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Abstract

The purpose of this article is to give a general introduction to the basic ideas and techniques from model theory.

I begin with some general remarks concerning model theory and its relationship with algebra. There follows a "mini-course" on first order languages, structures and basic ideas in model theory. Then there is a series of subsections which describe briefly some topics from model theory.

Contents

Bibliography

1 Model theory and algebra

There is a variety of ways in which people have described the relationship between model theory, algebra and logic. Certainly, model theory fits naturally between, and overlaps, algebra and logic. Model theory itself has a "pure" aspect, where we investigate structures and classes of structures which are delineated using notions from within model theory, and it has an "applied" aspect, where we investigate structures and classes of structures which arise from outside model theory.

The first aspect is exemplified by stability theory where we assume just that we are dealing with a class of structures (cut out by some axioms) in which there is a "notion of independence" satisfying certain reasonable conditions. The investigation of such classes and the development of structure theory within such classes was a major project of Shelah and others (see [47], [48], [23]). Out of it have grown other projects and directions, in particular, "geometrical stability theory" which has close links with algebraic geometry (see, e.g., [39]).

The second aspect is exemplified by the model theory of fields (or groups, or modules, or ...). Here the techniques used arise mostly from the specific area but there is some input of model-theoretic ideas, techniques and theorems. The input from model theory is typically not from the most highly developed "internal" parts of the area but one can be fairly sure that at least the Compactness Theorem 2.10 will figure as well as a certain perspective. The model-theoretic perspective, of course, leads one to ask questions which may not be algebraically "natural" but it may also lead to fresh ideas on existing algebraic questions.

For example, within the model theory of modules one may aim to classify the complete theories of modules over a given ring. Model-theoretically this is a natural project because of the central role played in model theory by elementary classes. Algebraically it does not seem very natural, even though it can be described in purely algebraic terms (by making use of the notion of ultraproduct). Nevertheless, this project did lead to unexpected discoveries and algebraic applications (see my Handbook of Algebra article on model theory and modules for example).

In its development model theory has looked very much towards algebra and other areas outside logic. It has often taken ideas from these areas, extracted their content within a framework provided by logic, developed them within that context and applied the results back to various areas of algebra (as well as parts of analysis and geometry see, e.g., [8], [32], [51]). An example of this process is provided by the concept of being "algebraic over" a set of elements (see the subsection on this below). The inspirational example here is the notion of an element of a field being algebraic (as opposed to transcendental) over a subfield. This leads to a general and fundamental model theoretic notion which applies in many different contexts.

That phase of the development of model theory provided most of the ideas mentioned in this article. For some snapshots of model theory as it is now, one may look at the various (especially survey) articles mentioned above, below and in the bibliography.

2 The Basics

Most, but not all, model theory uses first-order, finitary logic. In this article I confine myself to the first-order, finitary context. By a "formal language", or just "language", I will, from now, mean a first-order finitary language.

> "First order" means that the quantifiers range over elements of a given structure (but also see the subsection on many-sorted structures). In a second-order language we also have quantifiers which range over arbitrary subsets of structures (there are intermediate languages where there are restrictions on the subsets). A second-order language is, of course, much more expressive than a first-order one but we lose the compactness theorem. One of the points of model theory is that if a property can be expressed by a first-order formula or sentence then we know that it is preserved by certain constructions.
> "Finitary" means that the formulas of the ($L_{\omega\omega}$-) language all are finite strings of symbols. An $L_{\infty\omega}$-language is one where arbitrarily large conjunctions and disjunctions of formulas are formulas. In such a language one can express the property of a group being torsion by $\forall x \bigvee_{i\geq 1} x^i = e$. In an $L_{\infty\infty}$-language one also allows infinite strings of quantifiers.
> See [17], [24, Section 2.8], for these languages and for some algebraic applications of them.
> From the great variety of languages that have been considered by logicians, it is the first-order finitary ones which have proved to be most useful for applications in algebra.

The concept of a first order, finitary formula is rather basic but is often rather quickly passed over in accounts written for non-logicians. Certainly it is possible to do a, perhaps surprising, amount of model theory without mentioning formulas and so, when writing an article for algebraists for instance, one may wish to minimise mention of, or even entirely avoid, talking about formulas because one knows that this will be a stumbling block to many readers. Although this is possible and sometimes even desirable, this is not the course that I take here.

A **formula** (by which I will always mean a formula belonging to a formal language) is a string of symbols which can be produced in accordance with certain rules of formation. In general a formula will contain occurrences of variables $(x, y, ...)$. Some of these will be **bound** (or **within the scope of**) a quantifier. For example in $\forall x \exists y (x \neq y)$ this is true of the occurrence of x and that of y (the "x" in $\forall x$ is counted as part of the quantifier, not as an occurrence of x). Some occurrences may be **free**. For example in $\forall x(\exists y(x \neq y)) \wedge (x = z)$ the unique occurrence of z is free as is the second occurrence of x. The **free variables** of a formula are those which occur free somewhere in the formula. A formula without free variables is called a **sentence** and such a formula is either true or false in a given structure for the relevant language (for instance $\forall x \exists y(x \neq y)$ is true in a structure iff that structure has at least two distinct elements). We write $M \models \phi$ if the sentence ϕ is true in the structure M. We write $\phi(x_1, ..., x_n)$ to indicate that the set of free variables of the formula ϕ is contained in $\{x_1, ..., x_n\}$ (it is useful not to insist that each of x_1 to x_n actually occurs free in ϕ). Given a formula $\phi(x_1, ..., x_n)$ of a language L, given a structure M for that language and given elements $a_1, ..., a_n \in M$ we may replace every free occurrence of x_i in ϕ by a_i - the result we denote by $\phi(a_1, ..., a_n)$ - and then we obtain a **formula with parameters**, which is now a statement that is either true or false in M: we write $M \models \phi(a_1, ..., a_n)$ if it is true in M. For example if $\phi(x, z)$ is $\forall x(\exists y(x \neq y)) \wedge x = z$ and if $a, b \in M$ then $M \models \phi(a, b)$ iff M has at least two elements and if $a = b$.

> One cannot literally replace an occurrence of a variable by an element of a structure. Rather, one enriches the language by "adding names (new constant symbols) for elements of the structure" and then, using the same notation for an element and for its name, $\phi(a_1, ..., a_n)$ becomes literally a sentence of a somewhat larger language. See elsewhere (for example [10], [24]) for details. Also see those references for the precise definition of the satisfaction relation, $M \models \phi$ and $M \models \phi(a_1, ..., a_n)$, between structures and sentences/formulas with parameters. It is a natural inductive definition and one does not normally have to refer to it in order to understand the content of the relation in particular cases.

A formal language has certain basic ingredients or building blocks. Some of these, such as the symbol $\wedge$ which represents the operation of conjunction ("and"), are common to all languages: others are chosen according to the intended application. Then one has certain rules which delimit exactly the ways in which the formulas of the language may be built up from these ingredients.

The ingredients common to all languages are: an infinite stock of **variables** (or **indeterminates**); the **logical connectives**, $\wedge$ (**conjunction** "and"), $\vee$ (**disjunction** "or"), $\neg$ (**negation** "not"), $\rightarrow$ (**implication** "implies"), $\leftrightarrow$ (**bi-implication** "iff"); the **universal quantifier** $\forall$ ("for all") and the **existential quantifier** $\exists$ ("there exists"); a symbol, $=$, for equality. We also need to use parentheses, (and), to avoid ambiguity but there are conventions which reduce the number of these and hence aid readability of formulas. The language which is built up from just this collection of symbols we denote by L_0 and call the **basic language** (**with equality**). The formulas of this language are built up in a natural way, as follows.

The basic language We use letters such as x, y, u, v and indexed letters such as $x_1, x_2, ...$ for variables. We also abuse notation (in the next few lines and in general) by allowing these letters to range over the set of variables, so x for instance is a "generic" variable.

The definition of the formulas of the language is inductive. First we define the atomic formulas (the most basic formulas) and then we say how the stock of formulas may be enlarged by inductively combining formulas already constructed.

If x and y are any two variables then $x = y$ is an **atomic formula** (so $x = x, u = x, ...$ are atomic formulas). There are no more atomic formulas (for this language).

If ϕ and ψ are formulas then the following also are formulas: $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\neg\phi)$, $(\phi \to \psi)$, $(\phi \leftrightarrow \psi)$. Any formula constructed from the atomic formulas using only these operations is said to be **quantifier-free**.

Examples 2.1 *:* $(x = y \wedge x = u)$ *is a formula, so is* $(x = y \to u = v)$*, as is* $((x = y \wedge x = u) \vee (x = y \to u = v))$*, as is* $(\neg((x = y \wedge x = u) \vee (x = y \to u = v)))$*. In order to increase readability we write, for instance,* $(x \neq y)$ *rather than* $(\neg x = y)$*. We may also drop pairs of parentheses when doing so does not lead to any ambiguity in reading a formula. Some conventions allow the removal of further parentheses. Just as* $\times$ *has higher priority than* $+$ *(so* $2+3\times 4$ *equals* 14 *not* 20*) we assign* $\neg$ *higher priority than* $\wedge$ *and* $\vee$*, which, in turn, have higher priority than* $\to$ *and* $\leftrightarrow$*. The assignment of priorities to* $\forall x$ *and* $\exists x$ *is rather less consistent.*

If ϕ is a formula and x is any variable then $(\forall x\phi)$ and $(\exists x\phi)$ are formulas.

