The classification and dynamics of the momentum polytopes of the SU(3) action on points in the complex projective plane with an application to point vortices

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THE CLASSIFICATION AND DYNAMICS OF THE MOMENTUM POLYTOPES OF THE SU(3) ACTION ON POINTS IN THE COMPLEX PROJECTIVE PLANE WITH AN APPLICATION TO POINT VORTICES

A THESIS SUBMITTED TO THE UNIVERSITY OF MANCHESTER FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE FACULTY OF ENGINEERING AND PHYSICAL SCIENCE

2018

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The Classification and Dynamics of the Momentum Polytopes of the $\mathsf{SU}(3)$ Action on Points in the Complex Projective Plane with an Application to Point Vortices

We have fully classified the momentum polytopes of the $\mathsf{SU}(3)$ action on $\mathbb{C}\mathsf{P}^2 \times \mathbb{C}\mathsf{P}^2$ and $\mathbb{C}\mathsf{P}^2 \times \mathbb{C}\mathsf{P}^2 \times \mathbb{C}\mathsf{P}^2$, both actions with weighted symplectic forms, and their corresponding transition momentum polytopes. For $\mathbb{C}\mathsf{P}^2 \times \mathbb{C}\mathsf{P}^2$ the momentum polytopes are distinct line segments. The action on $\mathbb{C}\mathsf{P}^2 \times \mathbb{C}\mathsf{P}^2 \times \mathbb{C}\mathsf{P}^2$ has 8 different momentum polytopes. The vertices of the momentum polytopes of the $\mathsf{SU}(3)$ action on $\mathbb{C}\mathsf{P}^2 \times \mathbb{C}\mathsf{P}^2 \times \mathbb{C}\mathsf{P}^2$ fall into two categories: definite and indefinite vertices. The reduced space corresponding to momentum map image values at definite vertices is isomorphic to the 2-sphere. We show that these results can be applied to assess the dynamics by introducing and computing the space of allowed velocity vectors for the different configurations of two-vortex systems.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

Chapter 1

Introduction

The Atiyah [9], Guillemin and Sternberg [34], [35]and then Kirwan [46] theory for the momentum polytope has become well known.

Let A be an $n \times n$ Hermitian matrix, with diagonal elements $(\delta_1, \ldots, \delta_n)$, Issai Schur [87] showed that these δ_i, along with eigenvalues $(\lambda_1, \ldots, \lambda_n)$ satisfy a particular collection of linear inequalities. Let S_n be the symmetric group that permutes points in $\mathbb{R}^n$ which we will consider δ and λ as coordinate points in. The convex hull of the points in $S_n.\lambda$ contains δ. In fact the two equate as A. Horn [41] proved the converse.

Now for $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$, and $\mathfrak{h}$ the Lie algebra of maximal torus $H \subset G$, then the projection map $\mathfrak{g}^* \to \mathfrak{h}^*$ is the restriction of the coadjoint action of G on $\mathfrak{g}^*$ to $\mathfrak{h}$. The Weyl group W acts on $\mathfrak{h}$ and its dual. According to this Kostant [49] generalised the above so it applies to any compact group G.

Theorem. *Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be a coadjoint orbit under G. Then the projection of $\mathcal{O}$ on $\mathfrak{h}^*$ is the convex hull of a W-orbit.*

Therefore Schur and Horn's result becomes only a particular restriction of this theorem for G the unitary group of dimension n and H the diagonal matrix subgroup, and $\mathfrak{g}$ is the set of anti-Hermitian matrices whose dual is established according to the scalar product $\mathrm{Tr}(AB)$ that is G-invariant. Therefore the diagonal of A defines the projection of $A \in \mathfrak{g}^*$ on $\mathfrak{h}^*$.

The standard framework for the convexity theorem is according to Atiyah [9], Guillemin-Sternberg [34], Kirwan [46]'s versions which work for the more general Hamiltonian action of Lie group H on symplectic manifold M. Where the momentum map $M \to \mathfrak{h}^*$ can be made more specific to the projection $\mathcal{O} \to$

$\mathfrak{h}^*$. And Kirwan showed that the *momentum polytope* is convex polytope that results from the intersection of the momentum map image with the positive Weyl chamber.

For given Hermitian matrices A and B, their respective eigenvalues that are in a system of linear inequalities that bound the eigenvalues of their sum $A + B$ thanks to the non-abelian convexity theorem, if we think of it in terms of Schur and Horn's specifications. We go into later developments of convexity theory in section 2.5.6.

