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To my parents, with thanks.

Abstract

In this paper the diameter of the commuting involution graph for the Monster simple group on the $2B$ involution conjugacy class is shown to be 3. As a consequence we deduce that the Monster graph has diameter 5 or 6.

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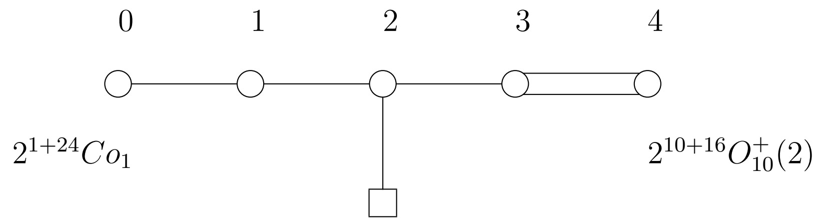
1 Introduction

In seeking paradigms for the finite sporadic simple groups, not surprisingly, eyes were cast toward the rich theory of buildings. Out of this arose geometries for many of the sporadic simple groups modelled, in various ways, upon buildings. Early harvests of these geometries are to be found in Buekenhout [4],[5], Ronan and Smith [13] and Ronan and Stroth [14]. The Monster simple group $\mathbb{M}$, so christened because it is the largest sporadic simple group, possesses a number of p-local geometries for various primes p. Such geometries are constructed making use of certain p-local subgroups which contain a fixed Sylow p-subgroup of $\mathbb{M}$. Here we look at one of these geometries - the maximal 2-local geometry for $\mathbb{M}$, which we denote by Γ. This geometry, which has rank 5, made its debut in [13] and, by analogy with the Coxeter diagram of a building, has the following diagram.



Above the diagram nodes we give the types of the objects of Γ. The objects of type 0 will be referred to as points (of Γ) and those of type 1 lines (of Γ). Also indicated above are the stabilizers in $\mathbb{M}$ of a point and of an object of type 4 (for stabilizers of the other objects see [13]). We let $\mathcal{G}$ denote the point-line collinearity graph of Γ - the vertices of $\mathcal{M}$ are the points of Γ with two (distinct) vertices of $\mathcal{G}$ adjacent in $\mathcal{G}$ if they are incident with a common line in Γ. The number of vertices in $\mathcal{G}$ is

$$5,791,748,068,511,982,636,944,259,375,$$

and, on account of its origin and size, call it the Monster graph.

As part of a long term programme, [15] and [16] accumulate intricate details of the structure of a small part of $\mathcal{G}$. This enterprise seems, at times, to be like hacking through the dense undergrowth of a dark equatorial forest. Here we fly high above the canopy and prove

Theorem 1.1. *The diameter of* $\mathcal{G}$ *is either* 5 *or* 6*.*

Recall that $\mathbb{M}$ has two involution conjugacy classes, $2A_{\mathbb{M}}$ and $2B_{\mathbb{M}}$. We shall use the ATLAS[7] names for conjugacy classes of $\mathbb{M}$ and other groups that we encounter, though often with the addition of a subscript. This extra notational baggage is necessary as we will sometimes be dealing with several groups at the same time.

An alternative description of $\mathcal{G}$ which doesn't mention geometries is to take $2B_{\mathbb{M}}$ as the vertex set and two (different) involutions x and y in $2B_{\mathbb{M}}$ are adjacent whenever $y \in O_2(C_{\mathbb{M}}(x))$. Interestingly $\mathcal{G}$ appears implicitly in the computer construction by Holmes and Wilson [10] of $\mathbb{M}$ as a group of matrices over $GF(3)$.

Theorem 1.1 will be deduced from Theorem 1.2 in which we investigate a particular commuting involution graph. So we next discuss such graphs. Suppose G is a finite group and X a G-conjugacy class of involutions. The commuting involution graph, $\mathcal{C}(G,X)$, is the graph whose vertex set is X and two (distinct) vertices x and y are adjacent in $\mathcal{C}(G,X)$ if x and y commute in G. Letting $\mathrm{Diam}\mathcal{C}(G,X)$ denote the diameter of $\mathcal{C}(G,X)$, we establish the following result.

