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LOCAL-GLOBAL PRINCIPLE FOR THE BAUM-CONNES CONJECTURE WITH COEFFICIENTS

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Abstract. We establish the Hasse principle (local-global principle) in the context of the Baum-Connes conjecture with coefficients. We illustrate this principle with the discrete group $GL(2,F)$ where F is any global field.

1. Introduction

Let G be a second countable locally compact Hausdorff topological group. We shall say that G satisfies BCC, or BCC is true for G, if the Baum-Connes conjecture with coefficients in an arbitrary $G-C^*$-algebra is true for G.

Our main result is:

Theorem 1.1. *Let G be the ascending union of open subgroups G_n, and let A be a $G-C^*$-algebra. If each subgroup G_n satisfies the Baum-Connes conjecture with coefficients A, then G satisfies the Baum-Connes conjecture with coefficients A.*

One application is to the following new permanence property.

Theorem 1.2. *Let F be a global field, $\mathbb{A}$ its ring of adeles, G a linear algebraic group defined over F. Let F_v denote a place of F. If BCC is true for each local group $G(F_v)$ then BCC is true for the adelic group $G(\mathbb{A})$.*

Another application of Theorem 1.1 is the proof of the Baum-Connes conjecture for reductive adelic groups [2].

To derive Theorem 1.2 from Theorem 1.1 we note that the adelic group $G(\mathbb{A})$ admits an ascending union of open subgroups. We then make use of a crucial permanence property due to Chabert-Echterhoff [5], namely that BCC is stable under direct product of finitely many groups.

If G satisfies BCC, then any closed subgroup of G also satisfies BCC [5, Theorem 2.5]. Since $G(F)$ is a discrete subgroup of $G(\mathbb{A})$, we have the following result:

Theorem 1.3. *If BCC is true for each local group $G(F_v)$ then BCC is true for the discrete group $G(F)$.*

There is, at present, a limited supply of local groups for which BCC is known to be true. Nevertheless, some examples are known.

For the group $SO(n,1)$, Kasparov [12] proved that $\gamma = 1_G$ in the Kasparov representation ring $R(G) = KK_G(\mathbb{C},\mathbb{C})$. For the group $SU(n,1)$, Julg-Kasparov [11] and Higson-Kasparov [8] proved that $\gamma = 1_G$ in the ring $R(G)$. For the group $Sp(n,1)$, Julg [9] has recently proved that, for any $G-C^*$-algebra A, the image of γ via the map:

$$R(G) \to KK_G(A,A) \to KK(A \rtimes_r G, A \rtimes_r G) \to End\, K_*(A \rtimes_r G)$$

is the identity element even though $\gamma \neq 1$ in the ring $R(G)$. Therefore the following rank-one Lie groups satisfy BCC:

$$SO(n,1),\ SU(n,1),\ Sp(n,1).$$

The *Haagerup property* (or *a-T-menability* in the sense of Gromov) is discussed in detail in [6]. We quote the main result in [8, Theorem 1.1].

Theorem 1.4. *(Higson-Kasparov) If G is a second countable locally compact Hausdorff group which has the Haagerup property then G satisfies BCC.*

Let F be a global field and consider the group $GL(2,F)$. At each place v of F we have the local group $GL(2,F_v)$. Each local group $GL(2,F_v)$ is a-T-menable by [6, p.91]. By Theorem 1.4, each local group satisfies BCC.

Let $\mathbb{A}_F$ be the adele ring attached to F. By Theorem 1.2, we have

Theorem 1.5. *The adelic group $GL(2,\mathbb{A}_F)$ satisfies BCC.*

By Theorem 1.3, we have

Theorem 1.6. *The discrete group $GL(2,F)$, and each of its subgroups, satisfies BCC.*

Our method is therefore an example of the Hasse principle (local-global principle) applied to the *local* groups $G(F_v)$ and the *discrete* group $G(F)$.

It is worth noting that if the Baum-Connes conjecture fails for the discrete group $SL(n,\mathbb{Z})$, then BCC fails for $SL(n,\mathbb{R})$.

We note that in [6, 6.1.2] it is proved that the adelic group $SL(2,\mathbb{A})$ is a-T-menable. It then follows from [6, 6.1.6] and the short exact sequence

$$1 \to SL(2,\mathbb{A}) \to GL(2,\mathbb{A}) \to^{det} \mathbb{A}^\times \to 1$$

that $GL(2, \mathbb{A})$ is a-T-menable, and therefore satisfies BCC.

Our method is new, but does not at present create new examples of groups which satisfy BCC. It does, however, raise the following question.

Consider the group $SO(n, 1, \mathbb{Q})$ and the local groups $SO(n, 1, \mathbb{Q}_p)$ with $2 \leq p \leq \infty$. We know from [12] that $SO(n, 1, \mathbb{R})$ satisfies BCC.

QUESTION. Do the p-adic groups $SO(n, 1, \mathbb{Q}_p)$ satisfy BCC ?

A positive answer to this would prove (due to Theorem 1.3) that the discrete group $SO(n, 1, \mathbb{Q})$ satisfies BCC.

If $\mathbb{Q}_p$ admits a square root of -1 then the split-rank of $SO(n, 1, \mathbb{Q}_p)$ is equal to the Witt index (the dimension of a maximal isotropic subspace) of the quadratic form

$$x_0^2 - x_1^2 + x_2^2 - x_3^2 + \cdots ,$$

see [7, p. 222]; thanks to Alain Valette for this remark. The Witt index is $[(n+1)/2]$, the integer part of $(n+1)/2$. Therefore, if $n \geq 3$, the affine building of $SO(n, 1, \mathbb{Q}_p)$ is not a tree. In this case, the proof in [10] does not apply.

EXAMPLE. The affine building of $SO(3, 1, \mathbb{Q}_5)$ is of dimension 2.

In the course of our work, we find it necessary to use a model $P_c(G)$ of the universal example for proper actions of G which is itself a direct limit of compact spaces. This model $P_c(G)$ is paracompact, Hausdorff and separable, but not metrizable, and so falls outside the discussion of proper actions in [1]. We have therefore to choose a different starting point for the theory of proper actions: we use the definition of Bourbaki [4].