A **formula** is any string of symbols which is formed in accordance with these rules.

A further convention sometimes used is to write $\forall x(\phi)$ or just $\forall x\phi$ for $(\forall x\phi)$ and similarly for $\exists$ and even for a string of quantifiers.

Examples 2.2 *:* $(\forall x \exists y(x = y \vee x = z)) \vee (x \neq y \wedge \forall u\, u = z)$ *is a formula which, with all parentheses shown, would be* $((\forall x(\exists y(x = y \vee x = z))) \vee (x \neq y \wedge (\forall u\, u = z)))$*.*

We remark that only $\neg$, $\wedge$ and $\exists$ (say) are strictly necessary since one has, for instance, that $\phi \vee \psi$ is logically equivalent to $\neg(\neg\phi \wedge \neg\psi)$ and that $\forall x\phi$ is logically equivalent to $\neg\exists x\neg\phi$. This allows proofs which go by induction on complexity (of formation) of formulas to be shortened somewhat since fewer cases need be considered.

The optional extras from which we may select to build up a more general language, L, are the following: function symbols; relation symbols; constant symbols. Each function symbol and each relation symbol has a fixed **arity** (number of arguments). These optional symbols are sometimes referred to as the **signature** of the particular language.

Example 2.3 *Suppose that we want a language appropriate for groups. We could take the basic language L_0 and select, in addition, just one binary function symbol with which to express the multiplication in the group. In this case it would be natural to use operation, $x*y$, rather than function, $f(x,y)$, notation and that is what we do in practice. Since inverse and identity are determined once we add the group axioms we need select no more. For instance the group axiom which says that every element has a right inverse could be written $\forall x \exists y \forall z((x*y)*z = z \wedge z*(x*y) = z)$. But it would make for more easily readable formulas if we give ourselves a unary (= 1-ary) function symbol with which to express the function $x \mapsto x^{-1}$ and a constant symbol with which to "name" the identity element of the group. Again, we use the natural notation and so would have, among the axioms for a group written in this language, $\forall x(x * x^{-1} = e)$.*

For many purposes the choice of language is not an issue so long as the collection of definable sets (see the subsection on these) remains unchanged. But change of language does change the notion of substructure and it is also crucial for the question of quantifier-elimination. For instance, we may consider the p-adic field $\mathbb{Q}_p$ as a structure for the language of ordered fields supplemented by predicates, P_n for each integer $n \geq 2$, by interpreting $P_n(\mathbb{Q}_p)$ to be the set of elements of $\mathbb{Q}_p$ which are n-th powers. In this language every formula is equivalent, modulo the theory of this structure, to one without quantifiers [31]: (we say that $\mathbb{Q}_p$ has elimination of quantifiers in this language) but this is certainly not true of $\mathbb{Q}_p$ regarded as a structure just for the language of ordered fields (note that the property of being an n-th power in this latter language requires an existential quantifier for its expression).

Example 2.4 *Suppose that we want a language appropriate for ordered rings (such as the reals). We could take a minimal set consisting just of two binary function symbols, $+$ and $\times$, (for addition and multiplication) together with a binary relation symbol, $\leq$, for the order on the ring. But we could also have constant symbols, 0 and 1, a unary function symbol, $-$, for negative (and/or a symbol for subtraction), and binary relation symbols, $\geq$, $<$ and $>$, for the relations "associated" to $\leq$. If we are dealing with ordered fields note that we cannot introduce a unary function symbol for multiplicative inverse because that operation is not total. We could, however, introduce the symbol as an informal abbreviation in formulas since this partial operation is certainly definable by a formula of our language.*

The inductive part of the definition of formula for a general language is identical to that for L_0 and it is in the atomic formulas that the difference lies. First, we say that a **term** of the language L is any expression built up from the variables, the constant symbols (if there are any) and any already constructed terms by using the function symbols (if there are any).

Any variable is a term. Any constant symbol is a term. If f is an n-ary function symbol and if $t_1, ..., t_n$ are terms then $f(t_1, ..., t_n)$ is a term. For example, if L is a language for rings, with function and constant symbols $+, \times, -, 0, 1$, and if we use the axioms for rings to replace some terms by equivalent terms, then a term may be identified with a non-commutative polynomial, with integer coefficients, in the variables.

An **atomic formula** is any expression of the form $t_1 = t_2$ where t_1, t_2 are terms or of the form $R(t_1, ..., t_n)$ where R is an n-ary relation symbol and $t_1, ..., t_n$ are terms.

Example 2.5 *Examples of atomic formulas in a language for ordered rings are* $xy + z \leq x^2y - 1$ *and* $xy - yx = 0$.

One point which we have rather glossed over is: exactly what is a language? (That is, as a mathematical object, what is it?) The simplest answer is to regard a formal language simply as the set of all formulas of the language and that is what we shall do. In practice, however, one usually describes a language by giving the building blocks and the ways in which these may be combined.

Suppose that L is a language. An L-**structure** is a set M together with, for each optional (constant, function, relation) symbol of the language, a specific element of, function on, relation on the set (of course the functions and relations must have the correct arity). Just as we refer to "the group G" rather than "the group $(G, *)$" usually we refer to "the structure M". When we need to be more careful (for example if a certain set is being considered as the underlying set of two different structures for the same language or of structures for two different languages) then we may use appropriate notation, such as that just below.

An L-structure is $\mathbf{M} = (M; c^{\mathbf{M}}, ..., f^{\mathbf{M}}, ..., R^{\mathbf{M}}, ...)$ where: M is a set; for each constant symbol c of L, $c^{\mathbf{M}}$ is an element of M; for each function symbol f of L with arity, say, n, $f^{\mathbf{M}} : M^n \longrightarrow M$ is an n-ary function on M; for each relation symbol R of L with arity, say, n, $R^{\mathbf{M}} \subseteq M^n$ is an n-ary relation on M.

Suppose that L is a language. A property of L-structures is **elementary**, or **axiomatisable** or **first-order expressible**, if there is a set, T, of sentences of L such that an L-structure M has that property iff M satisfies (every sentence in) T.

Example 2.6 *The property of being abelian is an elementary property of groups since it is equivalent to satisfying the sentence* $\forall x \forall y (x * y = y * x)$.

The property of being torsion free is an elementary property of groups since it is equivalent to satisfying the set $\{\forall x (x^n = e \rightarrow x = e) : n \geq 1\}$ *of sentences. Here* x^n *is just an abbreviation for* $x * x * ... * x$ *(n "x"s). (Strictly speaking, we should say "The property of being an abelian group is an elementary property of L-structures, where L is a language appropriate for groups." but, having noted* *that being a group is an elementary property, it is harmless and natural to extend the terminology in such ways.)*

The property of being torsion is not an elementary property of groups. Note that the formal rendition, $\forall x \exists n(x^n = e)$, of this property is not a sentence of the language for groups since the quantifier $\exists n$ should range over the elements of the group, not over the integers. The fact that it is not an elementary property will be proved later (3.8).

Example 2.7 *Let L be a language for rings. The property of being a field of characteristic $p \neq 0$ is* **finitely axiomatisable** *(meaning that there is a finite number of sentences, equivalently a single sentence (their conjunction), which axiomatises it).*

The property of being (a field) of characteristic zero needs infinitely many sentences of the form $1+1+...+1 \neq 0$. It follows from the Compactness Theorem 2.10 that if σ is any sentence in L which is true in all fields of characteristic zero then there is an integer N such that all fields of characteristic greater than N satisfy σ.

Let T be a set of sentences of L (briefly, an L-**theory**). We say that an L-structure M is a **model of** T and write $M \models T$ if M satisfies every sentence of T. A class $\mathcal{C}$ of L-structures is **elementary**, or **axiomatisable**, if there is some L-theory T such that $\mathcal{C} = \mathrm{Mod}(T) = \{M : M \models T\}$. If $\mathcal{C}$ is any class of L-structures then the **theory of** $\mathcal{C}$, $\mathrm{Th}(\mathcal{C}) = \{\sigma \in L : M \models \sigma \, \forall M \in \mathcal{C}\}$, is the set of sentences of L satisfied by every member of $\mathcal{C}$. We write $\mathrm{Th}(M)$ for $\mathrm{Th}(\{M\})$ and call this the (**complete**) **theory of** M ("complete" since for every sentence σ either σ or $\neg\sigma$ is in $\mathrm{Th}(M)$).

Example 2.8 *We show that the class of algebraically closed fields is an elementary one (in the language for rings). A field K is algebraically closed iff every non-constant polynomial with coefficients in K in one indeterminate has a root in K. At first sight it may seem that there is a problem since we cannot refer to particular polynomials (the coefficients are elements of a field which are, apart from 0, 1 and other integers, not represented by constant symbols or even terms of the language) but then we note that it is enough to have general coefficients. So for each integer $n \geq 1$ let τ_n be the sentence $\forall u_0, ..., u_n(u_n \neq 0 \rightarrow \exists x(u_n x^n + ... + u_1 x + u_0 = 0))$. Let $S = \{\sigma_f\} \cup \{\tau_n : n \geq 1\}$ where σ_f is a sentence axiomatising the property of being a field. Then a structure K for the language of rings is an algebraically closed field iff $K \models S$.*

Theorem 2.9 *(***Completeness Theorem***) Let T be a set of sentences of L. Then $\mathrm{Th}(\mathrm{Mod}(T))$ is the deductive closure of T. In particular, if $\sigma \in \mathrm{Th}(\mathrm{Mod}(T))$ then there are finitely many sentences $\tau_1, ..., \tau_k \in T$ such that every L-structure M satisfies $\tau_1 \wedge ... \wedge \tau_k \rightarrow \sigma$.*

The term **deductive closure** comes from logic and refers to some notion of formal proof within a precisely defined system. The "completeness" in the name of the theorem refers to the fact that this deductive system is strong enough to capture all consequences of any set of axioms. One can set up such a system, independent of L, in various alternative ways and then the deductive closure of a set, T, of sentences consists of all sentences which can be generated from T using this system (see, for example, [18] for details). Since a formal proof is of finite length the second statement follows immediately and is one of many ways of expressing the Compactness Theorem. Here is another.