Theorem 1.2. Diam$\mathcal{C}(\mathbb{M}, 2B_{\mathbb{M}}) = 3$

Throughout this paper we let X denote the $2B_{\mathbb{M}}$ involution conjugacy class of $\mathbb{M}$ and t a fixed involution in X. For C a conjugacy class of $\mathbb{M}$ we define

$$X_C = \{x \in X \mid tx \in C\}.$$

Clearly X is the union of the sets X_C where C runs over all the conjugacy classes of $\mathbb{M}$. Moreover each (non-empty) X_C will be a union of certain $C_G(t)$-orbits of X. Also, $|X_C|$ may be calculated using the well known formula

$$|X_C| = \frac{|\mathbb{M}|}{|C_{\mathbb{M}}(t)||C_{\mathbb{M}}(h)|} \sum_{r=1}^{k} \frac{\chi_r(h)\chi_r(t)\overline{\chi_r(t)}}{\chi_r(1)},$$

where h is a representative from C and $\chi_1, \ldots, \chi_k$ the complex irreducible characters of $\mathbb{M}$. The sets X_C make frequent appearances in the proof of Theorem 1.2 and their sizes may be quickly calculated with the aid of GAP[8].

The well-known fact that two involutions in a group always generate a dihedral group will be useful here. For $g \in \mathbb{M}$ we define

$$C^*_{\mathbb{M}}(g) := \{\, h \in \mathbb{M} \mid g^h = g \text{ or } g^h = g^{-1}\}.$$

Clearly, $[C^*_{\mathbb{M}}(g) : C_{\mathbb{M}}(g)] = 1$ or 2. Observe that for $x \in X$ and $z = tx$, we have $t, x \in C^*_{\mathbb{M}}(z) \setminus C_{\mathbb{M}}(z)$ if the order of z is greater than 2.

For $x, y \in X$, $\partial(x, y)$ will denote the graph theoretical distance between x and y in the graph $\mathcal{C}(\mathbb{M}, X)$ while $d(x, y)$ denotes the distance between x and y in $\mathcal{G}$.

We shall make extensive use of the ATLAS[7] for information on $\mathbb{M}$-conjugacy classes, and will follow the ATLAS notation and conventions with a few perturbations. For $n \in \mathbb{N}$, $Sym(n)$ and $Dih(n)$ will denote, repectively, the symmetric group of degree n and the dihedral group of order n. We also use BM (rather than B) to denote the Baby Monster simple group. As mentioned earlier, we shall append a subscript G to an ATLAS conjugacy class name when the conjugacy class is a conjugacy class of the group G.

2 Preliminary Results

Lemma 2.1. *Suppose that G is a finite group and $g, h \in G$. Set $z = gh$. Then*

$$C_{C_G(g)}(h) = C_G(g) \cap C_G(h) = C_{C_G(z)}(g) = C_{C_G(z)}(h).$$

Proof. Clearly $C_{C_G(g)}(h) = C_G(g) \cap C_G(h)$. If $x \in C_G(g) \cap C_G(h)$, then x centralizes $gh = z$, and so $x \in C_{C_G(z)}(g)$. And if $x \in C_{C_G(z)}(g)$, then x centralizes $g^{-1}z = h$, whence $x \in C_G(g) \cap C_G(h)$. Thus $C_G(g) \cap C_G(h) = C_{C_G(z)}(g)$.Similarly we see that $C_G(g) \cap C_G(h) = C_{C_G(z)}(h)$. □

Lemma 2.2. *Suppose that G is a finite group with x_1, x_2, y_1, y_2 involutions in G. Set $z = x_1x_2$, $w = y_1y_2$. If $\langle z \rangle$ and $\langle w \rangle$ are G-conjugate and $C_G(z)$ has odd order, then $\langle x_1, x_2 \rangle$ and $\langle y_1, y_2 \rangle$ are G-conjugate.*

Proof. By conjugating we may suppose that $\langle y_1y_2 \rangle = \langle z \rangle$ and so $\langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \leqslant C^*_G(z) := \{ g \in G \mid z^g = z \text{ or } z^g = z^{-1} \}$. Since $|C_G(z)|$ is odd, the lemma follows by Sylow's theorems.

□

Lemma 2.3. *Suppose that $a \in 2A_{\mathbb{M}}$, $K = C_{\mathbb{M}}(a)$, $\overline{K} = K/\langle a \rangle$ and $\xi \in K$. Set $X = 2B_{\mathbb{M}}$ and $E = \langle \xi, a \rangle$.*

(i) $\overline{X \cap K} = 2B_{\overline{K}} \cup 2D_{\overline{K}}$.