In this thesis we explore the action of $SU(3)$ on $\mathbb{C}P^2$ and extend the convexity theorem according to this action. Specifically, for $g \in SU(3)$ and $[v] \in \mathbb{C}P^2$, with action $A[v] = [Av]$. Our compact manifold M is the product of 2 or 3 copies of $\mathbb{C}P^2$, with invariant Fubini-Study symplectic form on it and $SU(3)$ acts diagonally on it. We assign a scalar Γ_i (for $i = 1, 2, 3$) to each invariant symplectic form on each copy of $\mathbb{C}P^2$. This is a Hamiltonian action and the momentum map depends on the Γ_i chosen. We classify the momentum polytopes in full for each of these actions.

Chapter 2 includes the background theory establishing the convex polytope including properties of the momentum map of the Lie group action on its coadjoint orbit. The positive Weyl chamber is a result of the orbit momentum map. The bifurcation lemma provides a rank-nullity correspondence that is essential in dictating the gradient of the edges of the momentum polytope. We then introduce the Witt-Artin decomposition which we use in calculations of the reduced spaces. The dynamics of the action are explored through their relative equilibria.

In Chapter 3 we confirm the action that the results of this thesis are based on: the $SU(3)$ action on its coadjoint orbit $\mathbb{C}P^2$, namely determining the isotropy subgroups and the fixed point sets of the action on two and three copies of $\mathbb{C}P^2$. $SU(3)$ has two types of non-trivial coadjoint orbits (up to diffeomorphism): the 4-dimensional $\mathbb{C}P^2$ and the 6-dimensional complex flag manifold $F(2, 1)$. Like all coadjoint orbits, these are symplectic manifolds with a transitive G-action. For $\mathbb{C}P^2$ the invariant symplectic form is unique up to a scalar multiple, while for $F(2, 1)$ there is a 2-parameter family of invariant symplectic forms. In chapter 3 we confirm the action that the results of this thesis are based on: the $SU(3)$ action on its coadjoint orbit $\mathbb{C}P^2$, namely determining the isotropy subgroups and the fixed point sets of the action on two and three copies of $\mathbb{C}P^2$. The isotropy subgroup of a point in $\mathbb{C}P^2$ is $S[U(2) \times U(1)] \simeq U(2)$. For two points that are perpendicular in $\mathbb{C}P^2$ the isotropy subgroup is isomorphic to $\mathbb{T}^2$. And the isotropy subgroup of two points that are neither perpendicular or parallel is $\mathbb{T}^1$, the fixed point set of which consists of the $z_3 = 0$ subspace - which is a $\mathbb{C}P^1 \subset \mathbb{C}P^2$: in this case the dynamics on this subspace restricts to the dynamics of the two points on the sphere.

In chapter 4 we indicate the symplectic form on the coadjoint orbit and show that it satisfies certain properties as required and after exhibiting the appropriate properties of the momentum map of the $SU(3)$ action on $\mathbb{C}P^2$. For the action of $SU(3)$ on products of $\mathbb{C}P^2$, we introduced the *weighted* symplectic form. The symplectic form on N-copies of $\mathbb{C}P^2$ is

$$\omega = \Gamma_1\omega_{FS} \oplus \ldots \oplus \Gamma_N\omega_{FS} \quad \text{where} \quad \Gamma_1, \ldots \Gamma_N \in \mathbb{R}$$

so the symplectic form is the sum of scalar multiples of the Fubini-Study form on each $\mathbb{C}P^2$.We establish the momentum map on that symplectic manifold $J : \prod_{i=1}^{N} \mathbb{C}P^2 \to \mathfrak{su}(3)^*$ of the $SU(3)$ action on N-products of $\mathbb{C}P^2$ is

$$J : Z_1, \ldots Z_N \to \sum_{i=1}^{N} i\Gamma_i \frac{Z_i \otimes \bar{Z}_i}{|Z_i|^2} - \frac{1}{3}\Gamma_i I, \quad Z_i \in \mathbb{C}P^2.$$

The intersection of the image of the momentum map with the positive Weyl chamber is a convex polytope and the vertices are contained in the set of the image of the momentum map of the fixed point set of the action of the Lie group. The coadjoint orbits through points in the boundary of the Weyl chamber are $\mathbb{C}P^2$s while those through the interior points are the flag manifolds. Each coadjoint orbit corresponds to an isospectral submanifold.

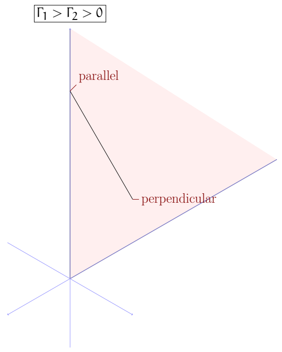
Chapter 5 includes most of our results. After exploring the uniqueness of the fixed point sets according to the respective weightings assigned to each copy of $\mathbb{C}P^2$ we draw up and classify the momentum polytopes of the $SU(3)$ action on two and three copies of $\mathbb{C}P^2$. These momentum polytopes are separated by transition polytopes which are defined and classified for each action. Section 5.5 uses the bifurcation lemma to prove that the gradient of each edge of the momentum polytopes must be perpendicular to one of the walls of the Weyl chambers.