This paper is a sequel to our Note [2]. We thank Siegfried Echterhoff, Pierre Julg, Ryszard Nest for valuable conversations, and the two referees for their detailed comments.

2. The ring of adeles

A *local field* is a non discrete locally compact topological field. It is shown in [15] that a local field F must be of the following form. If $\mathrm{char}(F) = 0$, then $F = \mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ for some prime p. If $\mathrm{char}(F) = p > 0$, then F is the field $\mathbb{F}_q((X))$ of formal Laurent series (with finite tail) in one variable with coefficients in a finite field $\mathbb{F}_q$.

The fields $\mathbb{R}, \mathbb{C}$ are known as *archimedean fields*. All other local fields are known as *nonarchimedean fields*. The topology on a nonarchimedean field is always totally disconnected.

Let $\mathbb{F}_p(t)$ denote the field of fractions of the polynomial ring $\mathbb{F}_p[t]$. A *global field* is a finite extension of $\mathbb{Q}$, or a finite extension of the function field $\mathbb{F}_p(t)$. A *completion* (v, F_v) of F is a dense isomorphic embedding v of F into a local field F_v. Two completions (v, F_v), (u, F_u) are said to be equivalent if there is an isomorphism ρ of F_v onto F_u such that $u = \rho \circ v$. A *place* of F is an equivalence class of completions. We say the place (v, F_v) is *infinite* if F_v is an archimedean field and *finite* otherwise. If $\mathrm{char}(F) = p > 0$, then F has countably many finite places and *no* infinite places. If $\mathrm{char}(F) = 0$ then F has countably many finite places and finitely many (but at least one) infinite places.

Suppose we have an ascending sequence of topological spaces

$$X_1 \subset X_2 \subset X_3 \subset \ldots$$

Then we can give the union $X = \cup X_n$ the *direct limit topology*: that is a set is open in X if and only if it has open intersection with each X_n.

The following Lemma is immediate.

Lemma 2.1. *Let X be a topological space and let $X_1, X_2, X_3, \ldots$ be open subsets of X such that $X_1 \subset X_2 \subset X_3 \subset \ldots$ and $X = \cup_n X_n$. Then any compact subset of X lies entirely within some X_n.*

Suppose that for each $n = 1, 2, 3, \ldots$ we have a locally compact, second countable and Hausdorff topological group G_n, such that G_m is an *open* subgroup of G_n for $m \leq n$:

$$G_1 \subset G_2 \subset G_3 \subset \ldots$$

Let $G = \cup_{n=1}^{\infty} G_n$ and furnish this with the direct limit topology. Then G is a topological group which is locally compact, second countable and Hausdorff.

Any nonarchimedean local field F_v contains a unique maximal compact open subring $\mathcal{O}_v$. Let S denote any finite set of places of F which contains all the infinite places. By an *adele* we mean an element $a = (a)_v$ of the product $\prod_v F_v$ such that $a \in A_S = \prod_{v \in S} F_v \times \prod_{v \notin S} \mathcal{O}_v$ for some S. The adeles of F form a ring $\mathbb{A}_F$, addition and multiplication being defined componentwise. Each A_S has its natural topology and $\mathbb{A}_F = \cup_S A_S$ is topologized as the inductive limit with respect to S. There is an obvious embedding of F in $\mathbb{A}_F$, by means of which we identify F with a subring of $\mathbb{A}_F$. The field F is a discrete cocompact subfield of the non-discrete locally compact semisimple commutative ring $\mathbb{A}_F$.

Now suppose G is a linear algebraic group defined over F. We shall be interested in the *adelic group* $G(\mathbb{A}_F)$ of $\mathbb{A}_F$-rational points of G. For a finite place v of F let $G(\mathcal{O}_v)$ denote the group $G(F_v) \cap GL_n(\mathcal{O}_v)$. We set

$$G_S = \prod_{v \in S} G(F_v) \times \prod_{v \notin S} G(\mathcal{O}_v)$$

Then $G(\mathbb{A})$ is equal, by definition, to the direct limit of the groups G_S. We now equip $G(\mathbb{A})$ with the direct limit topology, following Weil [14, p.2]. Then $G(\mathbb{A})$ is a locally compact second countable Hausdorff group. The map $x \mapsto (x, x, x, \ldots)$ embeds $G(F)$ as a *discrete* subgroup of $G(\mathbb{A})$.

3. Proper actions and universal examples

We recall that a topological space X is *completely regular* if it satisfies the following separation axiom: If B is a closed subset of X and $p \in X \setminus B$ then there exists a continuous function $f : X \to [0, 1]$ such that $f(p) = 0$ and $f(B) = \{1\}$. Let G be a locally compact, Hausdorff, second countable group. A *G-space* is a topological space X with a given continuous action of G such that

- X is completely regular and any compact subset of X is metrizable.
- The quotient space X/G is paracompact and Hausdorff.

Note that the definition of a G-space in [1] uses the slightly more restrictive conditions that X and X/G be metrizable. Nothing is altered in [1] if this is replaced throughout with the above relaxed conditions.

The following definition may be found in Bourbaki [4, Definition 1, p.250, Prop.7, p.255].

Definition 3.1. The action of G on a G-space X is *proper* if given any two points $x, y \in X$ there are open neighbourhoods U_x, U_y of x and y respectively such that the set

$$\{g \in G : gU_x \cap U_y \neq \emptyset\}$$

has compact closure in G. A proper G-space X is said to be *G-compact* if the quotient X/G is compact.

In appendix A we give the full statement of Theorem 3.8 in Biller [3]. This theorem, on the existence of slices, reconciles the Bourbaki definition of proper actions with the definition in [1]. In particular, if X is a proper G-space such that X and and X/G are metrizable then X satisfies the condition which was taken in [1] to be the definition of proper. The reverse implication is also valid. If X is a metrizable G-space which is proper in the sense of [1], then X is a proper G-space.