(ii) If $\overline{\xi} \in 2A_{\overline{K}}$, then E is a $2A_{\mathbb{M}}$-pure fours group.

(iii) If $\overline{\xi} \in 2B_{\overline{K}}$, then $|E \cap 2A_{\mathbb{M}}| = 2$ and $|E \cap 2B_{\mathbb{M}}| = 1$.

(iv) If $\overline{\xi} \in 2D_{\overline{K}}$, then $|E \cap 2A_{\mathbb{M}}| = 1$ and $|E \cap 2B_{\mathbb{M}}| = 2$.

Proof. □

Lemma 2.4. *Suppose $G \cong BM$ and $Y = 2B_G$. Then* $\text{Diam}\, \mathcal{C}(G, Y) = 2$.

Proof. See Theorem 1.1 in [2]. □

To avoid clutter, we suspend our subscripting convention in the next result.

Lemma 2.5. *Suppose that $G \cong BM$ and set $Y = 2B_G$ and $Z = 2D_G$.*

(i) For $y_1, y_2 \in Y$, $y_1y_2 \in C$ where C is one of the following G-conjugacy classes:-

$1A, 2B, 2D, 3A, 4B, 4E, 4G, 5A, 6C.$

(ii) For $y \in Y$ and $z \in Z$, $yz \in C$ where C is one of the following G-conjugacy classes:-

$$2A, 2B, 2D, 4A, 4B, 4C, 4D, 4E, 4F, 4G, 4H, 6A, 6B, 6C, 6E, 6H, 8B,$$
$$8C, 8E, 8J, 10A, 10B, 10E, 12E, 12H, 12L.$$

(iii) In part (ii), the unique involution in $\langle yz\rangle$ is either in $Y \cup Z$ or $yz \in 2A \cup 6A \cup 6B \cup 10A$.

Proof. Parts (i) and (ii) may be verified by calculating the appropriate structure constants with the aid of [8]. Consulting [7] yields (iii).

□

Lemma 2.6. *Suppose that $G \cong BM$.*

(i) Let $\xi \in 3A_G$ and set $F = C_G(\xi)'$. Then $F \cong Fi_{22}$ and $2B_F \subseteqq 2B_G$.

(ii) Let $\eta \in 5A_G$ and set $H = C_G(\eta)'$. Then $H \cong HS$ and $2A_H \subseteqq 2B_G$.

Proof. (i) From the [7] $C_G(\xi) \cong 3 \times Fi_{22} : 2$ and so $F = Fi_{22}$. Suppose $g \in G$ with g of order 6, $g^2 \in 3A_G$ and $g^3 \in 2B_G$. Then, using [**?**] again, we see that $g \in 6C_G$ and $|C_G(g)| = 2^{18}.3^5.5$. From [7] we have, for τ an involution of $Fi_{22} : 2$, the following centralizer data.

Class of τ	$\lvert C_F(\tau)\rvert$
$2A_F$	$2^{16}.3^6.5, 7, 11$
$2B_F$	$2^{17}.3^4.5$
$2C_F$	$2^{16}.3^3$
$2D_F$	$2^{13}.3^6.5^2.7$
$2E_F$	$2^{13}.3^4.5$
$2F_F$	$2^{10}.3^4.5.7$

Hence we deduce that $g^3 \in 2B_F$ and therefore $2B_F \subseteq 2B_G$.

(ii) Using [7] yet again gives $C_G(\eta) \cong 5 \times HS : 2$. So $H \cong HS$. Let $g \in G$ with g of order 10, $g^2 \in 5A_G$ and $g^5 \in 2B_G$. By [7] $g \in 10_G$. Then from $|C_{HS}(\tau)|$ where τ is an involution in $HS : 2$ we infer that $g^5 \in 2A_{HS}$. Hence $2A_H \subseteq 2B_G$.

