and $\alpha_{i,j}$'s are *positive* integers.

(ii) An *elementary divisor chain of length g* associated with $\pi(\lambda) \in \mathbb{F}[\lambda]$ is a list of the form $(\, \pi(\lambda)^{\alpha_1}, \ldots, \pi(\lambda)^{\alpha_g} \,)$, with $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_g$.

(iii) An *infinite elementary divisor chain of length g* is a list of the form $\mathcal{L}_\infty = (\, \omega^{\beta_1}, \ldots, \omega^{\beta_g} \,)$, with $0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_g$.

(iv) A *list $\mathcal{L}$ of elementary divisors and minimal indices* is of the form

$$\mathcal{L} = \left\{ \mathcal{L}_{fin} \,;\, \mathcal{L}_\infty \,;\, \mathcal{L}_{left} \,;\, \mathcal{L}_{right} \right\}, \tag{4}$$

where $\mathcal{L}_{fin}$ is a list of finite elementary divisors, $\mathcal{L}_\infty$ is an infinite elementary divisor chain, and $\mathcal{L}_{left} = \{\eta_1, \ldots, \eta_q\}$ and $\mathcal{L}_{right} = \{\varepsilon_1, \ldots, \varepsilon_p\}$ are lists of nonnegative integers. $\mathcal{L}$ is said to be *nontrivial* if at least one of the lists $\mathcal{L}_{fin}, \mathcal{L}_\infty, \mathcal{L}_{left}$, and $\mathcal{L}_{right}$ is nonempty.

(v) Given two lists $\mathcal{L}$ and $\widehat{\mathcal{L}}$ as in (4), the *concatenation* of $\mathcal{L}$ and $\widehat{\mathcal{L}}$, denoted by $c(\mathcal{L}, \widehat{\mathcal{L}})$, is the list of elementary divisors and minimal indices

$$c(\mathcal{L}, \widehat{\mathcal{L}}) := \left\{ \{\mathcal{L}_{fin}, \widehat{\mathcal{L}}_{fin}\}; \{\mathcal{L}_{\infty}, \widehat{\mathcal{L}}_{\infty}\}; \{\mathcal{L}_{left}, \widehat{\mathcal{L}}_{left}\}; \{\mathcal{L}_{right}, \widehat{\mathcal{L}}_{right}\} \right\}, \quad (5)$$

obtained by simply adjoining the corresponding lists as in (5), including all repetitions.

Several of the key quantities associated with a list $\mathcal{L}$ of elementary divisors and minimal indices that play a central role in the rest of the paper are introduced next.

Definition 3.2. ([9, Def. 3.3]) Let $\mathcal{L}$ be a list as in (4).

(i) The *total finite degree* and the *total infinite degree* of $\mathcal{L}$, denoted by $\delta_{fin}(\mathcal{L})$ and $\delta_{\infty}(\mathcal{L})$, respectively, are defined by

$$\delta_{fin}(\mathcal{L}) := \sum_{i=1}^{s} \sum_{j=1}^{g_i} \alpha_{i,j} \qquad \text{and} \qquad \delta_{\infty}(\mathcal{L}) := \beta_1 + \cdots + \beta_g ,$$

where $\alpha_{i,1}, \ldots, \alpha_{i,g_i}$, for $i = 1, \ldots, s$, are the (nonzero) finite partial multiplicities in $\mathcal{L}$, and $\beta_1, \ldots, \beta_g$ are the (nonzero) infinite partial multiplicities in $\mathcal{L}$.

(ii) The *total degree of* $\mathcal{L}$ is the number given by $\delta(\mathcal{L}) := \delta_{fin}(\mathcal{L}) + \delta_{\infty}(\mathcal{L})$.

(iii) The *sum of all minimal indices of* $\mathcal{L}$ is defined as $\mu(\mathcal{L}) := \sum_{i=1}^{p} \varepsilon_i + \sum_{j=1}^{q} \eta_j$.

(iv) The *length of the longest elementary divisor chain in* $\mathcal{L}$ (finite or infinite) is denoted by $\gamma(\mathcal{L})$.

For the rest of this paper, we adopt a *convention* that when the list $\mathcal{L}$ under consideration is clear from the context, the quantities (ii)–(iv) from Definition 3.2 will simply be denoted by δ, μ, and γ, respectively. A list $\mathcal{L}$ is said to be the list of elementary divisors and minimal indices of some matrix polynomial P when all the elementary divisors and minimal indices of P are precisely those in $\mathcal{L}$; when necessary such a list is denoted as $\mathcal{L}(P)$. Finally, there is a simple but powerful result, known as the *Index Sum Theorem*, that relates the quantities $\delta(\mathcal{L}), \mu(\mathcal{L})$, grade$(P)$, and rank$(P)$.

Theorem 3.3. [8, Thm. 6.5] (Index Sum Theorem).
Let $P(\lambda)$ be an arbitrary matrix polynomial over an arbitrary field, and let $\mathcal{L}$ denote the list of elementary divisors and minimal indices of P, i.e., $\mathcal{L} = \mathcal{L}(P)$. Then:

$$\delta(\mathcal{L}) + \mu(\mathcal{L}) = \operatorname{grade}(P) \cdot \operatorname{rank} P. \quad (6)$$

Now with the clearly defined notions of a list of elementary divisors and minimal indices and related quantities, we are ready to tackle (SP-1) of the T-palindromic QRP over $\mathbb{R}$. A natural place to start is the following notion.

Definition 3.4. [9, Def. 3.5] (p-quad realizability).
A list $\mathcal{L}$ of elementary divisors and minimal indices is said to be *p-quad realizable over the field* $\mathbb{F}$ if there exists some T-palindromic quadratic matrix polynomial over $\mathbb{F}$, with grade of palindromicity 2, whose elementary divisors and minimal indices are exactly the ones in $\mathcal{L}$.

Note that throughout the rest of the paper the phrases *"Q is a (quadratic) realization of $\mathcal{L}$"*, and *"Q realizes $\mathcal{L}$"*, are used interchangeably to mean that Q is a quadratic matrix polynomial whose elementary divisors and minimal indices are precisely those in $\mathcal{L}$. Since this paper concerns quadratic realizations that are T-palindromic over $\mathbb{R}$, we introduce several new concepts that capture the special spectral and singular structure particular to *real* quadratic T-palindromic matrix polynomials (see Section 2.1). These are adaptations of analogous concepts defined in [9], where the underlying field is algebraically closed.

Definition 3.5. (p-quad symmetry over $\mathbb{R}$)
A list $\mathcal{L}$ of elementary divisors and minimal indices over $\mathbb{R}$ is said to have *p-quad symmetry over* $\mathbb{R}$ (or to have *real p-quad symmetry*) if the following conditions are satisfied:

(1) (a) For any $a \in \mathbb{R}$ with $a \neq 0, \pm 1$, and $\beta \geq 1$, the elementary divisor $(\lambda - a)^\beta$ appears in $\mathcal{L}$ with the same multiplicity as $(\lambda - \frac{1}{a})^\beta$ (i.e., they appear exactly the same number of times, perhaps zero).

(b) For any $\beta \geq 1$, the elementary divisors λ^β and ω^β appear in $\mathcal{L}$ with the same multiplicity.

(c) Any odd degree elementary divisor in $\mathcal{L}$ associated with either eigenvalue $a = +1$ or $a = -1$ has even multiplicity.

(d) An elementary divisor $(\lambda^2 + b\lambda + 1)^\beta$, where $\lambda^2 + b\lambda + 1$ is any *palindromic irreducible quadratic* scalar polynomial over $\mathbb{R}$ and $\beta \geq 1$, may occur with either odd or even multiplicity.

(e) For any elementary divisor $(\lambda^2 + b\lambda + c)^\beta$, where $\lambda^2 + b\lambda + c$ is a *non-palindromic irreducible quadratic* scalar polynomial over $\mathbb{R}$ and $\beta \geq 1$, the elementary divisor $\frac{1}{c^\beta}(c\lambda^2 + b\lambda + 1)^\beta = \frac{1}{c^\beta}\big(\mathrm{rev}_2(\lambda^2 + b\lambda + c)\big)^\beta$ appears in $\mathcal{L}$ with the same multiplicity.

(2) The ordered sublists of left and right minimal indices are identical.

Note that if in Definition 3.5 the underlying field $\mathbb{F}$ had been algebraically closed rather than $\mathbb{R}$, then conditions (1d) and (1e) would be vacuously satisfied for any spectral data list $\mathcal{L}$; over such a field $\mathbb{F}$ all finite elementary divisors are of the form $(\lambda - a)^\beta$ for some $a \in \mathbb{F}$. Thus (1d) and (1e) are exactly what is required to accommodate the additional spectral structure arising from the underlying field being $\mathbb{R}$, see [28, Thm. 7.6]. These two conditions in Definition 3.5 also highlight the importance of *"quadratic irreducibles"*, i.e., *$\mathbb{R}$-irreducible scalar polynomials of degree two*, when solving the T-palindromic QRP over $\mathbb{R}$.

Conditions (1d) and (1e) in Definition 3.5 also clearly indicate the importance of distinguishing between *palindromic* and *non-palindromic* quadratic irreducibles. Consequently, throughout this paper we adopt the *notational convention* that $p(\lambda)$ and $q(\lambda)$ stand for quadratic irreducibles over $\mathbb{R}$ that are, respectively, palindromic and non-palindromic. Equivalently, scalar polynomials $p(\lambda)$ and $q(\lambda)$ are of the form

$$p(\lambda) := \lambda^2 + b\lambda + 1 \qquad \text{and} \qquad q(\lambda) := \lambda^2 + b\lambda + c\,, \;\; c \neq 1\,. \tag{7}$$

Observe that as a consequence of the quadratic formula we have that $p(\lambda)$ and $q(\lambda)$ are $\mathbb{R}$-irreducible if and only if $-2 < b < 2$ for $p(\lambda)$, and $0 \leq b^2 < 4c$ for $q(\lambda)$. The final piece of notation concerns the (scaled) reversals of $p(\lambda)$ and $q(\lambda)$, namely

$$
\begin{aligned}
\widehat{p}(\lambda) &:= \operatorname{rev}_2 p(\lambda) = \lambda^2 + b\lambda + 1\,, \\
\widehat{q}(\lambda) &:= \operatorname{rev}_2 q(\lambda) = c\lambda^2 + b\lambda + 1\,, \quad \text{and} \\
\widetilde{q}(\lambda) &:= (1/c)\cdot \operatorname{rev}_2 q(\lambda) = \lambda^2 + (b/c)\lambda + (1/c)\,.
\end{aligned} \tag{8}
$$

Next we introduce the concept that plays a pivotal role in the solution of (SP-1) for the T-palindromic QRP over $\mathbb{R}$. In fact, we will see in Theorem 3.13 that this concept comprises the necessary and sufficient conditions for a list of elementary divisors and minimal indices to be p-quad realizable over $\mathbb{R}$.

Definition 3.6. (p-quad admissibility over $\mathbb{R}$)
A list $\mathcal{L}$ of elementary divisors and minimal indices is said to be *p-quad admissible over* $\mathbb{R}$ (or to be *real p-quad admissible*) if the following two conditions are satisfied:

(a) $\gamma \leq \frac{1}{2}\big(\delta + \mu\big)$, and
(b) $\mathcal{L}$ has p-quad symmetry over $\mathbb{R}$.

It should be pointed out that the conditions in Definition 3.6 are in fact all of the previously known necessary conditions for a list $\mathcal{L}$ to be p-quad realizable over $\mathbb{R}$. More specifically, condition (a) follows from Theorem 3.3, while condition (b) for p-quad symmetry over $\mathbb{R}$ is a consequence of [28, Cor. 8.1–8.2] for the elementary divisors and [6, Thm. 3.6] for the minimal indices. Also, note that in terms of Definitions 3.4 and 3.6, the main result of this paper states that a list $\mathcal{L}$ is p-quad realizable over $\mathbb{R}$ if and only if $\mathcal{L}$ is p-quad admissible over $\mathbb{R}$ (see Theorem 3.17).

One of the key properties of real p-quad admissibility is that it is preserved by the operation of list concatenation as defined in Definition 3.1(v) – the next lemma describes this for two real p-quad admissible lists, though the result immediately extends to the concatenation of any finite number of such lists.

Lemma 3.7. *Let $\mathcal{L}$ and $\widehat{\mathcal{L}}$ be any two lists of elementary divisors and minimal indices that are p-quad admissible over $\mathbb{R}$. Then the concatenated list $c(\mathcal{L}, \widehat{\mathcal{L}})$ is also p-quad admissible over $\mathbb{R}$.*

Proof. Suppose $\mathcal{L}$ and $\widehat{\mathcal{L}}$ are real p-quad admissible lists, so that both lists have real p-quad symmetry. Then it is straightforward to see that the concatenated list $c(\mathcal{L}, \widehat{\mathcal{L}})$ still has real p-quad symmetry. But $\mathcal{L}$ and $\widehat{\mathcal{L}}$ also both satisfy condition (a) in Definition 3.6, so we claim that $c(\mathcal{L}, \widehat{\mathcal{L}})$ does as well. This follows from the sub-additivity of γ, i.e., that $\gamma\big(c(\mathcal{L}, \widehat{\mathcal{L}})\big) \leq \gamma(\mathcal{L}) + \gamma(\widehat{\mathcal{L}})$, together with the additivity of δ and μ, i.e., that $\delta\big(c(\mathcal{L}, \widehat{\mathcal{L}})\big) = \delta(\mathcal{L}) + \delta(\widehat{\mathcal{L}})$ and $\mu\big(c(\mathcal{L}, \widehat{\mathcal{L}})\big) = \mu(\mathcal{L}) + \mu(\widehat{\mathcal{L}})$. Altogether this shows that concatenation preserves real p-quad admissibility, as desired. □

A notion complementary to the concatenation of lists of elementary divisors and minimal indices is that of *partition*. More specifically, a *partition* of a list $\mathcal{L}$ consists of lists $\mathcal{L}_1, \ldots, \mathcal{L}_m$, $m > 1$, such that $\mathcal{L} = c(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m)$. A partition is said to be nontrivial whenever at least two of the lists $\mathcal{L}_1, \ldots, \mathcal{L}_m$ are non-empty. In the context of the T-palindromic QRP over $\mathbb{R}$, we now define the simplest, most primitive lists having real p-quad symmetry.

Definition 3.8. (p-quad Irreducibility over $\mathbb{R}$)
A list $\mathcal{L}$ is *p-quad irreducible over* $\mathbb{R}$ (or *real p-quad irreducible*) if it is p-quad admissible over $\mathbb{R}$, and there is no nontrivial partition of $\mathcal{L}$ into real p-quad admissible sublists.

One of the main results in this section consists of identifying a *complete set* of all possible p-quad irreducible lists over $\mathbb{R}$ (see Tables 1-3). In our recent work [9], we have identified an equivalent notion of p-quad irreducible lists when the underlying field is algebraically closed. In fact, Tables 1-2 correspond to Tables 1-2 in [9], respectively, and are included here for the sake of completeness. Note that all of the lists in these two tables are still p-quad irreducible over $\mathbb{R}$, with the understanding that the constant a in these lists is an element of $\mathbb{R}$. However, a closer examination of Definition 3.5 immediately indicates that the lists in Tables 1-2 can *not* be sufficient to describe all possible real p-quad irreducible lists. We will show that the lists in Table 3, all ones that are new to this work, constitute all of the additional real p-quad irreducible lists that are needed to accommodate the additional spectral structure specific to *real* T-palindromic matrix polynomials. Now one can easily verify in a direct manner, first that each list in Tables 1-3 is p-quad admissible over $\mathbb{R}$, and second that any nontrivial partition of any of these lists into two sublists will violate at least one of the conditions in Definition 3.6; in other words, lists in Tables 1-3 are all p-quad irreducible over $\mathbb{R}$.