Theorem 2.10 *(***Compactness Theorem***) Let T be a set of sentences of L. If every finite subset of T has a model then T has a model.*

Another reformulation of the Completeness Theorem is that if a set of sentences is **consistent** (meaning that no contradiction can be formally derived from it) then it has a model.

The Compactness Theorem can be derived independently of the Completeness Theorem by using the ultraproduct construction and Łos' Theorem (and thus this cornerstone of model theory can be obtained without recourse to logic per se).

If two L-structures, M, N satisfy exactly the same sentences of L (that is if $\mathrm{Th}(M) = \mathrm{Th}(N)$) then we say that they are **elementarily equivalent**, and write $M \equiv N$. This is a much coarser relation than that of isomorphism (isomorphic structures are, indeed, elementarily equivalent) but it allows the task of classifying structures up to isomorphism to be split as classification up to elementary equivalence (sometimes much more tractable than classification to isomorphism) and then classification up to isomorphism within elementary equivalence classes (model theory is particularly suited to working within such classes and so, within such a context, one may be able to develop some structure theory). One may also be led to work within an elementary equivalence class even if one is investigating a specific structure M. For M shares many properties with the structures in its elementary equivalence class (which is, note, $\mathrm{Mod}(\mathrm{Th}(M))$) but some of these structures may have useful properties (existence of "non-standard" elements, many automorphisms, ...) that M does not. This will be pointless if M is finite since, in this case, any structure elementarily equivalent to M must be isomorphic to M. But otherwise the elementary equivalence class of M contains arbitrarily large structures.

Theorem 2.11 *(***Upwards Löwenheim-Skolem Theorem***) Let M be an infinite L-structure. Let κ be any cardinal greater than or equal to the cardinality of M and the cardinality of L. Then there is an L-structure, elementarily equivalent to M, of cardinality κ. Indeed M has an elementary extension of size κ.*

(The cardinality of L is equal to the larger of $\aleph_0$ and the number of optional symbols in L.) If M is a substructure of the L-structure N then we say that M is an **elementary substructure** of N (and N is an **elementary extension** of M), and we write $M \prec N$, if for every formula $\phi = \phi(x_1, ..., x_n)$ of L and every $a_1, ..., a_n \in M$ we have $M \models \phi(a_1, ..., a_n)$ iff $N \models \phi(a_1, ..., a_n)$. A way to see this definition is as follows. Let $\bar{a}$ be any finite tuple of elements of M. When regarded as a tuple of elements of N, as opposed to M, the L-expressible properties of $\bar{a}$ may change (those expressible by quantifier-free formulas will not but consider, e.g., the even integers as a subgroup of the group of all integers and look at divisibility by 2). If they do not, and if this is true for all tuples $\bar{a}$, then $M \prec N$.

The upwards Löwenheim-Skolem Theorem says that if M is an infinite structure then there are arbitrarily large structures with the same first-order properties as M. The next theorem goes in the other direction.

Theorem 2.12 *(***Downwards Löwenheim-Skolem Theorem***) Let M be an L-structure and let κ be an infinite cardinal with $\mathrm{card}(L) \leq \kappa \leq \mathrm{card}(M)$. Then M has an elementary substructure of cardinality κ. If $A \subseteq M$ and if $\mathrm{card}(A) \leq \kappa$ then there is an elementary substructure $N \prec M$ of cardinality κ and with $A \subseteq N$.*

Example 2.13 *Consider the reals $\mathbb{R}$ as a structure for the language, L, of ordered rings. Then $\mathbb{R}$ has a countable elementary substructure $\mathbb{R}'$. Among the properties that $\mathbb{R}'$ shares with $\mathbb{R}$ are the Intermediate Value Theorem for polynomials with coefficients in the structure and factorisability of polynomials into linear and quadratic terms, both these being expressible (at least indirectly) by sets of sentences of the language.*

So, for example, if L is a countable language and if T is any L-theory with an infinite model then T has a model of every infinite cardinality.

The second part of the above result can be used as follows. Start with a structure M. Produce an elementary extension M' of M which contains some element or set, B, of elements with some desired properties. Use the result (with $A = M \cup B$) to cut down to a "small" elementary extension of M containing the set B.

Example 2.14 *All algebraically closed fields of a given characteristic are elementarily equivalent. For, any two algebraically closed fields of the same characteristic and the same uncountable cardinality, say $\aleph_1$ for definiteness, are isomorphic (being copies of the algebraic closure of the rational function field in $\aleph_1$ indeterminates over the prime subfield). If K, L are algebraically closed then, by the Löwenheim-Sko lem theorems, there are fields K_1, L_1 elementarily equivalent to K, L respectively and of cardinality $\aleph_1$. So if K and L, and hence K_1 and L_1, have the same characteristic then $K_1 \simeq L_1$ and so $K \equiv K_1 \equiv L_1 \equiv L$ as claimed.*

A composition of elementary embeddings is elementary. Furthermore, one as the following.

Theorem 2.15 (**Elementary Chain Theorem**) *Suppose that $M_0 \prec M_1 \prec \ldots \prec M_i \prec \ldots$ are L-structures, each elementarily embedded in the next. Then the union carries an L-structure induced by the structures on the various M_i and, with this structure, it is an elementary extension of each M_i.*

More generally, the direct limit of a directed system of elementary embeddings is an elementary extension of each structure in the system.

Proposition 2.16 *(Criterion for elementary substructure) Let M be a substructure of the L-structure N. Then $M \prec N$ iff for every formula $\phi(x, \bar{y}) \in L$ and for every tuple $\bar{b}$ of elements of M (of the same length as $\bar{y}$) if $N \models \exists x \phi(x, \bar{b})$ then there is $a \in M$ such that $N \models \phi(a, \bar{b})$.*

This is proved by induction on the complexity (of formation) of formulas.

3 Topics

3.1 Applications of the Compactness Theorem

The Compactness Theorem pervades model theory, directly and through the proofs of numerous other theorems. A common direct use has the following form. We want to produce a structure with a specified set of properties. For some reasons we know that every finite subset of this set of properties can be realised in some structure. The Compactness Theorem guarantees the existence of a structure which satisfies all these properties simultaneously.

Example 3.1 *Suppose that T is the theory of (algebraically closed) fields and let σ be a sentence of the language of rings. Suppose that there is, for each of infinitely many distinct prime integers p, a (algebraically closed) field K_p which satisfies σ. Then there is a (algebraically closed) field of characteristic zero which satisfies σ. For consider the set $T \cup \{\sigma\} \cup \{1 + \ldots + 1 (n \text{ “1”s }) \neq 0 : n \geq 1\}$. By assumption, every finite subset of this set has a model (K_p for p large enough) and so, by compactness, this set has a model K, as required.*

Note the corollary: if σ is a sentence in the language of rings then there is an integer N such that for all primes $p > N$, and for $p = 0$, every algebraically closed field of characteristic p satisfies σ or else every such field satisfies $\neg\sigma$. For otherwise there would be infinitely many primes p such that there is an algebraically closed field of characteristic p satisfying σ and the same would be true for $\neg\sigma$. So there would, by the first paragraph, be an algebraically closed field of characteristic 0 which satisfies σ and also one which satisfies $\neg\sigma$, contradicting the fact, 2.14, that all algebraically closed fields of characteristic 0 are elementarily equivalent.

Another type of use has the following form (in fact it is just the kind of use already introduced but omitting explicit enrichment of the language with new constant symbols). In this case we want to produce an element, tuple or even an infinite set of elements with some specified properties. Again, we know for some reasons that these properties are finitely satisfiable. The Compactness Theorem says that they are simultaneously satisfiable. This type of use often takes place within the context of the models of a complete theory.