□

Again, we temporarily, leave off the subscript from the conjugacy class names. Let $\mathcal{E}$ denote the set of all $\mathbb{M}$- conjugacy classes with exception of the following:- $8C, 8F, 23A, 23B{*}{*}, 24F, 24G, 24H, 24J, 31A, 31B{*}{*}, 32A, 32B, 39C, 39D{*}{*}, 40A, 40C, 40D{*}{*}, 44A, 44B{*}{*}, 46A, 46B{*}{*}, 46C, 46D{*}{*}, 47A, 47B{*}{*}, 48A, 56B, 56C{*}{*}, 59A, 59B{*}{*}, 62A, 62B{*}{*}, 69A, 69B{*}{*}, 71A, 71B{*}{*}, 78B, 78C{*}{*}, 87A, 87B{*}{*}, 88A, 88B{*}{*}, 92A, 92B{*}{*}, 93A, 93B{*}{*}, 94A, 94B{*}{*}, 95A, 95B{*}{*}, 104A, 104B{*}{*}, 119A, 119B{*}{*}$.

We also define a further set of $\mathbb{M}$-conjugacy classes

$$\mathcal{F} = \{27B, 29A, 39B, 41A, 45A, 51A, 57A, 105A\},$$

which we observe is a subset of $\mathcal{E}$.

Lemma 2.7. *We have $X_C \neq \emptyset$ for $\mathbb{M}$-conjugacy classes C if and only if $C \in \mathcal{E}$.*

Proof. Using [8] we see which conjugacy classes C have $|X_C| \neq 0$, so giving the lemma.

$\square$

Lemma 2.8. *Suppose that C is an $\mathbb{M}$-conjugacy class with $X_C \neq \emptyset$, and let $z \in C$. Then $|C_{\mathbb{M}}(z)|$ is odd if and only if $C \in \mathcal{F}$.*

Proof. This follows from Lemma 2.7 and [7].

$\square$

3 Proof of the Theorems

We begin this section examining the commuting involution graph $\mathcal{C}(\mathbb{M}, X)$. Recall that t is a fixed element of X.

Lemma 3.1. *Suppose that $t, x \in X$ and that $C_{\mathbb{M}}(t) \cap C_{\mathbb{M}}(x)$ has even order. Then $\partial(t, x) \leq 3$*

Proof. Since $C_{\mathbb{M}}(t) \cap C_{\mathbb{M}}(x)$ has even order, we may select an involution a which commutes with both t and x. If $a \in X$, then we have $\partial(t, x) \leq 2$. So, for the remainder of the proof of the lemma, we may assume that $a \in 2A_{\mathbb{M}}$. Set $K = C_{\mathbb{M}}(a)$ and $\overline{K} = K/\langle a \rangle \cong BM$. Appealing to Table 2 of Norton [12] gives

(3.1) $C_{\mathbb{M}}(t) \cap K \sim 2^{1+23}Co_2$ (if $ta \in 2A_{\mathbb{M}}$) or $C_{\mathbb{M}}(t) \cap K \sim 2^{2+8+16}O_8^+(2)$ (if $ta \in 2B_{\mathbb{M}}$).

(3.2) There exists $b \in X \cap C_{\mathbb{M}}(t) \cap K$ such that $\bar{b} \in 2B_{\overline{K}}$.

By 3.1 we have $\overline{C_{\mathbb{M}}(t) \cap K} \cong 2^{1+22}Co_2$ or $2^{1+8+16}O_8^+(2)$ which are, respectively, centralizers in $\overline{K}$ of a $2B_{\overline{K}}$ and a $2D_{\overline{K}}$-involution and therefore must contain a $2B_{\overline{K}}$-involution. Hence, by Lemma 2.3(iii), 3.2 holds.

(3.3) Let $x_1, x_2, x_3 \in K \cap X$.

(i) If $\overline{E} = \langle \overline{x_1}, \overline{x_2} \rangle$ is a fours subgroup of $\overline{K}$ with $\overline{E} \cap 2C_{\overline{K}} = \emptyset$, then $\partial(x_1, x_2) = 1$.

(ii) Assume that $\overline{E} = \langle \overline{x_1}, \overline{x_2} \rangle$ and $\overline{F} = \langle \overline{x_2}, \overline{x_3} \rangle$ are fours subgroups of $\overline{K}$ with $\overline{E} \cap 2C_{\overline{K}} = \emptyset = \overline{F} \cap 2C_{\overline{K}}$. Then $\partial(x_1, x_3) \leq 2$.

For (i), let E be the inverse image in K of $\overline{E}$. Since $\overline{E} \cap 2C_{\overline{K}} = \emptyset$, all cosets of $\langle a \rangle$ in E contain an involution by Lemma 2.3 (ii), (iii), (iv) and so E is elementary abelian. Hence $[x_1, x_2] = 1$ and therefore $\partial(x_1, x_2) = 1$.