Type	Subtype	Elementary Divisors/Minimal Indices	Conditions
$\boxed{\mathcal{X}}$	$\mathcal{X}_1$	$(\lambda-a)^m, (\lambda-\frac{1}{a})^m$	$m\geq 1, a\neq 0, \pm 1$
	$\mathcal{X}_2$	λ^m, ω^m	$m \geq 1$
$\boxed{\mathcal{Y}}$	$\mathcal{Y}_1$	$(\lambda-1)^{2m}$	$m \geq 1$
	$\mathcal{Y}_1'$	$(\lambda+1)^{2m}$	$m \geq 1$
	$\mathcal{Y}_2$	$(\lambda-1)^{2m+3}, (\lambda-1)^{2m+3}$	$m \geq 0$
	$\mathcal{Y}_2'$	$(\lambda+1)^{2m+3}, (\lambda+1)^{2m+3}$	$m \geq 0$
$\boxed{\mathcal{S}}$	$\mathcal{S}_1$	$\varepsilon = 2k\,,\ \eta = 2k$	$k \geq 0$
	$\mathcal{S}_2$	$\varepsilon = 2k+1\,,\ \eta = 2k+1$	$k \geq 0$

Table 1.: The irreducible NoDO lists over $\mathbb{F} = \overline{\mathbb{F}}$

Remark 3.9. Note that for any of the lists of elementary divisors given in Table 3, it is easy to determine the Smith form of any possible quadratic matrix polynomial realizing that list. For example, consider the $\mathcal{D}_1$-type list of elementary divisors

$$\mathcal{D}_1 = \Big\{ \underbrace{\lambda-1, \ldots, \lambda-1}_{2m}\ ,\ p(\lambda)^k \Big\},$$

where $p(\lambda)$ is an $\mathbb{R}$-irreducible quadratic palindromic scalar polynomial, and $k \geq m > 0$. Now if $Q(\lambda)$ is any quadratic matrix polynomial that has exactly this spectral data, then it must be regular (there are no minimal indices in $\mathcal{D}_1$), and so by the Index Sum Theorem 3.3 must have rank $= m+k$, and hence size $(m+k)\times(m+k)$. Consequently, the Smith form of $Q(\lambda)$ would be

$$\operatorname{diag}\Big(I_{k-m}\,,\, (\lambda-1)\cdot I_{2m-1}\,,\, (\lambda-1)\cdot p^k(\lambda) \Big)\,.$$

Type	Subtype	Elementary Divisors/Minimal Indices	Conditions
$\boxed{\mathcal{A}}$	$\mathcal{A}_1$	$\underbrace{\lambda-1,\ldots,\lambda-1}_{2m},\ (\lambda-a)^n,\ (\lambda-\frac{1}{a})^n$	$n\geq m>0, a\neq 0,\pm 1$
	$\mathcal{A}_1'$	$\underbrace{\lambda+1,\ldots,\lambda+1}_{2m},\ (\lambda-a)^n,\ (\lambda-\frac{1}{a})^n$	$n\geq m>0, a\neq 0,\pm 1$
	$\mathcal{A}_2$	$\underbrace{\lambda-1,\ldots,\lambda-1}_{2m},\ \lambda^n,\ \omega^n$	$n\geq m>0$
	$\mathcal{A}_2'$	$\underbrace{\lambda+1,\ldots,\lambda+1}_{2m},\ \lambda^n,\ \omega^n$	$n\geq m>0$
$\boxed{\mathcal{B}}$	$\mathcal{B}_1$	$\underbrace{\lambda-1,\ldots,\lambda-1}_{2m},\ (\lambda+1)^{2n}$	$n\geq m>0$
	$\mathcal{B}_1'$	$\underbrace{\lambda+1,\ldots,\lambda+1}_{2m},\ (\lambda-1)^{2n}$	$n\geq m>0$
	$\mathcal{B}_2$	$\underbrace{\lambda-1,\ldots,\lambda-1}_{2m},\ (\lambda-1)^{2n}$	$n>m>0$
	$\mathcal{B}_2'$	$\underbrace{\lambda+1,\ldots,\lambda+1}_{2m},\ (\lambda+1)^{2n}$	$n>m>0$
$\boxed{\mathcal{C}}$	$\mathcal{C}_1$	$\underbrace{\lambda-1,\ldots,\lambda-1}_{2m},\ (\lambda+1)^n,\ (\lambda+1)^n$	n odd, $0<m\leq n$
	$\mathcal{C}_1'$	$\underbrace{\lambda+1,\ldots,\lambda+1}_{2m},\ (\lambda-1)^n,\ (\lambda-1)^n$	n odd, $0<m\leq n$
	$\widetilde{\mathcal{C}}_1$	$\lambda-1,\lambda-1,\ \lambda+1,\ \lambda+1$	
	$\mathcal{C}_2$	$\underbrace{\lambda-1,\ldots,\lambda-1}_{2m},\ (\lambda-1)^n,\ (\lambda-1)^n$	n odd, $m\geq 0$ $2n-2m\geq 4$
	$\mathcal{C}_2'$	$\underbrace{\lambda+1,\ldots,\lambda+1}_{2m},\ (\lambda+1)^n,\ (\lambda+1)^n$	n odd, $m\geq 0$ $2n-2m\geq 4$
$\boxed{\mathcal{M}}$	$\mathcal{M}_1$	$\underbrace{\lambda-1,\ldots,\lambda-1}_{2m},\ \varepsilon=2k,\ \eta=2k$	$2k\geq m>0$
	$\mathcal{M}_1'$	$\underbrace{\lambda+1,\ldots,\lambda+1}_{2m},\ \varepsilon=2k,\ \eta=2k$	$2k\geq m>0$
	$\mathcal{M}_2$	$\underbrace{\lambda-1,\ldots,\lambda-1}_{2m},\ \varepsilon=2k+1,\ \eta=2k+1$	$2k+1\geq m>0$
	$\mathcal{M}_2'$	$\underbrace{\lambda+1,\ldots,\lambda+1}_{2m},\ \varepsilon=2k+1,\ \eta=2k+1$	$2k+1\geq m>0$

Table 2.: The irreducible "degree-one" lists over $\mathbb{F}=\overline{\mathbb{F}}$

In the remainder of this section the bulk of our effort goes into showing that every real p-quad admissible list of elementary divisors and minimal indices can be partitioned into p-quad irreducible lists over $\mathbb{R}$, in particular into lists only of the types appearing in our three tables.

Definition 3.10. (p-quad partitioning over $\mathbb{R}$)
A list of elementary divisors and minimal indices is *p-quad partitionable over* $\mathbb{R}$ (or *real p-quad partitionable*) if it can be partitioned into real p-quad irreducible sublists *of the types appearing in Tables 1-3.*

We are now almost ready to state and prove the Palindromic Quadratic Partitioning

Type	Subtype	Elementary Divisors/Minimal Indices	Conditions
$\boxed{\mathcal{Z}}$	$\mathcal{Z}_1$	$p^k(\lambda)$	$k \geq 1$
	$\mathcal{Z}_2$	$q^k(\lambda)\,,\,\widetilde{q}^{\,k}(\lambda)$	$k \geq 1$
$\boxed{\mathcal{D}}$	$\mathcal{D}_1$	$\underbrace{\lambda-1,\ldots,\lambda-1}_{2m}\,,\,p^k(\lambda)\,,$	$k \geq m > 0$
	$\mathcal{D}_1'$	$\underbrace{\lambda+1,\ldots,\lambda+1}_{2m}\,,\,p^k(\lambda)$	$k \geq m > 0$
	$\mathcal{D}_2$	$\underbrace{\lambda-1,\ldots,\lambda-1}_{2m}\,,\,q^k(\lambda)\,,\,\widetilde{q}^{\,k}(\lambda)$	$2k \geq m > 0$
	$\mathcal{D}_2'$	$\underbrace{\lambda+1,\ldots,\lambda+1}_{2m}\,,\,q^k(\lambda)\,,\,\widetilde{q}^{\,k}(\lambda)$	$2k \geq m > 0$

Table 3.: Additional p-quad irreducible lists over $\mathbb{R}$, where $p(\lambda), q(\lambda)$, and $\widetilde{q}(\lambda)$ are $\mathbb{R}$-irreducibles as in (7) and (8).

Theorem over $\mathbb{R}$, but we still need one more result, namely, the NoDO Lemma over $\mathbb{R}$. This lemma is a mild adaptation of [9, Lem. 4.2], a result which in [9] assumed that the underlying field is algebraically closed.

Lemma 3.11. (NoDO Lemma over $\mathbb{R}$)
Let $\mathcal{L}$ be a list of spectral data containing No Degree One *elementary divisors of the form $(\lambda \pm 1)^\alpha$ with $\alpha > 0$; for short, we call such a list $\mathcal{L}$ a* NoDO list. *If a NoDO list $\mathcal{L}$ has real p-quad symmetry, then $\mathcal{L}$ is p-quad partitionable over $\mathbb{R}$. In particular, $\mathcal{L}$ can be partitioned into lists of types $\mathcal{S}_1, \mathcal{S}_2, \mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_1', \mathcal{Y}_2, \mathcal{Y}_2', \mathcal{Z}_1$, and $\mathcal{Z}_2$.*

Proof. The proof closely follows the argument in [9, Lem. 4.2]. We start by partitioning $\mathcal{L}$ into two sublists $\mathcal{E}$ and $\mathcal{T}$, where $\mathcal{T}$ contains all minimal indices and $\mathcal{E}$ all elementary divisors; note that both sublists inherit real p-quad symmetry from $\mathcal{L}$. In addition, $\mathcal{T}$ is also a p-quad admissible list over $\mathbb{R}$, since $\gamma(\mathcal{T}) = 0$ due to the fact that $\mathcal{T}$ contains only minimal indices, and thus condition (a) in Definition 3.6 is satisfied. Furthermore, real p-quad symmetry implies that left and right minimal indices come in pairs of equal value, and therefore $\mathcal{T}$ can be partitioned into sublists of type $\mathcal{S}_1$ and $\mathcal{S}_2$ from Table 1.

On the other hand, list $\mathcal{E}$ is a NoDO list, and due to its p-quad symmetry over $\mathbb{R}$, elementary divisors can be partitioned into four groups:

$$\begin{array}{rl} \text{(i)} & \text{all } (\lambda - a)^\beta \text{ with } \beta \geq 1, \text{ and } a \in \mathbb{R} \text{ with } a \neq 0, \pm 1, \\ \text{(ii)} & \text{all } \lambda^\alpha \text{ and } \omega^\beta \text{ with } \alpha, \beta \geq 1, \\ \text{(iii)} & \text{all } (\lambda \pm 1)^\beta \text{ with } \beta \geq 2 \\ \text{(iv)} & \text{all } t(\lambda)^\beta \text{ with } \beta \geq 1 \text{ and } t(\lambda) \in \mathbb{R}[\lambda] \text{ is quadratic and } \mathbb{R}\text{-irreducible.} \end{array} \tag{9}$$

Now conditions (1a) and (1b) in Definition 3.5 guarantee that the elementary divisors in groups (i) and (ii) can all be paired up to form lists of type $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. The elementary divisors in group (iii) of even degree individually form lists of type $\mathcal{Y}_1$ or $\mathcal{Y}_1'$; those of odd degree can be paired up to form lists of type $\mathcal{Y}_2$ or $\mathcal{Y}_2'$.

Finally, condition (1d) in Definition 3.5 implies that all elementary divisors in group (iv) with palindromic $t(\lambda)$ can be used one at a time to form lists of type $\mathcal{Z}_1$. For all elementary divisors in (iv) in which $t(\lambda)$ is not palindromic, condition (1e) in Definition 3.5 implies that these can all be paired up to form lists of type $\mathcal{Z}_2$. □

Remark 3.12. Careful analysis of the proof of Lemma 3.11 shows that *no mixed lists* of quadratic irreducibles and minimal indices are needed in a p-quad partitioning of any NoDO list. In such a list the spectral and singular structures can be kept completely separate and disentangled from each other, while respecting the requirements of a palindromic, quadratic partitioning.

Finally, the stage is set to *state* and *prove* one of the key results in this section, which can be viewed as an extension of [9, Thm. 3.16].

Theorem 3.13. (Palindromic Quadratic Partitioning Theorem over $\mathbb{R}$) *Let $\mathcal{L}$ be a list of elementary divisors and minimal indices over $\mathbb{R}$. Then*

$$\mathcal{L} \text{ is p-quad partitionable over } \mathbb{R} \iff \mathcal{L} \text{ is p-quad admissible over } \mathbb{R}\,.$$

Proof. ($\Rightarrow$) Assume that $\mathcal{L}$ is p-quad partitionable over $\mathbb{R}$. Then from Definition 3.10 it follows that $\mathcal{L}$ is a concatenation of real p-quad irreducible, and hence real p-quad admissible lists. Since real p-quad admissibility is preserved by concatenation (Lemma 3.7), the desired conclusion follows.

($\Leftarrow$) Assume that $\mathcal{L}$ is p-quad admissible over $\mathbb{R}$. If $\mathcal{L}$ contains any zero minimal indices, then they can be paired together to form sublists of type $\mathcal{S}_1$. It is easy to see that after partitioning away any type $\mathcal{S}_1$ sublist from $\mathcal{L}$, the remaining list is still p-quad admissible over $\mathbb{R}$. Thus for the rest of this proof we assume that $\mathcal{L}$ contains no zero minimal indicies.

Next we examine different possibilities for $\mathcal{L}$ with respect to the number and types of degree-one elementary divisors associated with the eigenvalues $\lambda_0 = \pm 1$. To that end, let r and s be the number of degree-one elementary divisors $(\lambda - 1)$ and $(\lambda + 1)$, respectively, contained in $\mathcal{L}$. There are four cases to consider:

$$\text{(i) } r = s = 0, \qquad \text{(ii) } r = s > 0, \qquad \text{(iii) } r > s \geq 0, \qquad \text{(iv) } s > r \geq 0.$$

In case (i), $\mathcal{L}$ has no degree-one elementary divisors $(\lambda \pm 1)$, so the desired conclusion follows from the NoDO Lemma 3.11. In case (ii), $r/2$ lists of type $\widetilde{\mathcal{C}}_1$ can be partitioned away from $\mathcal{L}$ so that the remaining list $\mathcal{L}'$ still has p-quad symmetry over $\mathbb{R}$, but no degree-one elementary divisors $(\lambda \pm 1)$. Once again, Lemma 3.11 comes to the rescue to conclude that $\mathcal{L}'$ can be p-quad partitioned over $\mathbb{R}$.

Undoubtedly, the hardest case to consider is (iii). Let $r - s =: \ell > 0$. Note that by condition (1c) in Definition 3.5, we know that r and s, and hence also ℓ, must be even. The partitioning of the list $\mathcal{L}$ now proceeds in two phases.

Phase 1: We start by grouping all s elementary divisors $(\lambda + 1)$ with s elementary divisors $(\lambda - 1)$, four at a time, to form $\frac{1}{2}s$ lists of type $\widetilde{\mathcal{C}}_1$. Partitioning away these lists from $\mathcal{L}$ leaves a new list $\mathcal{L}'$ that we claim is still p-quad admissible over $\mathbb{R}$, and has exactly ℓ degree-one elementary divisors, all of which are $(\lambda - 1)$. It is easy to see that $\mathcal{L}'$ still has real p-quad symmetry, but why does it possess the first property required for real p-quad admissibility, i.e., condition (a) in Definition 3.6? To see why this is so, consider the partitioning away of *just one* $\widetilde{\mathcal{C}}_1$ list from $\mathcal{L}$, leaving behind the list $\widetilde{\mathcal{L}}$. For convenience, we define $\rho(\mathcal{L}) := \frac{1}{2}\big(\delta(\mathcal{L}) + \mu(\mathcal{L})\big)$. Note that $2\rho(\mathcal{L})$ is just the index sum of $\mathcal{L}$, so $\rho(\mathcal{L})$ equals the rank of any possible quadratic realization of the list $\mathcal{L}$, and $\gamma(\mathcal{L}) \leq \rho(\mathcal{L})$ is condition (a) in Definition 3.6. Now observe that the longest elementary divisor chain in $\mathcal{L}$ that does *not* involve $(\lambda \pm 1)$ can have length that is at most $\rho(\mathcal{L}) - 2$; any longer such chain would involve irreducible quadratics p or q, or pairs $(\lambda - a), (\lambda - \frac{1}{a})$ or λ, ω, and thus by itself contribute too much to the index sum to leave room for the degree 4 contribution from the $\widetilde{\mathcal{C}}_1$ list being partitioned away. The same length bound on elementary divisor chains not involving $(\lambda \pm 1)$ must also then hold for the list $\widetilde{\mathcal{L}}$. Now in the passage from $\mathcal{L}$ to $\widetilde{\mathcal{L}}$, the length of elementary divisor chains involving $(\lambda \pm 1)$ must decrease by 2, so their length in $\widetilde{\mathcal{L}}$ must also be bounded by $\rho(\mathcal{L}) - 2$. Hence

$$\gamma(\widetilde{\mathcal{L}}) \leq \rho(\mathcal{L}) - 2 = \rho(\widetilde{\mathcal{L}}),$$

the latter equality holding since $\delta(\widetilde{\mathcal{L}}) = \delta(\mathcal{L}) - 4$, due to the loss of the $\widetilde{\mathcal{C}}_1$ sublist. This is the condition that guarantees real p-quad admissibility for $\widetilde{\mathcal{L}}$, and by repeated application also for $\mathcal{L}'$.

Phase 2: The second step of the partitioning of $\mathcal{L}$ is to conjoin as many as possible of the $(\lambda - 1)$'s with both (nonzero) minimal indices in $\mathcal{L}'$ and elementary divisors in $\mathcal{L}'$ that are not associated with the eigenvalue $\lambda_0 = 1$, forming lists of type

$$\mathcal{M}_1, \mathcal{M}_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1, \text{ and/or } \mathcal{D}_2. \tag{10}$$

Now each of the lists in (10) has the capacity to contain a number of $(\lambda-1)$ elementary divisors, up to the total degree of all the *other* elementary divisors (or the sum of the minimal indices) contained in that list. For example, list $\mathcal{D}_1$ can contain up to $2k$ copies of $(\lambda - 1)$, One still has to verify that after partitioning away lists of type (10) from $\mathcal{L}'$, that the remaining list $\mathcal{L}''$ is real p-quad admissible, and p-quad partitionable over $\mathbb{R}$. It turns out that this is the case, as long as the conjoining of $(\lambda - 1)$'s into the lists (10) is done in such a way that the *maximum* possible number of $(\lambda - 1)$'s is absorbed by each of the lists in (10). We do not include the argument justifying this last claim here; although it requires a rather substantial amount of tedious bookkeeping, it is essentially identical to the proof given for [9, Thm. 3.14]. This concludes the proof of case (iii).

Finally, in case (iv), where $\mathcal{L}$ contains more $(\lambda+1)$ elementary divisors than $(\lambda-1)$'s, the proof proceeds in the same way as in case (iii), but with the roles of $(\lambda - 1)$ and $(\lambda + 1)$ reversed. To achieve this role reversal, the lists $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{D}_1$, and $\mathcal{D}_2$ in (10) are simply replaced with their primed counterparts. □

One of the consequences of Theorem 3.13 worth highlighting is that we now know what all of the real p-quad irreducible structural data lists are.

Corollary 3.14. (All p-quad irreducible lists over $\mathbb{R}$)
The structural data lists in Tables 1, 2, and 3 together include all of the possible real p-quad irreducible lists.

Proof. By Theorem 3.13, any real p-quad admissible list that is *not* in Tables 1-3 can be partitioned into lists from those tables, and hence is not irreducible. □

A result about the *uniqueness* of the real p-quad partitioning now follows in turn from Corollary 3.14.

Corollary 3.15. (Unique partitioning of NoDO lists)
Let $\mathcal{L}$ be any NoDO list with p-quad symmetry. Then the real p-quad partitioning of $\mathcal{L}$ described in Lemma 3.11 is the unique way to partition $\mathcal{L}$ into real p-quad irreducible sublists.

Proof. For each elementary divisor or minimal index in a NoDO list $\mathcal{L}$, there is *only one* type of list in Tables 1-3 that contains that particular kind of elementary divisor or minimal index. Since there are *NO* other types of real p-quad irreducible lists that exist, that implies that the partitioning obtained in Lemma 3.11 must be unique. $\square$

Remark 3.16. The uniqueness of partitioning of p-quad symmetric NoDO lists has further ramifications. We will see in Section 4 that each of the NoDO lists in Tables 1 and 3, i.e., those of types $\boxed{\mathcal{X}}$, $\boxed{\mathcal{Y}}$, $\boxed{\mathcal{Z}}$, and $\boxed{\mathcal{S}}$, has a realization that can reasonably be viewed as canonical. This unique partitioning together with these canonical realizations essentially constitutes a quadratic palindromic canonical form for every structural data list that forms a p-quad symmetric NoDO list.