Note that if X is locally compact then this is equivalent to the following condition: if K_1, K_2 are compact subsets of X then the set

$$\{g \in G : gK_1 \cap K_2 \neq \emptyset\} \text{ is compact.}$$

It is easy to see any locally compact Hausdorff group acts properly on itself. Also note if X is a proper G-space and Y is any G-invariant subset of X then Y is a proper G-space when equipped with the subspace topology and obvious action of G. We shall want to define KK groups for the algebra $C_0(X)$ of a G-compact proper G-space and the following ensures that these algebras are separable.

Proposition 3.2. *Suppose G acts properly on X and X/G is compact. Then X is locally compact and second countable.*

Proof. The space X is locally compact by [4, (e), p.310]. There exists a compact set $S \subset X$ such that $X = G \cdot S$.

By assumption any compact subset of X is metrizable, and we may deduce that S is second countable when given the subspace topology from X.

The action of G on X gives rise to a map

$$G \times S \to GS = X,$$

which is continuous, open and surjective. By assumption G is second countable and so clearly X must be second countable. □

Definition 3.3. Let X be a proper G-space. A *cutoff function* on X is a function $c : X \to \mathbb{R}_+$ such that the support of c has compact intersection with GK for any compact subset K of X and

$$\int c(g^{-1}x)dg = 1, \qquad \text{for each } x \in X$$

Note the set of all such functions is convex. For the proof of the next Proposition, see Tu [13, 6.11].

Proposition 3.4. *Let X be a G-compact proper G-space. Then there exists a cutoff function on G.*

Definition 3.5. Let X, Y be proper G-spaces. A continuous map $\varphi : X \to Y$ is a called a *G-map* if

$$\varphi(gx) = g\varphi(x) \text{ for all } g \in G, x \in X$$

Two G-maps are *G-homotopic* if they are homotopic through G-maps.

Definition 3.6. A universal example for proper actions of G, denoted $\underline{E}G$, is a proper G-space with the following property: If X is any proper G-space, then there exists a G-map $f : X \to \underline{E}G$, and any two such maps are G-homotopic.

Let K be any compact subset of G. The set of all probability measures on K — denoted $\mathcal{P}(K)$ — is a separable compact Hausdorff space in the topology induced from the weak* topology on $C(K)$. Recall $\mu_i \to \mu$ in the weak* topology if and only if

$$\int f d\mu_i \to \int f d\mu \quad \text{for any } f \in C(K).$$

If $K_1 \subset K_2$ are compact subgroups of G then define the following map

$$\iota : \mathcal{P}(K_1) \to \mathcal{P}(K_2), \quad (\iota\mu)(U) = \mu(U \cap K_1) \text{ for each Borel set } U \subseteq K_2$$

This is clearly injective and continuous. So ι is an injective map from a compact space to a Hausdorff space and hence gives rise to a homeomorphism between $\mathcal{P}(K_1)$ and its image in $\mathcal{P}(K_2)$ with the subspace topology. To simplify notation we identify $\mathcal{P}(K_1)$ with its image in $\mathcal{P}(K_2)$.

As G is locally compact and second countable it is clearly σ-compact and by [4, p.94, Prop. 15 and Cor. 2] we may find compact subsets $K_1 \subset K_2 \subset K_3 \subset \dots$ with $G = \cup_{i=1}^{\infty} K_i$ such that *any* compact subset of G is contained within some K_i. Now define

$$\mathcal{P}_c(G) = \cup_{i=1}^{\infty} \mathcal{P}(K_i).$$

Clearly $\mathcal{P}_c(G)$ consists of all compactly supported probability measures on G. We topologize $\mathcal{P}_c(G)$ by giving it the direct limit topology with respect to the sequence $\mathcal{P}(K_i)$. This topology is independent of the sequence K_i.

The group G acts on $\mathcal{P}_c(G)$ by setting

$$(g\mu)(U) = \mu(g^{-1}U) \text{ for any Borel set } U \subseteq G.$$

Lemma 3.7. *The action of G on $\mathcal{P}_c(G)$ is continuous.*

Proof. Let $(g_\alpha, \mu_\alpha) \to (g, \mu)$, $\alpha \in A$, be a convergent net in $G \times \mathcal{P}_c(G)$. By definition of the direct limit topology on $\mathcal{P}_c(G)$, we must have $\mu_\alpha \to \mu$ in $\mathcal{P}(K_i)$ for some i. Given $f \in C(K_i)$ and $\epsilon > 0$ we can find α_0 with the property that

$$\alpha > \alpha_0 \Rightarrow \left| \int f \, d\mu_\alpha - \int f \, d\mu \right| \leq \epsilon/2$$

Furthermore $g_\alpha \to g$ and so we may choose α_0 in such a way that we also have

$$\alpha > \alpha_0 \Rightarrow \|f_{g_\alpha} - f_g\| \leq \epsilon/2$$

where f_g denotes the function $x \mapsto f(g^{-1}x)$. Finally note

$$\begin{aligned}\left|\int f\, d(g_\alpha\mu_\alpha) - \int f\, d(g\mu)\right| &= \left|\int f_{g_\alpha}\, d\mu_\alpha - \int f_g\, d\mu\right| \\ &\leq \left|\int (f_{g_\alpha} - f_g)\, d\mu_\alpha\right| + \left|\int f_g\, d\mu_\alpha - \int f_g\, d\mu\right| \\ &\leq \epsilon/2 + \epsilon/2\end{aligned}$$

This shows that $g_\alpha\mu_\alpha \to g\mu$ in $\mathcal{P}(K_i)$ and therefore $g_\alpha\mu_\alpha \to g\mu$ in $\mathcal{P}_c(G)$. □

Lemma 3.8. *The action of G on $\mathcal{P}_c(G)$ is proper.*

Proof. Take $\mu \in \mathcal{P}_c(G)$ and let f_μ be a continuous compactly supported function $f_\mu : G \to [0,1]$ with $f_\mu \equiv 1$ on $\mathsf{supp}\,(\mu)$.

