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ABSTRACT

Most reconstruction algorithms in electrical impedance tomography (EIT, ERT and ECT) are known to suffer from the computational costs of calculating the Jacobian (also known as sensitivity matrix). In this paper we review and slightly modify the adjoint fields technique already widely used in electromagnetic imaging, diffuse optical and ultrasound tomography, which enables the reconstruction of nonlinear solutions without the explicit calculation of the Jacobian. This has enormous computational advantages in the efficiency of image reconstruction. Here we address the complex isotropic EIT problem.

Keywords Adjoint operator, electrical impedance tomography, computational efficiency.

1 THE ADJOINT PROBLEM

The aim of this paper is to explain the computational advantages associated to the adjoint operator, in that it calculates the product of the transpose of the Jacobian matrix with a vector of suitable dimensions without calculating the Jacobian matrix explicitly. This operator can be introduced to a variety of inverse solvers, linear and nonlinear alike; in particular to the so-called "low-storage" methods, which only require first derivatives for the computation of their descent directions. The nonlinear conjugate gradients algorithm for instance, has a regularised direction p_i of the form

$$p_i = J^*_{i-1}(V^{meas} - V_i^{sim}) + \beta_i p_{i-1} \tag{1}$$

where J_{i-1} is the Jacobian matrix based on the previous model update, $V^{meas} - V_i^{sim}$ is the i'th residual in the measurements and β_i a real scalar parameter. Throughout this paper X^* denotes the Hermitian transpose of X. The method offers a substantial computational advantage to problems with excessive degrees of freedom and numerous measurements, that is in the cases where the computation and storage of the Jacobian matrix is computationally demanding. Thus, in the tendency of solving realistic problems where accurate modelling is essential, apart from reducing the computational time for image reconstruction, the method offers a memory saving alternative as well.

The adaptation of the method for electrical impedance tomography (EIT) is relatively easy in that the framework is quite similar to that of the electromagnetic imaging (EM) case extensively described by Dorn et al. in (Dorn 1999; Dorn 2002). Nonetheless, there are still some critical differences, mainly emerging from the boundary conditions in the complete electrode model (Vauhkonen 1999). In broad terms, some extra care is required due to the mismatch between the direct sources (the actual current patterns) and the adjoint sources, which are simply hypothetical sources situated at the positions of the measuring electrodes. This means that direct fields are defined by boundary current densities that are not constant on the surfaces of the excited electrodes, while the measurements gathered; which are then to be used as adjoint sources, are known to be constant on the surfaces of the measuring electrodes (Paulson 1992). Furthermore, current patterns and boundary measurements usually involve two or more electrodes with a finite surface rather than single point sources and detectors as used in the EM case, which poses some concerns in deriving the adjoint sources, as these are defined on electrodes while the measurement residuals are defined on electrode pairs.

In general adjoint problems are usually based upon the reciprocity principle that holds in the appropriate physical system. Considering for instance the low-frequency, Maxwell's time harmonic equations on a conductive domain Ω, with boundary $\partial\Omega$ and $\Gamma_1, \Gamma_2 \subset \partial\Omega$ where Γ_1, Γ_2 are disjoint then if $\mathbf{r}$ and $\mathbf{s}$ are the position and direction vectors respectively on Γ_1, then the potential measurement collected at $\mathbf{r}$ in the direction $\mathbf{s}$ due to a current source q_0 situated at $\mathbf{r_0} \in \Gamma_2$ is the same as the potential that would have been measured at $\mathbf{r_0}$ due to a source q_0 this time situated at

$\mathbf{r}$ in direction $-\mathbf{s}$. To derive the adjoint method for EIT we consider a system consisting of a conductive domain Ω in a Euclidean 3-space and L electrodes attached to its boundary.