On the other hand, the uniqueness of partitioning of NoDO lists also immediately raises the question of whether this uniqueness extends to *all* real p-quad admissible lists. The answer to this is a definite NO. There are many admissible lists for which partitioning is not unique, but they have one thing in common – the presence of the elementary divisors $(\lambda - 1)$ and/or $(\lambda + 1)$. Let us try to convey an intuitive sense for why these elementary divisors can cause a problem. The primary source of non-unique partitioning lies in Phase 2 of the partitioning algorithm described in the proof of Theorem 3.13, in which "excess" $(\lambda \pm 1)$'s are being "absorbed" by other elementary divisors and nonzero minimal indices. At this stage, if the number of excess $(\lambda \pm 1)$ elementary divisors is *less* than the total capacity of the rest of the *non*-$(\lambda \pm 1)$ elementary divisors (and minimal indices) to absorb, then the distribution of these excess $(\lambda \pm 1)$'s can be done in more than one way, leading to qualitatively distinct partitionings. To illustrate this phenomenon, consider the real p-quad admissible list

$$\mathcal{L} \;=\; \big\{\lambda - 1,\, \lambda - 1,\, \lambda - 1,\, \lambda - 1,\, \lambda + 1,\, \lambda + 1,\, p_1(\lambda),\, p_2(\lambda),\, p_3(\lambda)\big\}\,, \tag{11}$$

where the three $p_i(\lambda)$'s are *distinct* quadratic irreducible palindromic scalar polynomials. Then after partitioning off a $\widetilde{\mathcal{C}}_1$ list we are left with

$$\mathcal{L}' \;=\; \big\{\lambda - 1,\, \lambda - 1,\, p_1(\lambda),\, p_2(\lambda),\, p_3(\lambda)\big\}\,.$$

The remaining two "excess" $(\lambda - 1)$'s can now be grouped (as a pair) with either p_1, p_2, or p_3, leading to three qualitatively distinct real p-quad partitionings of $\mathcal{L}$. (For further discussion and examples of non-unique partitioning, see [9, Remark 4.5].)

There is an additional mechanism (present in this example) that can sometimes lead to further non-uniqueness of partitioning. And that is to skip Phase 1 of the partitioning procedure (in Theorem 3.13) altogether. Note, however, that doing this will not always lead to a valid real p-quad partitioning. But in this example it is a feasible pathway to additional partitionings. Observe that in the original list $\mathcal{L}$ one can pair up two $(\lambda-1)$'s, the other two $(\lambda-1)$'s, and the two $(\lambda+1)$'s. Then each of these pairs can be conjoined to one of the p_i's. This can be done in three ways, thus leading to three additional qualitatively distinct real p-quad partitionings of the list $\mathcal{L}$.

It is worth emphasizing, though, that the mere presence of $(\lambda \pm 1)$'s in a list $\mathcal{L}$ does *not* automatically lead to non-uniqueness of partitioning. If the number of excess $(\lambda \pm 1)$'s are greater than the absorption capacity of the non-$(\lambda \pm 1)$'s, then uniqueness of partitioning may now return. As an example of this, consider the list $\widetilde{\mathcal{L}}$ formed by removing p_2 and p_3 from the list $\mathcal{L}$ in (11). For $\widetilde{\mathcal{L}}$, it is not hard to see that there is *only one way* to group the four $(\lambda-1)$'s: two with the two $(\lambda+1)$'s in a $\widetilde{\mathcal{C}}_1$ list and the other two with $p_1(\lambda)$ in a $\mathcal{D}_1$ list. In other words, $\widetilde{\mathcal{L}}$ has a unique real p-quad partition.

Beyond these Corollaries, the most important role for Theorem 3.13 is as one of the two key ingredients in proving one of the main results in this paper, the solution of (SP-1) for the T-palindromic QRP. The other key ingredient in this proof is the explicit real T-palindromic quadratic realization of each of the real p-quad irreducible lists in Tables 1-3. The construction of these realizations will be completed in Section 4.

Theorem 3.17. ((SP-1) for the T-palindromic QRP)
A structural data list $\mathcal{L}$ of elementary divisors and minimal indices is p-quad realizable over $\mathbb{R}$ if and only if it is p-quad admissible over $\mathbb{R}$.

Proof. ($\Rightarrow$) Let $Q(\lambda)$ be a real p-quad realization of $\mathcal{L}$, i.e., $\mathcal{L} = \mathcal{L}(Q)$. Then from Remark 2.6 it follows that $\mathcal{L}$ has p-quad symmetry over $\mathbb{R}$. The desired conclusion will follow once we verify that $\mathcal{L}$ satisfies condition (a) in Definition 3.6. From the Smith form of Q we know that $\gamma(\mathcal{L}) \leq \text{rank}(Q)$, where $\gamma(\mathcal{L})$ is the length of the longest elementary divisor chain in $\mathcal{L}$. Combining this inequality with the Index Sum Theorem 3.3 and equation (6) gives us $\gamma(\mathcal{L}) \leq \text{rank}(Q) = \frac{1}{2}\big(\delta(\mathcal{L}) + \mu(\mathcal{L})\big)$, as desired.

($\Leftarrow$) Assume $\mathcal{L}$ is p-quad admissible over $\mathbb{R}$. Then Theorem 3.13 implies that $\mathcal{L}$ can be partitioned into a finite number of real p-quad irreducible lists, let's say $\mathcal{L}_1, \mathcal{L}_1, \ldots, \mathcal{L}_m$. Once we show that each of the real p-quad irreducible lists in Tables 1-3 can be realized by a T-palindromic quadratic matrix polynomial over $\mathbb{R}$ – this is Theorem 5.1 – the desired conclusion will follow by taking a direct sum of the real p-quad realizations of $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m$. □

Theorem 3.17 now provides us with a very simple characterization of p-quad realizability over $\mathbb{R}$. All that is needed is to verify the two easy-to-check conditions for real p-quad admissibility given in Definition 3.6. Now although this theorem is itself very simple to state and in the end has a rather short proof, that brevity is somewhat misleading. Underpinning Theorem 3.17 are *many* technical results, both from this paper and from [9] underlying Theorem 3.13, as well as Theorem 5.1 in the penultimate section, where it is shown how to concretely realize each of the new real p-quad irreducible lists that have been introduced in this paper in order to handle the *real* T-palindromic QRP. Note that *any* real T-palindromic quadratic realizations of real p-quad irreducible lists from Tables 1-3 would suffice for the purpose of proving Theorem 3.17. However, we will use the whole next section to build the infrastructure needed to show how each of the real p-quad irreducible lists can in fact be realized by a "canonical" real T-palindromic quadratic block that *transparently* reveals its spectral and singular structures. Then taking a direct sum of such blocks will produce not just a quadratic realization for any given real p-quad admissible list $\mathcal{L}$, but in fact a *structured Kronecker-like real quasi-canonical form* for any real T-palindromic quadratic matrix polynomial having structural data list $\mathcal{L}$. In other words, we will have solved (SP-2) of the T-palindromic QRP over $\mathbb{R}$.

4. Transparent real p-quad realizations of lists in Table 3

In the previous section, we have defined the notion of real p-quad irreducible lists of elementary divisors and minimal indices and have shown how they can be used to partition an arbitrary p-quad admissible list over $\mathbb{R}$. That in itself brought us a step closer to solving (SP-1) of the T-palindromic QRP over $\mathbb{R}$, i.e., showing that a real p-quad admissible list is in fact p-quad realizable over $\mathbb{R}$. The missing part needed to complete the solution of (SP-1), and at the same time solve (SP-2), is to show how to explicitly construct a real T-palindromic quadratic matrix polynomial whose spectral and singular structures are transparently displayed and realize each of the real p-quad irreducible lists from Tables 1-3; the current section accomplishes exactly this.

4.1. Background technical results

We start by introducing notation for two types of matrices commonly encountered throughout the rest of this paper. With $\widetilde{I}_k$ and $\widetilde{N}_k$ we denote the $k \times k$ constant matrices given by

$$\widetilde{I}_k := \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}_{k\times k} \qquad \text{and} \qquad \widetilde{N}_k := \begin{bmatrix} & & & 0 \\ & & 0 & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & \end{bmatrix}_{k\times k} . \tag{12}$$

Of particular interest to us will be block-matrix polynomials of the form

$$\widetilde{I}_k \otimes A(\lambda) \;+\; \widetilde{N}_k \otimes B(\lambda) \;=\; \begin{bmatrix} & & & A(\lambda) \\ & & A(\lambda) & B(\lambda) \\ & \iddots & \iddots & \\ A(\lambda) & B(\lambda) & & \end{bmatrix}_{nk\times nk} ,$$

where $A(\lambda)$ and $B(\lambda)$ are $n \times n$ matrix polynomials, possibly constant. Of course, all of the omitted entries/blocks are assumed to be zero.

Next we recall two fundamental lemmas from our earlier work [9] that also play an important role in this paper – they are included here for the sake of completeness and without proof. As a reminder, the notation $P(\lambda) \sim Q(\lambda)$ is used to denote that matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are unimodularly equivalent, Row_i and Col_j denote the i^{th} row and j^{th} column of a general matrix, the notation $(\star) \to (\bullet)$ corresponds to the elementary row/column operation that replaces the row/column $(\star)$ by the row/column $(\bullet)$, and $(\star) \leftrightarrow (\bullet)$ denotes row/column swap between $(\star)$ and $(\bullet)$. Finally, we assume that the gcd of two scalar polynomials is always taken to be monic.

Lemma 4.1. ([9, Lem. 5.2]) *Let $f, g, h \in \mathbb{F}[\lambda]$ and let $r := \gcd(f, h)$. Then:*

(a) $\begin{bmatrix} 0 & g \\ f & h \end{bmatrix} \cdot U = \begin{bmatrix} t & s \\ r & 0 \end{bmatrix}$, *where U is a unimodular matrix, both s and t are polynomial multiples of g, and the relation $rs = fg$ holds.*

(b) *Let $r, s, t \in \mathbb{F}[\lambda]$ be such that r divides t. Then* $\begin{bmatrix} t & s \\ r & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & s \\ r & 0 \end{bmatrix}$, *using exactly one elementary row operation of the form* $\mathrm{Row}_1 \to \mathrm{Row}_1 + k \cdot \mathrm{Row}_2$, *where $k \in \mathbb{F}[\lambda]$.*

(b) *Let $r, s, t \in \mathbb{F}[\lambda]$ be such that $\gcd(r, s) \equiv 1$. Then* $\begin{bmatrix} t & s \\ r & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & s \\ r & 0 \end{bmatrix}$, *using exactly one elementary column operation of the form* $\mathrm{Col}_1 \to \mathrm{Col}_1 + \beta \cdot \mathrm{Col}_2$ *and one elementary row operation of the form* $\mathrm{Row}_1 \to \mathrm{Row}_1 + k \cdot \mathrm{Row}_2$, *where $\beta, k \in \mathbb{F}[\lambda]$.*

Lemma 4.2. (Bi-antidiagonal Collapsing Lemma, [9, Lem. 5.4]).
Let $B(\lambda)$ be an $n \times n$ matrix polynomial over an arbitrary field of the form

$$B(\lambda) = \begin{bmatrix} & & & & a_n(\lambda) \\ & & & a_{n-1}(\lambda) & b_{n-1}(\lambda) \\ & & \iddots & \iddots & \\ & a_2(\lambda) & b_2(\lambda) & & \\ a_1(\lambda) & b_1(\lambda) & & & \end{bmatrix}.$$

Let $r(\lambda) := \gcd(a_1, b_1)$ *and assume the following:*

(a) r *divides each of the polynomials* $a_1, a_2, \ldots, a_n$*, and*

(b) $\gcd\left(\frac{a_1 a_2 \cdots a_j}{r^{j-1}}, b_j\right) = r$*, for* $j = 1, \ldots, n-1$.

Then $B(\lambda)$ *is unimodularly equivalent to the anti-diagonal matrix* $W(\lambda)$*, where* $W(\lambda) := \widetilde{I}_n \cdot \operatorname{diag}\Big(\underbrace{r(\lambda), \ldots, r(\lambda)}_{n-1}, p(\lambda)\Big)$ *and*

$$p(\lambda) := r(\lambda) \cdot \left(\frac{a_1(\lambda)a_2(\lambda)\cdots a_n(\lambda)}{r^n(\lambda)}\right) = \frac{a_1(\lambda)a_2(\lambda)\cdots a_n(\lambda)}{r^{n-1}(\lambda)}.$$

Moreover, the unimodular equivalence $B(\lambda) \sim W(\lambda)$ *can be achieved in such a way that the only elementary row operation involving the first row is of the form* $\mathrm{Row}_1 \to \mathrm{Row}_1 + h(\lambda) \cdot \mathrm{Row}_2$*, for some polynomial* $h(\lambda)$*.*

Remark 4.3. It is important to note that a "downwards version" of Lemma 4.2 is also available. More specifically, if $r(\lambda) := \gcd(a_n, b_{n-1})$, condition (b) is replaced by

$$\gcd\left(\frac{a_n a_{n-1} \cdots a_{n-j+1}}{r^{j-1}}, b_{n-j}\right) = r(\lambda), \quad \text{for} \quad j = 1, \ldots, n-1,$$

and $W(\lambda)$ is replaced by $\operatorname{diag}(r(\lambda), \ldots, r(\lambda), p(\lambda)) \cdot \widetilde{I}_n = W^T(\lambda)$, then Lemma 4.2 still holds. Moreover, the only elementary column operation in the unimodular reduction of $B(\lambda)$ that involves the first column of $B(\lambda)$ is of the form $\mathrm{Col}_1 \to \mathrm{Col}_1 + h(\lambda) \cdot \mathrm{Col}_2$, for some polynomial $h(\lambda)$ – for additional details see [9, Rem. 5.5].

4.2. Basic building blocks and auxiliary results

In this section we introduce the essential building blocks used to construct real p-quad realizations for the lists from Table 3, and examine the structural data of these blocks. Furthermore, we establish several auxiliary results that play a key role in our solution of (SP-2).

Lemma 4.4. *For an* $\mathbb{R}$-irreducible palindromic scalar polynomial $p(\lambda) = \lambda^2 + b\lambda + 1$, *define* $J(p)$ *and* $J'(p)$ *to be the* 2×2 *matrix polynomials*

$$\begin{aligned} J(p) &:= \begin{bmatrix} (\beta+1)(\lambda-1)^2 & (\lambda-1)(\beta\lambda-1) \\ (\lambda-1)(\lambda-\beta) & (\lambda-1)^2 \end{bmatrix} = (\lambda-1)\begin{bmatrix} (\beta+1)(\lambda-1) & (\beta\lambda-1) \\ (\lambda-\beta) & (\lambda-1) \end{bmatrix}, \\ J'(p) &:= \begin{bmatrix} (\delta+1)(\lambda+1)^2 & (\lambda+1)(\delta\lambda+1) \\ (\lambda+1)(\lambda+\delta) & (\lambda+1)^2 \end{bmatrix} = (\lambda+1)\begin{bmatrix} (\delta+1)(\lambda+1) & (\delta\lambda+1) \\ (\lambda+\delta) & (\lambda+1) \end{bmatrix}, \end{aligned} \tag{13}$$

where β and δ are scalars such that

$$(\beta-1)^2 = b+2 \qquad \text{and} \qquad (\delta-1)^2 = 2-b\,. \tag{14}$$

Then:

(a) *$J(p)$ and $J'(p)$ are T-palindromic quadratic matrix polynomials over $\mathbb{R}$.*

(b) *$J(p)$ and $J'(p)$ are unimodularly equivalent to $\widetilde{J}(p)$ and $\widetilde{J}'(p)$, respectively, where*

$$\widetilde{J}(p) := (\lambda-1)\begin{bmatrix} 0 & -1 \\ p(\lambda) & \lambda-1 \end{bmatrix} \qquad \text{and} \qquad \widetilde{J}'(p) := (\lambda+1)\begin{bmatrix} 0 & -1 \\ p(\lambda) & \lambda+1 \end{bmatrix}.$$

Proof. The assumption that $p(\lambda) = \lambda^2 + b\lambda + 1$ is an $\mathbb{R}$-irreducible quadratic polynomial is equivalent to the condition $-2 < b < 2$, which together with (14) implies that $\beta - 1$ and $\delta - 1$ are *nonzero real* constants. Consequently, $\beta, \delta \in \mathbb{R}$ and so, by construction, $J(P)$ and $J'(p)$ are *real* matrix polynomials. Now direct inspection easily confirms that $J(p)$ and $J'(p)$ are in fact T-palindromic and quadratic, thus proving statement (a). To see why (b) is true, consider the matrix polynomials

$$\begin{aligned}
F(\lambda) &:= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \zeta(\lambda) & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{1-\beta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \zeta(\lambda)-1 & \frac{1}{1-\beta} \end{bmatrix}, \\
F'(\lambda) &:= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \zeta'(\lambda) & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\delta-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \zeta'(\lambda)-1 & \frac{1}{\delta-1} \end{bmatrix}, \\
E &:= \begin{bmatrix} 1 & -\beta \\ 0 & 1-\beta \end{bmatrix}, \qquad \text{and} \qquad E' := \begin{bmatrix} 1 & -\delta \\ 0 & \delta-1 \end{bmatrix},
\end{aligned} \tag{15}$$

where

$$\zeta(\lambda) := \frac{(\lambda-1)+(\beta-1)^2}{1-\beta} \qquad \text{and} \qquad \zeta'(\lambda) := \frac{(\lambda+1)-(\delta-1)^2}{\delta-1}.$$

Observe that matrix polynomials $F(\lambda), F'(\lambda), E$, and E' all have nonzero constant determinants and hence are *unimodular*. Finally, it is easy to verify that

$$E \cdot J(p) \cdot F(\lambda) = \widetilde{J}(p) \qquad \text{and} \qquad E' \cdot J'(p) \cdot F'(\lambda) = \widetilde{J}'(p)\,, \tag{16}$$

which proves part (b). ☐

Remark 4.5. Matrices $E, F(\lambda), E'$, and $F'(\lambda)$ from (15) have several notable features. First, all of them are just products of elementary unimodular matrix polynomials corresponding to row and column operations, and hence they are also unimodular. Second, $F(\lambda)$ and $F'(\lambda)$ are *lower triangular*, while E and E' are *upper triangular* matrices; we will see later that this observation is essential when investigating spectral structures of block matrix polynomials having $J(p)$ and $J'(p)$ as their blocks.

