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1 Introduction

Here is yet another "non-commutative geometry". It can be described briefly as follows: take a commutative noetherian ring R (e.g. the coordinate ring of an affine variety); describe $\mathrm{Spec}(R)$ in terms of the category, Mod-R, of representations of R; now use the same description with any small pre-additive category $\mathcal{A}$ in place of R. The result is a topological space (with points the "primes" of $\mathcal{A}$) with associated presheaf (of "localisations" of $\mathcal{A}$).

We may, in particular, apply this definition with mod-R (the category of finitely presented right R-modules) in place of R to obtain a non-commutative geometry associated with R. Admittedly this has moved us one 'representation-level' up since it is rather the spectrum of mod-R than of R. Nevertheless the "usual spectrum" of R, if it has one, sits inside this richer structure (more accurately, inside that with $(R\text{-mod})^{\mathrm{op}}$ replacing R). For example if R is commutative noetherian then the (Zariski) spectrum of R is a subspace of the larger space, 2.7, and the associated sheaf of rings is a part of the larger presheaf, see 7.11.

If discovery could peer at itself through hindsight this, no doubt, would have been how I had come to the definition. In fact I first defined this space as the dual of another, the Ziegler spectrum, and only later realised that it could be presented as a natural generalisation of the Zariski spectrum [36].

Here is the exact definition. Let $\mathcal{A}$ be a small preadditive category and let Mod-$\mathcal{A}$ denote the category of right $\mathcal{A}$-modules (that is, the category, $(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$, of contravariant additive functors from $\mathcal{A}$ to the category, $\mathbf{Ab}$, of abelian groups). Let inj-$\mathcal{A}$ denote the set of isomorphism classes of indecomposable injective $\mathcal{A}$-modules: this is the set underlying our topological space. The topology is determined by declaring the following sets to be open: $[F] = \{E \in \text{inj-}\mathcal{A} : (F, E) = 0\}$ where F ranges over (isomorphism classes of) finitely presented $\mathcal{A}$-modules. Since $F \oplus G$ is finitely presented if F and G are, these basic open sets are closed under finite intersection, so an arbitrary open set will just be a union of sets of the form $[F]$.

We will call this the **Gabriel-Zariski spectrum** of $\mathcal{A}$ because the idea of representing prime ideals by the corresponding indecomposable injective representations goes back to Gabriel's thesis [10]. We write $\mathrm{GZspec}(\mathcal{A})$. If $\mathcal{A}$ itself has the form $(R\text{-mod})^{\mathrm{op}}$ for some ring (or small preadditive category) R then we will refer to this as the **rep-Zariski spectrum** of R and we write just Zar_R. This means that if R is commutative noetherian then we write $\mathrm{Spec}(R)$ for the usual spectrum and Zar_R for this larger space in which, as we will see, the former embeds.

We have yet to describe the ring(oid)ed structure on this space: the description requires a little setting up.

For $\mathrm{GZspec}(\mathcal{A})$ this is the presheaf of localisations defined as follows. Given $M \in \text{mod-}\mathcal{A}$, the indecomposable objects in the basic open set $[M] = \{E \in \text{inj-}\mathcal{A} : (M, E) = 0\}$ together cogenerate a hereditary torsion theory on Mod-$\mathcal{A}$: the torsionfree objects are those which embed in some direct product of copies of members of $[M]$ and the torsion objects are those with no non-zero morphism to any member of $[M]$. Recall, [55], that each of the classes, $\mathcal{T}$, of torsion and, $\mathcal{F}$, of torsionfree objects determines the other and the pair $\tau = (\mathcal{T}, \mathcal{F})$ is referred to as a hereditary torsion theory. Throughout this paper when we say "torsion theory" we always mean a hereditary torsion theory.

In fact we want a torsion theory which is determined by the *finitely presented* torsion objects, that is, one **of finite type**, so we use instead the torsion theory whose torsion class, which we denote $\mathcal{T}_{[M]}$, is generated as such by the finitely presented objects with no non-zero morphism to any member of $[M]$ (so $\mathcal{T}_{[M]} \subseteq \mathcal{T}$). Denote by $\mathcal{F}_{[M]}$ ($\supseteq \mathcal{F}$) the corresponding torsionfree class. The localisation $\text{Mod-}\mathcal{A} \longrightarrow (\text{Mod-}\mathcal{A})_{\tau_{[M]}}$ (the latter category we also write just as $(\text{Mod-}\mathcal{A})_{[M]}$) at this finite type torsion theory, $\tau_{[M]} = (\mathcal{T}_{[M]}, \mathcal{F}_{[M]})$, corresponding to $[M]$, is the Grothendieck category which is obtained from Mod-$\mathcal{A}$ by forcing all objects of $\mathcal{T}_{[M]}$ to become zero. The image of $\mathcal{A}$ in $(\text{Mod-}\mathcal{A})_{\tau_{[M]}}$ (*via* the Yoneda embedding of $\mathcal{A}$ in Mod-$\mathcal{A}$), we denote it $\mathcal{A}_{\tau_{[M]}}$, is the localisation of $\mathcal{A}$ which is assigned to the basic open set $[M]$. With the natural restriction maps, this gives a presheaf defined on the given basis (which is enough to define the corresponding sheaf).

We would also like to say explicitly how the above definition reads for the rep-Zariski spectrum, Zar_R, of a ring R or, more generally, for $\mathrm{Zar}_{\mathcal{A}}$ where $\mathcal{A}$ is a small preadditive category.

So let mod-$\mathcal{A}$ denote (a small version of) the category of finitely presented right $\mathcal{A}$-modules. Recall that an $\mathcal{A}$-module M is **finitely presented** if there is an exact sequence $P \longrightarrow Q \longrightarrow M \longrightarrow 0$ where P, Q are finitely generated projective $\mathcal{A}$-modules (that is, are finite direct sums of representable functors from $\mathcal{A}^{\mathrm{op}}$ to $\mathbf{Ab}$). It is equivalent to require that the hom-functor $(M, -)$ commute with direct limits. Similarly $\mathcal{A}$-mod denotes the category of finitely presented left $\mathcal{A}$-modules. Now consider the category $(\mathcal{A}\text{-mod}, \mathbf{Ab})$ (of left ($\mathcal{A}$-mod)-modules). There is a full embedding [16] of Mod-$\mathcal{A}$ into $(\mathcal{A}\text{-mod}, \mathbf{Ab})$ given by sending the right $\mathcal{A}$-module M to the functor $M \otimes_{\mathcal{A}} - : \mathcal{A}\text{-mod} \longrightarrow \mathbf{Ab}$ (this functor is determined by its being right exact and having the action, $(M \otimes -) : (A, -) \mapsto MA$ ($A \in \mathcal{A}$) on this generating set of projectives in

$\mathcal{A}$-Mod) and having the obvious effect on morphisms. In particular, since $\mathcal{A}$ embeds as a full subcategory of mod-$\mathcal{A}$ (*via* the Yoneda embedding $A \mapsto (-, A)$ for A an object of $\mathcal{A}$), there is also a copy of $\mathcal{A}$ sitting as a full subcategory of the functor category $(\mathcal{A}\text{-mod}, \mathbf{Ab})$. This may be identified with the Yoneda-image of the Yoneda-image of $\mathcal{A}$ in $\mathcal{A}$-Mod. That is, map $A \in \mathcal{A}$ to $(A, -) \in \mathcal{A}$-mod and then this to the representable functor $((A, -), -)$ in $(\mathcal{A}\text{-mod}, \mathbf{Ab})$. This latter doesn't look quite so bad in the case where $\mathcal{A}$ is a ring R since the usual practice is to denote the projective left R-module $(R, -)$ also by R (or ${}_RR$) and then the image of this in the functor category is denoted $({}_RR, -)$ (and is just the forgetful functor).

Now let $F \in (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ (the full subcategory of finitely presented functors, which we could, though won't, write as ($\mathcal{A}$-mod)-mod). As described above, the indecomposable injective functors in the basic open set $[F]$ together cogenerate a torsion theory and we take the largest torsion theory of finite type smaller (in the sense of inclusion of torsion classes) than this, denoting it $\tau_{[F]}$. Let $(\mathcal{A}\text{-mod}, \mathbf{Ab})_{[F]}$ denote the quotient category of $(\mathcal{A}\text{-mod}, \mathbf{Ab})$ with respect to this torsion theory. Denote by $\mathcal{A}_{[F]}$ the image of $\mathcal{A}$ (regarded as a full subcategory of the functor category) in this localisation. It is a category with the same objects as $\mathcal{A}$ but, in general, modified morphism groups and we can think of it as a kind of localisation of $\mathcal{A}$. Assign to the basic open set $[F]$ this localisation of $\mathcal{A}$.

Note that in the case where $\mathcal{A}$ is a ring R (identified with the forgetful functor $({}_RR, -)$ sitting in $(R\text{-mod}, \mathbf{Ab})$) the localisation at F is again a one-point category, that is, a ring which, *qua* ring, is the endomorphism ring $R_{[F]} = \mathrm{End}(({}_RR, -)_{[F]})$. The ring morphism $R \longrightarrow R_{[F]}$ is not necessarily a localisation in the usual sense since we used a torsion theory on the functor category rather than on the module category but we remark that this notion of localisation in the functor category includes all finite type localisations in the module category (7.11), as well as all ring epimorphisms with domain R ([39]), so, for a right noetherian ring, this is literally a more general notion of localisation.

In this way, to every basic open set $[M]$ of $\mathrm{GZspec}(\mathcal{A})$ we have associated a localisation $\mathcal{A}_{\tau_{[M]}}$ of $\mathcal{A}$, and, at the 'rep'-level, to every basic open set $[F]$ of $\mathrm{Zar}_{\mathcal{A}}$ we have associated a "localisation", $\mathcal{A}_{[F]}$, of $\mathcal{A}$. In each case we have a presheaf defined on a basis, that is, a presheaf on the lattice of basic open sets. To see this let, say, F, G be such that $[F] \supseteq [G]$. Then the corresponding torsionfree classes satisfy $\mathcal{F}_{[F]} \supseteq \mathcal{F}_{[G]}$ and hence $\mathcal{T}_{[F]} \subseteq \mathcal{T}_{[G]}$, so localisation at $\tau_{[G]}$ factors canonically through localisation at $\tau_{[F]}$. Therefore there is a canonical "restriction of scalars" functor $\mathcal{A}_{[F]} \longrightarrow \mathcal{A}_{[G]}$.

Each of these presheaves extends to a presheaf defined on the whole space, see Section 7.2, which we will denote, in the case of $\mathrm{GZspec}(\mathcal{A})$, as $\mathrm{FT}_{\mathcal{A}}$ and refer to as the **finite-type structure presheaf** and, in the case $\mathrm{Zar}_{\mathcal{A}}$, by $\mathrm{Def}_{\mathcal{A}}$ and call it the **presheaf of definable scalars** of $\mathcal{A}$ (the terminology is defined below). We will see, 7.7, that if R is commutative noetherian then

$\mathrm{FT}_R = \mathcal{O}_{\mathrm{Spec}(R)}$, the usual structure sheaf.

We emphasise that, even in 'nice' cases, Zar_R will be a presheaf, rather than a sheaf, though it will always be separated (i.e. a monopresheaf) and, in many cases, interesting parts of it will be have the gluing property, so will be subsheaves.

The terminology "definable scalar" comes from a completely different way of arriving at this structure, which we describe. To do so we must first defined another space.

The Ziegler spectrum of a ring was defined in Ziegler's work [57] on the model theory of modules. The points of this space are the isomorphism classes of indecomposable pure-injective (also called algebraically compact; the definition is given later) right R-modules; let us denote that set by pinj-R. A basis for the topology is defined in terms of pairs of "pp conditions" but may be defined in other (purely algebraic) ways. The resulting space, the (right) Ziegler spectrum of R, denoted Zg_R, also makes sense if we start with any preadditive category $\mathcal{A}$ in place of R and the result is denoted $\mathrm{Zg}_{\mathcal{A}}$.

To every closed subset, X, of Zg_R there corresponds a definable subcategory $\mathcal{X}$ of Mod-R and conversely (5.2). Definable subcategories were originally described model-theoretically but they have the algebraic characterisation of being those subcatgories of Mod-R which are closed under direct limits, direct products and pure submodules (and $\mathcal{X}$ is the closure of the set X of modules under these operations). To each definable subcategory $\mathcal{X}$ there corresponds the ring of all additive relations which are definable in the natural first-order language of R-modules and which are functional on (i.e. which define abelian group endomorphisms on) each member of $\mathcal{X}$ (equivalently, on each member of X). We refer to the ring (under addition and composition) of these as the **ring of definable scalars** of $\mathcal{X}$ and denote it by $R_{\mathcal{X}}$, also by R_X. Since for every element $r \in R$ the multiplication- by-r map is definable on every module there is a natural ring homomorphism $R \longrightarrow R_{\mathcal{X}}$.

Now, by the work of Gruson and Jensen [15] there is a bijection between pinj-R and (R-mod)-inj given by $N \mapsto N \otimes_R -$ (i.e. this is defined *via* the embedding of Mod-R into $(R\text{-mod}, \mathbf{Ab})$ which we have already introduced). Therefore the Ziegler spectrum of R and the rep-Zarski spectrum of R can be regarded as topologies on the same underlying set. They are not, however, the same topology. In fact, if we follow Hochster's definition [20] for spectral spaces (which these spaces are not, but never mind) and declare the complements of compact Ziegler-open sets to form a basis of open sets for a new, "dual", topology, then we obtain precisely the rep-Zariski topology. That is, and this is true for all small preadditive categories $\mathcal{A}$ (recall that these are also called 'ringoids' or 'rings with many objects') in place of R, $\mathrm{Zar}_{\mathcal{A}}$ is the dual of the Ziegler topology $\mathrm{Zg}_{\mathcal{A}}$. Furthermore, if $[F]$ is a basic open set of the rep-Zariski topology, hence is a closed set of the Ziegler topology, then the ring(oid) of definable scalars corresponding to $[F]$ coincides with the $[F]$-localisation, $\mathcal{A}_{[F]}$, of $\mathcal{A}$ defined above. Thus the presheaf of localisations of $\mathcal{A}$ has an entirely different interpretation in terms of rings of definable maps. It was as the dual-Ziegler topology that I first defined Zar_R in [36].

Definitions omitted or left incomplete in this introduction are given at appropriate places below.

This paper is a rewritten and updated version of (part of) [37], with material from [43] added.

2 The Gabriel-Zariski and rep-Zariski spectra

2.1 The Zariski spectrum through representations

A key idea of algebraic geometry is to replace a geometric object (such as an affine variety) by an algebraic object (its coordinate ring) which contains equivalent information. The rings which arise from affine algebraic varieties are commutative but the notion of localisation, which is so central in algebraic geometry, can be extended (though in more than one way) to non-commutative rings.