Then the set

$$\begin{aligned} U_\mu &= \left\{\lambda \in \mathcal{P}_c(G) : \left|\int f_\mu\, d\mu - \int f_\mu\, d\lambda\right| \leq 1/2\right\} \\ &= \left\{\lambda \in \mathcal{P}_c(G) : \int f_\mu\, d\lambda > 1/2\right\}\end{aligned}$$

is an open neighbourhood of μ in $\mathcal{P}_c(G)$.

Now take any $\mu, \nu \in \mathcal{P}_c(G)$ and assume $gU_\mu \cap U_\nu \neq \emptyset$ for some $g \in G$. Indeed let $\lambda \in gU_\mu \cap U_\nu$. Then

$$\int g^{-1} f_\mu\, d\lambda > 1/2 \quad \text{and} \quad \int f_\nu\, d\lambda > 1/2$$

If $\mathsf{supp}\,(g^{-1}f_\mu)$ and $\mathsf{supp}\,(f_\nu)$ are disjoint then we have $0 \leq g^{-1}f_\mu + f_\nu \leq 1$. However by the above

$$\int g^{-1} f_\mu + f_\nu\, d\lambda > 1$$

clearly contradicting the fact that λ is a probability measure. Hence we may conclude that $g\,\mathsf{supp}\,(\mu) \cap \mathsf{supp}\,(\nu) \neq \emptyset$ and so

$$\{g \in G : gU_\mu \cap U_\nu \neq \emptyset\} \subset \{g \in G : g\,\mathsf{supp}\,(\mu) \cap \mathsf{supp}\,(\nu) \neq \emptyset\}.$$

Now both μ and ν are compactly supported and since any group acts properly on itself the larger set here is compact. □

Theorem 3.9. *The space $\mathcal{P}_c(G)$ is a universal example for proper actions of G.*

Proof. Let X be *any* proper G-space, we aim to show there exists a G-equivariant map $X \to \mathcal{P}_c(G)$. Take any $x \in X$. By [3, Theorem 3.8] there is a G-invariant open neighbourhood U_x of x, a compact subgroup H of G, and a G-equivariant map

$$\rho : U_x \to G/H.$$

Let μ_H denote Haar measure on the compact subgroup H, normalized to have total mass 1. There is an obvious G-equivariant map

$$\Psi : G/H \to \mathcal{P}_c(G), \quad gH \mapsto g \cdot \mu_H$$

and let

$$\theta_x = \Psi \circ \rho : U_x \to \mathcal{P}_c(G).$$

Now X may be covered by such neighbourhoods and if π denotes the quotient map $X \to X/G$ then $\{\pi(U_x)\}$ is an open cover of X/G. Recall that by definition X/G is paracompact and Hausdorff and so there is a locally finite partition of unity subordinate to the cover $\{\pi(U_x)\}$. Precisely there are continuous maps

$$\omega_x : X/G \to [0,1], \quad \text{for each } x \in X$$

with $\mathsf{supp}\,(\omega_x) \subseteq \pi(U_x)$ and for each $[y] \in X/G$ we have

$$\omega_x([y]) = 0 \text{ for almost all } x \in X \text{ and } \sum_{x\in X} \omega_x([y]) = 1.$$

Now

$$\Xi : X \to \mathcal{P}_c(G), \quad \Xi(y) = \sum_{x\in X} \omega_x(\pi(y))\theta_x(y)$$

is the required map. Note that (as remarked above) $P_c(G)$ is a convex set and this convexity is being used in the construction of Ξ.

Finally, if $\varphi_1, \varphi_2 : X \to \mathcal{P}_c(G)$ are G-equivariant maps then they are G-homotopic via

$$t\varphi_1 + (1-t)\varphi_2, \quad t \in [0,1].$$

So $\mathcal{P}_c(G)$ is a universal example for G. □

Baum-Connes conjecture with coefficients. If A is any G-C^*-algebra then we may define

$$K_*^{\mathrm{top}}(G,A) = \varinjlim_{\substack{\text{G-invariant} \\ \text{G-compact} \\ Z\subseteq \underline{E}G}} KK_*^G(C_0(Z), A)$$

We say a group G satisfies the Baum-Connes conjecture with coefficients if for every G-C^*-algebra A the map

$$\mu_A : K_*^{\mathrm{top}}(G,A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

4. K-theory for ascending unions of groups

Theorem 4.1. *Let H be an open subgroup of G. Then the inclusion of H in G determines a homomorphism of abelian groups*

$$\mathcal{T}_H^G : K_*(A \rtimes_r H) \to K_*(A \rtimes_r G).$$

Furthermore suppose $G_1 \subset G_2 \subset G_3 \subset \dots$ is an ascending sequence of open subgroups , then there is an inductive system of abelian groups

$$K_*(A \rtimes_r G_1) \xrightarrow{\mathcal{T}_{G_1}^{G_2}} K_*(A \rtimes_r G_2) \xrightarrow{\mathcal{T}_{G_2}^{G_3}} K_*(A \rtimes_r G_3) \xrightarrow{\mathcal{T}_{G_3}^{G_4}} \dots$$

If $G = \cup G_n$ we have

$$K_*(A \rtimes_r G) = \varinjlim_n K_*(A \rtimes_r G_n).$$

Proof. If H is an open subgroup of G and A is a G–C^*–algebra then the canonical inclusion $\iota : C_c((H, A) \to C_c(G, A)$ embeds the $*$-algebra $C_c(H, A)$ into the $*$-algebra $C_c(G, A)$. Since it preserves convolution and involution, it extends to an isometric embedding of the Hilbert A-module $F := L^2(H, A)$ into the Hilbert A-module $E := L^2(G, A)$ with respect to the inner product $< \xi, \eta >_A = \xi^* * \eta(e)$ for $\xi, \eta \in C_c(G, A)$. It is then easy to check that

$$\begin{aligned} \|\iota f\|_{A \rtimes_r G} &= \sup \{ \|\iota f * \xi\|_E : \xi \in E, \|\xi\|_E \leq 1 \} \\ &= \sup \{ \|f * \xi\|_F : \xi \in F, \|\xi\|_F \leq 1 \} \\ &= \|f\|_{A \rtimes H} , \end{aligned}$$

which implies that ι extends to an injective $*$-homomorphism

$$\iota : A \rtimes_r H \to A \rtimes_r G.$$

We then define

$$\mathcal{T}_H^G = \iota_* : K_*(A \rtimes_r H) \to K_*(A \rtimes_r G).$$

If $G = \cup_n G_n$ for the ascending sequence of open subgroups G_n, then Lemma 2.1 implies that $C_c(G, A) = \cup_n C_c(G_n, A)$. But this implies that

$$A \rtimes_r G = \overline{\cup_n \iota(A \rtimes_r G_n)}$$

and hence

$$K_*(A \rtimes_r G) = K_*(\varinjlim_n (A \rtimes_r G_n) = \varinjlim_n K_*(A \rtimes_r G_n).$$