The matrix polynomials $J(p)$ and $J'(p)$ in Lemma 4.4 also have easily traceable spectral data. The characterization of invariant polynomials in terms of gcd's of minors (from Theorem 2.1(iii)) applied to $\widetilde{J}(p)$ and $\widetilde{J}'(p)$ implies that the Smith forms of $J(p)$ and $J'(p)$ are $\operatorname{diag}\big((\lambda-1), (\lambda-1)p(\lambda)\big)$ and $\operatorname{diag}\big((\lambda+1), (\lambda+1)p(\lambda)\big)$, respectively. Consequently, $\delta_{fin}(J(p)) = \delta_{fin}(J'(p)) = 4$. Coupling this information with Theorem 3.3 and the fact that rank and grade of both $J(p)$ and $J'(p)$ is two, implies that $\delta_\infty(J(p)) = \delta_\infty(J'(p)) = 0$. Hence, the spectral structures of $J(p)$ and $J'(p)$ consist solely of the *finite* elementary divisors $\{(\lambda-1), (\lambda-1), p(\lambda)\}$ and $\{(\lambda+1), (\lambda+1), p(\lambda)\}$, respectively.

The alert reader will have spotted an ambiguity in the definitions of the matrix polynomials $J(p)$ and $J'(p)$. The equations in (14) defining the constants β and δ each admit two possible values, $\beta = 1 \pm \sqrt{b+2}$ and $\delta = 1 \pm \sqrt{2-b}$. So it is natural to wonder whether it matters which values are used. Within the confines of Lemma 4.4, it does not matter; the claims in the statement are true regardless of which values are chosen. However, for the purposes of the rest of the paper, where $J(p)$ and $J'(p)$ will be used as elements of larger block constructions, it is *essential* that one value be picked for each of β and δ, and to stick with those values throughout. To that end, then, we will simply make the arbitrary choices $\beta := 1 + \sqrt{b+2}$ and $\delta := 1 + \sqrt{2-b}$, and keep those values fixed for the rest of the paper.

The following simple, but important, result will be used frequently throughout the rest of the paper, so we state it here for the sake of easier reference.

Lemma 4.6. *Let $E(\lambda)$ and $F(\lambda)$ be 2×2 upper triangular and lower triangular matrix polynomials, respectively, such that the $(1,1)$ entry of each is equal to one. Then*

$$E(\lambda)\cdot\begin{bmatrix} \bullet & 0 \\ 0 & 0 \end{bmatrix} = E(\lambda)\cdot\begin{bmatrix} \bullet & 0 \\ 0 & 0 \end{bmatrix}\cdot F(\lambda) = \begin{bmatrix} \bullet & 0 \\ 0 & 0 \end{bmatrix}\cdot F(\lambda) = \begin{bmatrix} \bullet & 0 \\ 0 & 0 \end{bmatrix}, \tag{17}$$

where $\bullet$ is an arbitrary scalar polynomial, possibly a constant.

Definition 4.7. Let $t(\lambda) \in \mathbb{R}[\lambda]$ and $G(\lambda) \in \mathbb{R}[\lambda]^{2\times 2}$. For a positive integer m, the $m\times m$ matrix $R_m(t)$ and the $2m\times 2m$ matrices $H_m(G)$ and $H'_m(G)$ are defined as

$$R_m(t) \;:=\; t(\lambda)\cdot\widetilde{I}_m \;+\; \lambda\widetilde{N}_m\,, \tag{18}$$

$$H_m(G) \;:=\; \widetilde{I}_m \otimes G(\lambda) \;+\; \widetilde{N}_m \otimes Q^-(\lambda)\,, \tag{19}$$

$$H'_m(G) \;:=\; \widetilde{I}_m \otimes G(\lambda) \;+\; \widetilde{N}_m \otimes Q^+(\lambda)\,, \tag{20}$$

where $\widetilde{I}_m$ and $\widetilde{N}_m$ are given in (12), $Q^-(\lambda) = \left[\begin{smallmatrix} (\lambda-1)^2 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and $Q^+(\lambda) = \left[\begin{smallmatrix} (\lambda+1)^2 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

The following lemma investigates properties of several special H_m and H'_m blocks.

Lemma 4.8. *Let $p(\lambda) = \lambda^2 + b\lambda + 1$ be an $\mathbb{R}$-irreducible quadratic palindromic scalar polynomial and let m be a positive integer. Then:*

(a) *$H_m(J(p))$ and $H'_m(J'(p))$ are real T-palindromic quadratic matrix polynomials.*

(b) *$H_m(\widetilde{J}(p))$ and $H'_m(\widetilde{J}'(p))$ are bi-antidiagonal with sub-antidiagonal entries $(\lambda-1)^2$ and $(\lambda+1)^2$, respectively.*

(c) $H_m(J(p)) \sim H_m(\widetilde{J}(p))$ *and* $H'_m(J'(p)) \sim H'_m(\widetilde{J}'(p))$.

(d) *The Smith form of $H_m(J(p))$ is* $\operatorname{diag}\big((\lambda-1)\cdot I_{2m-1},\, (\lambda-1)\cdot p^m\big)$.

(e) *The Smith form of $H'_m(J'(p))$ is* $\operatorname{diag}\big((\lambda+1)\cdot I_{2m-1},\, (\lambda+1)\cdot p^m\big)$.

Proof. Part (a) follows directly from the block structure of $H_m(J(P))$ and $H'_m(J(p))$, properties of block matrix transpose, and the fact that all $J(p)$, $Q^-(\lambda)$, and $Q^+(\lambda)$

are T-palindromic quadratic matrix polynomials. Part (b) is a consequence of the fact that $\widetilde{J}(p)$ and $\widetilde{J}'(p)$ are 2×2 bi-antidiagonal matrix polynomials whose $(2,2)$ entries are $(\lambda-1)^2$ and $(\lambda+1)^2$, respectively.

To prove part (c) we consider unimodular matrix polynomials $E, E', F(\lambda)$, and $F'(\lambda)$ from (15) and observe that the $(1,1)$ entry of each of them is equal to one. Using the properties of block multiplication, the special triangular structures of $E, E', F(\lambda)$, and $F'(\lambda)$, together with Lemma 4.6, one can verify directly that

$$\begin{aligned} \big(I_m \otimes E\big) \cdot H_m(J(p)) \cdot \big(I_m \otimes F(\lambda)\big) &= H_m(\widetilde{J}(p))\,, \\ \big(I_m \otimes E'\big) \cdot H'_m(J'(p)) \cdot \big(I_m \otimes F'(\lambda)\big) &= H'_m(\widetilde{J}'(p))\,. \end{aligned} \tag{21}$$

Now since the determinants of $I_m \otimes E, I_m \otimes E', I_m \otimes F(\lambda)$, and $I_m \otimes F'(\lambda)$ are the determinants of $E, E', F(\lambda)$, and $F'(\lambda)$ raised to the power of m, respectively, it follows that they are nonzero constants. Therefore, $I_m \otimes E, I_m \otimes E', I_m \otimes F(\lambda)$, and $I_m \otimes F'(\lambda)$ are unimodular matrix polynomials, and so (21) proves part (c).

Next we apply Lemma 4.1 to $H_m(\widetilde{J}(p))$ to obtain

$$H_m(J(p)) \sim H_m(\widetilde{J}(p)) \sim \begin{bmatrix} & (-1)^m(\lambda-1)p^m(\lambda) \\ (\lambda-1)\widetilde{I}_{2m-1} & \end{bmatrix}. \tag{22}$$

Finally, scaling the last column of (22) by $(-1)^m$, followed by an appropriate permutation of rows and columns implies part (d); mutatis mutandis, this argument also applies to part (e). ☐

We proceed with some further technical results.

Lemma 4.9. *Let $r(\lambda), s(\lambda), t(\lambda)$, and $u(\lambda)$ be* nonzero *scalar polynomials such that*

- *$r(\lambda)$ and $s(\lambda)$ are each relatively prime to λ and $(\lambda-1)$, and*
- *$t(\lambda)$ and $u(\lambda)$ are each relatively prime to λ and $(\lambda+1)$.*

Define the associated scalars

$$\rho := -r(1)s(1) \qquad \text{and} \qquad \rho' := t(-1)u(-1)\,, \tag{23}$$

and consider the 3×3 matrix polynomials $T(r,s)$ and $T'(t,u)$ given by

$$T(r,s) := \begin{bmatrix} \rho\cdot\lambda & 0 & s(\lambda) \\ 0 & (\lambda-1)^2 & \lambda(1-\lambda) \\ r(\lambda) & (\lambda-1) & -\lambda \end{bmatrix}, \; T'(t,u) := \begin{bmatrix} \rho'\cdot\lambda & 0 & u(\lambda) \\ 0 & (\lambda+1)^2 & \lambda(\lambda+1) \\ t(\lambda) & (\lambda+1) & \lambda \end{bmatrix}. \tag{24}$$

Then

(a) *$T(r,s)$ has the Smith form* $\operatorname{diag}\big(1\,,\,(\lambda-1)\,,\,(\lambda-1)\cdot\widetilde{r}(\lambda)\cdot\widetilde{s}(\lambda)\big)$*, and*

(b) *$T'(t,u)$ has the Smith form* $\operatorname{diag}\big(1\,,\,(\lambda+1)\,,\,(\lambda+1)\cdot\widetilde{t}(\lambda)\cdot\widetilde{u}(\lambda)\big)$*,*

where $\widetilde{r}(\lambda), \widetilde{s}(\lambda), \widetilde{t}(\lambda)$, and $\widetilde{u}(\lambda)$ denote the scalar multiples of $r(\lambda), s(\lambda), t(\lambda)$, and $u(\lambda)$, respectively, that are monic.

Proof. Let $G(\lambda)$ and $U(\lambda)$ be the matrix polynomials defined by

$$G(\lambda) := \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & s(\lambda) \\ 0 & -1 & (\lambda-1) \end{bmatrix} \quad \text{and} \quad U(\lambda) := \begin{bmatrix} 0 & 1 & 0 \\ 1 & r(\lambda) & \lambda \\ 1 & r(\lambda) & (\lambda-1) \end{bmatrix}. \tag{25}$$

Computing the determinants of $G(\lambda)$ and $U(\lambda)$ shows that both matrix polynomials are *unimodular*. Now let $\widehat{T}(\lambda)$ be the 2×2 matrix polynomial given by

$$\widehat{T}(\lambda) := \begin{bmatrix} \rho\cdot\lambda + r(\lambda)\cdot s(\lambda) & (\lambda-1)\cdot s(\lambda) \\ (\lambda-1)\cdot r(\lambda) & 0 \end{bmatrix} \tag{26}$$

and observe that

$$G(\lambda)\cdot T(r,s)\cdot U(\lambda) = \left[\begin{array}{c|c} 1 & \\ \hline & \widehat{T}(\lambda) \end{array}\right]. \tag{27}$$

But (27) implies that if $\widehat{D}(\lambda)$ is the Smith form of $\widehat{T}(\lambda)$, then $\operatorname{diag}\big(1, \widehat{D}(\lambda)\big)$ is the Smith form of $T(r,s)$. We now show that $\widehat{D}(\lambda) = \operatorname{diag}\big((\lambda-1), (\lambda-1)\cdot\widetilde{r}(\lambda)\cdot\widetilde{s}(\lambda)\big)$.

The specific choice of ρ in (23) implies that the polynomial $\rho\cdot\lambda + r(\lambda)\cdot s(\lambda)$, when evaluated at $\lambda = 1$, equals zero. Therefore, $(\lambda-1)$ is a factor of $\rho\cdot\lambda + r(\lambda)\cdot s(\lambda)$, i.e., there exists a scalar polynomial $h(\lambda)$ such that

$$\rho\cdot\lambda + r(\lambda)\cdot s(\lambda) \;=\; (\lambda-1)\cdot h(\lambda). \tag{28}$$

Combining (28) and (26) together shows that $(\lambda-1)$ is a common factor of *all* entries of $\widehat{T}(\lambda)$, and so $(\lambda-1)$ divides the first invariant polynomial of $\widehat{T}(\lambda)$. To prove that the first invariant polynomial is exactly $(\lambda-1)$, it suffices to show that $h(\lambda)$ is relatively prime to $s(\lambda)$ and/or $r(\lambda)$ (see Theorem 2.1(iii)).

From the assumption that both $r(\lambda)$ and $s(\lambda)$ are relatively prime to λ we know that $r(\lambda)\cdot s(\lambda)$ is also relatively prime to λ. Consequently, there exist scalar polynomials $a(\lambda)$ and $b(\lambda)$ such that

$$a(\lambda)\cdot\big(r(\lambda)\cdot s(\lambda)\big) + b(\lambda)\cdot\lambda \;=\; 1\,. \tag{29}$$

Furthermore, the assumption that both $r(\lambda)$ and $s(\lambda)$ are also relatively prime to $\lambda-1$ implies that $\rho\neq 0$. Hence, there exist polynomials $x_1(\lambda)$ and $x_2(\lambda)$ such that

$$b(\lambda) = \rho\cdot x_2(\lambda) \qquad \text{and} \qquad a(\lambda) = x_1(\lambda) + x_2(\lambda)\,. \tag{30}$$

From (29) and (30) we obtain

$$x_1(\lambda)\cdot\big(r(\lambda)\cdot s(\lambda)\big) + x_2(\lambda)\cdot\big(\rho\cdot\lambda + r(\lambda)\cdot s(\lambda)\big) \;=\; 1\,, \tag{31}$$

which in turn implies that $r(\lambda)\cdot s(\lambda)$ s relatively prime to $\rho\cdot\lambda + r(\lambda)\cdot s(\lambda)$. But this fact together with (28) implies that $h(\lambda)$ is relatively prime to $r(\lambda)\cdot s(\lambda)$, and therefore that the first invariant polynomial of $\widehat{T}(\lambda)$ is $(\lambda-1)$, as desired.

Now note that $\det(\widehat{T})$ is just the product of the invariant polynomials of $\widehat{T}(\lambda)$ up to a scalar multiple. Since $\det(\widehat{T}) = -(\lambda-1)^2 r(\lambda)s(\lambda)$, we conclude that the second invariant polynomial of $\widehat{T}(\lambda)$ is $(\lambda-1)\cdot\widetilde{r}(\lambda)\cdot\widetilde{s}(\lambda)$, which completes our proof of part (a).

Finally, part (b) can be verified by a similar argument, replacing the matrix polynomials $G(\lambda), U(\lambda)$, and $\widehat{T}(\lambda)$ with their primed counterparts, namely,

$$G'(\lambda) := \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & u(\lambda) \\ 0 & -1 & (\lambda+1) \end{bmatrix}, \quad U'(\lambda) := \begin{bmatrix} 0 & 1 & 0 \\ -1 & -t(\lambda) & -\lambda \\ 1 & t(\lambda) & (\lambda+1) \end{bmatrix},$$
$$\widehat{T}'(\lambda) := \begin{bmatrix} \rho' \cdot \lambda + t(\lambda)\cdot u(\lambda) & (\lambda+1)\cdot u(\lambda) \\ (\lambda+1)\cdot t(\lambda) & 0 \end{bmatrix}.$$

□

An interesting feature of our proof of Lemma 4.9 concerns the matrix polynomials $G(\lambda)$ and $U(\lambda)$. At first sight, it is unclear why these matrix polynomials are unimodular, where they come from, and why they give us the desired unimodular equivalence (27). But the answer is quite simple – $G(\lambda)$ and $U(\lambda)$ are products of the elementary unimodular matrix polynomials corresponding to elementary row and column operations, respectively, used to transform $T(r,s)$ into $\operatorname{diag}\big(1, \widehat{T}(\lambda)\big)$ as in (27). More specifically, pre- and post-multiplying $T(r,s)$ by $G(\lambda)$ and $U(\lambda)$ from (25), respectively, is the same as performing the following sequence of elementary unimodular operations on $T(r,s)$

①	$\mathrm{Row}_2 \longrightarrow (1-\lambda)\cdot\mathrm{Row}_3 + \mathrm{Row}_2$	⑦	$\mathrm{Row}_3 \longrightarrow (-1)\cdot\mathrm{Row}_3$
②	$\mathrm{Col}_3 \longrightarrow \mathrm{Col}_2 + \mathrm{Col}_3$	⑧	$\mathrm{Row}_3 \longleftrightarrow \mathrm{Row}_2$
③	$\mathrm{Col}_2 \longrightarrow (\lambda-1)\cdot\mathrm{Col}_3 + \mathrm{Col}_2$	⑨	$\mathrm{Row}_2 \longleftrightarrow \mathrm{Row}_1$
④	$\mathrm{Row}_1 \longrightarrow s(\lambda)\cdot\mathrm{Row}_3 + \mathrm{Row}_1$	⑩	$\mathrm{Col}_3 \longleftrightarrow \mathrm{Col}_2$
⑤	$\mathrm{Col}_1 \longrightarrow r(\lambda)\cdot\mathrm{Col}_3 + \mathrm{Col}_1$	⑪	$\mathrm{Col}_2 \longleftrightarrow \mathrm{Col}_1$
⑥	$\mathrm{Row}_2 \longrightarrow (-1)\cdot\mathrm{Row}_2$		

We now take a closer look at matrix polynomials T and T' from (24). Our initial motivation for considering these matrix polynomials was to construct T-palindromic quadratic realizations for special cases of lists of type $\mathcal{D}_1$ from Table 3. To that end, note that if $t(\lambda)$ is a real nonzero scalar polynomial of grade two, then $T(t, \mathrm{rev}_2 t)$ and $T'(t, \mathrm{rev}_2 t)$ from (24) are quadratic T-palindromic matrix polynomials over $\mathbb{R}$. The following result follows directly from Lemma 4.9.