Using part (i) gives $\partial(x_1, x_2) = 1 = \partial(x_2, x_3)$ and thus $\partial(x_1, x_3) \leq 2$, so proving (ii).

Now let b be as in 3.2. So $b, x \in K \cap X$.

(3.4) If $\overline{x} \in 2B_{\overline{K}}$, then $\partial(b, x) \leq 2$.

Let $x_1, x_2 \in K \cap X$ be such that $\overline{x_1}, \overline{x_2} \in 2B_{\overline{K}}$ and $\langle \overline{x_1}, \overline{x_2} \rangle$ is a fours group. Applying Lemma 2.5(i) gives $\langle \overline{x_1}, \overline{x_2} \rangle \cap 2C_{\overline{K}} = \emptyset$. From Lemma 2.4 Diam $\big(\mathcal{C}(\overline{K}, 2B_{\overline{K}})\big) = 2$ and so, as $\overline{x} \in 2B_{\overline{K}}$ by assumption, $\partial(b, x) \leq 2$ by 3.2.

(3.5) If $\overline{x} \in 2D_{\overline{K}}$, then $\partial(b, x) \leq 2$.

Let $\overline{y}$ be the unique involution in $\langle \bar{b}\overline{x} \rangle$. Observe that $\bar{b}\overline{x}$ and $\overline{xy}$ are both involutions in $\langle \bar{b}, \overline{x} \rangle \setminus \langle \bar{b}\overline{x} \rangle$. Also, as $\langle \bar{b}, \overline{x} \rangle \setminus \langle \bar{b}\overline{x} \rangle$ consists of two $\langle \bar{b}, \overline{x} \rangle$-classes of involutions, an involution in $\langle \bar{b}, \overline{x} \rangle \setminus \langle \bar{b}\overline{x} \rangle$ is either in $2B_{\overline{K}}$ or $2D_{\overline{K}}$ by Lemma 2.3(i). Suppose that $\overline{y} \in 2B_{\overline{K}} \cup 2D_{\overline{K}}$. Then $\langle \bar{b}, \overline{y} \rangle \cap 2C_{\overline{K}} = \emptyset = \langle \overline{y}, \overline{x} \rangle \cap 2C_{\overline{K}}$ and, as a consequence, $\partial(b, x) \leq 2$ by 3.3(ii). Therefore, by Lemma 2.3(iii), $\partial(b, x) \leq 2$ if $\bar{b}\overline{x} \notin 2A_{\overline{K}} \cup 6A_{\overline{K}} \cup 6B_{\overline{K}} \cup 10A_{\overline{K}}$. If $\bar{b}\overline{x} \in 2A_{\overline{K}}$, then $\partial(b, x) = 1$ by 3.3(i). So to complete the proof of 3.5 it remains to consider $\bar{b}\overline{x} \in 6A_{\overline{K}} \cup 6B_{\overline{K}} \cup 10A_{\overline{K}}$.

First we consider the case when $\bar{b}\overline{x} \in 6A_{\overline{K}} \cup 6B_{\overline{K}}$. Then $\xi = (\bar{b}\overline{x})^2 \in 3A_{\overline{K}}$ and $N_{\overline{K}}(\langle \xi \rangle) = D \times F$ where $D \cong Sym(3)$ and $F \cong Fi_{22} : 2$. Also $C_{\overline{K}}(\xi) = \langle \xi \rangle \times F$. We have $\bar{b}\overline{x} = \xi\tau$ where τ is an involution in F. Now $\overline{x} = \alpha\beta$ where $\alpha \in D$, $\beta \in F$ and α, β have order 1 or 2. Observe that

$$C_{C_{\overline{K}}(\bar{b}\overline{x})}(\overline{x}) = C_F(\langle \tau, \beta \rangle)$$