Corollary 3.2 *Let M be a structure, $B \subseteq M$, and let $n \geq 1$. Suppose that Φ is a set of formulas over B which is finitely satisfied in M. Then Φ is* **realised** *in an elementary extension of M. (That is, there is a tuple $\bar{c}$ of elements in some elementary extension of M such that $\bar{c}$ satisfies every formula in Φ.)*

Example 3.3 *Perhaps the best known example is the construction of infinitesimals ("construction" is not an accurate term: "pulled out of a hat" is closer to the truth). Consider the reals, $\mathbb{R}$, regarded as an ordered field (in a suitable language). Let $\Phi(x)$ be the set $\{x > 0\} \cup \{x + ... + x \leq 1(n$ "x"s$) : n \geq 1\}$ of formulas which, taken together, describe an element which is strictly greater than zero but less than $\frac{1}{n}$ for each $n \geq 1$. Of course, no element of $\mathbb{R}$ satisfies all these formulas but any finite subset of Φ does have a solution in $\mathbb{R}$. So, using the Compactness Theorem, there is an elementary extension $\mathbb{R}^*$ of $\mathbb{R}$ which contains a realisation of Φ: and such a realisation is an infinitesimal so far as the copy of $\mathbb{R}$ sitting inside $\mathbb{R}^*$ (as an elementary substructure) is concerned.*

Example 3.4 *(Bounds in polynomial rings) There is a host of questions concerning ideals in polynomial rings, of which the following is a basic example. Consider the polynomial ring $R = K[X_1, ..., X_t]$ where K is a field. Let $f, g_1, ..., g_n \in R$. If f belongs to the ideal $I = \langle g_1, ..., g_n \rangle$ then there are polynomials $h_1, ..., h_n \in R$ such that $f = h_1 g_1 + ... + h_n g_n$. There is no a priori bound on the (total) degrees of the h_i which might be needed but if f does belong to I then there are, in fact, such polynomials h_i with degree bounded above by a function which depends only on the degree of f and the degrees of the g_i. Similarly, if f belongs to the radical of I - that is, some power of f belongs to I - then the minimal such power can be bounded above by a function of the degree of f and the degrees of the g_i only. In many cases such bounds arise directly from explicit computation procedures but existence of such bounds often may be obtained, sometimes very easily, by use of the Compactness Theorem. See, e.g., [15] and references therein. Such methods are used and extended in [11] to obtain Lang-Weil-type estimates on the sizes of definable subsets of finite fields $\mathbb{F}_{p^n}$ as $n \to \infty$.*

Example 3.5 *(Polynomial maps) Suppose that V is an algebraic subvariety of $\mathbb{C}^n$ (that is, V is the set, $V_{\mathbb{C}}(g_1, ..., g_k)$, of common zeroes, in $\mathbb{C}^n$, of some set, $g_1, ..., g_k$, of polynomials in $\mathbb{C}[X_1, ..., X_n]$) and let $f : V \longrightarrow V$ be a polynomial map (that is $f(\bar{a}) = (f_1(\bar{a}), ..., f_n(\bar{a}))$ for some polynomials $f_1, ..., f_n$). Suppose that f is injective. Then f is onto [3]. This can be proved as follows.*

Notice that the assertion is true if we replace $\mathbb{C}$ by a finite field (simply because V is then a finite set). It follows that the assertion is true if $\mathbb{C}$ is replaced by the algebraic closure of any finite field (for this is a union of finite fields F and, if F is large enough to contain the coefficients of the polynomials g_i, f_j, then $V_F(g_1, ..., g_k)$ is closed under f).

Next observe that the assertion is expressible by a sentence in the language of fields. Of course the polynomials $g_1, ..., g_k$ have to be replaced by polynomials with general coefficients as do $f_1, ..., f_n$ (as in the argument that "algebraically closed" is axiomatisable) and the argument must be applied to each member of an infinite set of sentences (since any single sentence refers to polynomials of bounded total degree) but, having noted this, we may easily express the conditions "$\bar{x} \in V(g_1, ..., g_k)$" (meaning $\bar{x} \in V_K(g_1, ..., g_k)$, where now K can be any field), "f is surjective" and "f is injective".

The fact that I have not written down the relevant sentences is rather typical in model theory since, with some experience, it can be clear that certain conditions are expressible by sentences (which may be rather indigestible if actually written down) of the relevant language (it is also often clear "by compactness" that certain properties are not so expressible).

Thus we have our sentence, σ, true in each field $\widetilde{\mathbb{F}_p}$ which is the algebraic closure of the finite field $\mathbb{F}_p$. By 3.1 it follows that there is an algebraically closed field of characteristic 0 which satisfies σ. Finally we use the fact, proved earlier 2.14, that all algebraically closed fields of a given characteristic are elementarily equivalent and hence we deduce that $\mathbb{C}$ satisfies σ and, therefore, satisfies the original assertion (indeed, this argument shows that every algebraically closed field satisfies it).

3.2 Morphisms and the method of diagrams

A **morphism** $\alpha : \mathbf{M} \longrightarrow \mathbf{N}$ between L-structures is, of course, just a structure-preserving map.

> Precisely, we require that: for each constant symbol c of L we have $\alpha(c^{\mathbf{M}}) = c^{\mathbf{N}}$; for each n-ary function symbol f of L and n-tuple $\bar{a}$ from M we have $\alpha(f^{\mathbf{M}}(\bar{a})) = f^{\mathbf{N}}(\alpha\bar{a})$; for each n-ary relation symbol R of L and n-tuple $\bar{a}$ from M we have $R^{\mathbf{M}}(\bar{a})$ implies $R^{\mathbf{N}}(\alpha\bar{a})$ (where $\alpha\bar{a}$ denotes $(\alpha a_1, ..., \alpha a_n)$ if $\bar{a} = (a_1, ..., a_n)$).
> If the language contains relation symbols then a bijective morphism need not be an isomorphism (exercise: give a counterexample in posets) and for **isomorphism** one needs the stronger condition "$R^{\mathbf{M}}(\bar{a})$ iff $R^{\mathbf{N}}(\alpha\bar{a})$".

A **substructure**, N, of an L-structure $\mathbf{M}$ is given by a subset N of M which contains all the interpretations, $c^{\mathbf{M}}$, of constant symbols in M and which is closed under all the functions, $f^{\mathbf{M}}$, on M. Then we make it an L-structure by setting $c^{\mathbf{N}} = c^{\mathbf{M}}$, $f^{\mathbf{N}} = f^{\mathbf{M}} \upharpoonright N^n$ and $R^{\mathbf{N}} = R^{\mathbf{M}} \cap N^n$ for each constant symbol,

c, n-ary function symbol, f, and n-ary relation symbol R of L. More generally we have the notion of an **embedding** of L-structures: a monic morphism which satisfies the additional condition "$R^{\mathbf{M}}(\bar{a})$ iff $R^{\mathbf{N}}(\alpha\bar{a})$" seen in the definition of isomorphism above.

The "method of diagrams" is a means of producing morphisms between L-structures. Suppose that M is an L-structure. Enrich the language L by adding a new constant symbol $[a]$ for every element $a \in M$, thus obtaining the language denoted L_M. Of course M has a natural enrichment to an L_M-structure, given by interpreting $[a]$ as a for each $a \in M$. The **atomic diagram** of M is the collection of all L_M-sentences of the form $t_1 = t_2$, $t_1 \neq t_2$, $R(t_1, ..., t_n)$, $\neg R(t_1, ..., t_n)$ satisfied by M, where the t_i are terms of L_M and R is any (n-ary) relation symbol of L. This is the collection of all basic positive and negative relations (in the informal sense) which hold between the elements of M. The **positive atomic diagram** of M is the subset of the atomic diagram containing just the positive atomic sentences (i.e. those of the forms $t_1 = t_2$ and $R(t_1, ..., t_n)$). For instance, if M is a ring then the positive atomic diagram is (equivalent to) the multiplication and addition tables of M and the atomic diagram further contains a record of all polynomial combinations of elements of M which are non-zero. The **full diagram** of M is the L_M-theory of M (i.e. all sentences of L_M satisfied by M).

Let D be any of the above "diagrams" and let $T' = \mathrm{Th}(M) \cup D$ - a set of L_M-sentences (by $\mathrm{Th}(M)$ I mean the theory of the L-structure M, not the enriched L_M-structure). If N' is any model of T' then first note that the reduction of the L_M-structure N' to an L-structure N (i.e. forget which elements $[a]^{N'}$ interpret the extra constant symbols $[a]$) satisfies $\mathrm{Th}(M)$ and hence $N \equiv M$. Second, we have a natural map, α, from M to N ($= N'$ as a set) given by taking $a \in M$ to the interpretation, $[a]^{N'}$, of the corresponding constant symbol, $[a]$, in N'. Because N' satisfies at least the positive atomic diagram of M it is immediate that α is a morphism of L-structures.

Theorem 3.6 *Let M be an L-structure and let D be any of the above diagrams. Let N' be a model of $\mathrm{Th}(M) \cup D$ and let N be the reduct of N' to L. Then N is elementarily equivalent to M. Furthermore, if $\alpha : M \longrightarrow N$ is the map defined above then: (a) if D is the positive atomic diagram of M then α is a morphism; (b) if D is the atomic diagram of M then α is an embedding; (c) if D is the full diagram of M then α is an elementary embedding.*

There are variants of this. For instance, if D is the atomic diagram of M and if we take N' to be a model of just this (so drop $\mathrm{Th}(M)$) then we have that α is an embedding to the L-structure N (which need not be elementarily equivalent to M). Or we can add names just for elements of some substructure A of M and then we obtain a morphism α from that substructure to N.

3.3 Types and non-standard elements

Suppose that M is an L-structure. Let $\bar{a} = (a_1, ..., a_n)$ be a tuple of elements of M. The **type** of $\bar{a}$ in M is the set of all L-formulas satisfied by $\bar{a}$ in M. Thus it is the "L-description" of this tuple (or, better, of how this tuple sits in M). If $f \in \mathrm{Aut}(M)$ is an automorphism then the type of $f\bar{a} = (fa_1, ..., fa_n)$ is equal to the type of $\bar{a}$ since, for any formula ϕ, we have $M \models \phi(\bar{a})$ iff $M \models \phi(f\bar{a})$ because f is an isomorphism.