Localisation of non-commutative rings goes back to Ore [30] and was developed much further, and in different directions, by Gabriel [10] (torsion theory) and Cohn [6] (universal localisation). More recently a host of other non-commutative geometries have arisen, often bearing rather little resemblance to each other, reflecting the great variety of their origins and motivations. A feature of many of these is that a non-commutative ring is, or represents, a geometry even though it may not be obvious how to, or whether it is possible to, associate to this ring a geometry (in any classical sense) of which it is some sort of coordinate ring.

As was seen in the introduction, the "non-commutative geometry" described here is a rather direct generalisation of the usual Zariski geometry and involves first describing that geometry in terms of the category of representations and then simply applying this definition beyond the context of the category of modules over a commutative noetherian ring.

Also seen in the introduction was the point that this definition of a geometry can be applied at different levels. For instance, given a ring R, it can be applied to Mod-R, giving a topology on the space of (isomorphism types of) indecomposable injective R-modules. Or it can be applied to the functor category (R-mod, $\mathbf{Ab}$) and then this topology on the set of indecomposable injective functors (left (R-mod)-modules) may be viewed as a topology on the set of indecomposable pure-injective right R-modules. In principle one may go arbitrarily further: in this case the next stage would be to have a topology on the set of indecomposable injective functors on the functor category (i.e. on the set of indecomposable pure-injective functors) and so on, even transfinitely! But here at most three levels will concern us: a ring R (or small preadditive category $\mathcal{A}$); its module category Mod-R and 'the functor category' $(R\text{-mod}, \mathbf{Ab})$. One might feel uneasy about the fact that, though we started with *right* R-modules, this last is the category of left modules over the finitely presented *left* R-modules: we remark that there is a duality, see 3.1, between the finitely presented functors in $(R\text{-mod}, \mathbf{Ab})$ and those in $(\text{mod-}R, \mathbf{Ab})$ so, despite the switch from right to left finitely presented modules, we do stay close to the category of right modules.

Recall the definition of the **Zariski spectrum**, $\text{Spec}(R)$, of a commutative ring R. The points are the prime ideals of R and a basis of open sets for the topology is given by the sets $D(r) = \{P \in \text{Spec}(R) : r \notin P\}$ for $r \in R$. If R is the coordinate ring of an affine variety then the maximal primes of R correspond to the (usual, i.e. closed) points of the variety and the other primes correspond to (indeed, are generic points of) irreducible subvarieties.

Assuming now that R is commutative noetherian, we show that the spectrum may be defined purely in terms of the category, Mod-R, of representations of R. This is essentially Gabriel's approach [10] and in part it builds on earlier work of Matlis [28] (see also Roos [51]).

Each point $P \in \text{Spec}(R)$ is replaced by the injective hull, $E_P = E(R/P)$, of the corresponding quotient module R/P. Because P is prime, hence $\cap$-irreducible in the lattice of ideals (this does use commutativity of R), E_P is indecomposable. Furthermore each indecomposable injective R-module E has this form. To see this, let a and b be non-zero elements of E and let I, J be their respective annihilators in R. Since E is indecomposable injective, hence uniform, the intersection $aR \cap bR$ is non-zero, so choose a non-zero element c in this intersection. Then $c(I + J) = 0$. Since R is noetherian it follows that there is a unique maximal annihilator, P say, of non-zero elements (equivalently, submodules) of E. A line of calculation (which can be found at 9.2) shows that this is a prime ideal and hence that E is the injective hull of R/P (since E contains an element with annihilator exactly P). This ideal is usually called the **associated prime** of E and is denoted $\text{ass}(E)$. So we may, and will, take the underlying set of $\text{Spec}(R)$ (R commutative noetherian) to be the set, inj-R, of (isomorphism classes of) indecomposable injective R-modules.

As for the topology, we have $D(r) = \{E \in \text{inj-}R : \text{Hom}(R/rR, E) = 0\}$. For, if $r \in R \setminus P$ and if $f : R/rR \longrightarrow E_P$ then $\text{ann}_R(f(1+rR)) \geq \text{ann}_R(1+rR) = rR$ and so, since P is the *unique* maximal annihilator of non-zero elements of E_P (see the proof of 9.2), it must be that $f(1 + rR) = 0$ hence $f = 0$. For the converse, if $r \in P$ then the canonical surjection from R/rR to R/P followed by inclusion is a non-zero morphism from R/rR to E_P.

Thus a basis of open sets for the Zariski topology is given by those sets of the form $[M] = \{E \in \text{inj-}(R) : \text{Hom}(M, E) = 0\}$ as M ranges over modules of the form R/rR. This, however, is not yet a description of the topology in terms of the category Mod-R because the property of being of this form (that is, cyclic-projective modulo cyclic) is not invariant under equivalence of categories.

We show that if we allow M to range instead over arbitrary finitely presented R-modules then we do not change the topology.

First consider the case that $M = R/I$ is cyclic. The argument used to reinterpret $D(r)$ applies equally well to show that, for any ideal I, the set $[R/I] = \{E \in \text{inj-}R : \text{Hom}(R/I, E) = 0\}$ coincides with $\{E_P : P \in \text{Spec}(R), I \nleq P\}$ and, since I is finitely generated, say $I = \sum_1^n r_i R$, we have $[R/I] = \bigcup_1^n [R/r_i R]$ (the inclusion "$\subseteq$" uses commutativity of R, see 9.4) $= \bigcup_1^n D(r_i)$, which is indeed open. This is not yet enough because the property of being cyclic is not Morita-invariant.

So now consider the case that M is finitely presented, in particular, is finitely generated, by $b_1, \dots, b_n$ say. Set $M_k = \sum_{j \le k} b_j R$, $M_0 = 0$. Each factor $C_j = M_j/M_{j-1}$ is cyclic and, we claim, $[M] = [C_1] \cap \cdots \cap [C_n]$. For, if there is a non-zero morphism from C_j to E then, by injectivity of E, this extends to a morphism from M/M_{j-1} to E and hence there is induced a non-zero morphism from M to E. Conversely, if $f : M \longrightarrow E$ is non-zero let j be minimal such that the restriction of f to M_j is non-zero. Then f induces a non-zero morphism from C_j to E.

We have shown the following (outside the commutative situation, the same holds if R is fully bounded noetherian, see 2.8).

Theorem 2.1 *Let R be a commutative noetherian ring. Then the space whose points are the isomorphism classes of indecomposable injective R-modules, E, and which has, for a basis of open sets, those of the form $[M] = \{E : \text{Hom}(M, E) = 0\}$ as M ranges over finitely presented R-modules, is naturally homeomorphic to* Spec(R). *The bijection is given by $P \mapsto E(R/P)$ and $E \mapsto \text{ass}(E)$.*

The word "natural" can be taken in the categorical sense. If $\alpha : R \longrightarrow S$ is a morphism of commutative rings then there is induced the map $\text{Spec}(S) \longrightarrow \text{Spec}(R)$ which takes the prime ideal Q of S to $\alpha^{-1}Q = \{r \in R : \alpha r \in Q\}$. For instance the projection $k[X] \longrightarrow k \simeq k[X]/\langle X \rangle$ induces the embedding $\{0\} = \text{Spec}(k) \longrightarrow \{\langle X \rangle\} \subseteq \text{Spec}(k[X])$. In terms of indecomposable injectives it is almost as direct: the S-module $E(S/Q)$ is uniform, so is also uniform as an R-module and hence has indecomposable injective hull, $E_R(E(S/Q))$ which, since it has an element $(1_S + Q)$ with annihilator $\alpha^{-1}Q$ (and no non-zero element has larger annihilator), is isomorphic to $E(R/\alpha^{-1}Q)$. It is also easy to check that, assuming the rings are noetherian, the two topologies on the image of inj-(S) in inj-(R) (the quotient topology induced from inj-(S) and the subspace topology induced from inj-(R)) coincide. For non-noetherian commutative rings one has naturality for the "ideals", rather than the "fg-ideals" topology, see the discussion after 2.3.

The above statement rather begs the question of whether it applies to arbitrary commutative rings and we do consider these in Section 9 but now we proceed to the next stage, which is to turn this description into a definition.

2.2 Zariski-injective and Gabriel-Zariski spectra of categories

The reformulated definition of the Zariski spectrum of a ring that we obtained above may be applied to any abelian category but surely makes most sense when the category is Grothendieck (so has enough injectives) and is locally finitely presented (so has enough finitely presented objects). Throughout this section, therefore, $\mathcal{C}$ will be a locally finitely presented abelian, hence [9, 2.4] Grothendieck, category. Because every such category is a nice localisation of a functor category ([32]) one may, when it is convenient to do so, concentrate on the case where $\mathcal{C} = (\mathcal{A}, \mathbf{Ab})$ with $\mathcal{A}$ a small preadditive category.

Denote by $\mathrm{inj}(\mathcal{C})$ the set of isomorphism types of indecomposable injective objects of $\mathcal{C}$ (so, in this notation, $\mathrm{inj}\text{-}R = \mathrm{inj}(\mathrm{Mod}\text{-}R)$): that this is a set follows directly from the fact that $\mathcal{C}$ has a generating set of finitely generated objects (and the fact that each object of such a category has only a set of subobjects). Equip this set with the topology which has, for a basis of open sets, those of the form $[F] = \{E \in \mathrm{inj}(\mathcal{C}) : (F, E) = 0\}$ with F a finitely presented object of $\mathcal{C}$. This is indeed a basis for a topology since $[F] \cap [G] = [F \oplus G]$. Denote the resulting space by $\mathrm{Zarinj}(\mathcal{C})$ and call it the **Zariski-injective spectrum** of $\mathcal{C}$. If $\mathcal{C} = (\mathcal{A}, \mathbf{Ab})$ for some small preadditive category $\mathcal{A}$ we have already called this space the left **Gabriel-Zariski spectrum** of $\mathcal{A}$ and denoted it $\mathrm{GZspec}(\mathcal{A}^{\mathrm{op}})$. If $\mathcal{A}$ is itself of the form R-mod for some ring (or small preadditive category) R then we have used the notation Zar_R (and will use the notation ${}_R\mathrm{Zar}$ for $\mathrm{Zar}_{R^{\mathrm{op}}}$) and, reflecting the change of level, called it the **rep-Zariski spectrum** of R. In summary, our notation is as follows:

$\mathrm{GZspec}(\mathcal{A}) = \mathrm{Zarinj}(\mathrm{Mod}\text{-}\mathcal{A})$ and

$\mathrm{Zar}_{\mathcal{A}} = \mathrm{GZspec}((\mathcal{A}\text{-mod})^{\mathrm{op}}) = \mathrm{Zarinj}((\mathcal{A}\text{-mod}, \mathbf{Ab}))$.

Example 2.2 *Let Γ be the path category of the quiver A_∞.*

$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$

It is easily checked that the indecomposable injective objects of $\mathcal{C} = \Gamma\text{-}\mathrm{Mod}_k$ (the category of representations of Γ in the category of k-vectorspaces) are the E_n where, using the representation-theoretic description, $E_n(i) = k$ if $i \leq n$, $= 0$ if $i > n$, and where at each arrow is an isomorphism, together with E_∞, which is 1-dimensional at each vertex. Note that E_n is the injective hull of the simple representation S_n which is 0 everywhere except at n, where it is 1-dimensional. This gives us the points of $\mathrm{inj}(\mathcal{C})$.

Dually, the indecomposable projective objects are the P_n ($n \geq 1$), where P_n is 1-dimensional at each vertex $m \geq n$ and 0 elsewhere (note that $P_1 = E_\infty$). Clearly the indecomposable representation $M_{[n,m]}$ which is 1-dimensional at i for $n \leq i \leq m$ and 0 elsewhere is finitely presented.

For any representation M one has $(M, E_n) = 0$ iff M does not have S_n as a subquotient, that is, iff $M(n) = 0$. Therefore each indecomposable injective E_n

is an isolated (by $[S_1]\cap\cdots\cap[S_{n-1}]\cap[P_{n+1}]$*) point and a basis of open neighbourhoods of* E_∞ *consists of the cofinite sets which contain that point (clearly the latter are open and, from the description of the* P_n*, it is easily seen that every infinite-dimensional finitely presented object is eventually* > 0*-dimensional, so there are no other open neighbourhoods of* E_∞*).*

Therefore $\mathrm{Zarinj}(\Gamma\text{-}\mathrm{Mod}_k)$ *is the one-point compactification, by* E_∞*, of the discrete set* $\{E_n : n \geq 1\}$*.*

The category $\mathrm{Mod}_k\text{-}\Gamma$ *is the category of representations of the opposite quiver, which we regard as the same quiver but with arrows reversed. Now, one has a finite-dimensional indecomposable projective* P'_n *for each n and, for each n, an indecomposable injective* E'_n*. The dimension vector of* P'_n *is as for* E_n *above and that of* E'_n *is as for* P_n *above.*

It is similarly checked that the open sets (apart from $\emptyset$*) of* $\mathrm{Zarinj}(\mathrm{Mod}_k\text{-}\Gamma)$ *are the cofinite sets.*

2.3 Embedding the Zariski spectrum

If R is any ring then, as a set, inj-R embeds in pinj-R (every injective module is pure-injective). Here we compare the Zariski-injective topology on inj-R, that is $\mathrm{GZspec}(R)$, with the topology inherited from the larger space Zar_R. We will see that if R is a right noetherian ring (or even just right coherent) then they do coincide so, in this case, the injective spectrum really is part of the larger pure-injective spectrum.

First, observe that we have the following (the proof is just as before 2.1).

Lemma 2.3 *For any ring R the sets* $[M] = \{E \in \text{inj-}R : (M,E) = 0\}$ *for M a finitely presented R-module form a basis of open sets of a topology which coincides with that obtained by taking just those sets of the form* $[R/I]$*, where I is a finitely generated ideal of R, as (sub-)basic open.*

We will also use the following.

Lemma 2.4 *Let* $I \leq J$ *be right ideals of a ring R and let E be an injective right R-module. Then* $\mathrm{ann}_E I/\mathrm{ann}_E J \simeq \mathrm{Hom}(J/I, E)$*.*

Proof. Just apply $(-,E)$ to the short exact sequence $0 \longrightarrow J/I \longrightarrow R/I \longrightarrow R/J \longrightarrow 0$ and note that $\mathrm{Hom}(R/I,E) \simeq \mathrm{ann}_E I$ (and similarly for J) and $\mathrm{Ext}^1(R/J,E) = 0$. □

Proposition 2.5 *[47, 1.1] If E is any injective right R-module and F is any finitely presented functor from R-*mod *to* **Ab** *which is a subfunctor of a power of the forgetful functor then* $\overrightarrow{DF}(E) = \mathrm{ann}_E(F({}_RR))$*.*

The notation $\overrightarrow{DF}$ is explained in 3.1. Also, if $F({}_RR) \leq R^m$ with $m > 1$ then by $\mathrm{ann}_E(F({}_RR))$ we mean $\{(a_1, \ldots, a_m) \in E^m : \sum_{i=1}^m a_i r_i = 0$ for all $(r_1, \ldots, r_m) \in F({}_RR)\}$.