and $\langle \tau, \beta \rangle$ is elementary abelian of order 2 or 4. Let $S \in Syl_2(F)$ be such that $\langle \tau, \beta \rangle \leq S$. Then $\langle \tau, \beta \rangle$ acts upon $Z(S \cap F')$ and therefore $C_{Z(S \cap F')}(\langle \tau, \beta \rangle) \neq 1$. Because, by [7], the involutions in $Z(S \cap F')$ are in $2B_{F'}$, Lemma 2.6(i) implies that $C_{C_{\overline{K}}(\bar{b}\overline{x})}(\overline{x}) \cap 2B_{\overline{K}} \neq \emptyset$. Hence, by Lemmas 2.1 and 2.3(iii), there exists $y \in K \cap X$ such that $\overline{y} \in C_{\overline{K}}(\bar{b}) \cap C_{\overline{K}}(\overline{x}) \cap 2B_{\overline{K}}$. Using Lemma 2.5(i), (ii) gives $\langle \bar{b}, \overline{y} \rangle \cap 2C_{\overline{K}} = \emptyset = \langle \overline{y}, \overline{x} \rangle \cap 2C_{\overline{K}}$ and therefore $\partial(b, x) \leq 2$ by 3.3(ii), as required.

Finally suppose $\bar{b}\overline{x} \in 10A_{\overline{K}}$, and set $\eta = (\bar{b}\overline{x})^2$. By [7] $\eta \in 5A_{\overline{K}}$ and $N_{\overline{K}}(\langle \eta \rangle) = D \times H$ where $D \cong Dih(10)$ and $H \cong HS : 2$. We have $\bar{b}\overline{x} = \eta\tau$ where τ is an involution in H. Also $\overline{x} = \alpha\beta$ where $\alpha \in D$, $\beta \in H$ and α, β have order 1 or 2. Now arguing as in the $6A_{\overline{K}} \cup 6B_{\overline{K}}$ case, but using Lemma 2.6(ii), we also deduce that $\partial(b, x) \leq 2$.

This completes the proof of 3.5.

Since $\overline{K \cap X} = 2B_{\overline{K}} \cup 2D_{\overline{K}}$ by Lemma 2.3(i) and $\partial(t,b) \leq 1$, 3.4 and 3.5 together prove Lemma 3.1.

□

Lemma 3.2. *Suppose that C is an $\mathbb{M}$-conjugacy class which is not in $\mathcal{F}$. Then for $x \in X_C$, $\partial(t,x) \leq 3$.*

Proof. Set $z = tx$. Since $C \notin \mathcal{F}$, $|C_{\mathbb{M}}(z)|$ is even by Lemma 2.8, whence, by Lemmas 2.1 and 3.1, $\partial(t,x) \leq 3$.

□

Lemma 3.3. *Suppose that $L = H \times K \leqslant \mathbb{M}$ with $t, x \in L \cap X$. Let $t = t_H t_K$ and $x = x_H x_K$ where $t_H, x_H \in H$ and $t_K, x_K \in K$. If there exists $x_1, x_2 \in K \cap X$ such that $[t_K, x_1] = [x_1, x_2] = [x_2, x_K] = 1$, then $\partial(t,x) \leq 3$.*

Proof. Since $x_1, x_2 \in K$, x_1 centralizes t_H and x_2 centralizes x_H. So, as $[t_K, x_1] = 1 = [x_2, x_K]$, $[t, x_1] = 1 = [x, x_2]$ and hence $\{t, x_1, x_2, x\}$ is a path in $\mathcal{C}(\mathbb{M}, X)$. Thus $\partial(t,x) \leq 3$.

□

Lemma 3.4. *Suppose that $C \in \mathcal{F} \backslash \{27B_{\mathbb{M}}, 41A_{\mathbb{M}}\}$. If $x \in X_C$, then $\partial(t,x) \leq 3$.*

Proof. Let n be the order of $z = tx$. By Lemma 2.2, as $C_{\mathbb{M}}(z)$ has odd order, there is a unique $\mathbb{M}$-conjugacy class of subgroups isomorphic to $Dih(2n)$. Put $D = \langle t, x \rangle$.

(3.6) There exists a subgroup $L = H \times K$ of $\mathbb{M}$ which contains D with C, H and K as follows:-

(i) $C = 39B_{\mathbb{M}}$, $H \cong Sym(3)$ and $K = Th$.

(ii) $C = 45A_{\mathbb{M}}$, $H \cong Dih(10)$ and $K \cong HN$.

(iii) $C = 51A_{\mathbb{M}}$, $H \cong Sym(4))$ and $K \cong L_2(17)$.

(iv) $C = 57A_{\mathbb{M}}$, $H \cong Sym(3)$ and $K \cong Th$.