More generally, we may want a description of how $\bar{a}$ sits in M with respect to some fixed set, $B \subseteq M$, of parameters. So we define the **type** of $\bar{a}$ in M **over** B to be $\mathrm{tp}^M(\bar{a}/B) = \{\phi(\bar{x}, \bar{b}) : \phi \in L, \bar{b} \text{ in } B\}$ - the set of all formulas (in some fixed tuple, $\bar{x}$, of free variables matching $\bar{a}$) with parameters from B satisfied by $\bar{a}$ in M. We write $\mathrm{tp}(\bar{a})$ for $\mathrm{tp}(\bar{a}/\emptyset)$. It is immediate that if N is an elementary extension of M then $\mathrm{tp}^N(\bar{a}/B) = \mathrm{tp}^M(\bar{a}/B)$ and so we often drop the superscript.

Note that $\mathrm{tp}(\bar{a}/B)$ has the following properties: it is a set of formulas with parameters from B in a fixed sequence of n free variables; it is closed under conjunction and under implication; it is consistent (it does not contain any contradiction such as $x_1 \neq x_1$); it is maximal such (it is "complete", equivalently, for every formula $\phi(\bar{x}, \bar{b})$ with parameters from B either this formula or its negation is contained in $\mathrm{tp}(\bar{a}/B)$). Any set of formulas satisfying these conditions is called an **n-type over** B.

Theorem 3.7 *Let M be a structure, $B \subseteq M$, and let $n \geq 1$. Suppose that p is an n-type over B. Then p is* **realised** *in an elementary extension of M. That is, there is $N \succ M$ and $c_1, ..., c_n \in N$ such that $\mathrm{tp}(\bar{c}/B) = p$. (Then $\bar{c}$ is said to be a* **realisation** *of p.)*

We say "every type is realised in an elementary extension". If Φ is any **consistent** set of formulas (that is, the closure of Φ under conjunction and implication contains no contradiction) in $\bar{x}$ with parameters from B then, by Zorn's Lemma, Φ is contained in a maximal consistent such set, that is, in a type over B (sometimes one says that the **partial type** Φ is contained in a **complete type**). This gives 3.2 above.

All the above goes equally for infinite tuples. In this way one can use compactness to produce not just elements, but structures, as in the subsection on the Method of Diagrams, for instance.

Example 3.8 *We claimed earlier that the property of being a torsion group is not elementary: now we justify that claim. Let G be a torsion group for which there is no bound on the order of its elements (for example let G be the direct sum of the finite groups $\mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$). Let $\Phi = \Phi(x) = \{x^n \neq e : n \geq 1\}$ where e denotes the identity element of G and where x^n is an abbreviation for the term which is a product of n x's (a slightly dangerous abbreviation since the whole point is that we cannot refer to general integers n in our formulas!).*

Then Φ is a partial type (in G) since any given finite subset of Φ is realised by an element of G which has high enough order. Therefore Φ is realised by some element, c say, in some elementary extension, G' say, of G. In particular the group G' is elementarily equivalent to G but it is not a torsion group (since $\Phi(c)$ says that c has infinite order). Thus the property of being torsion is not an elementary one.

3.4 Algebraic elements

Suppose that M is a structure, that $a \in M$ and that $B \subseteq M$. We say that a is **algebraic over** B if there is a formula $\phi(x, \bar{b})$ with parameters $\bar{b}$ from B such that a satisfies this formula, we write $B \models \phi(a, \bar{b})$, and such that the solution set, $\phi(M, \bar{b}) = \{c \in B : B \models \phi(c, \bar{b})\}$, of this formula in M is finite. In this case, if M' is an elementary extension of M then the solution sets $\phi(M', \bar{b})$ and $\phi(M, \bar{b})$ are equal (exercise - use that M satisfies a sentence which gives the size of the solution set of $\phi(x, \bar{b})$) so the relation of being algebraic over a set is unchanged by moving to an elementary extension.

If $M \models \phi(a, \bar{b})$ and f is an automorphism of M which fixes B pointwise then $M \models \phi(fa, \bar{b})$. Hence if a is algebraic over B then a has only finitely many conjugates under the action of $\mathrm{Aut}_B M$, by which we denote the group of automorphisms of M which fix B pointwise. If M is sufficiently saturated (see the subsection on saturated structures), though not for general structures M, the converse is true.

A tighter relation is that of being **definable over** a set B: this is as "algebraic over" but with the stronger requirement that the element or tuple is the *unique* solution of some formula with parameters from B (equivalently is fixed by all elements of $\mathrm{Aut}_B(M)$ in a sufficiently saturated model M).

Example 3.9 *If M is a vector space over a field then a is algebraic over B iff a is in the linear span of B iff a is definable over B. If M is an algebraically closed field then a is algebraic (in the model-theoretic sense) over B iff a is algebraic (in the usual sense) over B. Any element in the subfield, $\langle B \rangle$, generated by B is definable over B but if the characteristic of the field is $p > 0$ then one also has that any p^n-th root of any element of $\langle B \rangle$ is definable over B.*

3.5 Isolated types and omitting types

If we fix an integer $n \geq 1$ and a subset B of an L-structure M then the set, $S_n(B)$, of all n-types over B carries a natural topology which has, for a basis of clopen sets, the sets of the form $\mathcal{O}_\phi(\bar{x}, \bar{b}) = \{p \in S_n(B) : \phi(\bar{x}, \bar{b}) \in p\}$. We also denote this set $S_n^T(B)$ where $T = \mathrm{Th}(M)$ to emphasise that the notion of "type" makes sense only relative to a complete theory. This is in fact the Stone space (the space of all ultrafilters) of the boolean algebra of equivalence classes of formulas with free variables $\bar{x}$ (formulas are equivalent if they define the same subset of M and the ordering is implication). This space is compact (by the Compactness Theorem).

A type p is **isolated** or **principal** if there is some formula ϕ in p which proves every formula in p: for every $\psi \in p$ we have $M \models \forall \bar{x}(\phi(\bar{x}) \rightarrow \psi(\bar{x}))$. In this case $\mathcal{O}_\phi = \{p\}$ and p is an isolated point of the relevant Stone space. Such a type must be realised in every model: for, by consistency of p, we have $M \models \exists \bar{x}\phi(\bar{x})$, say $M \models \phi(\bar{c})$, and then, since ϕ generates p, we have $M \models \psi(\bar{c})$ for every $\psi \in p$, that is $M \models p(\bar{c})$ and $\bar{c}$ realises p, as required. For countable languages there is a converse.

Theorem 3.10 *(***Omitting Types Theorem***) Let L be a countable language, let M be an L-structure and let $p \in S_n(\emptyset) = S_n^{\mathrm{Th}(M)}(\emptyset)$. If p is a non-isolated type then there is an L-structure N elementarily equivalent to M which* **omits** *p (i.e. which does not realise p).*

This is extended to cover the case of types over a subset B by enriching L by adding a name for each element of B and then applying the above result (assuming, of course, that B is countable so that the enriched language is countable). There are extensions of this result which allow sets of types to be omitted simultaneously [10, 2.2.15, 2.2.19], [24, 7.2.1]. The result is not true without the countability assumption [10, after 2.2.18].

3.6 Categoricity and the number of models

A theory is **$\aleph_0$-categorical** if it has just one countably infinite model up to isomorphism. More generally it is **κ-categorical** where κ is an infinite cardinal if it has, up to isomorphism, just one model of cardinality κ. If an L-theory is κ-categorical for any cardinal $\kappa \geq \mathrm{card}(L)$ then, by the Löwenheim-Skolem Theorems, it must be complete.

Example 3.11 *Any two atomless boolean algebras (equivalently, boolean rings with zero socle) are elementarily equivalent since there is, up to isomorphism, just one such structure of cardinality $\aleph_0$.*

Theorem 3.12 *(***Morley's Theorem***) Suppose that L is a countable language and that T is a complete L-theory which is κ-categorical for some uncountable cardinal κ. Then T is λ-categorical for every uncountable cardinal λ.*

The situation for $\kappa = \aleph_0$ is different. For instance, the theory of algebraically closed fields of characteristic zero is uncountably categorical but not $\aleph_0$-categorical. Indeed, examples show that $\aleph_0$-categoricity, uncountable categoricity and their negations can occur in all four combinations.

Theorem 3.13 *Suppose that L is a countable language and that T is a complete L-theory which has an infinite model. Then the following are equivalent.*

(i) T is $\aleph_0$-categorical;
(ii) for each $n \geq 1$ the Stone space $S_n(\emptyset)$ is finite;
(iii) each type in each Stone space $S_n(\emptyset)$ $(n \geq 1)$ is isolated;
(iv) for each countable model M of T and for each integer $n \geq 1$ there are just finitely many orbits of the action of $\mathrm{Aut}(M)$ on n-tuples;
(v) for each finite tuple, $\bar{x}$, of variables there are just finitely many formulas with free variables $\bar{x}$ up to equivalence modulo T.

For instance, if some space $S_n(\emptyset)$ is infinite then, by compactness, it contains a non-isolated type p. Then there will be a countably infinite model which realises p but also a countable infinite model which omits p and so the theory cannot be $\aleph_0$-categorical.