□

5. Equivariant K-homology for ascending sequences of groups

Theorem 5.1. *Let H be an open subgroup of G. Then the inclusion of H in G determines a homomorphism of abelian groups*

$$\mathcal{R}_H^G : K_*^{top}(H, A) \to K_*^{top}(G, A).$$

Furthermore suppose $G_1 \subset G_2 \subset G_3 \subset \dots$ is an open ascending sequence of groups , then there is an inductive system of abelian groups

$$K_*^{top}(G_1, A) \xrightarrow{\mathcal{R}_{G_1}^{G_2}} K_*^{top}(G_2, A) \xrightarrow{\mathcal{R}_{G_2}^{G_3}} K_*^{top}(G_3, A) \xrightarrow{\mathcal{R}_{G_3}^{G_4}} \dots$$

If $G = \cup G_n$ then we have

$$K_*^{top}(G, A) = \varinjlim_n K_*^{top}(G_n, A)$$

In the course of proving this result, we shall make use of the *reciprocity isomorphism* [5, p.157] in equivariant KK-theory. Let H be an open subgroup of G, let A be an $H - C^*$-algebra, let $Ind_H^G A$ denote the induced algebra and let B be a $G - C^*$-algebra. Then we have the reciprocity isomorphism:

$$Inf_H^G : KK_*^H(A, B) \cong KK_*^G(Ind_H^G A, B).$$

If A is commutative we have $A \cong C_0(X), Ind_H^G A \cong C_0(G \times_H X)$ and so we have

$$Inf_X : K_*^H(C_0(X), B) \to K_*^G(C_0(G \times_H X), B).$$

Lemma 5.2. *Let H be an open subgroup of G and let X be any H-compact subset of $\mathcal{P}_c(H) \subset \mathcal{P}_c(G)$ then $G \times_H X \cong G \cdot X$ as G-spaces.*

Proof. Recall the definition of the space $G \times_H X$. The group H acts on the product $G \times X$ by setting $h \cdot (g, x) = (gh^{-1}, hx)$ and $G \times_H X$ is the quotient. The action of G on $G \times_H X$ is given by $g' \cdot [g, x] = [g'g, x]$, where $[g, x]$ is the equivalence class of the pair (g, x).

The map F_X is defined as follows:

$$F_X : G \times_H X \to G \cdot X, \ [g, x] \mapsto gx.$$

The map F_X is clearly surjective and G-equivariant. To show this map is injective take $[g_1, x_1], [g_2, x_2] \in G \times_H X$ with $g_1 x_1 = g_2 x_2$. Recall that x_1, x_2 are in fact probability measures on H and so

$$1 = x_1(H) = g_1^{-1} g_2 x_2(H) = x_2(g_2^{-1} g_1 H)$$

As x_2 is a measure in $\mathcal{P}_c(H)$ it will clearly be equal to zero on the coset $g_2^{-1}g_1H$ unless $g_2^{-1}g_1 \in H$. Now $g_2^{-1}g_1 \cdot (g_1, x_1) = (g_1g_1^{-1}g_2, g_2^{-1}g_1x_1) = (g_2, x_2)$ as required.

It now remains to show this map is a homeomorphism. Let π denote the quotient map $G \times X \to G \times_H X$ and $\theta : G \times X \to G \cdot X$ denote the map given by the action of G on X. Both π and θ are open, continuous maps. Since $\theta = F_X \circ \pi$, this implies that F_X is open and continuous. ☐

Let $\mathcal{R}^G_{H,X}$ denote the composition

$$KK^H_*(C_0(X), A) \xrightarrow{\mathrm{Inf}_X} KK^G_*(C_0(G \times_H X), A) \xrightarrow{(F_X)_*} KK^G_*(C_0(G \cdot X), A)$$

If X and Y are H-compact subsets of $\mathcal{P}_c(H)$ with $X \subset Y$ then the following diagram commutes,

$$\begin{array}{ccc} KK^H_*(C_0(X), A) & \longrightarrow & KK^H_*(C_0(Y), A) \\ {\scriptstyle \mathcal{R}^G_{H,X}}\big\downarrow & & {\scriptstyle \mathcal{R}^G_{H,Y}}\big\downarrow \\ KK^G_*(C_0(G \cdot X), A) & \longrightarrow & KK^G_*(C_0(G \cdot Y), A) \end{array}$$

where each horizontal map is given by inclusion of the spaces involved.