(v) $C = 105A_{\mathbb{M}}$, $H \cong Dih(10)$ and $K \cong HN$.

For (i) and (iv) we have, repectively, $z^{13} \in 3A_{\mathbb{M}}$ and $z^{19} \in 3A_{\mathbb{M}}$. Since, for $\xi \in 3A_{\mathbb{M}}$, $C^*_{\mathbb{M}}(\xi) \cong Sym(3) \times Th$ by [7] and any subgroup of $\mathbb{M}$ is conjugate to D, we have $D \leq H \times K$ with $H \cong Sym(3)$ and $K \cong Th$. Also, for $\xi \in 5A_{\mathbb{M}}$, $C^*_{\mathbb{M}}(\xi) \cong Dih(10) \times HN$ by [7]. Using [7], if $C = 45A_{\mathbb{M}}$, then $z^9 \in 5A_{\mathbb{M}}$

and if $C = 105A_{\mathbb{M}}$, then $z^{21} \in 5A_{\mathbb{M}}$. Hence $D \leq H \times K$ with $H \cong Dih(10)$ and $K \cong HN$ when $C = 45A_{\mathbb{M}}$ or $105A_{\mathbb{M}}$. Finally for $C = 51A_{\mathbb{M}}$, surveying the Monstralizer pairs in Table 1 of [12] we see a subgroup of $\mathbb{M}$ isomorphic to $Sym(4) \times Sp_8(2)$. Now, by [7], $Sp_8(2)$ contains subgroups isomorphic to $L_2(17)$ which themselves contain subgroups isomorphic to $Dih(34)$. Thus we have obtained a subgroup $H \times K$ containing $D \cong Dih(51)$ with $H \cong Sym(4)$, $K \cong L_2(17)$, so establishing 3.6.

From [1] $\mathrm{Diam}(\mathcal{C}(K, W)) = 3$ if $K \cong L_2(17)$ and W is a K-conjugacy class of involutions of K. While from [2] we see that $\mathrm{Diam}(\mathcal{C}(K, W)) \leq 3$ if $K \cong HN$ or Th and W is a K-conjugacy class of involutions of K. Combining Lemma 3.3 and 3.6 yields that $\partial(t, x) \leq 3$ when $C \in \{39B_{\mathbb{M}}, 45A_{\mathbb{M}}, 51A_{\mathbb{M}}, 57A_{\mathbb{M}}, 105A_{\mathbb{M}}\}$. So to complete the proof of the lemma we need to examine the case $C = 29A_{\mathbb{M}}$. By [9] $\mathbb{M}$ contains a subgroup isomorphic to $L_2(29)$ inside of which there are subgroups isomorphic to $Dih(58)$ and therefore we get $D \leq L \leq \mathbb{M}$ with $L \cong L_2(29)$. Hence, using [1] again, we deduce that $\partial(t, x) \leq 3$, and the proof of Lemma 3.4 is complete.

□

Lemma 3.5. *Let $C = 41A_{\mathbb{M}}$. Then for $x \in X_C$, $\partial(t, x) = 3$.*

Proof. Put $z = tx$. From [7] $C_{\mathbb{M}}(z) = \langle z \rangle$ and so, by Lemma 2.2, there is a unique $\mathbb{M}$-conjugacy class of subgroups isomorphic to $Dih(82)$. Also, clearly $\partial(t, x) \geqslant 3$. Again using [7], $\mathbb{M}$ has a subgroup H of shape $3^8 O_8^-(3)2$. Put $\overline{H} = H/O_3(H)$. Since $O_8^-(3)$ contains a subgroup isomorphic to $L_2(81)$ we can find a subgroup L of H with $O_3(H) \leqslant L$ and $\overline{L} \cong L_2(81)$. Now $L_2(81)$ contains a subgroup isomorphic to $Dih(82)$, which must split over $O_3(H)$. Thus, without loss of generality, we may suppose $\langle t, x \rangle \leqslant L$. We further note, as $\overline{L}$ has one involution conjugacy class and z has order 41 that all involutions in L are in X. Moreover, $|X_C| = |C_{\mathbb{M}}(t)|$ by [8] and, using Lemma 2.1, $C_{C_{\mathbb{M}(t)}}(x) = 1$. So X_C is a $C_{\mathbb{M}}(t)$-orbit. Appealing to [1] gives $x_1, x_2 \in X$ such that $[\overline{t}, \overline{x_1}] = [\overline{x_1}, \overline{x_2}] = [\overline{x_2}, \overline{x}] = 1$. Hence t normalizes $O_3(H)\langle x_1 \rangle$ and therefore $[t, y_1] = 1$ for some $y_1 \in X \cap O_3(H)\langle x_1 \rangle$. Also y_1 normalizes $O_3(H)\langle x_2 \rangle$ and so $[y_1, y_2] = 1$ for some $y_2 \in X \cap O_3(H)\langle x_2 \rangle$. Likewise y_2 normalizes $O_3(H)\langle x \rangle$ and so $[y_2, y] = 1$ for some $y \in X \cap O_3(H)\langle x \rangle$. Thus $\partial(t, y) \leqslant 3$. Since $\overline{y} = \overline{x}$ and the largest element order in $\mathbb{M}$ is 119, the order of ty must be 41. Because X_C is a $C_{\mathbb{M}}(t)$-orbit we then conclude that $\partial(t, x) \leqslant 3$.