In the low-frequency range, the time harmonic Maxwell's equations reduce to the partial differential equation

$$\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega \qquad (2)$$

with $\gamma = \sigma + \mathrm{i}\omega\varepsilon$, $\sigma > 0$, $\varepsilon > 0$. The electrical conductivity σ in Sm^{-1}, and the electrical permittivity ε in Fm^{-1} are taken, for the sake of simplicity, to be isotropic within the domain of interest, however the extension of the following to the anisotropic case is trivial. According to the complete electrode model (Vauhkonen 1999), the boundary conditions for the elliptic partial differential equation (2) are

$$q_l = \int_{e_l} \gamma \frac{\partial u}{\partial \mathbf{n}} \text{ on driving electrodes } l \qquad (3)$$

where q_l is the current at the surface of the l'th electrode, and

$$U_l = u + z_l \gamma \frac{\partial u}{\partial \mathbf{n}} \text{ on measuring electrodes } l \qquad (4)$$

where U_l is the constant potential value at the surface of the l'th electrode. The model is known to have a unique solution (Somersalo 1992) when $\sum_{i=1}^{L} q_i = 0$ and $\sum_{k=1}^{L} U_k = 0$. In equations (2) – (4), u is the scalar electric potential, z_l is the contact impedance of the l'th electrode and $\mathbf{n}$ is the outward normal unit vector on the boundary of the domain.

For the derivation of the adjoint method in EIT the following space definitions are introduced:

- S : The space of the parameters γ.
- Ψ : The space of the current sources q_l.
- W_j : The space of the current patterns I_j.
- Y : The space of potential distributions u_j.
- V : The space of electrode potentials U_l.
- Z_j : The space of measurements corresponding to the j'th current pattern ζ_i for $i = 1,...,k$ with Z the direct sum of Z_j.

In a bounded conductive domain Ω with boundary $\partial\Omega$ and L boundary electrodes attached, the j'th drive current pattern $I_j \in W_j$ is given as a sequence of boundary current sources using the currents operator $C_j : \Psi \to W_j$ as

$$I_j = C_j(q_1, q_2, ..., q_L) = C_j\left(\int_{e_1} \hat{j}_1 ds_1, \int_{e_2} \hat{j}_2 ds_2, ..., \int_{e_L} \hat{j}_L ds_L\right) \qquad (5)$$

where $\hat{j}_l$ denotes the l'th orthogonal current density vector at the boundary. The j'th *direct* forward solution $u_j \in Y$ satisfies

$$\nabla \cdot (\gamma \, \nabla u_j) = 0 \qquad (6a)$$

in the interior of the domain under a boundary current pattern I_j. In a matrix form can be expressed as

$$A_\gamma u_j = I_j \qquad (6b)$$

where A_γ is the forward system matrix based on γ. The associated j'th *adjoint* problem is to find a solution $v_j \in Y$ that satisfies

$$\nabla \cdot (\bar{\gamma} \, \nabla v_j) = 0 \qquad (7a)$$

in the interior of the domain for an adjoint boundary current pattern I_j^+. In matrix form

$$A_{\bar{\gamma}} v_j = I_j^+ \qquad (7b)$$

where $\bar{\gamma} = \sigma - \mathrm{i}\omega\varepsilon$ and $A_{\bar{\gamma}} = A_{\gamma}^{*}$ is the adjoint system matrix based on $\bar{\gamma}$. The collection of differential boundary potential measurements usually involves pairs of nearby located electrodes. If there are p measurements to be gathered under I_j then from the data potential distribution $u_j \in Y$ in Ω one can easily extract an array of potential readings $U_{jl} \in V$ for $l = 1,...,L$ at $\partial\Omega$ and thereafter assemble the boundary measurements vector $\zeta_{jk} \in Z_j$ for $k = 1,..., p$. To calculate the electrode potentials from the forward solution the electrode potential operator $M_j : Y \to V$ is used, hence

$$U_{jl} = M_j \circ u_j = \left(U_{j1}, U_{j2},...,U_{jL}\right) \tag{8}$$

The differential measurements are then calculated using the measurements operator $D_j : V \to Z_j$ so that

$$\zeta_{jk} = U_{jl} \circ D_j = \left(\zeta_{j1}, \zeta_{j2},...,\zeta_{jp}\right) \tag{9}$$

and combining equations (8) and (9) yields

$$\left\{\zeta_{jk}\right\}_{k=1}^{p} = D_j \circ M_j \circ u_j \tag{10}$$

In fact, the measurements operator D_j is simply an $L \times p$ matrix, whose columns are the definitions of the measurement patterns.