Corollary 4.10. *Let $p(\lambda)$ and $q(\lambda)$ be $\mathbb{R}$-irreducible quadratic scalar polynomials, where $p(\lambda)$ is palindromic and $q(\lambda)$ is not. Consider the following two $\mathcal{D}_1$-type lists of elementary divisors,*

$$\mathcal{L}_1 = \big\{\lambda-1\,,\,\lambda-1\,,\,p^2(\lambda)\big\} \quad \text{and} \quad \mathcal{L}_2 = \{\lambda+1\,,\,\lambda+1\,,\,q(\lambda)\,,\,\widetilde{q}(\lambda)\}\,,$$

where $p(\lambda)$, $q(\lambda)$, and $\widetilde{q}(\lambda)$ are given by (7)-(8). Then $T(p,p)$ and $T'(q, \mathrm{rev}_2 q)$ are T-palindromic quadratic realizations of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

The next two results are also motivated by the desire to construct T-palindromic quadratic realizations for additional cases of $\mathcal{D}_1$-type lists of elementary divisors. The 5×5 matrix polynomials in these two results contain one or the other of the matrix polynomials $T(r,s)$ and $T(t,u)$ from Lemma 4.9 as their central 3×3 blocks. Furthermore, Lemma 4.11 (resp., Lemma 4.12) describes the effect on the Smith form when

$r^k(\lambda)$ and $s^k(\lambda)$ (resp., $t^k(\lambda)$ and $u^k(\lambda)$) are "attached" to the corners of $T(r,s)$ (resp., $T(t,u)$) and "glued" with λ and $\lambda \pm 1$.

Lemma 4.11. *Let the polynomials $r(\lambda), s(\lambda), t(\lambda), u(\lambda), \widetilde{r}(\lambda), \widetilde{s}(\lambda), \widetilde{t}(\lambda), \widetilde{u}(\lambda)$ and the scalars ρ, ρ' be as in Lemma 4.9. Then the 5×5 matrix polynomials $N_{m,n,k}(r,s)$ and $N'_{m,n,k}(t,u)$ given by*

$$N_{m,n,k}(r,s) \;:=\; \begin{bmatrix} (\lambda-1)^m\lambda^n & & & & s^k(\lambda) \\ & \rho\cdot\lambda & 0 & s(\lambda) & \lambda \\ & 0 & (\lambda-1)^2 & \lambda(1-\lambda) & \\ & r(\lambda) & (\lambda-1) & -\lambda & \\ r^k(\lambda) & \lambda & & & \end{bmatrix}, \tag{32}$$

$$N'_{m,n,k}(t,u) \;:=\; \begin{bmatrix} (\lambda+1)^m\lambda^n & & & & u^k(\lambda) \\ & \rho'\cdot\lambda & 0 & u(\lambda) & \lambda \\ & 0 & (\lambda+1)^2 & \lambda(\lambda+1) & \\ & t(\lambda) & (\lambda+1) & \lambda & \\ t^k(\lambda) & \lambda & & & \end{bmatrix}, \tag{33}$$

where integers m,k are positive and n is nonnegative, have the following properties:

(a) *$N_{m,n,k}(r,s)$ has the Smith form* $\operatorname{diag}\big(I_3,\,(\lambda-1),\,(\lambda-1)\cdot\widetilde{r}^{\,k+1}(\lambda)\cdot\widetilde{s}^{\,k+1}(\lambda)\big)$,

(b) *$N'_{m,n,k}(t,u)$ has the Smith form* $\operatorname{diag}\big(I_3,\,(\lambda+1),\,(\lambda+1)\cdot\widetilde{t}^{\,k+1}(\lambda)\cdot\widetilde{u}^{\,k+1}(\lambda)\big)$.

Lemma 4.12. *Let the polynomials $r(\lambda), s(\lambda), t(\lambda), u(\lambda), \widetilde{r}(\lambda), \widetilde{s}(\lambda), \widetilde{t}(\lambda), \widetilde{u}(\lambda)$ and the scalars ρ, ρ' be as in Lemma 4.9. Then the 5×5 matrix polynomials $L_{m,n,k}(r,s)$ and $L'_{m,n,k}(t,u)$ given by*

$$L_{m,n,k}(r,s) := \begin{bmatrix} (\lambda-1)^m\lambda^n & & & & (\lambda-1)\cdot s^k(\lambda) \\ & \rho\cdot\lambda & 0 & s(\lambda) & (\lambda-1)^2 \\ & 0 & (\lambda-1)^2 & \lambda(1-\lambda) & \\ & r(\lambda) & (\lambda-1) & -\lambda & \\ (\lambda-1)\cdot r^k(\lambda) & (\lambda-1)^2 & & & \end{bmatrix}, \tag{34}$$

$$L'_{m,n,k}(t,u) := \begin{bmatrix} (\lambda+1)^m\lambda^n & & & & (\lambda+1)\cdot u^k(\lambda) \\ & \rho'\cdot\lambda & 0 & u(\lambda) & (\lambda+1)^2 \\ & 0 & (\lambda+1)^2 & \lambda(\lambda+1) & \\ & t(\lambda) & (\lambda+1) & \lambda & \\ (\lambda+1)\cdot t^k(\lambda) & (\lambda+1)^2 & & & \end{bmatrix}, \tag{35}$$

where integers m,k are positive and n is nonnegative, have the following properties:

(a) *$L_{m,n,k}(r,s)$ has the Smith form* $\operatorname{diag}\big(1,\,(\lambda-1)\cdot I_3,\,(\lambda-1)\cdot\widetilde{r}^{\,k+1}(\lambda)\cdot\widetilde{s}^{\,k+1}(\lambda)\big)$,

(b) *$L'_{m,n,k}(t,u)$ has the Smith form* $\operatorname{diag}\big(1,\,(\lambda+1)\cdot I_3,\,(\lambda+1)\cdot\widetilde{t}^{\,k+1}(\lambda)\cdot\widetilde{u}^{\,k+1}(\lambda)\big)$.

The proofs of Lemma 4.11 and Lemma 4.12 are left to Appendix A and Appendix B, respectively. For now we only emphasize that the key idea in our proofs is similar to the proof of Lemma 4.1 [9, Lem. 5.2], i.e., we perform carefully constructed elementary unimodular row/column operations on the 5×5 matrix polynomials while exploiting the assumptions of the relative primeness of $r(\lambda)$ and $s(\lambda)$ with λ and $(\lambda-1)$, and the relative primeness of $t(\lambda)$ and $u(\lambda)$ with λ and $(\lambda+1)$.

Next we turn our attention to establishing basic building blocks that will help us construct T-palindromic quadratic realizations for lists of elementary divisors of type $\mathcal{D}_2$. The following lemma is just the first step.

Lemma 4.13. *Let* $q(\lambda) = \lambda^2 + b\lambda + c$ *be an* $\mathbb{R}$-irreducible quadratic non-palindromic scalar polynomial*, and consider* $\widehat{q}(\lambda) := \mathrm{rev}_2(q) = c\lambda^2 + b\lambda + 1$*. Define the* 2×2 *matrix polynomials* $V(q)$*,* $V'(q)$*,* $W(\widehat{q})$*,* $W'(\widehat{q})$*,* $\widetilde{V}(q)$*,* $\widetilde{V}'(q)$*,* $\widetilde{W}(\widehat{q})$*, and* $\widetilde{W}'(\widehat{q})$ *as below:*

$$V(q) := (\lambda-1)\begin{bmatrix} (\lambda-1) & (b+2)\lambda+(c-1) \\ -1 & (\lambda-1) \end{bmatrix}, \qquad \widetilde{V}(q) := (\lambda-1)\begin{bmatrix} 0 & q(\lambda) \\ -1 & \lambda-1 \end{bmatrix},$$

$$V'(q) := (\lambda+1)\begin{bmatrix} (\lambda+1) & (b-2)\lambda+(c-1) \\ -1 & (\lambda+1) \end{bmatrix}, \qquad \widetilde{V}'(q) := (\lambda+1)\begin{bmatrix} 0 & q(\lambda) \\ -1 & \lambda+1 \end{bmatrix},$$

$$W(\widehat{q}) := (\lambda-1)\begin{bmatrix} (\lambda-1) & \lambda \\ -(c-1)\lambda-(b+2) & (\lambda-1) \end{bmatrix}, \quad \widetilde{W}(\widehat{q}) := (\lambda-1)\begin{bmatrix} 0 & 1 \\ -\widehat{q}(\lambda) & \lambda-1 \end{bmatrix},$$

$$W'(\widehat{q}) := (\lambda+1)\begin{bmatrix} (\lambda+1) & -\lambda \\ (c-1)\lambda+(b-2) & (\lambda+1) \end{bmatrix}, \quad \widetilde{W}'(\widehat{q}) := (\lambda+1)\begin{bmatrix} 0 & 1 \\ -\widehat{q}(\lambda) & \lambda+1 \end{bmatrix}.$$

Then

(a) $\big(\mathrm{rev}_2 V(q)\big)^T = W(\widehat{q})$ *and* $\big(\mathrm{rev}_2 V'(q)\big)^T = W'(\widehat{q})$*,*

(b) $V(q) \sim \widetilde{V}(q)$ *and* $V'(q) \sim \widetilde{V}'(q)$*,*

(c) $W(\widehat{q}) \sim \widetilde{W}(\widehat{q})$ *and* $W'(\widehat{q}) \sim \widetilde{W}'(\widehat{q})$*.*

Proof. The truth of statement (a) can be verified by a straightforward computation of appropriate reversals and transposes. For statements (b) and (c) note that

$$q(\lambda) = \lambda^2 + b\lambda + c = (\lambda-1)^2 + (b+2)\lambda + (c-1) = (\lambda+1)^2 + (b-2)\lambda + (c-1), \tag{36}$$

and consider the following 2×2 *unimodular* matrix polynomials

$$E_1(\lambda) := \begin{bmatrix} 1 & (\lambda-1) \\ 0 & 1 \end{bmatrix}, \; E_2(\lambda) := \begin{bmatrix} 1 & (\lambda+1) \\ 0 & 1 \end{bmatrix}, \; E_3 := \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \; E_4 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$
$$F_3(\lambda) := \begin{bmatrix} 1 & 0 \\ -(c\lambda+b+1) & 1 \end{bmatrix}, \quad \text{and} \quad F_4(\lambda) := \begin{bmatrix} 1 & 0 \\ -(c\lambda+b-1) & 1 \end{bmatrix}. \tag{37}$$

Then one can verify directly that

$$\begin{aligned} E_1(\lambda) \cdot V(q) &= \widetilde{V}(q), \quad & E_3 \cdot W(\widehat{q}) \cdot F_3(\lambda) &= \widetilde{W}(\widehat{q}), \\ E_2(\lambda) \cdot V'(q) &= \widetilde{V}'(q), \quad & E_4 \cdot W'(\widehat{q}) \cdot F_4(\lambda) &= \widetilde{W}'(\widehat{q}), \end{aligned} \tag{38}$$

which proves parts (b) and (c), as desired. □

Even though Lemma 4.13 is quite simple in its own right, we will see that it plays a crucial role when constructing real T-palindromic quadratic realizations of more complicated lists of elementary divisors of type $\mathcal{D}_2$. Furthermore, it is important to emphasize that the matrix polynomials E_i and F_j from (37) are in fact *upper* and *lower triangular*, respectively. We have already seen the relevance of a similar observation in Remark 4.5, Lemma 4.6, and Lemma 4.8; this fact is also essential in the following result.

Lemma 4.14. *Let* $q(\lambda) = \lambda^2 + b\lambda + c$ *be an* $\mathbb{R}$-irreducible quadratic non-palindromic scalar polynomial, *and consider* $\widehat{q}(\lambda) := \mathrm{rev}_2(q) = c\lambda^2 + b\lambda + 1$*. Then for a positive integer* m*, the following statements are true:*

(a) $\big(\mathrm{rev}_2\, H_m\big(V(q)\big)\big)^T = H_m\big(W(\widehat{q})\big)$ *and* $\big(\mathrm{rev}_2\, H'_m\big(V'(q)\big)\big)^T = H'_m\big(W'(\widehat{q})\big)$.

(b) $H_m(V(q)) \sim H_m(\widetilde{V}(q))$ *and* $H'_m(V'(q)) \sim H'_m(\widetilde{V}'(q))$.

(c) $H_m(W(\widehat{q})) \sim H_m(\widetilde{W}(\widehat{q}))$ *and* $H'_m(W'(\widehat{q})) \sim H'_m(\widetilde{W}'(\widehat{q}))$.

Proof. Statement (a) can be verified directly, while (b) and (c) follow from (38) along with the equalities in (39)

$$\begin{aligned} \big(I_m \otimes E_1(\lambda)\big) \cdot H_m\big(V(q)\big) &= H_m\big(\widetilde{V}(q)\big)\,, \\ \big(I_m \otimes E_2(\lambda)\big) \cdot H'_m\big(V'(q)\big) &= H'_m\big(\widetilde{V}'(q)\big)\,, \\ \big(I_m \otimes E_3\big) \cdot H_m\big(W(\widehat{q})\big) \cdot \big(I_m \otimes F_3(\lambda)\big) &= H_m\big(\widetilde{W}(\widehat{q})\big)\,, \\ \big(I_m \otimes E_4\big) \cdot H'_m\big(W'(\widehat{q})\big) \cdot \big(I_m \otimes F_4(\lambda)\big) &= H'_m\big(\widetilde{W}'(\widehat{q})\big)\,, \end{aligned} \tag{39}$$

where $E_1(\lambda)$, $E_2(\lambda)$, E_3, E_4, $F_3(\lambda)$, and $F_4(\lambda)$ are the unimodular matrix polynomials in (37). ☐

In the next section we will see how Lemmas 4.14 and 4.2 can be combined in order to construct real T-palindromic quadratic realizations for a large subclass of $\mathcal{D}_2$-type lists of elementary divisors from Table 3, in particular when $m = 2k$. But before we can realize the remaining $\mathcal{D}_2$-type lists we need the following two technical results.

Lemma 4.15. *For nonzero scalar polynomials* $r(\lambda)$ *and* $t(\lambda)$*, both of which are coprime to* $\lambda - 1$*, and an arbitrary* $(m-4) \times (m-4)$ *polynomial* $Z(\lambda)$*,* $m \geq 5$*, define the* $m \times m$ *matrix polynomial* $K(\lambda)$ *as*

$$K(\lambda) := \left[\begin{array}{cc|c|cc} \psi(\lambda) & 0 & & 0 & (\lambda-1)\,t(\lambda) \\ 0 & 0 & & (\lambda-1)\,u(\lambda) & (\lambda-1)^2 \\ \hline & & & \nu(\lambda) & \\ & & Z(\lambda) & & \\ \hline 0 & (\lambda-1)\,s(\lambda) & \eta(\lambda) & & \\ (\lambda-1)\,r(\lambda) & (\lambda-1)^2 & & & \end{array}\right],$$

where $\eta(\lambda), \nu(\lambda), s(\lambda), u(\lambda), \xi(\lambda)$ *are arbitrary scalar polynomials,* $\psi(\lambda) := (\lambda-1)\,\xi(\lambda)$*, and* $\eta(\lambda)$ *and* $\nu(\lambda)$ *are respectively the* $(m-1, 3)$ *and* $(3, m-1)$ *entries of* $K(\lambda)$*. Then* $K(\lambda)$ *is unimodularly equivalent to* $\widetilde{K}(\lambda)$*, where*

$$\widetilde{K}(\lambda) := \left[\begin{array}{cc|c|cc} 0 & 0 & & 0 & (\lambda-1) \\ 0 & (\lambda-1)^2\,\psi(\lambda) & & (\lambda-1)\,u(\lambda)\,t(\lambda) & 0 \\ \hline & & & \nu(\lambda) & \\ & & Z(\lambda) & & \\ \hline 0 & (\lambda-1)\,r(\lambda)\,s(\lambda) & \eta(\lambda) & & \\ (\lambda-1) & 0 & & & \end{array}\right].$$

Lemma 4.16. *For nonzero scalar polynomials* $r(\lambda)$ *and* $t(\lambda)$*, both of which are coprime to* λ*, and an arbitrary* $(m-4) \times (m-4)$ *polynomial* $Z(\lambda)$*,* $m \geq 5$*, define the* $m \times m$

matrix polynomial $O(\lambda)$ as

$$O(\lambda) := \left[\begin{array}{cc|c|cc} \psi(\lambda) & 0 & & 0 & t(\lambda) \\ 0 & 0 & & u(\lambda) & \lambda \\ \hline & & & \nu(\lambda) & \\ & & Z(\lambda) & & \\ \hline 0 & s(\lambda) & \eta(\lambda) & & \\ r(\lambda) & \lambda & & & \end{array}\right],$$

where $\eta(\lambda), \nu(\lambda), s(\lambda), u(\lambda), \psi(\lambda)$ are arbitrary scalar polynomials, and $\eta(\lambda)$ and $\nu(\lambda)$ are respectively the $(m-1,3)$ and $(3,m-1)$ entries of $O(\lambda)$. Then $O(\lambda)$ is unimodularly equivalent to $\widetilde{O}(\lambda)$, where

$$\widetilde{O}(\lambda) := \left[\begin{array}{cc|c|cc} 0 & 0 & & 0 & 1 \\ 0 & \lambda^2\,\psi(\lambda) & & t(\lambda)\,u(\lambda) & 0 \\ \hline & & & \nu(\lambda) & \\ & & Z(\lambda) & & \\ \hline 0 & r(\lambda)\,s(\lambda) & \eta(\lambda) & & \\ 1 & 0 & & & \end{array}\right].$$

Remark 4.17. Unimodular equivalences in Lemmas 4.15 and 4.16 can be written explicitly, and are very similar in their design; thus for the sake of limiting repetitiveness we only provide a complete proof of Lemma 4.15 in Appendix C. What is more important is just to remember that both of these results actually describe how the diagonal term $\psi(\lambda)$ is affected as Lemma 4.1 is applied along the anti-diagonal with different "gluing" entries, namely, $(\lambda-1)^2$ in Lemma 4.15 and λ in Lemma 4.16.

4.3. Real T-palindromic quadratic realizations of list types "Z" and "D"

In this section we provide the last missing piece of a solution to the T-palindromic QRP over $\mathbb{R}$, i.e., we explicitly construct real T-palindromic quadratic blocks (see Tables 4 and 5) that realize the p-quad irreducible lists of elementary divisors from Table 3. Note that, by design, these blocks have low bandwidth and transparently display their elementary divisor structure, thus resembling the Kronecker canonical blocks corresponding to individual elementary divisors of a matrix pencil.

We start by reminding the reader of some notation conventions adopted so far, and recall relevant definitions of the basic building blocks from Section 4.2 that appear as elements of the realizations in Tables 4 and 5.