□

Lemma 3.6. *Let $C = 27B_{\mathbb{M}}$. Then for $x \in X_C$, $\partial(t, x) = 3$.*

Proof. Let $x \in X_C$ and put $z = tx$. From $|C_{\mathbb{M}}(z)| = 243$, we note that $\partial(t, x) \geqq 3$. We shall calculate in a subgroup H of $\mathbb{M}$ of shape $3^2 3^5 3^{10}(M_{11} \times$

$2Sym(4)^+$) using MAGMA[6]. First we obtain an isomorphic copy of H from http://brauer.maths.qmul.ac.uk/Atlas/ as a permutation group of degree $34,992$. Sometimes, in what follows, we shall regard this copy of H as a subgroup of $\mathbb{M}$. Next we calculated all the conjugacy classes of H and then discovered that H has only two conjugacy classes C_1, C_2 of elements of order 27. Let $z_1 \in C_1$ and $z_2 \in C_2$. Note that, as H contains a Sylow 3-subgroup of $\mathbb{M}$, if z_1 and z_2 were $\mathbb{M}$-conjugate, then $\mathbb{M}$ would have only one conjugacy class of elements of order 27, which is not the case. We next find that $|C_H(z_1| = 486$ and $|C_H(z_2)| = 243$. Consequently, consulting [7], $C_1 \subseteq 27A_{\mathbb{M}}$ and $C_2 \subseteq 27B_{\mathbb{M}}$. Further calculation reveals that $|N_H(\langle z_2 \rangle)| = 4374$ and that $N_H(\langle z_2 \rangle)$ contains 81 involutions which invert z_2 (and they must all be in X). Let t_2 denote one of these involutions. Using [8] gives $|X_C| = |C_{\mathbb{M}}(T)|/3$. After seeing that $|C_{C_H(z_2)}(t_2)| = 3$ and observing by orders that $C_H(z_2) = C_{\mathbb{M}}(z_2)$, we deduce that X_C is a $C_{\mathbb{M}}(t)$- orbit by Lemma 2.1. Let $x_2 \in X \cap N_H(\langle z_2 \rangle)$ with $t_2 x_2 = z_2$. We check that an involution τ of H is in X by choosing random elements h of H and seeing if the order of $\tau\tau^h$ is greater than 6. By these means we find $y_1 \in X \cap C_H(t_2)$ and $y_2 \in X \cap C_H(x_2)$ with $[y_1, y_2] = 1$. As a consequence we deduce that $\partial(t_2, x_2) \leqslant 3$, which as X_C is a $C_{\mathbb{M}}(t)$-orbit, proves Lemma 3.6.

□

Combining Lemmas 3.2, 3.4, 3.5 and 3.6 yields Theorem 1.2.

Moving onto the proof of Theorem 1.1, we establish the following lemma.

Lemma 3.7. *Let $x, y \in X$. If $\partial(x, y) = 1$, then $d(x, y) \leq 2$.*

Proof. The set of $x \in X$ for which $\partial(t, x) = 1$ is equal to $\Delta_1(t) \cup \Delta_2^1(t) \cup \Delta_2^2 \cup \Delta_2^4$ (using the notation of [15]). Hence Lemma 3.7 holds.

□

Now Theorem 1.1 is a consequence of Theorem 1.2 and Lemma 3.7.

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