The forward operator $F : S \to Z$, which relates the parameters of interest (admittivities) to the boundary measurements (voltages) can be derived. Using F which is Frechét differentiable and nonlinear in γ, the aim of the inverse problem is to reconstruct a solution $\gamma \in S$ so that for all j

$$F_j(\gamma) = \zeta_j \tag{11}$$

In reality though, the measurements are contaminated with noise so the right hand side of (11) is essentially replaced with a perturbed measurements vector $\tilde{\zeta}_j \in Z$, where $\left\|\zeta_j - \tilde{\zeta}_j\right\| \le \kappa$, for some noise level κ. In this context, the aim is to minimise the quadratic $\frac{1}{2}\left\|F_j(\gamma) - \tilde{\zeta}_j\right\|_2^2$. Taking perturbations on the parameters $\gamma \to \gamma + \delta\gamma$ and recording the perturbations in the forward solution $u_j \to u_j + \delta u_j$ the Frechet derivative of F_j can be calculated, yielding the discrete linearised form of the forward problem

$$F_j'(\gamma)\delta\gamma = -D_j \circ M_j \circ \delta u_j \tag{12}$$

where $F_j' : S \to Z$ is a compact linear operator and $F_j'(\gamma)$ is the Jacobian matrix evaluated at γ. If $\xi_j \in Z_j$ is the residual in the measurements of the j'th current pattern so that $\xi_j = F_j(\gamma) - \tilde{\zeta}_j$, then for an appropriately selected adjoint current pattern I_j^{+}, the product of the transpose of the Jacobian times a residuals vector $F_j'(\gamma)^{*}\xi_j$ can be calculated using one direct and one adjoint forward solution. If there are p measurement residuals in ξ_j, then expanding $F_j'(\gamma)^{*}\xi_j$ yields

$$F_j'(\gamma)^{*}\xi_j = \sum_{q=1}^{p} F_{jq}'(\gamma)^{*}\xi_{jq} \tag{13}$$

From the integral derivation of the Jacobian in the complete electrode model (Polydorides 2002), the above can be expressed as

$$F_j'(\gamma)^{*}\xi_j = -\int_{\Omega} (\nabla\hat{u}_j)^{*}\left(\sum_{q=1}^{p}\nabla\hat{v}_{jq}\cdot\xi_q\right)\mathrm{dV} \tag{14}$$

where dV is a volume metric. The following refer explicitly to the j'th current pattern so to eliminate clutter the subscript is dropped in equations (15) – (18). Taking the gradients of the finite element representations of the potentials gives

$$\nabla\hat{u} = \sum_{i=1}^{n} u_i \nabla\phi_i \quad \text{and} \quad \nabla\hat{v} = \sum_{k=1}^{n} v_k \nabla\phi_k \tag{15}$$

where ϕ_i, ϕ_k are linear nodal shape functions and u_i, v_k the vectors of nodal potential values, and substituting (15) into (14) yields