Now $\mathcal{P}_c(H), \mathcal{P}_c(G)$ are universal examples for H, G respectively. If X is an H-compact subset of $\mathcal{P}_c(H)$ then $G \cdot X$ is a G-compact subset of $\mathcal{P}_c(G)$. Due to the above the following map is well defined

$$\mathcal{R}^G_H : K^{\mathrm{top}}_*(H, A) \to K^{\mathrm{top}}_*(G, A).$$

Now let $G_1 \subset G_2 \subset G_3 \subset \dots$ be an ascending sequence of open subgroups and let $G = \cup G_n$. There is then an inductive system of abelian groups

$$K^{\mathrm{top}}_*(G_1, A) \xrightarrow{\mathcal{R}^{G_2}_{G_1}} K^{\mathrm{top}}_*(G_2, A) \xrightarrow{\mathcal{R}^{G_3}_{G_2}} K^{\mathrm{top}}_*(G_3, A) \xrightarrow{\mathcal{R}^{G_4}_{G_3}} \dots$$

Lemma 5.3. *There exist compact sets $\Delta_1 \subset \Delta_2 \subset \Delta_3 \subset \dots$ such that $\Delta_n \subset G_n$ and $\cup_n$ Interior$(\Delta_n) = G$. Set $Z^{n,m} = G_n \cdot P_c(G_n \cap \Delta_m)$. Then*

- $Z^{n,m} \subset P_c(G_n)$
- *$Z^{n,m}$ is preserved by G_n and is G_n-compact*
- $Z^{n,m} \subset Z^{n,m+1}$
- $\cup_m Z^{n,m} = P_c(G_n)$
- $G_{n+1} \cdot Z^{n,m} \subset Z^{n+1,m}$

Proof. Since G is σ-compact, there exists an ascending sequence K_n of compact subsets of G such that $G = \cup_n K_n$. Let V be a fixed compact neighbourhood of the identity and put $\Delta_n = (K_n \cdot V) \cap G_n$. Then the sequence (Δ_n) has all the desired properties.

Any compact set in G is contained in some Δ_n.

□

Lemma 5.4. *Let $\{A^{m,n}\}$ be a commutative diagram of abelian groups in which the typical commutative square is*

$$\begin{array}{ccc} A^{n,m} & \longrightarrow & A^{n,m+1} \\ \downarrow & & \downarrow \\ A^{n+1,m} & \longrightarrow & A^{n+1,m+1} \end{array}$$

with $n, m = 1, 2, 3, \ldots$. Then there is a canonical isomorphism of abelian groups:

$$lim_{m\to\infty}(lim_{n\to\infty} A^{m,n}) \cong lim_{n\to\infty}(lim_{m\to\infty} A^{m,n}).$$

Proof. Each side is canonically isomorphic to the direct limit $\varinjlim A^{n,n}$ of the directed system $\{A^{n,n}\}$. □

Theorem 5.5. *$lim_{n\to\infty} K_*^{top}(G_n, A) = K_*^{top}(G, A)$.*

Proof. Let $K_j^G(Z^{n,m}, A) = KK_j^G(C_0(Z^{n,m}, A)$ and consider the commutative diagram of abelian groups in which the typical commutative square is:

$$\begin{array}{ccc} K_j^{G_n}(Z^{n,m}, A) & \longrightarrow & K_j^{G_n}(Z^{n,m+1}, A) \\ {\scriptstyle \rho_n}\downarrow & & {\scriptstyle \rho_n}\downarrow \\ K_j^{G_{n+1}}(Z^{n+1,m}, A) & \longrightarrow & K_j^{G_{n+1}}(Z^{n+1,m+1}, A) \end{array}$$

with $n, m = 1, 2, 3, \ldots$. Each horizontal arrow

$$K_j^{G_n}(Z^{n,m}, A) \to K_j^{G_n}(Z^{n,m+1}, A)$$

is the map of abelian groups determined by the inclusion $Z^{n,m} \to Z^{n,m+1}$. Each vertical map $\rho_n : K_j^{G_n}(Z^{n,m}, A) \to K_j^{G_{n+1}}(Z^{n+1,m}, A)$ is the map $\mathcal{R}_{G_n}^{G_{n+1}}$ followed by the map of abelian groups induced by the inclusion $G_{n+1} \cdot Z^{n,m} \to Z^{n+1,m}$.

We will write

$$A^{n,m} = K_j^{G_n}(Z^{n,m}, A).$$

Each $Z^{n,m}$ is G_n-compact, $\cup_m Z^{n,m} = P_c(G_n)$ and any G_n-compact set in $P_c(G_n)$ is contained in some $Z^{n,m}$. Taking the direct limit along the nth row, we have

$$lim_{m\to\infty} A^{m,n} = K_j^{top}(G_n, A).$$

If we now take the direct limit in a vertically downward direction, we obtain

$$lim_{n\to\infty} K_j^{top}(G_n, A).$$

Now we fix attention on the mth column. If $n \geq m$ then $G_n \supset \Delta_m$ and so $G_n \cap \Delta_m = \Delta_m$. Then we have

$$A^{n,m} = K_j^{G_n}(G_n \cdot P_c(\Delta_m), A).$$

Now the G_n-saturation of $P_c(\Delta_m)$ is equal to the G-saturation of $P_c(\Delta_m)$. Now we apply the reciprocity isomorphism and we have

$$A^{n,m} \cong K_j^G(G \cdot P_c(\Delta_m), A).$$

if $n \geq m$. Therefore the mth column *stabilizes* as soon as $n \geq m$. Therefore the direct limit down the mth column is given by

$$lim_{n\to\infty} A^{m,n} = K_j^G(G \cdot P_c(\Delta_m), A)).$$

Now the sets $G \cdot P_c(\Delta_m)$ are cofinal in G-compact sets in $P_c(G)$. Taking the direct limit in the horizontal direction, we have

$$lim_{m\to\infty} lim_{n\to\infty} A^{m,n} \cong K_j^{top}(G, A).$$

By Lemma 5.4 we have

$$lim_{n\to\infty} K_j^{top}(G_n, A) \cong K_j^{top}(G, A).$$

□

6. Adelic groups

For H an open subgroup of G we have constructed a homomorphism from $K_*^{\mathrm{top}}(H, A)$ to $K_*^{\mathrm{top}}(G, A)$, and likewise for the K theory of the reduced crossed product C^*-algebras. We wish to check that these maps are compatible with the Baum–Connes μ map, i. e. that the following diagram commutes.

$$\begin{array}{ccc} K_*^{\mathrm{top}}(H, A) & \xrightarrow{\mu_H} & K_*(A \rtimes_r H) \\ \mathcal{R}_H^G \downarrow & & \downarrow \mathcal{I}_H^G \\ K_*^{\mathrm{top}}(G, A) & \xrightarrow{\mu_G} & K_*(A \rtimes_r G) \end{array}$$

As a first step, we prove that the reciprocity isomorphism Inf_X is compatible with the Baum-Connes μ map.