The statement at [47, 1.1], as are various of our other references, is in terms of pp formulas. We give some explanation of that terminology in Section 10.

Theorem 2.6 *[58, 1.3], [52, Prop. 7] A ring R is right coherent iff every right ideal of the form $F({}_RR)$ with F a finitely presented functor from R-mod to **Ab** is finitely generated.*

Proposition 2.7 *Let R be right coherent. Then the Gabriel-Zariski topology on inj-R coincides with the topology induced from the rep-Zariski topology on pinj-R. That is, we may regard $\mathrm{GZspec}(R)$ as a subspace of Zar_R.*

Proof. One direction needs no assumption on R: given $M \in$ mod-R, the basic Gabriel-Zariski-open set $[M] = \{E \in \text{inj-}R : (M, E) = 0\}$ (see just after 3.1) is just the intersection of the basic rep-Zariski-open set $[(M, -)]$ with the image of inj-R in pinj-R.

For the other direction, given $F \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ we have the basic rep-Zariski-open set $[F] = \{N \in \text{pinj-}R : (F, N \otimes -) = 0\} = \{N \in \text{pinj-}R : \overrightarrow{DF}(N) = 0\}$ by 3.1. Since every finitely presented functor is the quotient of two finitely generated subfunctors of some power of the forgetful functor it follows from 2.5 that for $E \in$ inj-R, $\overrightarrow{DF}(E) = \mathrm{ann}_E I / \mathrm{ann}_E J$ for some right ideals I, J which, by 2.6 are finitely generated. Hence, by 2.4, $\{E \in \text{inj-}R : \overrightarrow{DF}(E) = 0\} = [I/J]$ in the notation established in Section 2.1. Since $[I/J]$ is finitely presented this is a basic Gabriel-Zariski open set, as required. □

The same holds also with any small preadditive category $\mathcal{A}$ in place of R (i.e., provided that Mod-$\mathcal{A}$ is a locally coherent category, the two topologies on inj-$\mathcal{A}$ coincide).

In general inj-R is neither an open nor closed subset of Zar_R. Take $R = \mathbb{Z}$. To see that inj-$\mathbb{Z}$ is not closed, just note (refer to the list after 4.2 or use 6.1 and the fact that $\mathbb{Q} \in \text{Zg-cl}(\overline{\mathbb{Z}_p})$) that $\overline{\mathbb{Z}_p} \in \text{Zar-cl}(\mathbb{Q})$. To see that it is not open, again just check the list after 4.2, and note that it is not the case that every injective point has an open neighbourhood completely contained in inj-$\mathbb{Z}$ (and the same will be true for any PI Dedekind domain with infinitely many primes).

If R is not noetherian then there is the question of whether it is more appropriate to use all right ideals rather than only the finitely generated ones to define a topology on the set of indecomposable injectives. We may, for contrast, refer to the Zariski-injective topology (i.e. that defined using finitely presented modules) also as the **fg-ideals** topology and refer to the space with basis of open sets the $D(I) = \{E \in \text{inj-}R : (R/I, E) = 0\}$, where now I is *any* ideal, as the **ideals** topology. By the same sort of argument that we used with the Zariski-injective topology, this is the same topology as that with basis of open sets the sets $[M]$ where now M is finitely *generated*.

Proposition 2.8 *Let R be a fully bounded noetherian (FBN) ring. Then the following spaces are naturally homeomorphic:*
Spec(R) *(defined exactly as in the commutative case);*
GZspec(R) = Zarinj(Mod-R)

Proof. The main difference between this and the commutative case is that if P is a prime ideal then $E(R/P)$ need not be indecomposable. It will, however, be a direct sum of finitely many copies of a unique indecomposable injective which, in this context, we denote by E_P. Then everything goes more or less as in the commutative case (e.g. cf. [33] which can be regarded as dealing with the, dual, Ziegler spectrum). $\square$

3 Dualities

In the section after this we describe the rep-Zariski topology over various classes of rings. We will make use of the following alternative description of basic open sets.

Recall that the points of Zar_R are the indecomposable pure-injective right R-modules and that if F is a finitely presented functor in $(R\text{-mod}, \mathbf{Ab})$ then $[F] = \{N \in \mathrm{Zar}_R : (F, N \otimes -) = 0\}$ is a typical basic open set. We use the following theorem.

Theorem 3.1 *([2], [16] for the first "$\simeq$", [36, p. 193] for the second) There is a duality $D : (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \simeq ((\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}})^{\mathrm{op}}$ such that, if M is any right R-module and $G \in (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ then $(DG, M \otimes -) \simeq \overrightarrow{G}M$, where $\overrightarrow{G}$ denotes the unique extension of G to a functor from* Mod-R *to* **Ab** *which commutes with direct limits (sometimes we write just G for this extension).*

Regarding the extension $\overrightarrow{G}$; if $M = \varinjlim M_\lambda$ with the M_λ finitely presented then $\overrightarrow{G}M = \varinjlim GM_\lambda$ and, if $(B, -) \longrightarrow (A, -) \longrightarrow G \longrightarrow 0$ with $A, B \in$ mod-R is a projective presentation of G then, interpreting the representable functors as functors on Mod-R, this can also be read as a projective presentation of this canonical extension of G.

It follows from this theorem that an alternative form for the basic open sets is what we will still write as $[G] = \{N \in \mathrm{Zar}_R : GN = 0\}$ as G ranges over the finitely presented functors in $(\text{mod-}R, \mathbf{Ab})$ (really, their canonical extensions to $(\text{Mod-}R, \mathbf{Ab})$).

There is, in general almost and in many cases actually, a homeomorphism between the right and left rep-Zariski spectra of a ring. What we mean by "almost a homeomorphism" is that the lattices (indeed complete Heyting algebras) of open sets are isomorphic.

Theorem 3.2 *[18, Section 4] (for the case of rings) Let $\mathcal{A}$ be a small preadditive category. Then there is a bijection between the open subsets of* $\mathrm{Zar}_{\mathcal{A}}$ *and those of* ${}_{\mathcal{A}}\mathrm{Zar}$ *which preserves containment, intersection and arbitrary union (we will say "a homeomorphism at the level of topology"). If $\mathcal{A}$ is countable (alternatively, under various conditions, such as $\mathcal{A}$ having Krull-Gabriel dimension, see [18, 4.10]) this is induced by a homeomorphism* $(\mathrm{Zar}_{\mathcal{A}}/\sim) \simeq ({}_{\mathcal{A}}\mathrm{Zar}/\sim)$ *where $\sim$ denotes the equivalence relation on a topological space which identifies topologically indistinguishable points (i.e. those which belong to exactly the same open sets).*

This duality of spaces is an expression of a duality which exists at various levels. At the level of functor categories, we have already mentioned the Gruson-Jensen/Auslander duality $((\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}})^{\mathrm{op}} \simeq (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}}$. At the level of pp conditions it is due to the author [34]. At the level of theories of modules it is due to Herzog [18] and is a refinement of his duality, like that in 3.2 above, for Ziegler spectra (for which see Section 5) and hence, since 'rep-Zariski=dual-Ziegler' (Section 5.1), the above result follows. The duality also exists at the level of Serre subcategories of the functor categories (see [19], [24]).

Corollary 3.3 *If R is countable and if there is an equivalence* $\text{Mod-}\mathcal{A} \simeq \mathcal{A}\text{-Mod}$ *(for example, if R is countable and commutative) then this induces a self-homeomorphism* $(\mathrm{Zar}_{\mathcal{A}}/\sim) \simeq (\mathrm{Zar}_{\mathcal{A}}/\sim)$.

Proof. Combine the homeomorphism $(\mathrm{Zar}_{\mathcal{A}}/\sim) \simeq ({}_{\mathcal{A}}\mathrm{Zar}/\sim)$ of 3.2 with that induced by the assumed equivalence. □

In the case of a commutative Dedekind domain R (which, even if not countable, does satisfy the other parenthetically referred-to condition, in 3.2, of having Krull-Gabriel dimension ($= 2$ in fact, or $= 0$ if R is a field)) the self-homeomorphism of Zar_R fixes all finite length points, interchanges, for each prime P, the P-adic and P-Prüfer points, and fixes the generic point (the quotient field of R).

4 Examples

4.1 The rep-Zariski spectrum of a PI Dedekind domain

The class of Dedekind domains includes both the ring of integers $\mathbb{Z}$ and the archetypal tame algebra $k[X]$ (where k, as always in this paper, denotes a field). What we say here extends with essentially no extra work to those non-commutative Dedekind domains which satisfy a polynomial identity, so we work in that generality.

First we have to list the points of the space, that is, the indecomposable pure-injective modules. For $R = \mathbb{Z}$ this goes back to Kaplansky [22] and the general case is much the same (see, [57], [27], [40]).

It is the case (e.g. see the last reference above) that if N is an indecomposable pure-injective R-module then the elements of the centre, $C(R)$, of R which do not act as automorphisms of N form a prime ideal and so N is a module over the corresponding localisation of R. This allows 4.2 below to be proved by reducing to the 'local' case, since there is, for R as above, see [29, 13.7.9], a bijection between the (prime) ideals of the centre and the (prime) ideals of the ring. This bijection also gives the following.

Remark 4.1 *Let R be a PI Dedekind domain, with centre $C(R)$. Then $P \mapsto P \cap C(R)$ gives a homeomorphism of* $\mathrm{Spec}(R)$ *with* $\mathrm{Spec}(C(R))$ *(and hence, 2.8, induces* $\mathrm{Zarinj}(R) \simeq \mathrm{Zarinj}(C(R))$*).*

Theorem 4.2 *([27], [40, 1.6]) Let R be a PI Dedekind domain. The points of* Zar_R *are the following:*
- *the indecomposable modules, R/P^n, of finite length, for $n \geq 1$ and for P a maximal ideal of R;*
- *the completion, $\overline{R}_P = \varprojlim_n R/P^n$, of R in the P-adic topology, for P a maximal prime of R; we call these* **adic** *modules;*
- *the Prüfer modules $R_{P^\infty} = E(R/P)$ as P ranges over the maximal (equivalently non-zero) primes of R;*
- *the quotient division ring, $Q = Q(R)$, of R.*

Since any Dedekind prime ring is Morita equivalent to a Dedekind domain ([29, 5.2.12]) this and what we say below apply equally well to such rings, since everything involved is Morita-invariant.

Now we describe the topology on Zar_R by giving a basis of open neighbourhoods at each point. Of course, to do this we need to know something about the finitely presented functors. Such information is available, though in most sources it is expressed in terms of 'pp-formulas' (we explain what these are in Section 10).

Over these rings all finitely presented functors are, in a sense (which may be made precise e.g. see [35, 2.$\mathbb{Z}$.1] or [48, Section 2.2]), built up from annihilator and divisibility conditions. It is easy to check that, for each element $r \in R$, both the functors (given on objects by) $M \mapsto Mr$ and $M \mapsto \mathrm{ann}_M(r)$ are finitely presented. Note that these are subfunctors of the forgetful functor. If L is a finitely generated (by $s_1, \ldots, s_l$) left ideal then the functor $M \mapsto ML$ is finitely presented, being the sum of the functors $M \mapsto Ms_j$. If I is a finitely generated (by $r_1, \ldots, r_m$) right ideal then the functor $M \mapsto \mathrm{ann}_M(I)$ also is finitely presented, being the intersection of the functors $M \mapsto \mathrm{ann}_M(r_i)$ (since every finitely presented functor is coherent, such an intersection will again be finitely presented). We will use a fairly obvious (we hope) notation, based on that after 3.1, with M as dummy variable where this aids description, in referring to such functors and their quotients.

• R/P^n: This point is isolated by the open set $[MP^n] \cap [\text{ann}(P)/(MP^{n-1} \cap \text{ann}(P))]$. We go through the details. The open set $[MP^n]$ contains exactly those indecomposable pure-injectives N satisfying $NP^n = 0$, namely $R/P, R/P^2, \ldots, R/P^n$. The open set $[\text{ann}(P)/(MP^{n-1} \cap \text{ann}(P))]$ contains exactly those N with $\text{ann}_N(P) \leq NP^{n-1}$ and, on consulting the list, one sees that this defines the set $\{R/P^n, R/P^{n+1}, \ldots, R_{P^\infty}\}$. The intersection of these two open sets is exactly $\{R/P^n\}$, as claimed. (We leave similar checks to the reader.)

• $\overline{R}_P$: First, there is a neighbourhood which excludes all points associated to the prime P apart from $\overline{R}_P$ itself, namely $[\text{ann}(P)]$. Then, given finitely many non-zero primes $Q_1, \ldots, Q_k$ different from P there is a neighbourhood of $\overline{R}_P$ which excludes all points associated to those primes, namely $\bigcap_{i=1}^k([M/MQ_i] \cap [\text{ann}(Q_i)])$. We cannot exclude points associated to more than finitely many primes since, otherwise, looking at the complementary Zariski-closed=Ziegler-open set, we could express a basic (so, 5.1, compact) Ziegler-open set as a union of infinitely many proper open subsets (one for each of the excluded primes). Therefore a basis consists of the sets given by 'finite localisation' (i.e. removing all trace of finitely many other primes) then removing all other points associated to P.

If R has only finitely many primes then there is a minimal neighbourhood, $\{\overline{R}_P, Q\}$.

• R_{P^∞}: The comments for $\overline{R}_P$ apply here also (alternatively use the duality after 3.3) and the sets $[M/MP] \cap \bigcap_{i=1}^k([M/MQ_i] \cap [\text{ann}(Q_i)])$, where $Q_1, \ldots, Q_k$ are any non-zero primes of R different from P, form a basis of open neighbourhoods.

• Q: Again, 'finite localisation' allows us to remove all trace of any finitely many non-zero primes but, for the same reasons as before, no more.

So, if R has only finitely many primes then Q is an open point.

Observe that Zar_R, provided R is not a division ring, is not compact: it is the union of the sets $[M/MP], [\text{ann}(P)]$ and the $[MP^n] \cap [\text{ann}(P)/(MP^{n-1} \cap \text{ann}(P))]$ for $n \geq 1$ and there is no finite subcover.

With this description to hand one may check that the following is true.