(a) Real p-quad realizations of elementary divisor lists $\mathcal{Z}$ and $\mathcal{D}$ from Table 3 are denoted by "Z" and "D" in Tables 4 and 5, respectively.

(b) The entries $*, \dagger, \diamond$, and $\odot$ appearing between neighboring anti-diagonal blocks in Tables 4 and 5 are always assumed to be located in the upper left corner. In other words, if the first row of the lower block of a neighboring pair is the k_1^{th} row of the entire matrix, and the first column of the higher block in the pair is the k_2^{th} column of the entire matrix, then the entries $*, \dagger, \diamond, \odot$ are in the (k_1, k_2) position.

(c) The entry $\bullet$ appearing in the D_{1b} and D_{2b} blocks in Tables 4-5 is always located in the upper left corner of the whole matrix, i.e., it is the $(1,1)$ entry D_{1b} and D_{2b}.

(d) $p(\lambda), q(\lambda), \widehat{q}(\lambda)$, and $\widetilde{q}(\lambda)$ are the $\mathbb{R}$-irreducible quadratic polynomials in (7)-(8).

(e) $J(p)$ and $J'(p)$ are the 2×2 matrix polynomials defined in Lemma 4.4.

(f) $V(q)$ and $W(\widehat{q})$ are the 2×2 matrix polynomials defined in Lemma 4.13.
(g) $T(p,p)$ and $T(q,\widehat{q})$ are the 3×3 matrix polynomials defined in Lemma 4.9.
(h) R_k, H_m, and H'_m are the block matrices from Definition 4.7.

Subtype	Block	Conditions
Z_1	$R_k(p)$	$k \geq 1$
Z_2	$\left[\begin{array}{c\|c} 0 & R_k(\widehat{q}) \\ \hline R_k(q) & 0 \end{array}\right]$	$k \geq 1$
$\mathrm{D}_{1\mathrm{a}}$	$\begin{bmatrix} & & R_\ell(p) \\ & H_m(J(p)) & * \\ R_\ell(p) & * & \end{bmatrix}$	$\begin{array}{rcl} m & > & 0 \\ k & = & m+2\ell \\ \ell & \geq & 0 \\ * & = & \lambda \end{array}$
$\mathrm{D}_{1\mathrm{b}}$	$\begin{bmatrix} \bullet & & & & R_j(p) \\ & & & H_n(J(p)) & * \\ & & T(p,p) & \diamond & \\ & H_n(J(p)) & \diamond & & \\ R_j(p) & * & & & \end{bmatrix}$	$\begin{array}{rcl} m & = & 2n+1 \\ k & = & 2n+2j+2 \\ j & \geq & 0 \\ * & = & \lambda \\ \diamond & = & (\lambda-1)^2 \\ \bullet & = & (\lambda-1)^2 \end{array}$
$\mathrm{D}_{1\mathrm{c}}$	$\begin{bmatrix} & & H_n(J(p)) \\ & R_{2\ell+1}(p) & \odot \\ H_n(J(p)) & \dagger & \end{bmatrix}$	$\begin{array}{rcl} m & = & 2n > 0 \\ k & = & 2n+(2\ell+1) \\ \ell & \geq & 0 \\ \dagger & = & \lambda-1 \\ \odot & = & \mathrm{rev}_2(\dagger) \end{array}$
$\mathrm{D}'_{1\mathrm{a}}$	Replace the $H_m(J(p))$ block with the $H'_m(J'(p))$ block.	
$\mathrm{D}'_{1\mathrm{b}}$	Replace each $H_n(J(p))$ block with the $H'_n(J'(p))$ block, and replace the $T(p,p)$ block with the $T'(p,p)$ block.	$\diamond = (\lambda+1)^2$ $\bullet = (\lambda+1)^2$
$\mathrm{D}'_{1\mathrm{c}}$	Replace each $H_n(J(p))$ block with the $H'_n(J'(p))$ block.	$\dagger = \lambda+1$

Table 4.: Blocks of types Z and D_1

Remark 4.18. While reading the proof of Theorem 4.19, it is important to keep in mind that the values of the parameters m and k come from the description of the elementary divisor list that is to be realized. The values of any of the parameters j, ℓ, and n that appear in the block realization for this list are then easily determined from the "Conditions" in the right-most column of Table 4 or 5.

Theorem 4.19. *Any real p-quad irreducible list $\mathcal{L}$ from Table 3 is p-quad realizable over $\mathbb{R}$. Furthermore, $\mathcal{L}$ is realizable by the corresponding block in Tables 4 and 5.*

Subtype	Block	Conditions
D_{2a}	$\begin{bmatrix} & & & R_j(\widehat{q}) \\ & & H_n(W(\widehat{q})) & * \\ & H_n(V(q)) & & \\ R_j(q) & * & & \end{bmatrix}$	$m = 2n > 0$ $k = n+j$ $j \geq 0$ $* = \lambda$
D_{2b}	$\begin{bmatrix} \bullet & & & & R_j(\widehat{q}) \\ & & & H_n(W(\widehat{q})) & * \\ & & T(q,\widehat{q}) & \diamond & \\ & H_n(V(q)) & \diamond & & \\ R_j(q) & * & & & \end{bmatrix}$	$m = 2n+1$ $k = n+j+1$ $j \geq 0$ $* = \lambda$ $\diamond = (\lambda-1)^2$ $\bullet = (\lambda-1)^2$
D'_{2a}	Replace the $H_n(V(q))$ block with the $H'_n(V'(q))$, and replace the $H_n(W(\widehat{q}))$ block with the $H'_n(W'(\widehat{q}))$ block.	
D'_{2b}	Replace the $H_n(V(q))$ block with an $H'_n(V'(q))$ block , replace the $H_n(W(\widehat{q}))$ block with the $H'_n(W'(\widehat{q}))$ block, and replace the $T(q,\widehat{q})$ block with a $T'(q,\widehat{q})$ block.	$\diamond = (\lambda+1)^2$ $\bullet = (\lambda+1)^2$

Table 5.: Blocks of type D_2

Proof. The proof proceeds by individually considering each type of block in Tables 4 and 5, and showing that its spectral structure corresponds to one of the p-quad irreducible lists from Table 3. Note that for the rest of this proof we focus on analyzing the unprimed blocks, since the analysis of their primed counterparts is completely analogous and hence omitted.

To start, note that the blocks of types Z and D are clearly real and quadratic, while their T-palindromic structure is either obvious or follows from Lemmas 4.4(a), 4.8(a), 4.13(a), 4.14(a) and Corollary 4.10. Moreover, both Z and D blocks are regular, i.e., $\operatorname{rank}(Z) = \operatorname{size}(Z)$ and $\operatorname{rank}(D) = \operatorname{size}(D)$. Next we show that selected blocks from Tables 4 and 5 have the claimed elementary divisor lists, while the same conclusion about other blocks can be obtained analogously.

Z blocks Applying Lemma 4.2 to Z_1 shows that $Z_1 \sim \operatorname{diag}\left(I_{k-1}, p^k\right)$, and therefore, Z_1 is regular and its only finite elementary divisor is $p^k(\lambda)$. The fact that Z_1 has no infinite elementary divisors follows from the Index Sum Theorem 3.3, since

$$2 \cdot k = \operatorname{grade}\left(Z_1\right) \cdot \operatorname{rank}\left(Z_1\right) = \delta_{fin}\left(Z_1\right) + \delta_\infty\left(Z_1\right) = 2k + \delta_\infty\left(Z_1\right), \tag{40}$$

and hence $\delta_\infty\left(Z_1\right) = 0$. Thus Z_1 is in fact a real T-palindromic quadratic realization of the $\mathcal{Z}_1$-type list.

On the other hand, applying Lemma 4.2 to the upper-right and lower-left $k \times k$

submatrices of Z_2 shows that

$$Z_2 \sim \begin{bmatrix} & & & \widehat{q}^{\,k}(\lambda) \\ & & \widetilde{I}_{k-1} & \\ & q^k(\lambda) & & \\ \widetilde{I}_{k-1} & & & \end{bmatrix},$$

and so the only finite elementary divisors of the Z_2 block are $\{\, q^k(\lambda)\,,\, \widetilde{q}^{\,k}(\lambda)\,\}$. One can use an index sum argument similar to (40) to show that Z_2 has no infinite elementary divisors. Therefore Z_2 is a real T-palindromic quadratic realization of the $\mathcal{Z}_2$-type list of elementary divisors.

D_1 blocks We now show that any possible $\mathcal{D}_1$-list from Table 3 can in fact be p-quad realized over $\mathbb{R}$ using D_1 blocks from Table 4. The reason for having the three blocks D_{1a}, D_{1b}, and D_{1c} is so that all the possible relations between k and m are included, while still satisfying the condition $k \geq m > 0$ from Table 3. More specifically, we claim that:

(i) A $\mathcal{D}_1$-type list of elementary divisors with m and k having the same parity is p-quad realizable via the corresponding D_{1a} block.
(ii) A $\mathcal{D}_1$-type list of elementary divisors with m odd and k even is p-quad realizable via the corresponding D_{1b} block.
(iii) A $\mathcal{D}_1$-type list of elementary divisors with m even and k odd is p-quad realizable via the corresponding D_{1c} block.

D_{1a} block There are three cases we need to consider.

Case 1: If $\ell = 0$, then $k = m$ and $D_{1a} = \mathcal{H}_m(J(p))$. Then the desired conclusion follows from Lemma 4.8(d), together with the fact that D_{1a} has no infinite elementary divisors. To see why the latter is true, one can use either the Index Sum Theorem 3.3 or the fact that the leading coefficient of D_{1a}, when viewed as a matrix polynomial, is invertible.

Case 2: Assume $m = 1$ and $\ell > 0$, and let E and $F(\lambda)$ be the unimodular matrix polynomials from the proof of Lemma 4.4 as in (15). Pre-multiplying and post-multiplying D_{1a} by the unimodular matrix polynomials $\mathrm{diag}(I_\ell\,, E\,, I_\ell)$ and $\mathrm{diag}(I_\ell\,, F(\lambda)\,, I_\ell)$, respectively, together with Lemmas 4.8 and 4.6, gives

$$D_{1a} \sim \begin{bmatrix} & & \boxed{R_\ell(p)} \\ & \boxed{H_m(\widetilde{J}(p))} & * \\ \boxed{R_\ell(p)} & * & \end{bmatrix} =: D_{1a}^{(1)}.$$

Applying Lemma 4.2 and Remark 4.3 to the lower-left and the upper-right $(\ell+1) \times (\ell+1)$ submatrices of $D_{1a}^{(1)}$, respectively, gives

$$D_{1a} \sim D_{1a}^{(1)} \sim \begin{bmatrix} & & & \boxed{\widetilde{I}_\ell} \\ & & (\lambda-1)\cdot p^{\ell}(\lambda) & \\ & (\lambda-1)\cdot p^{\ell+1}(\lambda) & (\lambda-1)^2 & \\ \boxed{\widetilde{I}_\ell} & & & \end{bmatrix} =: D_{1a}^{(2)}.$$

But $D_{1a}^{(2)} \sim \operatorname{diag}\left(I_{2\ell}\,,\,(\lambda-1)\,,\,(\lambda-1)\cdot p^k\right)$, where $k = 2\ell + m = 2\ell + 1$. Again, one can show that D_{1a} has no infinite elementary divisors, and conclude that D_{1a} is a quadratic T-palindromic realization for one kind of $\mathcal{D}_1$-type list of elementary divisors.

<u>Case 3:</u> Assume $m \geq 2$ and $\ell > 0$, and let E and $F(\lambda)$ be the unimodular matrix polynomials as in (15). Pre-multiplying and post-multiplying D_{1a} by the unimodular matrix polynomials $\operatorname{diag}(I_\ell\,,\,I_m \otimes E\,,\,I_\ell)$ and $\operatorname{diag}(I_\ell\,,\,I_m \otimes F(\lambda)\,,\,I_\ell)$, respectively, together with Lemmas 4.8 and 4.6, gives

$$D_{1a} \sim \begin{bmatrix} & & \boxed{R_\ell(p)} \\ & \boxed{H_m(\widetilde{J}(p))} & * \\ \boxed{R_\ell(p)} & * & \end{bmatrix} =: D_{1a}^{(3)}\,,$$

where $D_{1a}^{(3)}$ is a bi-anti-diagonal matrix polynomial. Applying Lemma 4.2 and Remark 4.3 to the lower-left and the upper-right $(\ell+1)\times(\ell+1)$ submatrices of $D_{1a}^{(3)}$, respectively, we obtain

$$D_{1a}^{(3)} \sim \begin{bmatrix} & & & & \boxed{\widetilde{I}_\ell} \\ & & & \begin{matrix} & -(\lambda-1)\cdot p^{\ell}(\lambda) \\ (\lambda-1)\cdot p(\lambda) & (\lambda-1)^2 \end{matrix} & \\ & & \boxed{H_{m-2}(\widetilde{J}(p))} & (\lambda-1)^2 & \\ & \begin{matrix} 0 & -(\lambda-1) \\ (\lambda-1)\cdot p^{\ell+1}(\lambda) & (\lambda-1)^2 \end{matrix} & (\lambda-1)^2 & & \\ \boxed{\widetilde{I}_\ell} & & & & \end{bmatrix} =: D_{1a}^{(4)}\,.$$

Finally, applying Lemma 4.2 to the middle $2m \times 2m$ submatrix of $D_{1a}^{(4)}$ gives

$$D_{1a} \sim D_{1a}^{(3)} \sim D_{1a}^{(4)} \sim \begin{bmatrix} & & & \boxed{\widetilde{I}_\ell} \\ & & (\lambda-1)\cdot p^k(\lambda) & \\ & (\lambda-1)\cdot \widetilde{I}_{2m-1} & & \\ \boxed{\widetilde{I}_\ell} & & & \end{bmatrix},$$

where $k = m + 2\ell$. So the *finite* elementary divisors of D_{1a} comprise a $\mathcal{D}_1$-type list from Table 3. In order to conclude that D_{1a} is a T-palindromic quadratic realization of a $\mathcal{D}_1$-type list of elementary divisors, we must show that D_{1a} has no infinite elementary divisors. This follows from the Index Sum Theorem 3.3 and the following calculation:

$$\begin{aligned} \operatorname{grade}(D_{1a}) \cdot \operatorname{size}(D_{1a}) &= \delta_{fin}(D_{1a}) + \delta_\infty(D_{1a}) \\ 2\cdot(2m+2\ell) &= 2m + 2k + \delta_\infty(D_{1a}) \\ 2\cdot(2m+2\ell) &= 2m + 2\cdot(m+2\ell) + \delta_\infty(D_{1a}) \\ 0 &= \delta_\infty(D_{1a})\,. \end{aligned} \tag{41}$$

D_{1b} block There are three cases we need to consider.

Case 1: If $j = n = 0$, then $D_{1b} = T(p, p)$, which by Lemma 4.9(a) has the Smith form $\operatorname{diag}(1, (\lambda - 1), (\lambda - 1)p^2(\lambda))$. The fact that the leading coefficient of $T(p, p)$ is invertible implies that $T(p, p)$ has no infinite eigenvalues, and consequently, the list of elementary divisors of $D_{1b} = T(p, p)$ is exactly a $\mathcal{D}_1$-type.

Case 2: If $j = 0$ and $n > 0$, then D_{1b} is given by

$$D_{1b} = \left[\begin{array}{ccc} \bullet & & \boxed{H_n(J(p))} \\ & \boxed{T(p,p)} & \diamond \\ \boxed{H_n(J(p))} & \diamond & \end{array}\right].$$

Once again consider the unimodular matrix polynomials E and $F(\lambda)$ from (15). Pre-multiplication and post-multiplication of D_{1b} with the unimodular polynomials $\operatorname{diag}\big(I_n \otimes E\,,\, I_3\,,\, I_n \otimes E\big)$ and $\operatorname{diag}\big(I_n \otimes F(\lambda)\,,\, I_3\,,\, I_n \otimes F(\lambda)\big)$, respectively, together with Lemmas 4.8 and 4.6, bi-anti-diagonalizes the $H_n(J(p))$ blocks to obtain

$$D_{1b} \sim \left[\begin{array}{ccc} \bullet & & \boxed{H_n(\widetilde{J}(p))} \\ & \boxed{T(p,p)} & \diamond \\ \boxed{H_n(\widetilde{J}(p))} & \diamond & \end{array}\right] =: D_{1b}^{(1)}.$$

Repeated use of Lemma 4.15 on $D_{1b}^{(1)}$ shows that $D_{1b}^{(1)} \sim D_{1b}^{(2)}$, where

$$\left[\begin{array}{c|c|ccc|c|c} & & & & & & (\lambda-1)\widetilde{I}_{2n-1} \\ \hline & (\lambda-1)^{4n} & & & & (\lambda-1)p^n(\lambda) & \\ \hline & & -p^2(1)\cdot\lambda & 0 & p(\lambda) & (\lambda-1)^2 & \\ & & 0 & (\lambda-1)^2 & \lambda(1-\lambda) & & \\ & & p(\lambda) & (\lambda-1) & -\lambda & & \\ \hline & (\lambda-1)p^n(\lambda) & (\lambda-1)^2 & & & & \\ \hline (\lambda-1)\widetilde{I}_{2n-1} & & & & & & \end{array}\right] =: D_{1b}^{(2)}.$$

From Lemma 4.12(a) applied to the middle 5×5 block of $D_{1b}^{(2)}$, and several row and column permutations, we conclude that the Smith form of D_{1b} is

$$\operatorname{diag}\Big(1, (\lambda - 1) \cdot I_{4n+1}\,,\, (\lambda - 1) \cdot p^{2n+2}\Big)\,,$$

where $2n + 2 = k$ and $4n + 1 = 2m - 1$. Finally, in order to conclude that D_{1b} is a T-palindromic realization of a $\mathcal{D}_1$-type list of elementary divisors, we must show that D_{1b} has no infinite elementary divisors. But this follows from the Index Sum Theorem 3.3

and the following chain of equalities:

$$\begin{aligned}
\text{grade}(D_{1b}) \cdot \text{size}(D_{1b}) &= \delta_{fin}(D_{1b}) + \delta_\infty(D_{1b}) \\
2 \cdot (2n+3+2n) &= (4n+2) + 2\cdot(2n+2) + \delta_\infty(D_{1b}) \\
0 &= \delta_\infty(D_{1b})\,. \qquad (42)
\end{aligned}$$