Lemma 6.1. *Let A be a $G-C^*$-algebra, let H be an open subgroup of G and let X be a locally compact proper H-compact H-space. Then the following diagram commutes:*

$$\begin{array}{ccc} KK^H_*(C_0(X),A) & \xrightarrow{\mu_H} & K_*(A\rtimes_r H) \\ {\scriptstyle Inf_X}\downarrow & & {\scriptstyle \mathcal{I}^G_H}\downarrow \\ KK^G_*(C_0(G\times_H X),A) & \xrightarrow{\mu_G} & K_*(A\rtimes_r G) \end{array}$$

Proof. The inverse of the reciprocity isomorphism Inf is the *compression isomorphism* [5, p. 157] and is given by the composition $i_*\circ \mathrm{Res}_X$, where Res_X is the obvious restriction map $KK^G_*(C_0(X),A)\to KK^H_*(C_0(X),A)$ and i is the inclusion $C_0(X)\hookrightarrow C_0(G\times_H X)$ given by

$$i(f)[g,x]=\begin{cases} f(gx) & \text{if } g\in H \\ 0 & \text{otherwise.}\end{cases}$$

Each of these maps is clearly functorial.

There is a commutative diagram

$$\begin{array}{ccc} KK^H_*(C_0(X),A) & \xrightarrow{j_H} & KK_*(C_0(X)\rtimes_r H, A\rtimes_r H) \\ {\scriptstyle i_*}\uparrow & & {\scriptstyle i'_*}\uparrow \\ KK^H_*(C_0(G\times_H X),A) & \xrightarrow{j_H} & KK_*(C_0(G\times_H X)\rtimes_r H, A\rtimes_r H) \\ {\scriptstyle \mathrm{Res}_X}\uparrow & & {\scriptstyle p_*\circ q_*}\downarrow \\ KK^G_*(C_0(G\times_H X),A) & \xrightarrow{j_G} & KK_*(C_0(G\times_H X)\rtimes_r G, A\rtimes_r G) \end{array}$$

in which j_H, j_G are *descent* homomorphisms. Here p denotes the map

$$p: C_0(G\times_H X)\rtimes_r G\to C_0(G\times_H X)\rtimes_r H$$

induced from the obvious restriction map $C_c(G,A)\to C_c(H,A)$, with $A=C_0(G\times_H X)$. And q denotes the map

$$q: A\rtimes_r H\to A\rtimes_r G$$

of Theorem 4.1.

Now let c be a cutoff function on the proper H-space X, we claim $i(c)\in C_0(G\times_H X)$ is a cutoff function on the proper G-space $G\times_H X$. To see this take any $[g_0,x]\in G\times_H X$. Then by definition we have

$$i(c)([g_0,x])=0$$

unless $g_0\in H$ in which case

$$i(c)[g_0,x]=c(g_0x).$$

Up to a normalizing factor between the Haar measures on H and G

$$\begin{aligned}\int_G i(c)(g^{-1}[g_0,x])\,dg &= \int_G i(c)(g^{-1}g_0^{-1}[g_0,x])\,dg \\ &= \int_G i(c)([g^{-1},x])\,dg \\ &= \int_H c(h^{-1}x)\,dh = 1.\end{aligned}$$

If K is any compact subset of $G \times_H X$ and if F denotes the homeomorphism between $G \times_H X$ and $G \cdot X$ then $F(GK) = G \cdot F(K)$ and $F(K)$ is compact. Also note $F(\mathsf{supp}\,(i(c)) \subseteq H \cdot \mathsf{supp}\,(c)$ and so

$$F(GK \cap \mathsf{supp}\,(i(c))) \subseteq G \cdot F(K) \cap H \cdot \mathsf{supp}\,(c) \subseteq H \cdot F(K) \cap \mathsf{supp}\,(c)$$

which is compact and so $GK \cap \mathsf{supp}\,(i(c))$ is compact. So we have shown $i(c)$ is a cutoff function on $G \times_H X$.

Let λ_X denote the projection in the twisted convolution algebra $C_c(H \times X)$ arising from the cutoff function c:

$$\lambda_Z(g,x) = c_Z(x)^{1/2} c_Z(g^{-1}x)^{1/2} \Delta(g)^{-1/2}$$

and let $\lambda_{G \times_H X}$ denote the projection in $C_c(G \times G \times_H X)$ arising from the cutoff function $i(c)$.

Then $p(\lambda_{G \times_H X})$ is simply the restriction of $\lambda_{G \times_H X}$ to $H \times G \times_H X$, and for any h in G and any $[g,x] \in G \times_H X$

$$\begin{aligned} p(\lambda_{G\times_H X})(h,[g,x]) &= i(c)[g,x]^{1/2} i(c)(h^{-1}[g,x])^{1/2} \Delta_G(h)^{-1/2} \\ &= \begin{cases} c(gx)^{1/2} c(h^{-1}gx) \Delta_H(h)^{-1/2} & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases} \\ &= i'(\lambda_X) \end{aligned}$$

So the following diagram commutes

$$\begin{array}{ccc} KK_*(C_0(X) \rtimes_r H, A \rtimes_r H) & \xrightarrow{[\lambda_X]\otimes\cdot} & KK_*(\mathbb{C}, A \rtimes_r H) \\ {\scriptstyle i'_*}\uparrow & & \| \\ KK_*(C_0(G \times_H X) \rtimes_r H, A \rtimes_r H) & \xrightarrow{i'_*([\lambda_X])\otimes\cdot} & KK_*(\mathbb{C}, A \rtimes_r H) \\ {\scriptstyle p_*\circ q_*}\downarrow & & \downarrow{\scriptstyle q_*} \\ KK_*(C_0(G \times_H X) \rtimes_r G, A \rtimes_r G) & \xrightarrow{[\lambda_{G\times_H X}]\otimes\cdot} & KK_*(\mathbb{C}, A \rtimes_r G) \end{array}$$