Example 3.14 *Any $\aleph_0$-categorical structure is* **locally finite** *in the sense that every finitely generated substructure is finite. For, let M be $\aleph_0$-categorical, let $\bar{a}$ be a finite sequence of elements from M and let b, c be distinct elements of the substructure, $\langle \bar{a} \rangle$, generated by $\bar{a}$. Then $\mathrm{tp}(b, \bar{a}) \neq \mathrm{tp}(c, \bar{a})$ since the type "contains the expression of b (respectively c) in terms of $\bar{a}$". Therefore $\mathrm{tp}(b, \bar{a}) \neq \mathrm{tp}(c, \bar{a})$ are distinct $(1 + n)$-types, where n is the length of the tuple $\bar{a}$. But $S_{1+n}(\emptyset)$ is finite and, therefore, $\langle \bar{a} \rangle$ is finite. Indeed the argument shows that M is* **uniformly locally finite** *since if $s_n = \mid S_n(\emptyset) \mid$ then any substructure of M generated by n elements has cardinality bounded above by s_{n+1}.*

A more general question in model theory is: given a complete theory T and an infinite cardinal κ what is the number, $n(\kappa, T)$, up to isomorphism, of models of T of cardinality κ? Remarkably complete results on this question and on the connected question of the existence or otherwise of structure theorems for models of T have been obtained by Shelah and others (see [48]).

3.7 Prime and atomic models

Suppose that M is an L-structure. Set $T = \mathrm{Th}(M)$. The elementary equivalence class, $\mathrm{Mod}(T)$, of M, equipped with the elementary embeddings between members of the class, forms a category (not very algebraically interesting unless, say, T has elimination of quantifiers, since there are rather few morphisms). A model of T is a **prime** model if it embeds elementarily into every model of T and is **atomic** if every type realised in it is isolated (hence is a type which must be realised in every model).

Theorem 3.15 *Let T be a complete theory in the countable language L and suppose that T has an infinite model. Then the L-structure M is a prime model of T iff M is a countable atomic model of T. Such a model of T exists iff, for every integer $n \geq 1$, the set of isolated points in $S_n(\emptyset)$ is dense in $S_n(\emptyset)$ (in particular this will be so if each $S_n(\emptyset)$ is countable). If T has a prime model then this model is unique to isomorphism.*

The proof of the second statement involves a back-and-forth construction: we describe this construction in the next section.

3.8 Back-and-forth constructions

This is a method for producing morphisms between structures. Suppose that M is a countably infinite L-structure and that N is an L-structure. Enumerate the elements of M in a sequence $a_0, a_1, ..., a_i, ...$ indexed by the natural numbers. If we are to embed M in N then we need to find an element, b_0, of N such that the isomorphism type of the substructure, $\langle b_0 \rangle$, of N generated by b_0 is isomorphic to the substructure, $\langle a_0 \rangle$, of M generated by a_0. Supposing that there is such an element, fix it. So now we have a "partial embedding" from M to N (the map with domain $\{a_0\}$ (or, if one prefers, $\langle a_0 \rangle$) sending a_0 to b_0). Now we need an element b_1 of N to which to map a_1. It is necessary that the substructures $\langle b_0, b_1 \rangle$ and $\langle a_0, a_1 \rangle$ be isomorphic by the map sending a_0 (resp. a_1) to b_0 (resp. b_1). If there is such an element, b_1 say, fix it. Continue in this way. At the typical stage we have images $b_0, ..., b_n$ for $a_0, ..., a_n$ and we need to find an element b_{n+1} of N which "looks the same over $b_0, ..., b_n$ as a_{n+1} does over $a_0, ..., a_n$". In the limit we obtain an embedding of M into N.

This is the shape of a "forth" construction. Of course, the key ingredient is missing: how can we be sure that the elements b_i of the sort we want exist? And, of course, that must somehow flow from the hypotheses that surround any particular application of this construction.

We may want a stronger conclusion: that the constructed embedding of M into N be an elementary embedding. In that case the requirement that $\langle a_0, ..., a_{n+1} \rangle$ be isomorphic (via $a_i \mapsto b_i$) to $\langle b_0, ..., b_n \rangle$ must be replaced by the stronger requirement $\mathrm{tp}^M(a_0, ..., a_{n+1}) = \mathrm{tp}^N(b_0, ..., b_{n+1})$ (note that, in this case we must assume $M \equiv N$).

Example 3.16 *The random graph is formed (with probability 1) from a countably infinite set of vertices by joining each pair of vertices by an edge with probability $\frac{1}{2}$. It is characterised as the unique countable graph such that, given any finite, disjoint, sets, U, V, of vertices, there is a point not in $U \cup V$ which is joined to each vertex of U and to no vertex of V. An easy "forth" argument, using this characterising property, shows that every countable graph embeds in the random graph (and the corresponding back-and-forth argument shows the uniqueness of this graph up to isomorphism).*

For a back-and-forth construction, we suppose that both M and N are countably infinite and we want to produce an isomorphism from M to N. For this we interlace the forth construction with the same construction going in the other direction. That is, on, say, even-numbered steps, we work on constructing the map α and on odd-numbered steps we work on constructing its inverse α^{-1} (in order to ensure that the resulting map α is surjective).

Example 3.17 *Let $\mathbb{Q}$ denote the rationals regarded as a partially ordered set: as such it is an example of a densely linearly ordered set without endpoints. Finitely many axioms in a language which has just one binary relation symbol suffice to axiomatise this notion. Let T_{dlo} denote the theory of densely linearly ordered sets without endpoints. Then any countable (necessarily infinite) model of T_{dlo} is isomorphic to $\mathbb{Q}$. This is a straightforward back-and-forth argument. It follows that T_{dlo} is a complete theory: all densely linearly ordered sets without endpoints are elementarily equivalent. For let $M \models T_{\text{dlo}}$. By the Downwards Löwenheim-Skolem Theorem there is a countable M_0 elementarily equivalent to M. But then $M_0 \simeq \mathbb{Q}$ and so $M \equiv M_0 \equiv \mathbb{Q}$, which is enough.*

3.9 Saturated structures

A saturated structure is one which realises all types of a particular sort: a "fat" structure as opposed to the "thin" atomic structures which realise only those types which must be realised in every model. For instance, an L-structure M is **weakly saturated** if it realises every type (in every Stone space $S_n(\emptyset)$) over the empty set. An L-structure M is **κ-saturated**, where $\kappa \geq \aleph_0$ is a cardinal, if for every subset $A \subseteq M$ of cardinality strictly less than κ and every n (it is enough to ask this for $n = 1$) every type in $S_n(A)$ is realised in M. Such structures always exist by a (possibly transfinite) process of realising types in larger and larger models and they provide a context in which "every consistent situation (of a certain "size") can be found". More precisely we have the following (which is proved by a "forth" construction).

Theorem 3.18 *(saturated implies universal) Let M be a κ-saturated L-structure. Then every model of the theory of M of cardinality strictly less than κ elementarily embeds in M.*

An L-structure M is **saturated** if it is $\mathrm{card}(M)$-saturated.

Theorem 3.19 *(saturated implies homogeneous) Suppose that M is a saturated structure and that $\bar{a}, \bar{b}$ are matching, possibly infinite, sequences of elements of M of cardinality strictly less than $\mathrm{card}(M)$ and with $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^M(\bar{b})$. Then there is an automorphism, α, of M with $\alpha(\bar{a}) = \bar{b}$.*

Thus, in a saturated structure, types correspond to orbits of the automorphism group of M.

Corollary 3.20 *Let M be a saturated structure of cardinality κ and suppose that the theory of M has complete elimination of quantifiers (for this see later). Suppose that A, B are substructures of M each generated by strictly fewer than κ elements and suppose that $\beta : A \longrightarrow B$ is an isomorphism. Then β extends to an automorphism of M.*

Often in model theory it is convenient to work inside a ("monster") model which embeds all "small" models and which is (somewhat) homogeneous in the above sense. With some assumptions on $\mathrm{Th}(M)$ one knows that there are saturated models of $\mathrm{Th}(M)$ of arbitrarily large cardinality but, for an arbitrary theory, unless some additional (to ZFC) set-theoretic assumptions are made, such models might not exist. One does have, however, without any additional set-theoretic assumptions, arbitrarily large elementary extensions which are, for most purposes, sufficiently saturated (see [24, Section 10.2]) to serve as a "universal domain" within which to work.

Related ideas can make sense in contexts other than the category of models of a complete theory, indeed, even in non-elementary classes.

Example 3.21 *(Universal locally finite groups) A group G is* **locally finite** *if every finitely generated subgroup of G is finite. A group G is a* **universal locally finite** *group if G is locally finite, if every finite group embeds into G and if, whenever G_1, G_2 are finite subgroups of G and $f : G_1 \longrightarrow G_2$ is an isomorphism, then there is an inner automorphism of G which extends f.*

For each infinite cardinal κ there exists a universal locally finite group of cardinality κ. Up to isomorphism there is just one countable locally finite group (an easy back-and-forth argument). This structure is not, however, $\aleph_0$-categorical, since it is not uniformly locally finite, nor is it $\aleph_0$-categorical (it has elements of unbounded finite order but no element of infinite order). If κ is uncountable then, [33], there are many non-isomorphic locally finite groups of cardinality κ. Each locally finite group of cardinality κ can be embedded in a universal locally finite group of cardinality λ for each $\lambda \geq \kappa$.