Proposition 4.3 *Let R be a PI Dedekind domain. The isolated (i.e. open) points of Zar_R are precisely the points of finite length, except in the case where R has only finitely many primes, in which case the generic point Q also is isolated. Every point of Zar_R, apart from Q, is closed.*

Taking the simplest example, that of a local ring, say $k[X]_{(X)}$ or $k[[X]]$, one has the maximal ideal, which is closed, together with the generic point, which is open, with closure itself plus the closed point (X). The 'extra' points in Zar_R, the finite length modules $k[X]/(X^n)$, are all clopen.

The Zariski spectrum in the usual sense (since these rings are PI, hence fully bounded, there is a bijection between indecomposable injectives and prime ideals) is embedded via the indecomposable injective modules (see Section 2.3).

We also have the following conclusions where we denote by $\mathrm{Zar}_R^{\mathrm{f}}$ the open set of points of Zar_R of finite length.

Lemma 4.4 *If R is a PI Dedekind domain with infinitely many primes then $\mathrm{Zar}_R^{\mathrm{f}}$ is Zariski-dense in Zar_R. In particular, $\mathrm{Zar}_R^{\mathrm{f}}$ is exactly the set of isolated points of Zar_R.*

Proof. From the list above, every open neighbourhood of every infinite length point contains a point of finite length. $\square$

We denote by Zar_R^1 the set $\mathrm{Zar}_R \setminus \mathrm{Zar}_R^{\mathrm{f}}$ of points of infinite length and endow this with the topology inherited from Zar_R. Since $\mathrm{Zar}_R^{\mathrm{f}}$ coincides with the set of all isolated points, Zar_R^1 also equals the first Cantor-Bendixson derivative of Zar_R. From the description of open neighbourhoods we have the following.

Lemma 4.5 *Let R be a PI Dedekind domain, with division ring of quotients Q. Then the non-empty Zariski-open subsets of Zar_R^1 are exactly the cofinite sets which contain the generic point Q.*

4.2 The rep-Zariski spectrum of a PI hereditary order

A **PI hereditary order** is a hereditary ring which is an order in a simple artinian ring, equivalently a PI hereditary noetherian prime ring. Here we note that the results of the previous section generalise to such rings. There is little to check since, by [40, Section 3], the description of the Ziegler spectrum, points and topology, and hence of the Zariski topology, is just as in the case of a Dedekind domain. In particular to every point, N, of Zar_R is associated a prime ideal, $P(N)$, of R. The only significant difference is that if the ring R is not a Dedekind prime ring then the map from $\mathrm{Spec}(R)$ to $\mathrm{Spec}(C(R))$ given by intersecting a prime ideal with the centre is not 1-1. So it is essential here to use $\mathrm{Spec}(R)$, rather than $\mathrm{Spec}(C(R))$, to parametrise the primes.

Example 4.6 *Let R be the ring $\left(\begin{smallmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ - a non-maximal order in the simple artinian ring $A = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{smallmatrix}\right)$. For each non-zero prime $p \in \mathbb{Z}$, $p \neq 2$, we have the corresponding prime ideal $P_p = \left(\begin{smallmatrix} p\mathbb{Z} & 2p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} \end{smallmatrix}\right)$, and the corresponding p-adic and p-Prüfer modules, which may be regarded as $(\bar{\mathbb{Z}}_{(p)}, \bar{\mathbb{Z}}_{(p)})$ and $(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty})$ respectively, as well as the finite length indecomposable modules, R/P^n, associated to P.*

Corresponding to the prime $p = 2$, there are two prime ideals of R, $P_1 = \left(\begin{smallmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right)$ and $P_2 = \left(\begin{smallmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ with corresponding simple modules $S_i = R/P_i$ and we have $\mathrm{Ext}(S_i, S_{3-i}) \neq 0$ for $i = 1, 2$. The P_1-adic module has a unique infinite descending chain of submodules with simple composition factors S_1,S_2 alternating and starting with S_1. Dually the P_1-Prüfer module N is the injective module with socle S_1, $\mathrm{soc}(N/S_1) = S_2$, $(N/S_1)/\mathrm{soc}(N/S_1) \simeq N$. Similarly for P_2.

4.3 The rep-Zariski spectrum of a tame hereditary artin algebra

If R is an artin algebra then every indecomposable module of finite length is a point of Zar_R. Therefore, one should not expect to be able to give a complete description of Zar_R for algebras which are of wild representation type. Nevertheless, one may aim to describe parts of this space and over some (tame) rings one may hope to give a complete description of the topological space and of the associated (pre-)sheaf of rings.

Throughout this section let R be a tame hereditary artin algebra (not of finite representation type).

We recall some facts about the points of Zar_R and then we recall the description of the topology from [40], [50] (refer to these for more detail). We do mean "recall": those who have not seen this before will need to consult, say, [49].

First we have the fact that over every artin algebra every indecomposable module of finite length is an open and closed point of Zar_R (see 6.4).

In fact, the open points are exactly those of finite length. These fall into three disjoint sets: the set, $\mathbf{P}$, of preprojective points, the set, $\mathbf{R}$, of regular points, and the set, $\mathbf{I}$, of preinjective points.

The set $\mathbf{R}$, regarded as part of the Auslander-Reiten quiver of R, is a disjoint union of 'tubes', each containing a finite number of quasisimple modules and all but finitely many being homogeneous, that is, containing just one quasisimple. We denote the tube (regarded as a set of modules) to which the quasisimple module S belongs by $\mathbf{T}(S)$ and, similarly, the coray of epimorphisms in $\mathbf{T}(S)$ containing S by $\mathbf{E}(S)$ and the ray of monomorphisms in $\mathbf{T}(S)$ containing S by $\mathbf{M}(S)$ (the terms 'ray' and 'coray' refer to the structure of the Auslander-Reiten quiver). To each quasisimple S is associated the S-adic module $P(S)$ which is the inverse limit of $\mathbf{E}(S)$ and the S-Prüfer module $E(S)$ which is the direct limit of $\mathbf{M}(S)$. (In Example 4.6 the prime $p = 2$ gives, in an analogous context, a non-homogeneous tube, with two quasisimples, S_1 and S_2.)

The modules $P(S)$ and $E(S)$, for S a quasisimple regular module, are points of the spectrum and the only other infinite-dimensional point is the generic module Q. We denote by Zar_R^1 the set or space of infinite-dimensional (=non-isolated) points. Thus, to each point N of $\mathbf{R} \cup \mathrm{Zar}_R^1$ apart from the generic point, we have an associated quasisimple module which we denote $S(N)$. In this context the quasi-simples play the role that primes did in previous examples (this is more than an analogy, see [7]). As in that case, the process of 'finite localisation', i.e. removing all trace of finitely many primes/quasisimples lies behind the description of neighbourhood bases. Given a set, $\mathcal{S}$, of quasisimple modules, let $U(\mathcal{S})$ denote the set consisting of the generic point and all points of $\mathbf{R} \cup \mathrm{Zar}_R^1$ which are associated to some quasisimple *not* in $\mathcal{S}$.

The papers [40], [50] describe the Ziegler, rather than the rep-Zariski, topology but, since the latter may be defined in terms of the former (5.4), one may deduce the following.

Theorem 4.7 *[43] Let R be a tame hereditary artin algebra. A basis of open sets for Zar_R is as follows.*
As for every artin algebra, the finite-dimensional points are open.
If N is S-adic or S-Prüfer then the sets of the form $\{N\} \cup U(\mathcal{S})$ where $\mathcal{S}$ is a finite set of quasisimples, form a basis of open neighbourhoods for N.
The sets of the form $U(\mathcal{S})$ where $\mathcal{S}$ is a finite set of quasisimples, form a basis of open neighbourhoods for the generic point G.

In particular, the sets $\mathbf{P}$ and $\mathbf{I}$ are Zariski-closed (as well as Zariski-open) so do not figure in the description of neighbourhood bases of the infinite-dimensional points.

4.4 Other examples

Let R be the k-path algebra of one of the quivers Λ_n ($n \geq 2$) shown:

$$1 \underset{\alpha_1}{\overset{\beta_1}{\rightrightarrows}} 2 \xrightarrow{\gamma_1} 3 \underset{\alpha_2}{\overset{\beta_2}{\rightrightarrows}} \quad \cdots \quad \underset{\alpha_{n-1}}{\overset{\beta_{n-1}}{\rightrightarrows}} 2n-2 \xrightarrow{\gamma_{n-1}} 2n-1 \underset{\alpha_n}{\overset{\beta_n}{\rightrightarrows}} 2n$$

with relations $\beta_i\gamma_i = 0 = \gamma_i\alpha_{i+1}$

The Ziegler and rep-Zariski spectra of these algebras were described in [5], with Λ_2 being treated in full detail, the others more briefly. To give these details would take some technical setting up so we refer the reader to that paper and make only a few remarks.

Corresponding to the n subquivers isomorphic to the Kronecker quiver $\tilde{A}_1$ (the quiver with two vertices and two arrows from one to the other - a tame hereditary algebra, so covered by the previous section) there are n Zariski-open subsets each homeomorphic to $\mathrm{Zar}_{\tilde{A}_1}$. In particular there are the n corresponding 'generic' points (and no other generic points). There are some other, discretely parametrised, infinite-dimensional points which 'link' these n open subsets (in the same way that they are linked in the quiver). After the (open and closed) finite-dimensional points are removed, all other points are 'generic', 'linking' or adic or Prüfer. All these infinite-dimensional points, except the generics, are closed. Roughly, after removing the finite-dimensional points, we have n double-except-for-generics copies of the projective line over k with some $\mathbb{N}$-parametrised families of points linking them into a chain.

In particular, for each of these algebras the space, Zar^1, of infinite-dimensional points is 'one-dimensional' in the algebraic-geometric sense and we do, in that paper, conjecture (based on admittedly rather limited examples) that for a finite-dimensional algebra of infinite representation type the 'algebraic-geometric' dimension of the space Zar^1 will be either 1 or ∞. By the latter we mean that it embeds algebraic varieties of arbitrarily high dimension.

It is shown in [41, Section 9] that wild algebras do have algebraic-geometric dimension ∞ in this sense.

We also mention the first Weyl algebra, $A_1(k)$ where k is a field of characteristic zero: this is a wild algebra [1] and the comment just above applies. One may, however, look at parts of the spectrum, as is done, among other things, in [46], where the relative topologies on inj-R, on the torsionfree indecomposable pure-injective modules, and on closures of some tubes are described (in fact, in a somewhat more general context than just this algebra).

5 The Ziegler spectrum

We recall the main definitions but for more detail see, for example, [35], [19], [24], [44].

An embedding $A \longrightarrow B$ of right R-modules is said to be a **pure embedding** if for every left R-module L the induced map $A \otimes_R L \longrightarrow B \otimes_R L$ of abelian groups is monic (it is enough to test with L being finitely presented, see, e.g., [55]). A right R-module N is said to be **pure-injective** (=**algebraically compact**) if it is injective over pure embeddings in Mod-R, equivalently if every pure embedding with domain N is split, equivalently if the functor $(N \otimes -) \in (R\text{-mod}, \mathbf{Ab})$ is injective.

The set of (isomorphism classes of) indecomposable pure-injective right R-modules is topologised, and the result is called the (right) **Ziegler spectrum**, Zg_R, of R, by taking, for a basis of open sets, the sets of the form $(F) = \{N \in \mathrm{Zg}_R : \mathrm{Hom}(F, N \otimes -) \neq 0\}$ where F ranges over $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. This is a reformulation of Ziegler's original definition which is in terms of notions (pp formulas) from model theory.

Theorem 5.1 *[57, 4.9] The sets of the above form constitute a basis for a topology and they are all compact.*

To every module M_R is associated a closed subset, $\mathrm{supp}(M)$, of Zg_R, called the **support** of M and defined by $\mathrm{supp}(M) = \{N \in \mathrm{Zg}_R : \mathrm{Hom}(F, N \otimes -) = 0$ for all $F \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ such that $\mathrm{Hom}(F, M \otimes -) = 0\}$. That is, take the torsion theory, which we denote τ_M, of finite type, on the functor category, associated to M (in the sense that it is the largest torsion theory of finite type for which $M \otimes -$ is torsionfree): then the support of M consists of those $N \in \mathrm{Zar}_R$ such that $N \otimes -$ is (indecomposable injective and) τ_M-torsionfree. Notice that τ_M is quite different from $\tau_{[M]}$ defined in the Introduction: these torsion theories are defined on different categories and $\tau_{[M]}$ was defined only for M finitely presented (the two notations, at least derived notations, may be compared in the statement of 7.12).

All this works with a small preadditive category $\mathcal{A}$ in place of R. The only notable difference is that for a ring R, Zg_R is compact, since it is the open set defined by the forgetful functor $(R, -)$ but, if $\mathcal{A}$ has infinitely many objects, then $\mathrm{Zg}_{\mathcal{A}}$ is not a finite union of basic open sets, hence not compact.

In fact, the category Mod-$\mathcal{A}$ may be replaced by any locally finitely presented abelian category $\mathcal{C}$. One may prove this by developing the theory of purity directly in such categories but more convenient is to use the fact that they are all 'nice' localisations of categories of the form Mod-$\mathcal{A}$ (one may take $\mathcal{A}$ to be (a small version of) the full subcategory of finitely presented objects of $\mathcal{C}$), [32], and then the Ziegler spectrum, $\mathrm{Zg}(\mathcal{C})$, of $\mathcal{C}$ is (or may be defined to be) a closed subset of $\mathrm{Zg}_{\mathcal{A}}$ (see [45] for more detail).

A subcategory of Mod-$\mathcal{A}$ is **definable** if it is closed under direct products, direct limits and pure submodules. There is the following characterisation of the closed subsets of $\mathrm{Zg}_{\mathcal{A}}$.

Theorem 5.2 *[57, 4.10] A subset of* $\mathrm{Zg}_{\mathcal{A}}$ *is closed in the Ziegler topology iff it has the form* $\mathcal{D} \cap \mathrm{Zg}_{\mathcal{A}}$ *for some definable subcategory* $\mathcal{D}$ *of* Mod-$\mathcal{A}$ *(and this gives a bijection between closed subsets and definable subcategories).*

5.1 rep-Zariski = dual-Ziegler

In [36] the underlying set of Zg_R was re-topologised by taking, for a basis of open sets, the complements of the compact open sets, that is, those of the form $[F] = \mathrm{Zg}_R \setminus (F) = \{N \in \mathrm{Zg}_R : \mathrm{Hom}(F, N \otimes -) = 0\}$. We immediately recognise that this "**dual-Ziegler**" topology is the rep-Zariski topology, Zar_R and the notation $[F]$ has the same meaning as before.

More generally, we have the following.