We finish the proof by splicing together these two diagrams. □

Lemma 6.2. *Let X be an H-compact subset of $P_c(H)$. Then the following diagram commutes:*

$$\begin{array}{ccc} KK^H_*(C_0(X), A) & \xrightarrow{\mu_H} & K_*(A \rtimes_r H) \\ \mathcal{R}^G_{H,X} \Big\downarrow & & \mathcal{I}^G_H \Big\downarrow \\ KK^G_*(C_0(G \cdot X), A) & \xrightarrow{\mu_G} & K_*(A \rtimes_r G). \end{array}$$

Proof. By Lemma 5.2 we know that the induced space $G \times_H X$ is G-homeomorphic to the G-saturation $G \cdot X$. We now apply Lemma 6.1. □

Theorem 6.3. *Let G be a locally compact, second countable Hausdorff topological group and let A be a $G - C^*$-algebra. Let G be the union of open subgroups G_n such that the Baum-Connes conjecture with coefficients A is true for each G_n. Then the Baum-Connes conjecture with coefficients A is true for G.*

Proof. We start with the commutative diagram in Lemma 6.2 and take the direct limit over all H-compact subsets of $P_c(H)$. We then obtain the commutative diagram

$$\begin{array}{ccc} K^{\mathrm{top}}_*(H, A) & \xrightarrow{\mu_H} & K_*(A \rtimes_r H) \\ \mathcal{R}^G_H \Big\downarrow & & \Big\downarrow \mathcal{I}^G_H \\ K^{\mathrm{top}}_*(G, A) & \xrightarrow{\mu_G} & K_*(A \rtimes_r G) \end{array}$$

If G is the union of open subgroups G_i each of which satisfies BCC, then applying Theorem 4.1 and Theorem 5.1 along with the above commutative diagram is enough to show that G satisfies BCC. □

Theorem 6.4. *Let F be a global field, $\mathbb{A}$ its ring of adeles, G a linear algebraic group over F. Let F_v denote a place of F. If BCC is true for each local group $G(F_v)$ then BCC is true for the adelic group $G(\mathbb{A})$.*

Proof. Let $v_1, v_2, v_3, \ldots$ be an ordering of the finite places of F, let S_∞ denote the finite set of all infinite places of F, and let $S(n) = \{v_1, v_2, \ldots, v_n\} \cup S_\infty$. Let $G_n = G_{S(n)}$ in the notation of section 2.

If Γ is a compact group then $\underline{\mathrm{E}}\Gamma$ is a point, and we have

$$K^{\mathrm{top}}_*(\Gamma, B) \cong KK^\Gamma_*(B) \cong K_*(B \rtimes \Gamma)$$

by the Green-Julg theorem. Therefore BCC is true for any compact group. Now G_n is a product of finitely many local groups and one compact group. But BCC is stable under the direct product of finitely many groups, by an important result of Chabert-Echterhoff [5, Theorem 3.17]. Therefore BCC is true for each *open* subgroup G_n. Now the adelic group $G(\mathbb{A})$ is the ascending union of the open subgroups G_n, therefore BCC is true for $G(\mathbb{A})$, by Theorem 6.3. □

Appendix A. Biller's theorem

We give here the full statement of Theorem 3.8 in Biller[3]. The stabilizer of $x \in X$ is denoted G_x.

Theorem (Existence of slices). Let G be a locally compact group acting properly on a completely regular space X, and choose $x \in X$. Then there is a convergent filter basis $\mathcal{N}$ that consists of compact subgroups of G normalized by G_x such that for every $N \in \mathcal{N}$, the coset space G/G_xN is a manifold and x is contained in a G_xN-slice for the action of G on X. In particular, some neighbourhood of the orbit $G \cdot x$ is a locally trivial fibre bundle over the manifold G/G_xN.

The dimension of $G \cdot x$ is infinite if and only if $N \in \mathcal{N}$ may be chosen such that the dimension of G/G_xN is arbitrarily high. If the dimension of $G \cdot x$ is finite, then $N \in \mathcal{N}$ may be chosen in such a way that $dim\, G/G_xN = dim\, G \cdot x$.

References

[1] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^* algebras. *Contemporary Math.* 167 (1994) 241–291.

[2] P. Baum, S. Millington, and R. Plymen. A proof of the Baum-Connes conjecture for reductive adelic groups. *C. R. Math. Acad. Sci. Paris*, 332 (2001) 195–200.

[3] H. Biller, Proper actions on cohomology manifolds, *Trans. Amer. Math. Soc.* 355 (2003) 407 – 432.

[4] N. Bourbaki, *General Topology*, Springer-Verlag, 1989.

[5] J. Chabert and S. Echterhoff, Permanence properties of the Baum-Connes conjecture. *Doc. Math.* 6 (2001) 127–183.

[6] P-A. Cherix, M. Cowling, P. Jolissaint, P.Julg and A. Valette, Groups with the Haagerup property, Progress in Math. 197, Birkhauser 2001.

[7] J.E. Humphreys, Linear algebraic groups, Springer-Verlag, 1995.

[8] N. Higson and G. G. Kasparov. E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.* 144 (2001) 23–74

[9] P. Julg, The Baum-Connes conjecture with coefficients for the group $Sp(n, 1)$, C. R. Math. Acad. Sci. Paris 334 (2002) 533 –538.

[10] P. Julg and A. Valette, K-theoretic amenability for $SL_2(\mathbb{Q}_p)$, and the action on the associated tree, J. Functional Analysis 58 (1984) 194 –215.

[11] P. Julg and G.G. Kasparov, Operator K-theory for the group $SU(n, 1)$, J. Reine Angew. Math. 463 (1995) 99 –152.

[12] G.G. Kasparov, Lorentz groups: K-theory of unitary representations and crossed products, Soviet Math. Dokl. 27 (1983) 105 – 109.

[13] J.L. Tu, The Novikov conjecture for hyperbolic foliations, K-Theory 16 (1999) 129 – 184.

[14] A. Weil, *Adeles and algebraic groups.* Birkhauser, 1982.

[15] A. Weil, Basic Number Theory, Springer Classics in Math. 1995.

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