$$\begin{aligned} F'(\gamma)^* \xi &= -\int_\Omega \left(\nabla \sum_{i=1}^{n} \overline{u}_i \phi_i \right) \left[\sum_{q=1}^{p} \left(\nabla \sum_{k=1}^{n} v_{k,q} \phi_k \right) \xi_q \right] \mathrm{dV} \\ &= -\sum_{i=1}^{n} \overline{u}_i \int_\Omega \nabla \phi_i \left[\sum_{q=1}^{p} \sum_{k=1}^{n} v_{k,q} \nabla \phi_k \ \xi_q \right] \mathrm{dV} \\ &= -\sum_{i=1}^{n} \sum_{k=1}^{n} \overline{u}_i \left(\sum_{q=1}^{p} v_{k,q} \xi_q \right) \int_\Omega \nabla \phi_i \nabla \phi_k \ \mathrm{dV} \end{aligned} \tag{16}$$

where $\overline{u}$ is the conjugate of u and the quantity in parenthesis is simply the solution of the adjoint problem. At this stage, in order to formulate the adjoint problem, one needs to trace the corresponding adjoint sources. In contrast to the already developed adjoint formulations as in electromagnetic imaging case described in (Dorn 1992) and (Dorn 2002), the diffuse optical imaging in (Arridge 1999) and the ultrasound tomography (Natterer 1995), here the measurement residuals refer to electrode pairs rather than single point detectors therefore the definition of the adjoint sources using the residuals of the measurements is not so trivial. Expanding the parenthesised quantity in equation (16) yields

$$\sum_{q=1}^{p} v_q \xi_q = v_1 \xi_1 + ,..., + v_p \xi_p \tag{17a}$$

For clarity in the following we introduce the notation $v(I_n)$ to denote the linear operation $v_n = A_\gamma^{-1} I_n$. Here we make use of the following two properties of linear operators

$$v(I_n) \pm v(I_m) = v(I_n \pm I_m) \tag{17b}$$

$$c \cdot v(I_n) = v(c \cdot I_n) \tag{17c}$$

with $c \in \Re$. With (17b) and (17c) the sum in (17a) takes the form of

$$\sum_{q=1}^{p} v_q \xi_q = v(I_1 \xi_1) + ,..., + v(I_p \xi_p) = v(I_1 \xi_1 +, ..., + I_p \xi_p) \tag{17d}$$

Hence,

$$\sum_{q=1}^{p} v_q \xi_q = v\left(\sum_{q=1}^{p} I_p \xi_p \right) = A_{\overline{\gamma}}^{-1} I_j^+ = v_j \text{ where } I_j^+ = \sum_{q=1}^{p} I_p \xi_p \tag{17e}$$

and from the definition of the adjoint problem in (7), equation (16) can also be written as

$$F_j'(\gamma)^* \xi_j = -\sum_{i=1}^{n} \sum_{k=1}^{n} \overline{u}_i v_k \int_\Omega \nabla \phi_i \nabla \phi_k \ \mathrm{dV} \tag{18}$$

The exact form of I_j^+ can be obtained via the differential measurements operator, this time aiming to decouple the residuals in a way that each electrode is assigned an 'individual' residual error. Thus if D_j is a real $L \times p$ matrix and ξ_j an array of complex measurement residuals then

$$I_j^+ = D_j \circ \xi_{jq} \tag{19}$$

The formation of the adjoint sources may at first seem as merely a computational trick, but it also has a rational physical interpretation, in that `the sum of the fields developed when *l* electrodes are excited individually in turn, is equal to the field that would have been deployed if all *l* electrodes were excited simultaneously'. Thus

$$F_j'(\gamma)^* \xi_j = -\left[\overline{A_\gamma^{-1} I_j} \right] [C] \left[A_{\overline{\gamma}}^{-1} I_j^+ \right] \tag{20}$$

where C is a fixed by the mesh topology matrix holding the integrals of the gradients of the shape functions.