Theorem 5.3 *Suppose that* $\mathcal{C}$ *is locally coherent. Then the Zariski topology on* $\mathrm{inj}(\mathcal{C})$ *coincides with the dual-Ziegler topology on* $\mathrm{inj}(\mathcal{C})$ *(regarded as a subset of* $\mathrm{pinj}(\mathcal{C})$*).*

Proof. This follows directly from 2.7 (and the extensions of that result indicated just after it and in the previous section). □

We state the, already noted, special case.

Corollary 5.4 *Suppose that* $\mathcal{A}$ *is a small preadditive category. Then the dual-Ziegler topology on* pinj-$\mathcal{A}$ *coincides with the rep-Zariski topology, via the identification of* pinj-$\mathcal{A}$ *with* $\mathrm{inj}(\mathcal{A}\text{-mod}, \mathbf{Ab})$.

6 Topological properties of Zar_R

In this section we point out various properties of this space.

• Although the rep-Zariski topology can be defined in terms of the Ziegler topology, the reverse is not true. This was shown in [5, 3.1] where one sees a homeomorphism of Zar_{Λ_2} which is not a homeomorphism with respect to the Ziegler topology.

• By definition of the topologies, we have the following.

Lemma 6.1 *$N' \in \text{Zar-cl}(N)$ iff $N \in \text{Zg-cl}(N')$.*

• The next result concerns (rep-)Zariski-closure.

Proposition 6.2 *Let R be any ring. Suppose that $X \subseteq \text{Zar}_R$. If $N \in \text{Zar-cl}(X)$ then there is $N' \in \text{Zg-cl}(X)$ such that $N \in \text{Zar-cl}(N')$.*

Proof. Let $\mathcal{F}$ be the filter of basic Zariski-open neighbourhoods of N. By assumption, for each $[F] \in \mathcal{F}$, $X \cap [F] \neq \emptyset$. Since Zg_R is compact so is its closed subset $\text{Zg-cl}(X)$, therefore there is $N' \in \text{Zg-cl}(X) \cap \bigcap \mathcal{F}$, as required. □

• A closed set is said to be **irreducible** if it is not the union of two proper closed subsets. A **generic point** of a closed set is one whose closure is that set.

Proposition 6.3 *If the basic Zariski-closed set (F) is irreducible then it has a generic point.*

Proof. Suppose, for a contradiction, that (F) does not have a Zariski-generic point. Then for each $N \in (F)$ there is $N' \in (F)$ such that $N' \notin \text{Zar-cl}(N)$, that is, such that $N \notin \text{Zg-cl}(N')$ and so there is a Ziegler-open neighbourhood, (F_N), of N which is a proper subset of (F). Thus we obtain a representation of (F) as a union of proper Ziegler-open subsets and so, since (F) is Ziegler-compact, there is a finite subcover. But this gives a covering of (F) by finitely many Zariski-closed sets so, by Zariski-irreducibility of (F), there is a single one of these, (F_N) say, which is equal to (F) - contradiction, as required. □

• Recall that a point, N, of a topological space is said to be **isolated** if $\{N\}$ is open. We also say that a point N is **closed**, resp. **clopen**, if $\{N\}$ is.

Proposition 6.4 *[42, 2.21] Let N be an indecomposable finitely generated module over the artin algebra R. Then N is both open and closed in the Zariski topology.*

Proof. It is known [35, 13.1] (following from the existence of almost split sequences over artin algebras) that N is Ziegler-isolated (so $\{N\}$ is basic Ziegler-open) and hence, since rep-Zariski=dual-Ziegler, N is Zariski-closed.

It is also the case that (over any ring) every point **of finite endolength** (i.e. of finite length when considered as a module over its endomorphism ring) is Ziegler-closed (essentially this goes back to [11, Theorem 13]) so $\{N\}^c$ is both Ziegler-open and Ziegler-closed, hence compact, hence, by the description of the

Ziegler topology, of the form (F) for some F. Therefore $\{N\} = [F]$, as required. (This very short argument was pointed out by Henning Krause.) $\square$

Proposition 6.5 *(a) Let R be a countable artin algebra and let $N \in \mathrm{Zar}_R$ be Zariski-open. Then N is of finite endolength.*

(b) Let R be a finite-dimensional algebra over a countable algebraically closed field and let $N \in \mathrm{Zar}_R$ be Zariski-open. Then N is of finite length.

Proof. (a) Say $\{N\} = [F]$. Then N has Krull-Gabriel dimension: otherwise, by [57, 8.3] (also see [45] for this said in non-model-theoretic language), there would be $2^{\aleph_0}$ points, rather than just one point, in $\mathrm{supp}(N) = [F]$ (this is where we use the assumption that R is countable). Since N has Krull-Gabriel dimension $\mathrm{supp}(N)$ contains a point of finite-endolength (by [57], see [35, 10.17]) - which must be N, as required.

(b) For the second statement, if N is not of finite length then it is a generic point in the sense of [8] and so, by [8], there is a Dedekind domain D with infinitely many primes and a representation embedding from Mod-D to Mod-R with N being the image of the generic D-module Q (=the quotient field of D). By [38] this induces a homeomorphism of the rep-Zariski spectrum of D into that of R. Since Q is not isolated in Zar_D, N cannot be isolated in Zar_R. $\square$

It is an open question whether, over an arbitrary ring, a Zariski-open (more generally, Ziegler-closed) point of the spectrum need be of finite endolength. This is related to the question (which has to be asked in localisations of the functor category as well as in the functor category itself) of whether for each Ziegler-open point N there is a simple functor F with $(F) = \{N\}$. See the discussion of the "isolation condition" in [44, p. 382] or [45]).

7 The presheaf structure

7.1 Rings of definable scalars

Recall, from the introduction, the full embedding of Mod-$\mathcal{A}$ into $(\mathcal{A}\text{-mod}, \mathbf{Ab})$ which is given on objects by $M \mapsto M \otimes_{\mathcal{A}} -$. Recall that we denote by τ_M the largest torsion theory of finite type with $M \otimes -$ torsionfree. This torsion theory depends only on the closed subset, $\mathrm{supp}(M)$, of $\mathrm{Zg}_{\mathcal{A}}$ (alternatively, on the definable subcategory of Mod-$\mathcal{A}$ generated by M). More generally, if $X \subseteq \mathrm{Zar}_{\mathcal{A}}$, we define the torsion theory τ_X corresponding to X to be that whose torsion class is generated by the Serre subcategory $\{F : (F, N \otimes -) = 0 \text{ for all } N \in X\}$ of $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$.

For the special case that $\mathcal{A}$ is a ring R we have the, already-mentioned interpretation of the 'localisation' at τ_M as the ring of definable scalars of M.

Theorem 7.1 *([37, A4.2], see [4, 4.6]) Let $M \in$ Mod-R and let τ_M be the torsion theory of finite type on* $(R\text{-mod}, \mathbf{Ab})$ *corresponding to M. Then the natural map $R \longrightarrow \mathrm{End}((R,-)_{\tau_M})$ is the ring of definable scalars, $R \longrightarrow R_M$, of M.*

Indeed, this interpretation is also valid for general $\mathcal{A}$ ([45]).

We may also obtain this ring as the biendomorphism ring of a suitably large module.

Theorem 7.2 *([37, A4.1/4.2] see [4, 4.3] (for the ring case), [4, 4.8] (for, in essence, the general case)) Let $\mathcal{A}$ be a small preadditive category and let $M \in$* Mod-$\mathcal{A}$. *Suppose that N is a (necessarily pure-injective) module such that $N \otimes -$ is an injective cogenerator for the finite type torsion theory, τ_M, corresponding to M. Then there is an index set I such that the biendomorphism ring of N^I is the ring of definable scalars of M: $\mathcal{A}_M \simeq \mathrm{Biend}(N^I)$.*

In summary we have the following, stated for the case of a ring but valid (with the obvious notion of "category of definable scalars" replacing "ring of definable scalars") for any small preadditive category.

Theorem 7.3 *([37, 4.3], see [4]) To each Ziegler-closed subset X of* pinj_R *there is associated an R-algebra, $R \longrightarrow R_X$, called the* **ring of definable scalars** *of X. This ring may be described by the following equivalent means.*

(i) R_X is the ring of pp-definable (i.e. definable in terms of projections of systems of linear equations, see Section 10) functions on any module with support equal to X.

(ii) R_X is the endomorphism ring of the image of the forgetful functor in the localisation of the category $(R\text{-mod}, \mathbf{Ab})$ *at the torsion theory, τ_X, corresponding to X.*

(iii) R_X is the biendomorphism ring of any suitably 'large' module N with support equal to X, "large" meaning that $N \otimes -$ is an injective cogenerator for τ_X and N is cyclic over its endomorphism ring.

For some modules every biendomorphism is a definable scalar.

Proposition 7.4 *([37, A1.5], [4, 3.6]) If M is a module of finite endolength then its ring of definable scalars coincides with its biendomorphism ring.*

Proposition 7.5 *([37, A.1.5′], see [4, after 3.6]) If M is a finitely presented module which is finitely generated over its endomorphism ring then its ring of definable scalars coincides with its biendomorphism ring.*

By way of contrast, the biendomorphism ring of the Prüfer group $\mathbb{Z}_{p^\infty}$ is the ring, $\overline{\mathbb{Z}_{(p)}}$, of p-adic integers whereas its ring of definable scalars is just the localisation $\mathbb{Z}_{(p)}$. The same goes for $\overline{\mathbb{Z}_{(p)}}$ regarded as a $\mathbb{Z}$-module (though, if regarded as a $\overline{\mathbb{Z}_{(p)}}$-module, the two rings will coincide).

Theorem 7.6 *([39], [37, A4.4] for the last statement) If $f : R \longrightarrow S$ is an epimorphism of rings then Zar_S may be identified with the (Ziegler-closed) subset "Mod-$S \cap \mathrm{Zar}_R$" of Zar_R and the 'localisation' of R corresponding to this closed subset (i.e. its ring of definable scalars) is just S (regarded as an R-algebra via f). Moreover, if S is regarded as an R-module via f then the ring of definable scalars of S_R is exactly S.*

Notice that we also have a larger presheaf of small categories which associates to each basic open set the whole localised category of finitely presented functors (which, [37, A3.16], equals the category of finitely presented functors of the localisation), and this makes sense as a presheaf over $\mathrm{inj}(\mathcal{C})$ for any locally coherent category $\mathcal{C}$.

7.2 The sheaf of locally definable scalars

Recalling the definition, in the introduction, of the "presheaf defined on a basis", we wish to extend this to a sheaf. One may go via a presheaf defined on the whole space (see later in this section) but, since we have local data sufficient to define a sheaf, we go straight for that. First, we check that what we described in the introduction really does extend the usual definition of the structure sheaf.

In the classical case, associated to an affine variety, X, equivalently to its coordinate ring R, is a sheaf, written $\mathcal{O}_{X=\mathrm{Spec}(R)}$, of rings which is defined on the standard basis $(D(r))_{r\in R\setminus\{0\}}$ of $\mathrm{Spec}(R)$ by $\mathcal{O}_X D(r) = R[r^{-1}]$ - the localisation of R obtained by inverting $r \in R$ (and the restriction maps are just the canonical localisation maps). Any open set is already a basic open set (note that $D(r) \cap D(s) = D(rs)$) and so this presheaf defined on a basis already is a presheaf indeed, see e.g. [17, Section II.2], a sheaf of local rings, with the stalk at a prime P being the localisation, $R_{(P)}$, of R at P. We must rewrite this sheaf in terms of Mod-R as we have done already for the base space $\mathrm{Spec}(R)$.

Given $r \in R$, consider the corresponding open set $D(r) = \{E \in \mathrm{inj}(R) : (R/rR, E) = 0\}$ and consider the torsion theory on Mod-R which these injective modules cogenerate. Since every hereditary torsion theory over a (right) noetherian ring is of finite type this is the torsion theory previously denoted by $\tau_{[R/rR]}$. Clearly the localisation of R at this torsion theory is precisely $R[r^{-1}]$, as required.

Proposition 7.7 *If R is commutative noetherian then the torsion-theoretic presheaf, FT_R defined earlier coincides with the usual structure (pre)sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$.*

Given any presheaf F on a topological space T and given any point $t \in T$ the **stalk** of F at t is defined to be $F_t = \varinjlim\{F(U) : t \in U \text{ and } U \text{ is open}\}$. If $\mathcal{U}_0$ is a basis for the topology then clearly $F_t = \varinjlim\{F(U) : t \in U \in \mathcal{U}_0\}$ so the notion of stalk is independent of any way that we might extend a presheaf on a basis to a presheaf (on arbitrary open sets).

Proposition 7.8 *([37, C1.1], see [41, 3.1]) Let $\mathcal{A}$ be a small preadditive category and let $E \in \text{inj-}\mathcal{A}$. Then the stalk of the presheaf,* $\mathrm{FT}_{\mathcal{A}}$ *at E is the localisation of $\mathcal{A}$ at the torsion theory of finite type corresponding to E.*

In particular if $\mathcal{A}$ is a ring and if $N \in \mathrm{Zar}_{\mathcal{A}}$ then the stalk of the presheaf, $\mathrm{Def}_{\mathcal{A}}$, *of definable scalars at N is the ring of definable scalars at N:* $(\mathrm{Def}_{\mathcal{A}})_N = \mathcal{A}_N$.

Strictly speaking, the proof in [37] is for the second statement. To show the first, for each point E of $\mathrm{GZspec}(\mathcal{A})$ we have the direct limit $\varinjlim_{\{F:E\in[F]\}} \mathcal{A}_{[F]}$ of small categories and we have to show that this limit is the image of $\mathcal{A}$ in the localisation of ($\mathcal{A}$-mod, **Ab**) by the finite type torsion theory corresponding to E. The proof in [41] more or less applies (after translating away the model-theoretic terminology), the key fact being that we are dealing with torsion theories of finite type.

Then we may define the topology on the stalk space in the usual way (see [56, 4.2.6]) and from that define the sheafification of $\mathrm{FT}_{\mathcal{A}}$, respectively $\mathrm{Def}_{\mathcal{A}}$, which we denote by $\mathrm{LFT}_{\mathcal{A}}$, respectively $\mathrm{LDef}_{\mathcal{A}}$. The latter we refer to as the **sheaf of locally definable scalars** of $\mathcal{A}$.

One has, in particular, that the centre of this sheaf is a sheaf of local rings. This follows from the fact (essentially [57, 5.4], see [35, 2.$\mathbb{Z}$.8] or [45]) that if N is any indecomposable pure-injective then the endomorphisms (in particular, the multiplications by elements of the centre, $C(R)$, of R) of N which are not automorphisms form a prime ideal and so there is a prime ideal, P, of $C(R)$ such that N is a module over the localisation of R at P (and the canonical morphism, $R \longrightarrow R_N$ to the stalk of the presheaf of definable scalars at N factors through this localisation).