Example 3.22 *(Saturated structures have injectivity-type properties) Suppose that M, N are L-structures for some language L. Suppose that N_0 is a substructure of N and that $f : N_0 \longrightarrow M$ is a morphism. Suppose that for every finite tuple $\bar{a}$ from N there is an extension of f to a morphism from the substructure, $\langle N_0, \bar{a}\rangle$, of N generated by N_0 together with $\bar{a}$, to M. Let M' be a sufficiently saturated elementary extension of M (precisely, M' should be $(\mid N \mid + \mid L \mid)^+$-saturated). Then there is an extension of f to a morphism from N to M'.*

To see this, enumerate N as $\{c_\alpha\}_{\alpha\in I} \cup \{d_\beta\}_{\beta\in J}$ with $\{c_\alpha\}_{\alpha\in I} = N_0$. Let $\Phi = \Phi(\{x_\beta\}_{\beta\in J}) = \{\phi(f(c_{\alpha_1}), ..., f(c_{\alpha_m}), x_{\beta_1}, ..., x_{\beta_n}) : \phi$ is atomic and $N \models \phi(c_{\alpha_1}, ..., c_{\alpha_m}, d_{\beta_1}, ..., d_{\beta_n})\}$. By assumption Φ is a partial type (any finite subset of Φ mentions only finitely many elements outside N_0 and then a morphism extending f and with domain including these elements provides us with a realisation of this finite subset since, as is easily seen, morphisms preserve the truth of atomic formulas). Since M' is sufficiently saturated it realises Φ, say $M' \models \Phi(\{b_\beta\}_{\beta\in J})$, and we extend f by mapping d_β to b_β for $\beta \in J$.

3.10 Ultraproducts

The ultraproduct construction has been extensively used in applications of model theory to algebra. In many, though by no means all, cases an appeal to the existence of suitably saturated extensions would serve equally well but the ultraproduct construction does have the advantage of being a purely algebraic one (although perhaps "construction" is not really the right word since it inevitably appeals to Zorn's lemma at the point where a filter is extended to an ultrafilter).

We start with a set, $\{M_i : i \in I\}$, of L-structures, M_i, indexed by a set I. The ultraproduct construction produces a kind of "average" of the M_i. Let D be an ultrafilter on I.

> A **filter** on a set I is a filter, D, in the boolean algebra, $\mathcal{P}(I)$, of all subsets of I. That is: $\emptyset \notin D$; $I \in D$; if $X, Y \in D$ then $X \cap Y \in D$; if $X \subseteq Y \subseteq I$ and $X \in D$ then $Y \in D$. An **ultrafilter** is a maximal filter and is characterised by the further condition: if $X \subseteq I$ then either $X \in D$ or $X^c \in D$. An ultrafilter, D, on I is **principal** if there is $i_0 \in I$ such that $D = \{X \subseteq I : i_0 \in X\}$. Any ultrafilter which contains a finite set must be principal. An example of a filter, sometimes called the **Fréchet filter**, on the infinite set I is the set of all cofinite sets ($X \subseteq I$ is **cofinite** if X^c is finite). Any filter can be extended to an ultrafilter: but, unless the ultrafilter is principal, there is no explicit way to describe its members (the existence of non-principal ultrafilters is just slightly weaker than the Axiom of Choice). Any ultrafilter containing the Fréchet filter is non-principal.

The **ultraproduct**, $\prod_{i\in I} M_i/D$, of the M_i $(i \in I)$ with respect to the ultrafilter D on I is, as a set, the product $\prod_{i\in I} M_i$ factored by the equivalence relation $\sim = \sim_D$ given by $(a_i)_{i\in I} \sim (b_i)_{i\in I}$ iff $a_i = b_i$ "D-almost-everywhere", that is, if $\{i \in I : a_i = b_i\} \in D$. Then the set $\prod_{i\in I} M_i/D$ is made into an L-structure by defining the constants, functions and relations pointwise almost everywhere (the defining properties of a filter give that this is well-defined).

Example 3.23 *Suppose that the M_i are groups. Then $\prod_i M_i/D$ is the product group factored by the normal subgroup consisting of all tuples $(a_i)_i$ which are equal to the identity on D-almost-all coordinates: $\prod_i M_i/D \equiv (\prod_i M_i)/\{(a_i)_i : \{i \in I : a_i = e_i\} \in D\}$ (where e_i denotes the identity element of M_i).*

Example 3.24 *Suppose that the M_i are fields. Let $a = (a_i)_i/\sim \in \prod_i M_i/D$ be a non-zero element of the ultraproduct. Define the element $b = (b_i)_i/\sim$ by setting $b_i = a_i^{-1}$ for each i with $a_i \neq 0$ and setting b_i to be, say, 0 on all other coordinates. Since $J = \{i \in I : a_i \neq 0\} \in D$ we have $\{i \in I : a_ib_i = 1\} \in D$ and hence $ab = 1$. That is, $\prod_i M_i/D$ is a field.*

If $D = \{X \subseteq I : i_0 \in X\}$ is a principal ultrafilter then the ultraproduct $\prod_i M_i/D$ is isomorphic to M_{i_0} so the interesting case is when D is non-principal.

Theorem 3.25 *(**Łos' Theorem**) Let M_i $(i \in I)$ be a set of L-structures and let D be an ultrafilter on I. Set $M^* = \prod_{i\in I} M_i/D$ to be the ultraproduct. If σ is a sentence of L then $M^* \models \sigma$ iff $\{i \in I : M_i \models \sigma\} \in D$ (that is, iff "D-almost all" coordinate structures satisfy σ).*

More generally, if $\phi(x_1, ..., x_n)$ is a formula and if $a^1, ..., a^n \in M^*$ with $a^j = (a_i^j)_i/\sim$ $(j = 1, ..., n)$ then $M^* \models \phi(a^1, ..., a^n)$ iff $\{i \in I : M_i \models \phi(a_i^1, ..., a_i^n)\} \in D$.

Example 3.26 *Let $\mathcal{P}$ be an infinite set of non-zero prime integers and let K_p be a finite field of characteristic p for $p \in \mathcal{P}$. Let D be a non-principal ultrafilter on $\mathcal{P}$ and let K be the corresponding ultraproduct $\prod_p K_p/D$. Then K has characteristic zero and is an infinite model of the theory of finite fields (for such* **pseudofinite** *fields see [2]). For example it has, for each integer $n \geq 1$, just one field extension of degree n since this is true, and can be expressed (with some work) in a first-order way of finite fields.*

Example 3.27 *(Ultraproduct proof of the compactness theorem) Let T be a set of sentences, each finite subset of which has a model. Let X be the set of all finite subsets of T. For each $S \in X$ choose a model M_S of S. Given any $\sigma \in T$ let $\langle\sigma\rangle = \{S \in X : \sigma \in S\}$. Note that the intersection of any finitely many of these sets is non-empty and so $F = \{S \in X : \langle\sigma\rangle \subseteq S$ for some $\sigma \in T\}$ is a filter. Let D be any ultrafilter containing F. Then $\prod_{S\in X} M_S/D$ is a model of T. For let $\sigma \in T$. Then $\langle\sigma\rangle \in D$ and, since $\sigma \in S$ implies $M_S \models \sigma$, we have $\{S : M_S \models \sigma\} \in D$ and so $\prod_{S\in X} M_S/D \models \sigma$ (by Łos Theorem), as required.*

If all M_i are isomorphic to some fixed L-structure M then we denote the ultraproduct by M^I/D and call it an **ultrapower** of M. In this case the construction does not produce an "average structure" but creates "non-standard" elements of M. For instance, any ultrapower of the real field $\mathbb{R}$ by a non-principal ultrafilter will contain infinitesimals.

A variant of the construction is to allow D to be an arbitrary filter in $\mathcal{P}(I)$: the result is then called a **reduced power**. For reduced products there is a (considerably) weaker version of Łos' Theorem (see [10, Section 6.2]).

Example 3.28 *(Embeddings into general linear groups) The following example of the use of ultraproducts is from [34]. It makes use of the fact that the ultraproduct construction, when extended in the obvious way to morphisms, is functorial. A group G is* **linear of degree** *n if there is an embedding of G into the general linear group $\mathrm{GL}(n, K)$ for some field K.*

Suppose that G is a group such that every finitely generated subgroup is linear of degree n. Then G is linear of degree n.

For the proof, let G_i $(i \in I)$ be the collection of finitely generated subgroups of G. For each i choose an embedding $f_i : G_i \longrightarrow \mathrm{GL}(n, K_i)$ for some field K_i. Let D be a non-principal ultrafilter on I and let $f = \prod_i f_i/D : \prod_i G_i/D \longrightarrow$

$\prod_i \mathrm{GL}(n, K_i)/D$. It is easy to see (for example think in terms of matrix representations of elements of $\mathrm{GL}(n, -)$) that $\prod_i \mathrm{GL}(n, K_i)/D \equiv \mathrm{GL}(n, \prod_i K_i/D)$. By Łos Theorem $\prod_i K_i/D$ is a field. It remains to see an embedding of G into $\prod_i G_i/D$: at this point we realise that the ultrafilter D should not be arbitrary (non-principal). Given an element $g \in G$ let $[g]$ denote the set of all $i \in I$ such that $g \in G_i$ and let $F = \{[g] : g \in G\}$. Since $[g_1] \cap ... \cap [g_t] = \{i : g_1, ..., g_t \in G_i\}$ the set F has the finite intersection property (the intersection of any finitely many elements of F is non-empty and hence the collection of those subsets of I which contain an element of F forms a filter). Take D to be any ultrafilter containing F. Now we can define the morphism from G to $\prod_i G_i/D$. Given $g \in G$ let $\bar{g}$ be the element of $\prod_i G_i$ which has i-th coordinate equal to g if $g \in G_i$ and equal to the identity element of G otherwise. Map g to $\bar{g}/D$. Our choice of D (to contain each set $[g]$) ensures that this map is an injective homomorphism, as required.