2 THE NONLINEAR ADJOINT FIELDS INVERSION ALGORITHM

A reconstruction algorithm based on adjoint fields has already been developed in (Dorn 1999; Dorn 2002) for electromagnetic imaging, in (Arridge 1999) for diffuse optical tomography and (Natterer 1995) for ultrasound tomography. This nonlinear method uses an iteration of the form

$$\gamma_{k+1} = \gamma_k + RF_j'(\gamma)^*\left(\tilde{\zeta}_j - F_j(\gamma)\right) = \gamma_k + RF_j'(\gamma)^*\xi_j \quad (21)$$

where R is a positive definite regularization matrix and

$$F_j'(\gamma)^*\xi_j = -\sum_{i=1}^{n}\sum_{k=1}^{n}\left(\overline{u}_i\right)_j\left(v_k\right)_j\int_\Omega \nabla\phi_i\nabla\phi_k \,\mathrm{dV} \quad (22)$$

The method reconstructs in turn subsets of the data, i.e. those corresponding from the same current pattern, each time updating the model using equation (21). Note that for the anisotropic problem one should keep separated the *x*, *y* and *z* components in the gradients of the shape functions in equation (22). Practically, in EIT the implementation of the method in its original format is rather problematic. This is mainly due to the fact that usually the measurements captured from a single current pattern, are either too few or too noisy to reconstruct a solution of a reasonable spatial resolution. To rectify this limitation, we calculate a block adjoint operator in the form of a sparse block diagonal matrix, so that to accommodate several adjoint operators corresponding to a number of current patterns rather than just a single one. If $g \subseteq W_j$ is a subset of the current patterns, with $\xi_g \in Z_j$ the concatenated vector of the residuals in the corresponding measurements then if $\overline{U}_j = \sum_{i=1}^{n}(\overline{u}_i)_j$ and $V_j = \sum_{k=1}^{n}(v_k)_j$

$$F_g'(\gamma)^*\xi_g = -\sum_{j=1}^{c}\overline{U}_j V_j\int_\Omega \nabla\phi_i\nabla\phi_k \,\mathrm{dV} \quad (23)$$

which implies that for c current patterns in g, the block adjoint operator $F_g'(\gamma)^T\xi_g$ can be calculated after g direct and g adjoint forward solutions. To implement the algorithm, one must first split the current patterns d and their corresponding measurements $\tilde{\zeta}$ in p groups, such as $d = \{d_1,...,d_p\}$ and $\tilde{\zeta} = \{\zeta_1,...,\zeta_p\}$. In addition, allow $R: S \to S$ being a positive definite regularization matrix, here incorporating some smoothing assumptions on the solution. If $\kappa \in \Re$ is the estimated error in the measurements, the adjoint fields algorithm is

- **WHILE** $\frac{1}{2}\left\|F_g(\gamma) - \tilde{\zeta}_g\right\|_2^2 > \kappa$
  - **FOR** $g = 1 : q$
  - $\gamma \leftarrow \gamma + R\left(-\sum_{\mathrm{j=1}}^{\mathrm{c}}\overline{U}_j V_j \int_\Omega \nabla\phi_{\mathrm{i}}\nabla\phi_k \,\mathrm{dV}\right)$
- **ENDFOR** **ENDWHILE**.

Performance statistics of the adjoint fields' algorithm for a number of problems with different parameters are shown in table 1. In the same table, the relevant figures for the nonlinear conjugate gradients algorithm are also supplied for comparison. The selection of the NLCG algorithm is based on the fact that the computational cost of the method is roughly that of calculating the Jacobian. In this sense the numbers in the NLCG column correspond to the amount of forward solutions required for the calculation of the Jacobian matrix via the efficient integral formulation.

D	ζ_d	DoF	NLCG	AFM
20	50	5000	1020	40
20	500	5000	10020	40
20	5000	5000	100020	40
100	50	5000	1500	200

Table 1: Performance statistics for the adjoint fields algorithm compared to the nonlinear conjugate gradients algorithm. In the above D is the number of current patterns, ζ_d is the number of measurements per current pattern and DoF is the number of degrees of freedom in the inverse problem. In the NLCG column the number of forward solutions required for the nonlinear CG iteration are listed next to the relevant figures for the adjoint fields iteration under the AFM column.

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