Case 3: Assume $j>0$ and $n>0$, and let E and $F(\lambda)$ be the unimodular matrices from the proof of Lemma 4.4 in (15). Pre-multiplying and post-multiplying D_{1b} by $\text{diag}\big(I_j, I_n\otimes E, I_3, I_n\otimes E, I_j\big)$ and $\text{diag}\big(I_j, I_n\otimes F, I_3, I_n\otimes F, I_j\big)$, respectively, together with Lemmas 4.8 and 4.6, bi-anti-diagonalizes the $H_n(J(p))$ blocks to obtain

$$\left[\begin{array}{ccccc}
\bullet & & & & R_j(p) \\
 & & & H_n(\widetilde{J}(p)) & * \\
 & & T(p,p) & \diamond & \\
 & H_n(\widetilde{J}(p)) & \diamond & & \\
R_j(p) & * & & &
\end{array}\right] =: D_{1b}^{(3)}\,.$$

Applying Lemma 4.16 repeatedly to $D_{1b}^{(3)}$ gives

$$\left[\begin{array}{ccccccc}
 & & & & & & \widetilde{I}_j \\
 & \widetilde{\bullet} & & & & \widetilde{P}(\lambda) & \\
 & & & & H_{n-1}(\widetilde{J}(p)) & \diamond & \\
 & & & T(p,p) & \diamond & & \\
 & & H_{n-1}(\widetilde{J}(p)) & \diamond & & & \\
 & P(\lambda) & \diamond & & & & \\
\widetilde{I}_j & & & & & &
\end{array}\right] =: D_{1b}^{(4)}\,,$$

where $\widetilde{\bullet} = (\lambda-1)^2\lambda^{2j}$ and

$$\widetilde{P}(\lambda) := (\lambda-1)\begin{bmatrix} 0 & -p^j(\lambda) \\ p(\lambda) & \lambda-1 \end{bmatrix} \quad \text{and} \quad P(\lambda) := (\lambda-1)\begin{bmatrix} 0 & -1 \\ p^{j+1}(\lambda) & \lambda-1 \end{bmatrix}.$$

Next apply Lemma 4.15 repeatedly to $D_{1b}^{(4)}$ to obtain $D_{1b}^{(4)} \sim D_{1b}^{(5)}$, where

$$D_{1b}^{(5)} := \left[\begin{array}{cc|c|ccc|c|cc}
 & & & & & & & & \widetilde{I}_j \\
 & & & & & & & (\lambda-1)\widetilde{I}_{2n-1} & \\
 & & (\lambda-1)^{4n}\lambda^{2j} & 0 & 0 & 0 & (\lambda-1)p^{n+j}(\lambda) & & \\
 & & \begin{matrix}0\\0\\0\end{matrix} & & T(p,p) & & \begin{matrix}(\lambda-1)^2\\0\\0\end{matrix} & & \\
 & & (\lambda-1)p^{n+j}(\lambda) & (\lambda-1)^2 & 0 & 0 & 0 & & \\
 & (\lambda-1)\widetilde{I}_{2n-1} & & & & & & & \\
\widetilde{I}_j & & & & & & & &
\end{array}\right].$$

Now observe that the middle 5×5 block of $D_{1b}^{(5)}$ is just the $L_{4n,2j,1}(p^{n+j}, p^{n+j})$ block from Lemma 4.12. With several simple row and column swaps we obtain that

$$D_{1b} \sim D_{1b}^{(3)} \sim D_{1b}^{(4)} \sim D_{1b}^{(5)} \sim \operatorname{diag}\Big(I_{2j} \,,\, (\lambda - 1) \cdot I_{4n-2} \,,\, L_{4n,2j,1}(p^{n+j} \,,\, p^{n+j}) \Big) . \tag{43}$$

Finally, from (43) and Lemma 4.12 we conclude that the Smith form of D_{1b} is given by

$$\operatorname{diag}\Big(I_{2j+1} \,,\, (\lambda - 1) \cdot I_{4n+1} \,,\, (\lambda - 1) \cdot p^{2n+2j} \Big) ,$$

which in turn implies that the finite elementary divisor list of D_{1b} is of $\mathcal{D}_1$-type. In order to conclude that D_{1b} is a T-palindromic quadratic realization of a $\mathcal{D}_1$-type list of elementary divisors from Table 3, one must show that D_{1b} has no infinite elementary divisors; this can be done analogously to (42), and hence is omitted.

D_{1c} block Let E and $F(\lambda)$ be the unimodular matrix polynomials from (15). Pre- and post-multiplication of D_{1c} by $\operatorname{diag}\big(I_n \otimes E, I_{2\ell+1}, I_n \otimes E\big)$ and $\operatorname{diag}\big(I_n \otimes F(\lambda), I_{2\ell+1}, I_n \otimes F(\lambda)\big)$, respectively, together with Lemmas 4.8 and 4.6, bi-anti-diagonalizes the $H_n(J(p))$ blocks to obtain

$$D_{1c} \sim \begin{bmatrix} & & H_n(\widetilde{J}(p)) \\ & R_{2\ell+1}(p) & \odot \\ H_n(\widetilde{J}(p)) & \dagger & \end{bmatrix} =: D_{1c}^{(1)} .$$

Applying Lemma 4.2 and Remark 4.3 to the lower-left and the upper-right $(2n+1) \times (2n+1)$ submatrices of $D_{1c}^{(1)}$, respectively, gives

$$D_{1c} \sim D_{1c}^{(1)} \sim \begin{bmatrix} & & (\lambda - 1) \cdot \widetilde{I}_{2n} \\ & \begin{matrix} & & & p^{n+1} \\ & & p & \lambda \\ & \begin{matrix} \cdot^{\cdot^{\cdot}} \end{matrix} & \begin{matrix} \cdot^{\cdot^{\cdot}} \end{matrix} & \\ & p & \lambda & \\ p^{n+1} & \lambda & & \end{matrix} & \\ (\lambda - 1) \cdot \widetilde{I}_{2n} & & \end{bmatrix} =: D_{1c}^{(2)}$$

Finally, applying Lemma 4.2 to the middle $(2\ell+1) \times (2\ell+1)$ block of $D_{1c}^{(2)}$ shows that

$$D_{1c} \sim D_{1c}^{(1)} \sim D_{1c}^{(2)} \sim \begin{bmatrix} & & & (\lambda - 1) \cdot \widetilde{I}_{2n} \\ & & p^k & \\ & \widetilde{I}_{2\ell} & & \\ (\lambda - 1) \cdot \widetilde{I}_{2n} & & & \end{bmatrix} ,$$

where $k = 2\ell + 2n + 1$. Hence, the finite elementary divisors of D_{1c} comprise a $\mathcal{D}_1$-type list. Finally, to conclude that D_{1c} is in fact a T-palindromic quadratic realization of a

$\mathcal{D}_1$-type list from Table 3, one must show that D_{1c} has no infinite elementary divisors. But that can be done analogously to (41), and so we omit the details.

D_2 blocks We now show that any possible $\mathcal{D}_2$-list from Table 3 has in fact a real p-quad realization using D_2 blocks from Table 4. Similarly to the D_1 blocks, we consider two blocks D_{2a} and D_{2b} in order to account for all the possible relations between k and m, while still satisfying the condition $2k \geq m > 0$ from Table 3. In particular, we consider the following two cases:

(i) A $\mathcal{D}_2$-type list of elementary divisors with m even, and k even or odd; such a list is p-quad realizable via the corresponding D_{2a} block. Note that if $m > 0$ is even, then $m = 2n$ for some positive integer n. But the condition $2k \geq m$ implies that $k \geq n$, or that there exists a non-negative integer j such that $k = n + j$.

(ii) A $\mathcal{D}_2$-type list of elementary divisors with m odd, and k even or odd; such a list is p-quad realizable via the corresponding D_{2b} block. If $m > 0$ is odd, then $m = 2n + 1$ for some non-negative integer n. But the condition $2k \geq m$ implies that $k \geq n + \frac{1}{2}$. Since k is an integer, it must be that $k \geq n + 1$. Consequently, there exists a non-negative integer j such that $k = n + j + 1$.

D_{2a} block There are two cases to consider.

Case 1: If $j = 0$, then D_{2a} is given by

$$D_{2a} = \left[\begin{array}{c|c} 0 & H_n(W(\widehat{q})) \\ \hline H_n(V(q)) & 0 \end{array}\right].$$

Now consider the unimodular matrix polynomials $E_1(\lambda)$, E_3, and $F_3(\lambda)$ from the proof of Lemma 4.13, as in (37). Pre- and post-multiplication of D_{2a} by the unimodular matrix polynomials $\operatorname{diag}\big(I_n \otimes E_3\,,\, I_n \otimes E_1(\lambda)\big)$ and $\operatorname{diag}\big(I_{2n}\,,\, I_n \otimes F_3(\lambda)\big)$, respectively, gives

$$D_{2a} \sim \left[\begin{array}{c|c} 0 & H_n(\widetilde{W}(\widehat{q})) \\ \hline H_n(\widetilde{V}(q)) & 0 \end{array}\right] =: D_{2a}^{(1)}.$$

Applying Lemma 4.2 and Remark 4.3 to the lower-left and upper-right $2n \times 2n$ sub-matrices of $D_{2a}^{(1)}$, respectively, gives

$$\begin{bmatrix} & & & (\lambda - 1)\cdot \widetilde{I}_{2n-1} \\ & & (\lambda-1)\cdot \widehat{q}^{\,n}(\lambda) & \\ & (\lambda-1)\cdot q^n(\lambda) & & \\ (\lambda - 1)\cdot \widetilde{I}_{2n-1} & & & \end{bmatrix}. \tag{44}$$

One can argue that D_{2a} has no infinite elementary divisors by using an index sum argument as in (41), and consequently, from (44) conclude that D_{2a} is a T-palindromic realization of a $\mathcal{D}_2$-type list of elementary divisors from Table 3.

Case 2: Assume $j > 0$ and $n > 0$, and again consider the unimodular matrix polynomials $E_1(\lambda)$, E_3, and $F_3(\lambda)$ from (37). Pre- and post-multiplying D_{2a} by the unimodular matrix polynomials diag $\big(I_j\,,\,I_n\otimes E_3\,,\,I_n\otimes E_1(\lambda)\,,\,I_j\big)$ and diag $\big(I_j\,,\,I_{2n}\,,\,I_n\otimes F_3(\lambda)\,,\,I_j\big)$, respectively, together with Lemma 4.6, gives

$$D_{2a}\sim\left[\begin{array}{cccc} & & & \boxed{R_j(\widehat{q})} \\ & & \boxed{H_n(\widetilde{W}(\widehat{q}))} & * \\ & \boxed{H_n(\widetilde{V}(q))} & & \\ \boxed{R_j(q)} & * & & \end{array}\right] =: D_{2a}^{(1)}\,.$$

Applying Lemma 4.2 and Remark 4.3 to the lower-left and upper-right $(j+1)\times(j+1)$ submatrices of $D_{2a}^{(1)}$, respectively, gives

$$\left[\begin{array}{cccccc} & & & & & \widetilde{I}_j \\ & & & & \begin{array}{cc} 0 & (\lambda-1)\,\widehat{q}^{\,j} \\ -(\lambda-1)\,\widehat{q} & (\lambda-1)^2 \end{array} & \\ & & & H_{n-1}(\widetilde{W}(\widehat{q})) & (\lambda-1)^2 & \\ & & H_{n-1}(\widetilde{V}(q)) & & & \\ & \begin{array}{cc} 0 & (\lambda-1)\,q \\ -(\lambda-1)\,q^{\,j} & (\lambda-1)^2 \end{array} & (\lambda-1)^2 & & & \\ \widetilde{I}_j & & & & & \end{array}\right] =: D_{2a}^{(2)}\,.$$

Finally, applying Lemma 4.2 on the middle $2n\times 2n$ submatrices of $D_{2a}^{(2)}$ produces the unimodular equivalnce $D_{2a}\sim D_{2a}^{(2)}\sim D_{2a}^{(3)}$, where

$$D_{2a}^{(3)} := \left[\begin{array}{cccccc} & & & & & \boxed{\widetilde{I}_j} \\ & & & & \boxed{(\lambda-1)\cdot\widetilde{I}_{2n-1}} & \\ & & & \boxed{(\lambda-1)\cdot\widehat{q}^{\,n}} & & \\ & & \boxed{(\lambda-1)\cdot q^{n}} & & & \\ & \boxed{(\lambda-1)\cdot\widetilde{I}_{2n-1}} & & & & \\ \boxed{\widetilde{I}_j} & & & & & \end{array}\right]. \qquad (45)$$

From (45) we conclude that the elementary divisor list of D_{2a} is exactly of $\mathcal{D}_2$-type. which can be done analogously to (42).

$\boxed{D_{2b}\ \textbf{block}}$ The proof that D_{2b} is a T-palindromic quadratic realization for a $\mathcal{D}_2$-type list of elementary divisors from Table 3 is completely analogous to the case of the block D_{1b}, except that the initial bi-anti-diagonalization of $H_n(V(q))$ and $H_n(W(\widehat{q}))$ blocks is achieved by pre- and post-multiplication of D_{2b} with the unimodular matrix polynomials diag $\big(I_j, I_n\otimes E_3, I_3, I_n\otimes E_1(\lambda), I_j\big)$ and diag $\big(I_j, I_{2n}, I_3, I_n\otimes F_3(\lambda), I_j\big)$, respectively, where $E_1(\lambda)$, E_3, and $F_3(\lambda)$ are as in the proof of Lemma 4.13 in (37). $\square$

5. Solution of the second part of the real T-palindromic QRP

All the work of Sections 3 and 4 now comes together to bring us to the culmination of this paper in the following theorem.

Theorem 5.1. ((SP-2) for the T-palindromic QRP)
Suppose a structural data list $\mathcal{L}$ of elementary divisors and minimal indices is p-quad admissible over $\mathbb{R}$. Then $\mathcal{L}$ has a transparent realization by a real T-palindromic quadratic matrix polynomial that is a direct sum of canonical blocks, each block realizing one of the real p-quad irreducible lists in Tables 1-3. The converse also holds.

Proof. By Theorem 3.13, any real p-quad admissible list $\mathcal{L}$ can be partitioned (sometimes uniquely and sometimes not) into real p-quad *irreducible* sublists from Tables 1-3, which by Corollary 3.14 contain all of the possible real p-quad irreducible lists. From the results in [9] for realizations of the real p-quad irreducible lists in Tables 1 and 2, and the realizations in Section 4 for real p-quad irreducible lists in Table 3, we know that *every* real p-quad irreducible list has a transparent realization by a sparse real T-palindromic matrix polynomial. Invoking Lemma 2.4, we now see that the list $\mathcal{L}$ can be realized by a direct sum of real T-palindromic blocks, one block for each real p-quad irreducible sublist in the real palindromic quadratic partitioning of $\mathcal{L}$.

That the converse also holds follows from the discussion surrounding Definition 3.6, where it is shown why real p-quad realizability implies real p-quad admissibility. ☐

With this theorem, we have now achieved the two main goals of this paper: to develop simple necessary and sufficient conditions to detect the real quadratic T-palindromic realizability of a given list of structural data, and to provide an explicit, non-numerical, finite procedure to construct such a structured realization whenever it exists, one that sparsely but transparently displays the given structural data.

6. Conclusion

This paper has extended the results of [9] on the T-palindromic quadratic realizability problem, from algebraically closed fields (in [9]) to the real field $\mathbb{R}$. These results can be viewed as providing a structured Kronecker-like *real* quasi-canonical form for real T-palindromic *quadratic* matrix polynomials, essentially a degree 2 analog for the palindromic Kronecker canonical form for T-palindromic pencils in [34]. In order to achieve these results, we have extended and adapted the concepts in [9] to handle the real case, and also developed constructions to obtain canonical realizations of the new irreducible structural data lists that appear when the underlying field is $\mathbb{R}$, rather than algebraically closed. Although many of these canonical realizations have (anti-diagonal) bandwidth of three or less, it has sometimes been necessary to extend the bandwidth beyond three. However, in all cases the realizations are very sparse, and transparently display their structural data.

There are two natural directions to consider for further extension of this work. One is to the T-palindromic realizability problem for degree three, or degree four (or higher), or even to consider the problem for arbitrary degree. Another natural direction is to consider classes of underlying field which support irreducible scalar polynomials of degree higher than two. However, in our view it is unlikely that the strategy used in this paper (i.e., forming a direct sum of canonical blocks that each realize irreducible lists of structural data), will produce a reasonable solution for any such extension. As the degree of the desired realization or the complexity of the field increases, there is likely to be a combinatorial explosion in the number of primitive, irreducible cases to consider, so that the complexity of any solution of this type becomes unmanageable. It is likely that a new approach will be needed in order to find a good solution to any of these extended realizability problems. One such approach [11] is currently under development.

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Appendix A. Proof of Lemma 4.11

(a) The key idea in our proofs is similar to the proof of Lemma 4.1 [9, Lem. 5.2], i.e., we perform carefully constructed elementary unimodular row/column operations on the 5×5 matrix polynomials while exploiting the assumptions of the relative primeness of $r(\lambda)$ and $s(\lambda)$ with λ and $(\lambda - 1)$, and the relative primeness of $t(\lambda)$ and $u(\lambda)$ with λ and $(\lambda + 1)$.

We start by introducing notation for quantities that play an important role in our unimodular reduction of $N_{m,n,k}(r,s)$. The assumption that both $r(\lambda)$ and $s(\lambda)$ are coprime to λ implies that the scalars $r_0 := r^k(0)$ and $s_0 := s^k(0)$ are *nonzero*. Furthermore, there exist scalar polynomials $q_r(\lambda)$ and $q_s(\lambda)$ such that $r^k(\lambda) = q_r(\lambda)\cdot\lambda + r_0$ and $s^k(\lambda) = q_s(\lambda)\cdot\lambda + s_0$. We now claim that $N_{m,n,k}(r,s) \sim N^{(1)}_{m,n,k}(r,s)$, where

$$N^{(1)}_{m,n,k}(r,s) := \left[\begin{array}{c|ccc} I_2 & & & \\ \hline & (\lambda-1)^m\lambda^{n+2} + \rho\,\lambda\, r^k(\lambda)s^k(\lambda) & 0 & s^{k+1}(\lambda) \\ & 0 & (\lambda-1)^2 & \lambda(1-\lambda) \\ & r^{k+1}(\lambda) & (\lambda-1) & -\lambda \end{array}\right]. \tag{A1}$$

An explicit unimodular equivalence that achieves this can be obtained by starting with $N_{m,n,k}(r,s)$ and performing the sequence of elementary unimodular row/column operations given in Figure A1, where $\eta(\lambda) = (\lambda-1)^m\cdot\lambda^{n+1} + q_s\cdot\rho\cdot\lambda\cdot r^k$.