The next result is an algebraic criterion for elementary equivalence. The result after that often lends itself to algebraic applications.

Theorem 3.29 *Two L-structures are elementarily equivalent iff they have isomorphic ultrapowers.*

Theorem 3.30 *A class of L-structures is elementary iff it is closed under ultraproducts and elementary substructures.*

3.11 Structure of definable sets and quantifier elimination

Suppose that M is an L-structure. A **definable** subset of M is one of the form $\phi(M) = \{a \in M : M \models \phi(a)\}$ for some formula $\phi = \phi(x) \in L$. More generally if $A \subseteq M$ then an A-**definable** subset of M is one which is definable by a formula $\phi = \phi(x, \bar{a})$ with parameters from A. Yet more generally one may consider subsets of powers, M^n, of M definable by formulas with n free variables. The logical operations on formulas correspond to set-theoretic operations on these sets: for instance conjunction, negation and existential quantification correspond, respectively, to intersection, complementation and projection.

For many questions it is important to understand something of the structure of these sets and the relations between them. Of particular importance are quantifier elimination results. A theory T has (**complete**) **elimination of quantifiers** if every formula is equivalent modulo T to the conjunction of a sentence and a quantifier-free formula (so if T is also a complete theory then every formula will be equivalent modulo T to one without quantifiers). In order to prove quantifier elimination for a complete theory T it is enough to show that any formula of the form $\exists y \phi(\bar{x}, y)$ with ϕ quantifier-free is itself equivalent to one which is quantifier-free. In other words, it is sufficient to show that any projection of any set which can be defined without quantifiers should itself be definable without quantifiers.

Examples 3.31 *The geometric content of elimination of quantifiers is illustrated by the case of the theory of algebraically closed fields. This theory does have elimination of quantifiers, a result due to Tarski and, in its geometric form (the image of a constructible set under a morphism is constructible), to Chevalley. The elimination comes down to showing that if X is a quantifier-free definable subset of some power K^n, where K is an algebraically closed field, then the projection along, say, the last coordinate is also quantifier-free definable (of course it is definable using an existential quantifier).*

Tarski also proved the considerably more difficult result that the real field and, therefore, all real-closed fields, have elimination of quantifiers in the language of ordered rings (so as well as $0, 1, +, \times, -$ there is a relation symbol $<$ for the order). The geometric form of this statement is fundamental in the study of real algebraic geometry. See, for example, [14].

Partial elimination of quantifiers may be useful. If every formula is equivalent, modulo the theory T, to an existential formula (equivalently, if every formula is equivalent, modulo T, to a universal formula) then T is said to be **model-complete**. This is equivalent to the condition that every embedding between models be an elementary embedding. See, for example [30] for more on this.

For another example, the theory of modules over any ring has a partial elimination of quantifiers: every formula is equivalent to the conjunction of a sentence and a boolean combination of "positive primitive" (certain positive existential) formulas and numerous consequences of this can be seen in [45].

A proof of model-completeness can be a stepping stone to a proof of full quantifier-elimination and has, in itself, geometric content (see, e.g., [19], [50]).

3.12 Many-sorted structures

A single-sorted structure is one in which all elements belong to the same set (or sort). Most model theory textbooks concern themselves with these. Yet many-sorted structures are very important within model theory and in its applications. Fortunately, there almost no difference between the model theory of single- and many-sorted structures.

Some structures are naturally many-sorted. For example, in the model-theoretic study of valued fields it is natural to have one sort for the (elements of the) field and another sort for the (elements of the) value group. One would also have a function symbol, representing the valuation, taking arguments in the field sort and values in the group sort.

Other structures can be usefully enriched to many-sorted structures. In fact, it is common now in model theory to work in the context of the many-sorted enrichment described in the following subsection.

3.13 Imaginaries and elimination of imaginaries

All the ideas that we have discussed up to now are quite "classical". What we describe next is more recent but now pervades work in pure model theory and in many areas of application. A precursor was the practice of treating n-tuples of elements from a structure M as "generalised elements" of the structure. Shelah went much further.

Let M be any L-structure, let $n \geq 1$ be an integer and let E be a ($\emptyset$-, that is, without extra parameters) definable equivalence relation on M^n. By that we mean that there is a formula, $\psi(\bar{x}, \bar{y}) \in L$, with $l(\bar{x}) = l(\bar{y}) = n$, such that for all n-tuples, $\bar{a}, \bar{b}$ of elements of M we have $M \models \psi(\bar{a}, \bar{b})$ iff $E(\bar{a}, \bar{b})$ holds. For example the relation of conjugacy of elements in a group is definable by the formula $\exists z(y = z^{-1}xz)$. The E-equivalence classes are regarded as generalised or **imaginary** elements of M.

Formally, one extends L to a multi-sorted language, denoted L^{eq}. This means that for each **sort** (set of the form M^n/E) we have a stock of variables and quantifiers which range just over the elements of that sort. One also adds to the language certain (definable) functions between sorts, such as the canonical projection from M^n to M^n/E for each n, E. The structure M, together with all its associated imaginary sorts M^n/E and morphisms between them, is an L^{eq}-structure, denoted M^{eq}. There is a natural equivalence between the category of models of $\mathrm{Th}(M)$ and the category of models of $\mathrm{Th}(M^{\mathrm{eq}})$ (we mean the categories where the morphisms are the elementary embeddings) and most model-theoretic properties are unchanged by moving to the much richer structure M^{eq}, a notable exception being not having elimination of quantifiers. It has proved to be enormously useful in model theory to treat these imaginary elements just as one would treat ordinary elements of a structure.

3.14 Interpretation

This is a long-standing theme in model theory which seems to have ever- growing uses and significance. The idea is that, "within" a structure, one may find, or "interpret", other structures (of the same kind or of quite different kinds). Then, for example, if the first structure has some good properties (finiteness conditions, dimensions, ranks...), these transfer to the interpreted structure and, conversely, if the interpreted structure has "bad" properties then this has consequences for the initial structure. Let us be somewhat more precise, using the notion of imaginary sorts that we introduced above.

Suppose that M is an L-structure and that M^n/E is some sort of M^{eq}. The structure on M induces structure on M^n/E (via reference to inverse images, in M^n, of elements in M^n/E). The set M^n/E equipped with some chosen part of all this induced structure is an L'-structure for some other language L' and is said to be **interpreted** in M. In fact it is convenient here to extend the structure M^{eq} to include, as additional sorts, all definable subsets of structures

M^n/E. See [24] for more on interpretation.

Example 3.32 *If K is a field, p an irreducible non-constant polynomial in $K[X]$ and L the corresponding finite extension field then L can be interpreted as K^n, where n is the degree of p, equipped with the obvious addition and with multiplication defined according to the polynomial p.*

Example 3.33 *The simplest example which uses quotient sorts is the interpretation of the set of un-ordered pairs of elements of a structure M. This is M^2/E where E is the equivalence relation on M^2 defined by $E((x,y),(x'y'))$ iff $M \models (x = x' \wedge y = y') \vee (x = y' \wedge y = x')$.*

For examples in the context of groups and fields see, for example, Chapter 3 of [6]. For very general results on finding interpretable groups and fields (the "group configuration") see [7], [38], [42].

3.15 Stability: ranks and notions of independence

There are various ranks which may be assigned to the definable subsets of a structure. These ranks give some measure of the complexity of the structure and are technically very useful since they allow one to have some measure of the extent to which one set depends on another. They also allow one to give meaning to the statement that an element a is no more dependent on a set B than on a subset $A \subseteq B$. The notion of independence that is referred to here, and which generalises linear independence in vector spaces and algebraic independence in algebraically closed fields, exists for all so-called stable theories (and beyond, see [29]) and is defined even when there is no global assignment of ranks to definable sets. To give an idea of one of these ranks we define Morley rank.

Let M be any structure. By a definable set we will mean one which is definable by a formula perhaps using as parameters some elements of M. A definable set has Morley rank 0 exactly if it is finite. Having defined what it means to have Morley rank $\geq n$ we say that a definable set X has Morley rank $\geq n+1$ if there is an infinite set X_i $(i \in I)$ of definable sets, each of which is a subset of X of rank n and with the X_i pairwise disjoint. The definition can be continued for arbitrary ordinals and can also be extended to types (of course a definable set or type may have Morley rank undefined or "∞"). Thus Morley rank is a measure of the extent to which a definable set may be chopped up into smaller definable sets. An L-structure M is said to be **totally transcendental** or, if L is countable, ω-**stable**, if every definable subset of M has Morley rank. See [6] for the rich theory of groups with finite Morley rank.

The origin of this notion of rank, and hence of model-theoretic stability theory, was Morley's Theorem 3.12, from the proof of which it follows that an $\aleph_1$-categorical theory in a countable language must be ω-stable.

Example 3.34 *Any algebraically closed field is ω-stable since, as we have seen, (2.14), any such field is $\aleph_1$-categorical. It follows that any structure interpretable in an algebraically closed field must be ω-stable: in particular this applies to affine algebraic groups. Cherlin conjectured (and Zilber has a similar conjecture) that any simple ω-stable group is an algebraic group over an algebraically closed field. For more on this influential conjecture, see [6].*

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