Theorem 7.9 *([37, D1.1], see [41, 6.1]) The centre of the presheaf of definable scalars is a presheaf of commutative local rings.*

Next we consider finite type localisation.

Proposition 7.10 *(e.g. [45]) Let τ be a torsion theory of finite type on the locally finitely presented abelian category $\mathcal{C}$. Then τ is cogenerated by the set of indecomposable injective torsionfree objects.*

The following result is [37, A1.7] and is also given proof avoiding model theory (but still using pp conditions) in [45]

Theorem 7.11 *Let τ be a torsion theory of finite type on* $\text{Mod-}R$ *and let $\mathcal{E} = \mathcal{F}_\tau \cap \text{inj-}R \subseteq \mathrm{Zar}_R$ be the corresponding set of indecomposable torsion-free injectives. Then the torsion-theoretic localisation, R_τ, of R at τ coincides (as an R-algebra) with the ring of definable scalars, $R_{\mathcal{E}}$, of $\mathcal{E}$.*

This, together with the comments at the beginning of the section, show that if R is commutative noetherian then the embedding of inj-R into pinj-R discussed in Section 2.3 extends to an embedding of the usual structure sheaf into the presheaf of definable scalars.

Although we went straight from the presheaf on a basis to the sheaf, we do now turn briefly to what we bypassed, namely extending the presheaf from the basis to all open sets. We follow a construction indicated in [56, after 4.2.6]. We may begin by extending the presheaf to finite unions of basic open sets as follows. Given basic open sets $[F]$ and $[G]$ consider the torsion theory whose torsion class is the intersection of $\mathcal{T}_{[F]} \cap \mathcal{T}_{[G]}$. This will be of finite type (if H' is a finitely presented functor and H' is $\mathcal{T}_{[F]} \cap \mathcal{T}_{[G]}$-dense in H then there is some finitely generated $H_1 \leq H'$ which is $\mathcal{T}_{[F]}$-dense in H' and there is some finitely generated $H_2 \leq H'$ which is $\mathcal{T}_{[G]}$-dense in H', then $H_1 + H_2$ is finitely generated and $\mathcal{T}_{[F]} \cap \mathcal{T}_{[G]}$-dense in H, which is enough by, e.g., [37, A3.2]). So we have the localisation, denote it $\mathcal{A}_{[F]\cup[G]}$, of $\mathcal{A}$ at that torsion theory. Then, to an arbitrary open set U we may assign the inverse limit (in the category of small preadditive categories) $\varprojlim_{\{V:V\subseteq U\}} \mathcal{A}_{[V]}$ where the limit is taken over finite unions of basic open sets contained in U.

We also have the following which generalises 7.8 (and has the same proof).

Lemma 7.12 *Let R be any ring and let Y be any subset of* Zar_R. *Then* $\varinjlim\{R_{[F]} : Y \subseteq [F]$ *and* $F \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}\} \simeq R_{\mathrm{Zg\text{-}cl}(Y)}$ *where the latter is the ring of definable scalars associated, in the sense of 7.3, with the closed subset* Zg-cl(Y) *of the Ziegler spectrum.*

8 Examples

We describe the (pre-)sheaf structure over some of the examples for which we calculated or described the space Zar_R. Tame hereditary artin algebras are treated in some detail in [43] (we deal with just one example here) and one could, with some work, do the computations for the domestic string algebras Λ_n.

8.1 The sheaf of locally definable scalars of a PI Dedekind domain

Recall, from Section 4.1, that to every point N of Zar_R (R a PI Dedekind domain) is associated a prime ideal, $P(N)$, of R, which may be obtained by taking the unique maximal ideal of $\mathrm{End}(N)$ (which is local), taking the inverse image of this under the canonical map from the centre, $C(R)$, of R to $\mathrm{End}(N)$, then taking $P(N)$ to be the prime ideal of R which is generated by this prime ideal of $C(R)$.

The rings of definable scalars of individual points (see 7.8) are as follows.

• The ring of definable scalars of the module R/P^n (P a non-zero prime) is the ring R/P^n: for the module has finite endolength and hence (7.4) its ring of definable scalars coincides with its biendomorphism ring, which is R/P^n.

• If N is the P-adic or P-Prüfer module then the ring of definable scalars, R_N, is the localisation, $R_{(P)}$, of R at P (by 7.8 for the Prüfer module and then directly, or using duality [18, 6.2] and 3.3, for the adic case).

• If N is the generic point, Q, the quotient division ring of R, then R_N is the ring Q (again by 7.4 since N has finite endolength and $\mathrm{Biend}(Q_R) = Q$ (because $R \longrightarrow Q$ is an epimorphism of rings)).

For any subset, $\mathcal{P}$, of the set, $\mathrm{maxspec}(R)$, of maximal ideals of R, let $U(\mathcal{P}) = \{N \in \mathrm{Zar}_R : P(N) \notin \mathcal{P}\}$. If we denote by $R[\mathcal{P}^{-1}]$ the localisation of R at $\mathrm{maxspec}(R) \setminus \mathcal{P}$ then, since the canonical map $R \longrightarrow R[\mathcal{P}^{-1}]$ is an epimorphism, we have, see 7.6, that $U(\mathcal{P})$ is a Ziegler-closed subset of Zg_R. In the case that $\mathcal{P}$ is a finite subset of $\mathrm{maxspec}(R)$ then $U(\mathcal{P})$ is Zariski-open, being, in the notation introduced earlier, $\bigcap_{P\in\mathcal{P}}([\mathrm{ann}(P)] \cap [M/MP])$.

We compute the presheaf of definable scalars. The following result is immediate from 7.11.

Lemma 8.1 *Let R be a PI Dedekind domain and let $\mathcal{P}$ be a finite subset of* $\mathrm{maxspec}(R)$. *Then the ring of definable scalars over the corresponding Zariski-open subset, $U(\mathcal{P})$, of* Zar_R *is the localisation, $R[\mathcal{P}^{-1}]$, of R. Furthermore, if $\mathcal{P} \subseteq \mathcal{P}'$, then the restriction map from $R_{U(\mathcal{P})}$ to $R_{U(\mathcal{P}')}$ is the natural embedding between these localisations of R.*

Lemma 8.1 gives all the information that we need to compute the sheafification, LDef_R, of the presheaf of definable scalars. We will describe, by way of example, the ring of definable scalars and the ring of sections, that is, the ring of *locally* definable scalars, over some basic open subsets of Zar_R.

• If $U = \{R/P_1^{n_1}, \dots, R/P_t^{n_t}\}$ where the primes $P_1, \dots, P_t$ are all distinct, then $\mathrm{LDef}_R(U) = R/P_1^{n_1} \times \cdots \times R/P_t^{n_t} = R/(P_1^{n_1} \dots P_t^{n_t})$.

• If $U = \{R/P^m, R/P^n\}$ with $n \geq m$ then $R_U = R/P^n$ (by 7.4 R_U is $\mathrm{Biend}(R/P^m \oplus R/P^n)_R$ and, noting that there is the endomorphism of this module projecting the second component on to the first, one easily computes that this is R/P^n). Since the set U has the discrete topology, $\mathrm{LDef}_R(U)$ is the direct product $R/P^m \times R/P^n$ and so we see that the presheaf of definable scalars is not a sheaf.

• R_U for U an arbitrary finite set of finite length points is given by combining the above observations.

• Let $U = \mathrm{Zar}_R \setminus \{N_1, \dots, N_t\}$ where each N_i is an adic or Prüfer module: then $R_U = R_V$ where V is the smallest set of the form $U(\mathcal{P})$ which contains U (i.e., provided at least one of the P-adic, P-Prüfer is in U then P cannot be inverted over U). The presence in U of infinitely many finite length, hence isolated, points means that the ring of locally definable scalars, $\mathrm{LDef}_R(U)$, is rather large. It makes sense, therefore, to throw away these isolated points (see later).

• Consider the special (but illustrative) case, $R = k[X]$, $U = U((X)) \cup \{R/(X)\}$. A module with support U (which is basic Zariski-open, so also Ziegler-closed) is $k[X, X^{-1}] \oplus (k[X]/(X))$ and so we have to compute the definable scalars on this module. By 7.3(iii) this is the biendomorphism ring of a module of the form $M \oplus M'$ where M, resp. M', is a 'large enough' module in the definable subcategory generated by $k[X, X^{-1}]$, resp. by $k[X]/(X)$. Since $\mathrm{Hom}(M, M') = 0 = \mathrm{Hom}(M', M)$ the endomorphism ring of this module is just the block-diagonal matrix ring $\mathrm{diag}(\mathrm{End}(M_R), \mathrm{End}(M'_R))$ and hence the biendomorphism ring is just the block-diagonal matrix ring $\mathrm{diag}(k[X, X^{-1}], k[X]/(X))$ - that is, the direct product of these rings.

Now we compute the sheaf of locally definable scalars restricted to the set Zar^1_R, obtained by throwing away the finite length points. Since Zar^1_R is not Zariski-open, we need the following observations concerning restriction. Define LDef^1_R to be the inverse image sheaf of LDef_R under the inclusion of Zar^1_R in Zar_R: by definition (e.g. see [17, p. 65]) this is the sheaf associated to the presheaf which assigns to a relatively open subset $U \cap \mathrm{Zar}^1_R$ of Zar^1_R (where U is a Zariski-open subset of Zar_R) the direct limit of the rings $\mathrm{LDef}_R(V)$ as V ranges over Zariski-open subsets of Zar_R with $V \supseteq U \cap \mathrm{Zar}^1_R$. If $U \cap \mathrm{Zar}^1_R = \mathrm{Zar}^1_R \setminus \{N_1, \ldots, N_t\}$ let V be the set of all points of Zar_R except those which belong to a prime P such that both the P-adic and P-Prüfer appear among $N_1, \ldots, N_t$, that is, V is the smallest set of the form $U(\mathcal{P})$ which contains U. Then, by the computations above, this limit is already equal to R_V and hence this presheaf is already a sheaf. That is, we have the following description of LDef^1.

Lemma 8.2 *Let R be a PI Dedekind domain and let Zar^1_R be the set of infinite-length points, regarded as a subspace of Zar_R. Let LDef_R denote the sheaf of locally definable scalars over Zar_R. Then the inverse image sheaf, LDef^1_R, on Zar^1_R may be computed as follows. Given a Zariski-open subset U of Zar_R, let V be smallest set of the form $U(\mathcal{P})$ which contains U. Then $\mathrm{LDef}^1_R(U \cap \mathrm{Zar}^1_R) = \mathrm{LDef}_R(V) = R[\mathcal{P}^{-1}]$ and the restriction maps are those of LDef_R, that is, the canonical localisation maps.*

In particular, the ring of definable scalars, $R_{\mathrm{Zar}^1_R}$, of Zar^1_R is R itself.

The sheaf LDef^1_R is 'unseparated' in the sense that it contains points N,N' such that U is an open neighbourhood of N iff $(U \setminus \{N\}) \cup \{N'\}$ is an open neighbourhood of N' (namely the Prüfer and adic associated to any maximal prime). In order to recover the 'classical' situation we have to identify corresponding adic and Prüfer points.

So let $\alpha : \mathrm{Zar}^1_R \longrightarrow \mathrm{Zar}^1_R$ be the map which interchanges the P-adic and P-Prüfer point for every P and which fixes the generic point.

Lemma 8.3 *The map* $\alpha : \mathrm{Zar}^1_R \longrightarrow \mathrm{Zar}^1_R$ *is a homeomorphism of order 2 and* $\mathrm{LDef}^1_R \simeq \alpha^\star \mathrm{LDef}^1_R \simeq \alpha_\star \mathrm{LDef}^1_R$ *where* $\alpha^\star$, $\alpha_\star$ *denote the inverse image and direct image sheaves respectively (see [17], [56]).*

Proof. From the description of the topology it is clear that α is a homeomorphism. For any basic Zariski-open set, U, of Zar^1_R we have, again by what has been said above, $R_U \simeq R_{\alpha U}$ and so the isomorphisms are direct from the definitions. $\square$

We can, therefore, form the quotient space Zar^1_R/α of α-orbits and the corresponding sheaf LDef^1_R/α over this space, thus obtaining a ringed space with centre isomorphic (*via* the identification of maxspec$R$ with maxspec$(C(R))$) to the structure sheaf over the commutative Dedekind domain $C(R)$.

8.2 The sheaf of locally definable scalars of a PI hereditary order

First we have to compute rings of definable scalars. These are obtained for primes P belonging to singleton cliques (in the sense of [7]) by localising just as in the Dedekind prime case. For the other primes we use universal localisation, as in [7] (alternatively, as mentioned there, Goodearl's localisation from [14]), to obtain the corresponding Prüfer and adic modules. We give an example below for illustration. Beyond this, the description of the presheaf of definable scalars and the corresponding sheaf, both on Zar_R and on Zar^1_R, is as before, using the fact that if $R \longrightarrow S$ is an epimorphism of rings then, in addition to the induced homeomorphic embedding of Zar_S into Zar_R (7.6), there is induced an embedding of LDef_S, 'up to Morita equivalence' into LDef_R. This follows from the argument of [41, Section 8]. (And every universal localisation is an epimorphism of rings.)

By way of example, we continue Example 4.6 by computing the various rings of definable scalars. We retain the notation of that example.

Corresponding to the prime P_p there is the ring of definable scalars $\left(\begin{smallmatrix} \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \end{smallmatrix}\right)$, which is a maximal order in A.

The rings of definable scalars corresponding to P_1 and P_2 may be computed using [14, first paragraphs of Section 2]. Explicitly, and adopting the notation of that paper, we remove the simple module S_1 by localising at $X_1 = \{P_2\} \cup \{P_p : p \neq 2\}$. Let $\mathcal{S}_1$ be the set of essential right ideals I of R such that none of R/P_2, R/P_p $(p \neq 2)$ occurs as a composition factor of R/I. Note that $P_1 \in \mathcal{S}_1$. Let $R_{(1)}$ denote the localisation of R at the torsion theory which has $\mathcal{S}_1$ as dense set of right ideals. Then, [14], $R_{(1)} = \{a \in M_2(\mathbb{Q}) : aI \leq R \text{ for some } I \in \mathcal{S}_1\}$. One checks that $R_{(1)} = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$. Similarly, if $R_{(2)}$ denotes the ring obtained by localising away S_2 then one checks that $R_{(2)} = \left(\begin{smallmatrix} \mathbb{Z} & 2\mathbb{Z} \\ (1/2)\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$. Hence the ring of definable scalars at P_1 is $\left(\begin{smallmatrix} \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \\ \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \end{smallmatrix}\right)$ and that at P_2 is $\left(\begin{smallmatrix} \mathbb{Z}_{(2)} & 2\mathbb{Z}_{(2)} \\ (1/2)\mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \end{smallmatrix}\right)$. Notice that, as rings, though not as R-algebras, these are isomorphic (by the map taking $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} a & 2b \\ c/2 & d \end{smallmatrix}\right)$).