① $\mathrm{Col}_1 \longrightarrow (-q_r)\cdot\mathrm{Col}_2 + \mathrm{Col}_1$	⑪ $\mathrm{Col}_5 \longrightarrow (1/s_0)\cdot\mathrm{Col}_5$
② $\mathrm{Col}_2 \longrightarrow (-1/r_0)\cdot\lambda\cdot\mathrm{Col}_1 + \mathrm{Col}_2$	⑫ $\mathrm{Col}_4 \longrightarrow (q_s\cdot s)\cdot\mathrm{Col}_5 + \mathrm{Col}_4$
③ $\mathrm{Col}_2 \longrightarrow (r_0)\cdot\mathrm{Col}_2$	⑬ $\mathrm{Col}_2 \longrightarrow (\eta(\lambda))\cdot\mathrm{Col}_5 + \mathrm{Col}_2$
④ $\mathrm{Row}_5 \longrightarrow (1/r_0)\cdot(\mathrm{Row})_5$	⑭ $\mathrm{Row}_1 \longleftrightarrow \mathrm{Row}_5$
⑤ $\mathrm{Row}_4 \longrightarrow (q_r\cdot r)\cdot\mathrm{Row}_5 + \mathrm{Row}_4$	⑮ $\mathrm{Row}_5 \longleftrightarrow \mathrm{Row}_4$
⑥ $\mathrm{Row}_2 \longrightarrow (q_r\cdot\rho\cdot\lambda)\cdot\mathrm{Row}_5 + \mathrm{Row}_2$	⑯ $\mathrm{Row}_4 \longleftrightarrow \mathrm{Row}_3$
⑦ $\mathrm{Row}_1 \longrightarrow (-(\lambda-1)^m\cdot\lambda^n)\cdot\mathrm{Row}_5 + \mathrm{Row}_1$	⑰ $\mathrm{Row}_3 \longleftrightarrow \mathrm{Row}_2$
⑧ $\mathrm{Row}_1 \longrightarrow (-q_s)\cdot\mathrm{Row}_2 + \mathrm{Row}_1$	⑱ $\mathrm{Col}_5 \longleftrightarrow \mathrm{Col}_4$
⑨ $\mathrm{Row}_2 \longrightarrow (-1/s_0)\cdot\lambda\cdot\mathrm{Row}_1 + \mathrm{Row}_2$	⑲ $\mathrm{Col}_4 \longleftrightarrow \mathrm{Col}_3$
⑩ $\mathrm{Row}_2 \longrightarrow (s_0)\cdot\mathrm{Row}_2$	⑳ $\mathrm{Col}_3 \longleftrightarrow \mathrm{Col}_2$

Figure A1.: Unimodular reduction of $N_{m,n,k}(r,s)$ to $N^{(1)}_{m,n,k}(r,s)$

Next, by slightly adapting matrices from (25) so they apply to $N^{(1)}_{m,n,k}(r,s)$, we define new *unimodular* matrix polynomials $G(\lambda)$ and $U(\lambda)$ given by

$$G(\lambda) = \left[\begin{array}{c|ccc} I_2 & & & \\ \hline & 0 & 0 & -1 \\ & 1 & 0 & s^{k+1} \\ & 0 & -1 & (\lambda-1) \end{array}\right] \quad\text{and}\quad U(\lambda) = \left[\begin{array}{c|ccc} I_2 & & & \\ \hline & 0 & 1 & 0 \\ & 1 & r^{k+1} & \lambda \\ & 1 & r^{k+1} & \lambda-1 \end{array}\right]. \tag{A2}$$

Then $G(\lambda)\cdot N^{(1)}_{m,n,k}(r,s)\cdot U(\lambda) = N^{(2)}_{m,n,k}(r,s)$, where

$$N^{(2)}_{m,n,k}(r,s) := \left[\begin{array}{c|cc} I_3 & & \\ \hline & (\lambda-1)^m\lambda^{n+2} + \rho\,\lambda\, r^k s^k + r^{k+1}s^{k+1} & (\lambda-1)\,s^{k+1} \\ & (\lambda-1)r^{k+1} & 0 \end{array}\right], \tag{A3}$$

implies $N_{m,n,k}(r,s) \sim N^{(1)}_{m,n,k}(r,s) \sim N^{(2)}_{m,n,k}(r,s)$. Consequently, in order to complete the proof of part (a) it suffices to show that the Smith form of the bottom right 2×2 block of $N^{(2)}_{m,n,k}(r,s)$ is $\mathrm{diag}\big((\lambda-1),\,(\lambda-1)\,\widetilde{r}^{\,k+1}(\lambda)\widetilde{s}^{\,k+1}(\lambda)\big)$.

Inspired by the proof of Lemma 4.9, we consider the $(4,4)$ entry of $N^{(2)}_{m,n,k}(r,s)$. The assumption that $\rho = -r(1)s(1)$ implies that $\rho\,\lambda\, r^k(\lambda)\, s^k(\lambda)\Big|_{\lambda=1} = -r^{k+1}(1)\,s^{k+1}(1)$, and so

$$\Big[(\lambda-1)^m\lambda^{n+2} \;+\; \rho\cdot\lambda\, r^k(\lambda)s^k(\lambda) \;+\; r^{k+1}(\lambda)s^{k+1}(\lambda)\Big]\Big|_{\lambda=1} \;=\; 0\,. \tag{A4}$$

From (A4) it follows that there exist a scalar polynomial $h(\lambda)$ such that

$$(\lambda-1)^m\lambda^{n+2} \;+\; \rho\,\lambda\, r^k(\lambda)\, s^k(\lambda) \;+\; r^{k+1}(\lambda)s^{k+1}(\lambda) = (\lambda-1)h(\lambda)\,. \tag{A5}$$

Combining (A5) and (A3) results in the following unimodular equivalence

$$N_{m,n,k}(r,s) \;\sim\; \begin{bmatrix} I_3 & & \\ & (\lambda-1)\,h(\lambda) & (\lambda-1)s^{k+1}\,(\lambda) \\ & (\lambda-1)\,r^{k+1}(\lambda) & 0 \end{bmatrix} \;=:\; N^{(3)}_{m,n,k}(r,s)\,. \tag{A6}$$

We now claim that $\gcd\{\,(\lambda-1)h(\lambda)\,,\,(\lambda-1)r^{k+1}(\lambda)\,,\,(\lambda-1)s^{k+1}(\lambda)\,\} \;=\; \lambda-1\,,$ or equivalently, that

$$\gcd\Big\{\,h(\lambda)\,,\; r^{k+1}(\lambda)\,,\; s^{k+1}(\lambda)\,\Big\} \;=\; 1\,. \tag{A7}$$

In order to verify (A7) it suffices to show that $h(\lambda)$ is coprime to either $r(\lambda)$ or $s(\lambda)$. Without loss of generality, assume that $h(\lambda)$ is not coprime to $r(\lambda)$. Since $r(\lambda)$ is an $\mathbb{R}$-irreducible nonzero scalar polynomial by assumption, it must be that $h(\lambda) = d(\lambda)r(\lambda)$ for some $d(\lambda) \in \mathbb{R}[\lambda]$. Then,

$$\begin{aligned}&(\lambda-1)^m\lambda^{n+2} + \rho\,\lambda\, r^k(\lambda)\, s^k(\lambda) + r^{k+1}(\lambda)s^{k+1}(\lambda) = (\lambda-1)d(\lambda)r(\lambda)\\ &\quad\Longleftrightarrow\quad (\lambda-1)^m\lambda^{n+2} = r(\lambda)\Big[(\lambda-1)\,d(\lambda) - \rho\,\lambda\, r^{k-1}(\lambda)\, s^k(\lambda) - r^k(\lambda)s^{k+1}(\lambda)\Big].\end{aligned}$$

But this implies that $r(\lambda)$ divides either $(\lambda-1)$ or λ, which is a contradiction to the assumption that $r(\lambda)$ is coprime to both $(\lambda-1)$ and λ. Thus it must be that (A7) holds, and therefore, the first non-constant invariant polynomial of $N^{(3)}_{m,n,k}(r,s)$ is $(\lambda-1)$.

Finally, we determine the second non-constant invariant polynomial of $N^{(3)}_{m,n,k}(r,s)$ by computing its determinant, dividing it by the first non-constant invariant polynomial $(\lambda-1)$, and scaling the resulting polynomial to make it monic. An easy calculation shows that the second non-constant invariant polynomial of $N^{(3)}_{m,n,k}(r,s)$ is $(\lambda-1)\cdot\widetilde{r}^{\,k+1}(\lambda)\cdot\widetilde{s}^{\,k+1}(\lambda)$ which together with (A6) reveal that the Smith form of $N_{m,n,k}(r,s)$ is $\operatorname{diag}\big(I_3\,,\,(\lambda-1)\,,\,(\lambda-1)\widetilde{r}^{\,k+1}\widetilde{s}^{\,k+1}\big)$, as desired.

(b) The proof is completely analogous to the one for part (a) and hence is omitted. □

Appendix B. Proof of Lemma 4.12

(a) We start by showing that $L_{m,n,k}(r,s)$ is unimodularly equivalent to $L^{(1)}_{m,n,k}(r,s)$, where

$$L^{(1)}_{m,n,k}(r,s) := \left[\begin{array}{cc|ccc} (\lambda-1) & & & & \\ & (\lambda-1) & & & \\ \hline & & (\lambda-1)^{m+2}\lambda^n + \rho\,\lambda\, r^k(\lambda)\, s^k(\lambda) & 0 & s^{k+1}(\lambda) \\ & & 0 & (\lambda-1)^2 & \lambda(1-\lambda) \\ & & r^{k+1}(\lambda) & (\lambda-1) & -\lambda \end{array}\right].$$

By an argument almost identical to that used for Lemma 4.11 (see Appendix A), we can conclude that the Smith form of the bottom right 3×3 block of $L^{(1)}_{m,n,k}(r,s)$ is $\operatorname{diag}\big(1,(\lambda-1),(\lambda-1)\widetilde{r}^{k+1}(\lambda)\widetilde{s}^{k+1}(\lambda)\big)$, which together with $L_{m,n,k}(r,s)\sim L^{(1)}_{m,n,k}(r,s)$ proves part (a).

Now it only remains to show that $L_{m,n,k}(r,s)\sim L^{(1)}_{m,n,k}(r,s)$. The assumption that both $r(\lambda)$ and $s(\lambda)$ are coprime to $(\lambda-1)$ implies that the scalars $r_1 := r^k(1)$ and $s_1 := s^k(1)$ are *nonzero*. Furthermore, there exist scalar polynomials $q_r(\lambda)$ and $q_s(\lambda)$ such that $r^k(\lambda) = q_r(\lambda)\cdot(\lambda-1)+r_1$ and $s^k(\lambda) = q_s(\lambda)\cdot(\lambda-1)+s_1$.

On the other hand, the assumption that both $r(\lambda)$ and $s(\lambda)$ are coprime to $(\lambda-1)$ also implies that $r^{k+1}(\lambda)$ and $s^{k+1}(\lambda)$ are coprime to $(\lambda-1)$. Consequently, there exist real scalar polynomials $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ and $d(\lambda)$ such that

$$\begin{aligned} a(\lambda)\cdot(\lambda-1)+b(\lambda)\cdot r^{k+1}(\lambda) &= 1\,,\\ c(\lambda)\cdot(\lambda-1)+d(\lambda)\cdot s^{k+1}(\lambda) &= 1\,. \end{aligned} \tag{B1}$$

Finally, the desired unimodular equivalence $L_{m,n,k}(r,s)\sim L^{(1)}_{m,n,k}(r,s)$ can be achieved by starting with $L_{m,n,k}(r,s)$ and performing the sequence of elementary unimodular row/column operations given in Figure B1, which are then followed by several row/column swaps identical to the ones in Figure A1. Here the scalar polynomials $\xi(\lambda)$ and $\psi(\lambda)$ in Figure B1 are given by $\xi(\lambda) = (\lambda-1)^m\cdot\lambda^n\cdot r\cdot q_r\cdot b$ and $\psi(\lambda) = (\lambda-1)^m\lambda^n - q_s\cdot s\cdot d\cdot(\lambda-1)^{m+1}\cdot\lambda^n + \rho\cdot\lambda\cdot q_s\cdot r^k\cdot c$. While this completes the proof of part (a), one additional point is worth emphasizing. Note that the operations in Figure B1 extensively utilize the auxiliary polynomials $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, and $d(\lambda)$ from (B1), which is in stark contrast to the analogous unimodular reduction in Lemma 4.11.

① $\mathrm{Col}_1 \longrightarrow (-q_r)\cdot\mathrm{Col}_2+\mathrm{Col}_1$	⑨ $\mathrm{Row}_2 \longrightarrow (q_r\cdot\rho\cdot\lambda\cdot a)\cdot\mathrm{Row}_5+\mathrm{Row}_2$
② $\mathrm{Col}_2 \longrightarrow (1/r_1)\cdot(1-\lambda)\cdot\mathrm{Col}_1+\mathrm{Col}_2$	⑩ $\mathrm{Row}_1 \longrightarrow (-q_s)\cdot\mathrm{Row}_2+\mathrm{Row}_1$
③ $\mathrm{Col}_2 \longrightarrow (r_1)\cdot\mathrm{Col}_2$	⑪ $\mathrm{Row}_2 \longrightarrow (1/s_1)\cdot(1-\lambda)\cdot\mathrm{Row}_1+\mathrm{Row}_2$
④ $\mathrm{Row}_5 \longrightarrow (1/r_1)\cdot\mathrm{Row}_5$	⑫ $\mathrm{Col}_5 \longrightarrow (1/s_1)\mathrm{Col}_5$
⑤ $\mathrm{Row}_1 \longrightarrow -(\lambda-1)^{m-1}\lambda^n\cdot\mathrm{Row}_5+\mathrm{Row}_1$	⑬ $\mathrm{Row}_2 \longrightarrow (s_1)\cdot\mathrm{Row}_2$
⑥ $\mathrm{Row}_4 \longrightarrow (r\cdot q_r\cdot a)\cdot\mathrm{Row}_5+\mathrm{Row}_4$	⑭ $\mathrm{Col}_4 \longrightarrow (q_s\cdot s\cdot c)\cdot\mathrm{Col}_5+\mathrm{Col}_4$
⑦ $\mathrm{Col}_1 \longrightarrow (r\cdot q_r\cdot b)\cdot\mathrm{Col}_2+\mathrm{Col}_1$	⑮ $\mathrm{Row}_1 \longrightarrow (q_s\cdot s\cdot d)\cdot\mathrm{Row}_2+\mathrm{Row}_1$
⑧ $\mathrm{Row}_1 \longrightarrow \xi(\lambda)\cdot\mathrm{Row}_5+\mathrm{Row}_1$	⑯ $\mathrm{Col}_2 \longrightarrow \psi(\lambda)\cdot\mathrm{Col}_5+\mathrm{Col}_2$

Figure B1.: Unimodular reduction of $L_{m,n,k}(r,s)$ to $L^{(1)}_{m,n,k}(r,s)$

(b) The proof is completely analogous to the one for part (a), and hence is omitted. □

Appendix C. Proof of Lemma 4.15

The assumption that both $r(\lambda)$ and $t(\lambda)$ are coprime to $(\lambda-1)$ implies that the scalars $r_1 := r(1)$ and $t_1 := t(1)$ are *nonzero*. Furthermore, there exist scalar polynomials $q_r(\lambda)$ and $q_t(\lambda)$ such that

$$r(\lambda) \;=\; q_r(\lambda)\cdot(\lambda-1)+r_1 \qquad \text{and} \qquad t(\lambda) \;=\; q_t(\lambda)\cdot(\lambda-1)+t_1\,.$$

Then the desired unimodular equivalence $K(\lambda)\sim\widetilde{K}(\lambda)$ can be achieved by starting with $K(\lambda)$ and performing the sequence of elementary unimodular row/column operations given in Figure C1.

①	$\mathrm{Col}_1 \longrightarrow (-q_r)\cdot\mathrm{Col}_2+\mathrm{Col}_1$	⑧	$\mathrm{Row}_2 \longrightarrow (t_1)\cdot\mathrm{Row}_2$
②	$\mathrm{Row}_1 \longrightarrow (-q_t)\cdot\mathrm{Row}_2+\mathrm{Row}_1$	⑨	$\mathrm{Row}_{m-1} \longrightarrow (q_r\cdot s)\cdot\mathrm{Row}_m+\mathrm{Row}_{m-1}$
③	$\mathrm{Col}_2 \longrightarrow (1/r_1)\cdot(1-\lambda)\cdot\mathrm{Col}_1+\mathrm{Col}_2$	⑩	$\mathrm{Col}_{m-1} \longrightarrow (q_t\cdot u)\cdot\mathrm{Col}_m+\mathrm{Col}_{m-1}$
④	$\mathrm{Row}_2 \longrightarrow (1/t_1)\cdot(1-\lambda)\cdot\mathrm{Row}_1+\mathrm{Row}_2$	⑪	$\mathrm{Row}_2 \longrightarrow (\psi)\cdot\mathrm{Row}_m+\mathrm{Row}_2$
⑤	$\mathrm{Col}_m \longrightarrow (1/t_1)\cdot\mathrm{Col}_m$	⑫	$\mathrm{Col}_2 \longrightarrow (\psi)\cdot\mathrm{Col}_m+\mathrm{Col}_2$
⑥	$\mathrm{Row}_m \longrightarrow (1/r_1)\cdot\mathrm{Row}_m$	⑬	$\mathrm{Col}_1 \longrightarrow (-\xi)\cdot\mathrm{Col}_m+\mathrm{Col}_1$
⑦	$\mathrm{Col}_2 \longrightarrow (r_1)\cdot\mathrm{Col}_2$		

Figure C1.: Unimodular reduction with $\lambda-1$ as the "gluing" entry

□