One way of regarding this is that we have two epimorphisms from R to the maximal order $M_2(\mathbb{Z})$. The corresponding Ziegler-closed and, one may check, Zariski-open sets cover Zar_R. So LDef_R is covered by two (very much) overlapping copies of '$M_2(\mathrm{LDef}_{\mathbb{Z}})$'.

Lemma 8.4 *Let R be a hereditary order. Then the presheaf of definable scalars over Zar^1_R is already a sheaf.*

Proof. Take an open cover $\{U_i\}_i$ of Zar^1_R, say $U_i = \mathrm{Zar}^1_R \setminus Y_i$ (where Y_i is any finite subset of Zar^1_R which does not contain the generic Q) and let elements $s_i \in R_i = R_{U_i}$ be such that on $U_i \cap U_j = \mathrm{Zar}^1_R \setminus \{Y_i \cup Y_j\}$ we have $s_i = s_j = s$, say, (we identify all the rings R_i with subrings of the full, simple artinian, quotient ring of R, so this equality makes sense). We have $s \in R_i \cap R_j$ and this equals $R_{U_i \cup U_j}$ since a prime P satisfies $P.R_i \cap R_j = R_i \cap R_j$ iff both the P-Prüfer and P-adic modules lie in both Y_i and Y_j, which is so iff $P.R_{U_i \cup U_j} = R_{U_i \cup U_j}$. So now taking any finite subcover, say $U_1, \ldots, U_n$, we deduce that $s_1 = \cdots = s_n = s \in R = R_{\mathrm{Zar}^1_R}$. Thus s is a global section which restricts on each U_i to s_i and is already in the presheaf, as required. $\square$

Proposition 8.5 *Suppose that R is a hereditary order. Let $\mathrm{Spec}R$ denote the space of prime ideals of R with the Zariski topology and let $\pi : \mathrm{Zar}^1_R \longrightarrow \mathrm{Spec}R$ be the map which sends $N \in \mathrm{Zar}^1_R$ to $P(N)$, the prime ideal of R associated with N. Then the direct image, $\pi_\star \mathrm{LDef}^1_R$, of LDef^1_R is a sheaf on $\mathrm{Spec}R$ which sends an open set $U = \mathrm{Spec}R \setminus Y$ (where Y is a finite subset of $\mathrm{maxspec}R$) to the localisation, in the sense of [14], of R at U and $\pi_\star \mathrm{LDef}^1_R$ may be identified with LDef^1_R/α where α is the homeomorphism interchanging corresponding Prüfer and adic points.*

Proof. The direct image of a sheaf is always a sheaf (e.g. [56]) and the description of the sheaf $\pi_\star \mathrm{LDef}R^1$ and its identification with $\mathrm{LDef}R^1/\alpha$ follows from the previous discussion. $\square$

The underlying space, $\mathrm{Spec}R$, of this sheaf may also be identified with the space $\mathrm{Spec}_s R$ (based on the set of simple modules) from [7] (*via* $S \mapsto \mathrm{ass}E(S)$). The sheaf $\mathrm{Spec}_s R$ in [7] may, therefore be identified with the centre of LDef^1_R/α.

8.3 The sheaf of locally definable scalars of the Kronecker algebra

The general tame hereditary case is considered in [43] and is based on the following result.

Theorem 8.6 *([54], [7]) Let R be a tame hereditary artin algebra.*

(a) Let $\mathcal{S}$ be any set of quasisimple modules. Then the universal localisation, $R_{\mathcal{S}}$, of R at $\mathcal{S}$ is a hereditary PI order which is a subring of the simple artinian ring A obtained by taking $\mathcal{S}$ to be the set of all quasisimple modules.

(b) The localisation $R \longrightarrow R_{\mathcal{S}}$ is an epimorphism of rings, and the image of the inclusion functor $\text{Mod-}R_{\mathcal{S}} \longrightarrow \text{Mod-}R$ *is the full subcategory of all modules M which are orthogonal to $\mathcal{S}$ (that is, modules M such that* $\text{Ext}^1(S, M) = 0 = \text{Hom}(S, M)$ *for all $S \in \mathcal{S}$).*

We have from 7.6 that the intersection of $\text{Mod-}R_{\mathcal{S}}$ with Zg_R is the closed subset $U(\mathcal{S})$ which, if $\mathcal{S}$ is finite, is basic Zariski-open, being defined by the conditions $\text{Ext}^1(S, -) = 0 = \text{Hom}(S, -)$ for $S \in \mathcal{S}$, and the ring of definable scalars for this Zariski-open set is just $R_{\mathcal{S}}$.

Using this, we may deduce the next result.

Proposition 8.7 *Let R be a tame hereditary artin algebra. Recall from Section 4.3 that a basis for the Zariski topology on* $\mathbf{R} \cup \text{Zar}_R^1$ *is the collection of sets of the form $U(\mathcal{S})$ where $\mathcal{S}$ ranges over those finite sets of quasisimple modules which, without loss of generality, contain all quasisimples from at least one tube, together with the sets $\{N\}$ where $N \in \mathbf{R}$.*

Then the restriction $\text{Def}_R \restriction (\mathbf{R} \cup \text{Zar}_R^1)$ *of the presheaf of definable scalars is given on this basis by sending $U(\mathcal{S})$ to the localisation $R \longrightarrow R_{\mathcal{S}}$ and sending any open subset of $U(\mathcal{S})$ to the ring of definable scalars of the corresponding subset, regarded as an open subset the rep-Zariski spectrum of the hereditary PI order (in A) $R_{\mathcal{S}}$ (such rings were discussed in the previous section).*

In particular, the ring of definable scalars of any member, N, of $\mathbf{R} \cup \text{Zar}_R^1$ may be computed by choosing a set $\mathcal{S}$ of quasisimples which contains all quasisimples from at least one tube and does not contain the associated quasisimple module $S(N)$, localising R at $\mathcal{S}$ to obtain the hereditary order $R_{\mathcal{S}}$ (which, by choosing $\mathcal{U}$ to contain all but at most one quasisimple from each inhomogeneous tube, may be assumed to be a Dedekind prime ring) and then computing the ring of definable scalars of N, regarded as an $R_{\mathcal{S}}$-module. (In particular the ring of definable scalars will be a matrix ring over either a uniserial artinian ring or a non-commutative discrete valuation ring, or is a simple artinian ring.)

Now let R be the Kronecker algebra $\tilde{A}_1(k)$. We give a more explicit description of the sheaf LDef_R^1 as a sheaf of hereditary orders (in fact, maximal orders).

By 8.6, the full quotient ring of $\tilde{A}_1(k)$ is the ring, $M_2(k(X))$, of 2×2 matrices over the function field $k(X)$. In order to maintain the symmetry between the arrows α and β of $\tilde{A}_1(k)$ we represent $k(X)$ in the form $k(X_0, X_1)_0$ where the subscript denotes the 0-grade part of the quotient field of the graded (with the usual grading) ring $k[X_0, X_1]$. Then there is a natural embedding of $\tilde{A}_1(k)$ into $M_2(k(X_0, X_1)_0)$ which takes e_1 to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, e_2 to $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, α to $\left(\begin{smallmatrix} 0 & X_0 \\ 0 & 0 \end{smallmatrix}\right)$ and β to

$\left(\begin{smallmatrix} 0 & X_1 \\ 0 & 0 \end{smallmatrix}\right)$ and, under this embedding, the quotient ring A may be identified with $M_2(k(X_0, X_1)_0)$.

Let S_0 (respectively S_1) be the quasisimple module which satisfies $S_0X_1 = 0$ (resp. $S_1X_0 = 0$). Let R_i denote the localisation of R at S_i ($i = 0, 1$). Let D_i be the Zariski-open subset of Zar^1_R, $D = \mathrm{Zar}^1_R \cap [M/MX_{1-i}] \cap [\mathrm{ann}(X_{1-i})]$. Then the localisation map $R \longrightarrow R_i$ identifies D_i with $\mathrm{Zar}^1_{R_i}$ and $\mathrm{LDef}^1_R \upharpoonright D_i$ with $\mathrm{LDef}^1_{R_i}$. Each of R_0,R_1 is isomorphic as a ring to the polynomial ring over k in one indeterminate and the localisation $R_{0,1}$ of R at $\{S_0, S_1\}$ (which corresponds to the intersection $D_0 \cap D_1$) is a ring isomorphic to $k[T, T^{-1}]$. It is straightforward to compute these as subalgebras of $M_2(k(X_0, X_1)_0)$ and one obtains:

$R_0 = \left(\begin{smallmatrix} k[X_0X_1^{-1}] & kX_1 \oplus X_0k[X_1^{-1}X_0] \\ X_1^{-1}k[X_0X_1^{-1}] & k[X_1^{-1}X_0] \end{smallmatrix}\right)$;

$R_1 = \left(\begin{smallmatrix} k[X_1X_0^{-1}] & kX_0 \oplus X_1k[X_0^{-1}X_1] \\ X_0^{-1}k[X_1X_0^{-1}] & k[X_0^{-1}X_1] \end{smallmatrix}\right)$;

$R_{0,1} = \left(\begin{smallmatrix} k[X_0X_1^{-1}, X_1X_0^{-1}] & X_1k[X_0^{-1}X_1] \oplus X_0k[X_1^{-1}X_0] \\ X_1^{-1}k[X_0X_1^{-1}] \oplus X_0^{-1}k[X_1X_0^{-1}] & k[X_1^{-1}X_0, X_0^{-1}X_1] \end{smallmatrix}\right) = \left(\begin{smallmatrix} k(X_0,X_1)_0 & k(X_0,X_1)_1 \\ k(X_0,X_1)_{-1} & k(X_0,X_1)_0 \end{smallmatrix}\right)$

(where subscript denotes degree in the graded ring $k(X_0, X_1)$).

9 The spectrum of a commutative coherent ring

Throughout this section R will be a commutative ring.

In our re-interpretation of $\mathrm{Spec}(R)$ as a topology on the set, inj_R, of indecomposable injective R-modules when R is commutative noetherian we used the noetherian hypothesis in the pairing up of indecomposable injectives with prime ideals (and also in that we did not have to choose between using finitely presented or finitely generated modules to define the topology).

In the general commutative case the association $P \mapsto E(R/P)$ gives only an injection of $\mathrm{Spec}(R)$ into inj-R.

The first example (which was pointed out to me by T. Kucera) shows that $\mathrm{Spec}(R) \longrightarrow$ inj-R need not be surjective.

Example 9.1 *Let $R = k[X_n (n \in \omega)]$ be a polynomial ring over a field k in infinitely many commuting indeterminates. It is easily checked that R is coherent. Let $I = (X_n^{n+1} : n \in \omega)$. Clearly I is not prime but $E = E(R/I)$ is an indecomposable injective. To see this, it is enough to show that R/I is uniform and this may be shown as follows. First note that a polynomial $\sum a_\nu X^\nu$ (each multi-index ν occurring at most once) is in I iff each of its monomial factors is in I. Let x_i denote the image in R/I of X_i. Let $p \in R \setminus I$. A short inductive (on the number of monomials) argument shows that there is a multiple of p whose image in R/I has the form $x_1x_2^2 \dots x_n^n$ (which, note, is non-zero) for some n. Hence any two non-zero elements of R/I have a common multiple of this form so R/I is uniform and $E(R/I)$ is indecomposable.*

On the other hand, E does not have the form $E(R/P)$ for any prime P. This follows from 9.2 below since it is easy to see that $P(E)$, as defined below, is the maximal ideal $(X_n : n \in \omega)$ and that $E(R/(X_n : n \in \omega))$ has non-zero socle whereas $E(R/I)$ has zero socle (if $p \in R \backslash I$, say $p \in k[X_0, \dots, X_n]$, then $(p+I)X_{n+1}$ generates a non-zero proper submodule of the submodule generated by $p+I$). Hence E is not isomorphic to $E(R/P(E))$ so, by 9.2, E does not have the form E_P for any prime ideal P.

Let E be any indecomposable injective R-module. Set $P = P(E)$ to be the sum of annihilator ideals of non-zero elements, equivalently submodules, of E. Since E is uniform the set of annihilator ideals of non-zero elements of E is closed under finite sum (see the proof below) so the only issue is whether the sum, $P(E)$, of them all is itself an annihilator ideal.

As before we use the notation E_P to denote $E(R/P)$.

Lemma 9.2 *If $E \in$ inj-R then $P(E)$ is a prime ideal. The module E has the form E_P for some prime ideal P iff the set of annihilator ideals of non-zero elements of E has a maximal member, namely $P(E)$, in which case $E = E_{P(E)}$.*

Proof. Suppose that $rs \in P(E)$. Then, by definition of $P(E)$ there is $a \in E$, $a \neq 0$ such that $ars = 0$. Then either $ar = 0$, in which case $r \in P(E)$, or $ar \neq 0$ and hence $s \in P(E)$. This shows that $P(E)$ is prime. So, if $P(E)$ is an annihilator ideal then $E = E_{P(E)}$.

If $E = E(R/P)$ then $P \leq P(E)$, by definition of the latter. If there were $r \in P(E) \setminus P$ let $b \in E$ be non-zero with $br = 0$ and let $a \in E$ be such that $\mathrm{ann}_R(a) = P$. By uniformity of E there is a non-zero element $c \in aR \cap bR$, say $c = at$ with $t \in R$. Since $cr = 0$ we have $atr = 0$, hence $tr \in P$ and hence $t \in P$ (impossible since $c = at \neq 0$) or $r \in P$ - contradiction. So $P = P(E)$. □

Remark 9.3 *The proof above shows that if P is a prime ideal and if $\mathrm{ann}_E P \neq 0$ then $E = E_P$.*

Before examining the relation between $E \in$ inj-R and $E_{P(E)} \in$ inj-R we address the issue of which topology we should be using on inj-R.

Extending our previous notation, if X a subset of R set $D(X) = \{P \in \mathrm{spec}(R) : X \not\subseteq P\}$. Since $D(X) = \bigcup_{r \in X} D(r)$ this is a Zariski-open subset of $\mathrm{Spec}(R)$.

For I an ideal of R let us set $D^{\mathrm{m}}(I) = \{E \in \text{inj-}R : (R/I, E) = 0\}$ ("m" for "morphism"). Since $D^{\mathrm{m}}(I) \cap D^{\mathrm{m}}(J) = D^{\mathrm{m}}(I \cap J)$ (for the non-immediate direction, note that any morphism from $R/(I \cap J)$ to E extends, by injectivity of E, to one from $R/I \oplus R/J$) these form a basis for what we earlier (Section 2.3) called the fg-ideals topology on inj-R. Note, however, that if $I = \sum_\lambda I_\lambda$ then clearly $D^{\mathrm{m}}(I) \supseteq \bigcup_\lambda D^{\mathrm{m}}(I_\lambda)$ but, as illustrated by Example 9.1, the inclusion may be proper (take E as there, take I to be $P(E)$ and take the I_λ to be the annihilators of non-zero elements of E).

Lemma 9.4 *If I is a finitely generated ideal of R and $I = \sum_1^n I_i$ then $D^{\mathrm{m}}(I) = \bigcup_1^n D^{\mathrm{m}}(I_i)$.*

Proof. Suppose that $E \notin \bigcup_1^n D^{\mathrm{m}}(I_i)$. Then for each i there is a non-zero morphism $f_i : R/I_i \longrightarrow E$. The intersection of the images of these morphisms is non-zero and, since R is commutative, any element in this intersection is annihilated by each I_i, hence by I, that is, $(R/I, E) \neq 0$ so $E \notin D^{\mathrm{m}}(I)$, as required. $\square$

In the early part of 2.1 we argued the (noetherian version of the) following, though there we could identify $\mathrm{Spec}(R)$ and inj-R; here we have only a containment of the former in the latter.

Corollary 9.5 *For any ideal I we have $D^{\mathrm{m}}(I) \cap \mathrm{Spec}(R) = D(I)$.*

Therefore the restrictions of both the ideals and the, in general coarser, fg-ideals topologies on inj-R to spec(R) give the usual Zariski topology. By the results at the beginning of Section 2.3 the topology induced on inj-R by the rep-Zariski=dual-Ziegler (5.4) (=induced rep-Zariski, by 2.7) topology is intermediate between these two (it uses just the (right) ideals of the form $DF({}_RR)$ and this includes all finitely generated ideals, but not necessarily all ideals). If R is coherent then this third topology coincides with the fg-ideals one. Of course, all four spaces; inj-R with its various topologies and spec(R) with the Zariski topology, coincide if R is noetherian.

In the remainder of this section we will mostly assume that R is coherent and concentrate on the fg-ideals = (therefore) rep-Zariski and Ziegler topologies.

Recall that for I any right ideal of a ring, R, and $r \in R$ we have an isomorphism $R/(I : r) \simeq (rR + I)/I$, where $(I : r) = \{s \in R : rs \in I\}$, induced by sending $1 + (I : r)$ to $r + I$.

Theorem 9.6 *Let R be commutative coherent, let E be an indecomposable injective module and let $P(E)$ be the prime ideal defined before. Then E and $E_{P(E)}$ are topologically indistinguishable in Zg_R and hence also in Zar_R.*

Proof. Let I be such that $E = E(R/I)$. For each $r \in R \setminus I$ we have, by the remark just above, that the annihilator of $rR + I \in E$ is $(I : r)$ and so, by definition of $P(E)$, we have $(I : r) \leq P(E)$. The natural projection $(rR + I)/I \simeq R/(I : r) \longrightarrow R/P(E)$ extends to a morphism from E to $E_{P(E)}$ which is non-zero on $r + I$. Forming the product of these morphisms as r varies over $R \setminus I$, we obtain a morphism from E to a product of copies of $E_{P(E)}$ which is monic on R/I and hence is monic. Therefore E is a direct summand of a product of copies of $E_{P(E)}$ and so is in the definable subcategory generated by $E_{P(E)}$. Therefore $E \in \text{Zg-cl}(E_{P(E)})$ (this conclusion required no assumption on R beyond commutativity).

For the converse, take a basic Ziegler-open neighbourhood of $E_{P(E)}$: by (the proof of) 2.7 this has the form (J/I) for a pair, $I < J$ of finitely generated ideals of R. Now, $E_{P(E)} \in (J/I)$ means that there is a non-zero morphism $f : J/I \longrightarrow E_{P(E)}$. Since $R/P(E)$ is essential in $E_{P(E)}$ the image of f has non-zero intersection with $R/P(E)$ so there is an ideal J', without loss of generality finitely generated, with $I < J' \leq J$ and such that the restriction, f', of f to J'/I is non-zero (and contained in $R/P(E)$). Since $R/P(E) = \varinjlim R/I_\lambda$, where I_λ ranges over the annihilators of non-zero elements of E, and J'/I is finitely presented, f' factorises through one of the maps $R/I_\lambda \longrightarrow R/P(E)$. In particular, there is a non-zero morphism $J'/I \longrightarrow E$ and hence, by injectivity of E, an extension to a morphism $J/I \longrightarrow E$, showing that $E \in (J/I)$. Therefore $E_{P(E)} \in \text{Zg-cl}(E)$, as required. $\square$

Remark 9.7 *If E is not isomorphic to $E_{P(E)}$ then E is not in the closure of $E_{P(E)}$ with respect to the ideals topology since we have the open neighbourhood $[R/P(E)]$ of $E_{P(E)}$ which does not contain E. Therefore the ideals topology is strictly finer than the fg-ideals topology whenever* inj-R *is strictly larger than* $\text{Spec}(R)$.

Example 9.8 *Coherence of R is not necessary (nor, as we saw in 9.1, sufficient) for equality of the prime and injective spectra. Take, for instance, $R = k[x_i\,(i \geq 1) : x_ix_j = 0\,(i,j \geq 1)]$ where k is a field. The Jacobson radical $J = \sum_{i\geq 1} x_iR$ is the only $\cap$-irreducible ideal so $E = E(R/I) = E(k)$ is the only point of* inj-R. *Since R/I embeds in R but is not finitely presented, R is not coherent.*

Corollary 9.9 *Let R be a commutative coherent ring and let $P \in \text{Spec}(R)$. Then the closure of E_P in the fg-ideals=Zariski topology on* inj-R *is $\{E \in \text{inj-}R : P(E) \geq P\}$.*

Proof. We have, for every prime Q, that $E_Q \in \text{Zar-cl}(E_P)$ iff $Q \geq P$ (9.5) and so, by 9.6, $E \in \text{Zar- cl}(E_P)$ iff $P(E) \geq P$. $\square$

If E is an injective module we denote by $\text{cog}(E)$ the (hereditary) torsionfree class cogenerated by E (i.e. all those modules which embed in a power of E). If E' is an indecomposable injective in $\text{cog}(E)$ then, since it is a direct summand of a direct product of copies of E, it is in the definable subcategory generated by E and hence is a member of $\text{supp}(E) \subseteq \text{Zg}_R$ (in particular, if E is indecomposable then $E' \in \text{Zg-cl}(E)$ and hence $E \in \text{Zar-cl}(E')$). The first half of the proof of 9.6 shows that $E \in \text{Zg-cl}(E_{P(E)})$ whether or not R is coherent. It also shows the following.

Lemma 9.10 *If $I \leq J$ are (right) ideals of R then $E(R/I) \in \text{cog}(E(R/J))$.*

Proposition 9.11 *Let E be an indecomposable injective module over the commutative coherent ring R. Then the torsion theory cogenerated by E is of finite type iff $E = E_P$ for some prime P.*

Proof. ($\Leftarrow$) By 9.5, for I any ideal of R we have $E_P \in D^{\mathrm{m}}(I)$ iff $E_P \in D(I)$, that is, iff $(R/I, E_P) = 0$ (i.e. R/I is E_P-torsion) iff $I \nleq P$ and the latter is so iff some finitely generated ideal $I' \leq I$ satisfies $I' \nleq P$. So each E_P-dense ideal contains a finitely generated E_P-dense ideal, which is (well-known to be) equivalent to the torsion theory cogenerated by E being of finite type.

($\Rightarrow$) If E cogenerates a torsion theory of finite type then, by the proof of the second half of 9.6, we have $E_{P(E)} \in \mathrm{cog}(E)$ (there, taking $J = R$, it is shown that if I is a finitely generated ideal with $\mathrm{Hom}(R/I, E) = 0$, (i.e. with R/I E-torsion) then $\mathrm{Hom}(R/I, E_{P(E)}) = 0$ so, by finite type, $E_{P(E)} \in \mathrm{cog}(E)$). Hence there is an embedding $R/P(E) \longrightarrow E^{\kappa}$ for some index set κ. It follows that $\mathrm{ann}_E P(E) \neq 0$ and hence, by 9.3, $E \simeq E_{P(E)}$, as required. □

Thus $\mathrm{Spec}(R)$ may be identified within inj-R as those points which cogenerate torsion theories of finite type.

Theorem 9.12 *If R is any right coherent ring then a subset of* inj-R *is Ziegler-closed iff it has the form $\mathcal{F} \cap$ inj-R where $\mathcal{F}$ is the torsionfree class for some torsion theory of finite type.*

Proof. A torsionfree class $\mathcal{F}$ is a definable subcategory of Mod-R iff the corresponding torsion theory is of finite type ([31], [23]) and so, by 5.2, any subset of the form given will be Ziegler-closed.

For the converse if $X \subseteq \mathrm{Zg}_R$ is closed then, by 5.2, it has the form $\mathcal{D} \cap$ inj-R for some definable subcategory, $\mathcal{D}$, of Mod-R. We may replace $\mathcal{D}$ by its closure, $\mathcal{D}'$, under arbitrary submodules which is, note, again a definable subcategory and also has the same intersection with inj-R as $\mathcal{D}$. The torsionfree class cogenerated by the members of X taken together clearly is contained in $\mathcal{D}'$. This torsionfree class might not be of finite type but (this is another characterisation of finite type) its closure under direct limits will be (since R is right coherent, so direct limits of injectives are injective, this still will be a torsionfree class). The intersection of this finite type torsionfree class with inj-R, being contained in the intersection of $\mathcal{D}'$ with inj-R, is X, as required. (We remark that this finite type torsion theory has been met already as the largest torsion theory of finite type less than or equal to that cogenerated by the (injective hull of the direct sum of the) members of X.) □

Theorem 9.13 *Let R be commutative coherent and let $X \subseteq$ inj-R be Ziegler-closed. Then X is irreducible iff $X = \mathrm{cog}(E_P) \cap$ inj-R for some prime ideal, P, of R.*

Proof. By 9.12 there is a torsionfree class, $\mathcal{F}$, of finite type with $\mathcal{F} \cap \text{inj-}R = X$. Let $\mathcal{I}$ be the set of annihilators of non-zero element of members of $\mathcal{F}$. If $\{I_\lambda\}_\lambda$ is a chain of members of $\mathcal{I}$ with their union=sum equal to I, say, then, since $\mathcal{F}$ is closed under direct limits (being of finite type), there is $M \in \mathcal{F}$ and $a \in M$ with $a \neq 0$ and $aI = 0$. So by Zorn's Lemma every $I \in \mathcal{I}$ is contained in a maximal member of $\mathcal{I}$. Denote the set of these maximal members by $\mathcal{P}$. The argument used in 9.2 shows that all ideals in $\mathcal{P}$ are prime.

Choose $P_0 \in \mathcal{P}$ and set $E_0 = E_{P_0}$ and $E' = \bigoplus\{E_P : P \in \mathcal{P}, P \neq P_0\}$. By 9.10 (and comments before that) $\text{supp}(E_0) \cup \text{supp}(E') = X$ (because R is coherent the support of any injective module will consist of injective modules, see [47, 4.4]). So, by irreducibility of X, either $X = \text{supp}(E_0)$, which equals $\text{cog}(E_0) \cap \text{inj-}R$ by 9.11 and 9.12, as required, or $X = \text{supp}(E')$. But, in the latter case we would have $E_0 \in \text{cog}E'$ and hence there would be an embedding of the form $f : R/P_0 \longrightarrow \prod\{E_P^{\kappa(P)} : P \in \mathcal{P}, P \neq P_0\}$ with, say $(1 + P_0) \longrightarrow (e_\lambda)_\lambda$. Some e_λ would be non-zero and so P_0 would be (properly!) contained in $\text{ann}_R(e_\lambda)$ - contradicting $P_0 \in \mathcal{P}$. □

Proposition 9.14 *Let R be commutative coherent. A subset V of* inj_R *is Zariski-closed and irreducible iff there is a prime ideal Q of R such that $V = \{E : P(E) \geq Q\}$.*

Proof. This is just the usual description of irreducible closed subsets of $\text{Spec}(R)$ combined with 9.6. □

Corollary 9.15 *Let R be commutative coherent. Then there are bijections between the following:*
(i) the set of irreducible Ziegler-closed subsets of inj_R*;*
(ii) the set of irreducible Zariski-closed subsets of $\text{Spec}(R)$*;*
(ii) the points of $\text{Spec}(R)$*;*
(iii) the set of irreducible (Gabriel-/induced rep-)Zariski-closed subsets of inj_R*;*
given by $\{E : P(E) \leq Q\} \sim \{P : P \geq Q\} \sim Q \sim \{E : P(E) \geq Q\}$.

10 Appendix: pp conditions

Many of our references use terminology derived from model theory: here we explain, briefly, the main item of terminology, namely "pp formula". In the context of this paper, it is most convenient to think of this as meaning simply a subfunctor of the forgetful functor (or of one of its finite powers).

Every finitely generated subfunctor of the n-th power, $(R^n, -)$, of the forgetful functor (from the category of R-modules to that of abelian groups) has the following form. Fix a homogeneous R-linear system of equations with $m \geq n$ indeterminates: to every module M we may associate the solution set in M of this system - this will be a subgroup of M^m; consider the image of this solution set under the projection of M^m onto the first n coordinates - this image is a subgroup of M^n. That's how the functor works on objects and the action on morphisms is the obvious one. That's all. (That is, a pp formula is basically such a system of equations, together with the specification of projecting on to (say) the first n coordinates.)

Every finitely presented functor is a quotient, F/F', of two such subfunctors of some power of the forgetful functor: the model-theoretic terminology corresponding to such a quotient is "pp-pair".

The functorial terminology is better in some regards: a formula is really a presentation rather than the functor being presented and for theorems (as opposed to calculations) one does not usually need to refer to presentations.

For more explanation, or for other terms, see the introductions to various of the references or, e.g. the expository paper [44]. A book, [45] on all this, and more, is in preparation but, for a fast introduction (as opposed to a comprehensive treatment), the existing literature is better.

We finish by mentioning some relevant papers in connection with derived and triangulated categories, namely [3] where the rep-Zariski spectrum appears in connection with the spectrum of the cohomology ring of the group algebra of a finite group, [26] where the Ziegler spectrum (and hence, implicitly the rep-Zariski spectrum) for compactly generated triangulated categories is defined and [12], [13] which continue in these directions.

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