The action $SU(3) \curvearrowright \mathbb{C}P^2 \times \mathbb{C}P^2$ is a cohomogeneity one action whose orbit is homeomorphic to the closed interval. We show that the momentum polytopes of this action fall into six different categories separated by the ratios between the

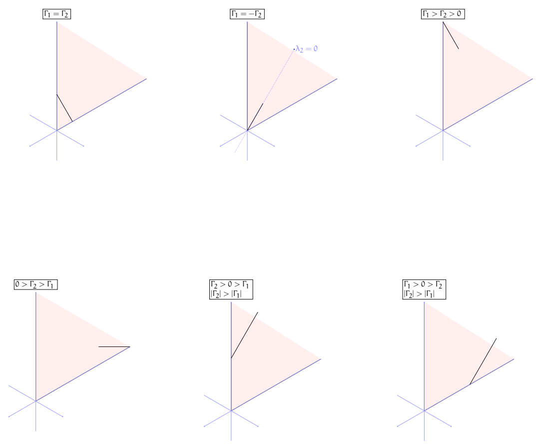
Γ_is. In section 5.3 we prove the following theorem:

Theorem. *The momentum polytopes of the* $SU(3)$ *action on* $\mathbb{C}P^2 \times \mathbb{C}P^2$ *with weighted symplectic form* $\Gamma_1\omega_{FS} \oplus \Gamma_2\omega_{FS}$ *fall into four different categories for which* $\Gamma_1 - \Gamma_2 \neq 0$, $\Gamma_1 + \Gamma_2 \neq 0$, $\Gamma_i \neq 0$ *where* $i, j = 1, 2$.

An example of these polytopes is shown in Figure 1.1a. All of the different polytopes corresponding to this action are shown in Figure 1.1b therefore describing in full, by their momentum polytopes, the different configurations of the two points according to their weights Γ_1 and Γ_2.



(a) Momentum polytope for $\Gamma_1 > \Gamma_2 > 0$ clearly indicating the regular and singular orbits in relation to parallel or perpendicular points. The region shaded in pink is the positive Weyl chamber for $SU(3)$.



(b) All of the different momentum polytopes for the $SU(3)$ action on two points $(z_1, z_2) \in \mathbb{C}P^2 \times \mathbb{C}P^2$.

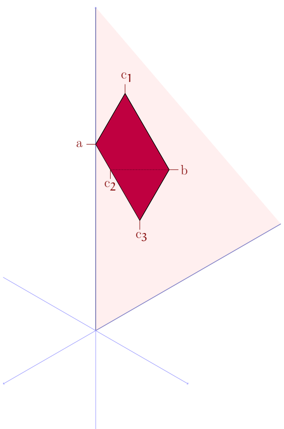
Figure 1.1: The momentum polytopes of $SU(3) \curvearrowright \mathbb{C}P^2 \times \mathbb{C}P^2$.

For the $SU(3)$ action on points in $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ there are only five distinct fixed points:

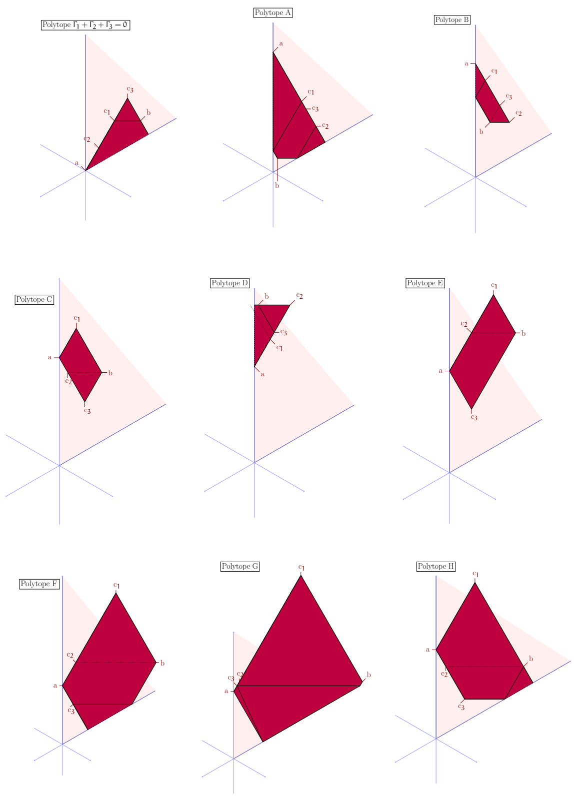
$$\Big\{[e_1, e_1, e_1],\ [e_2, e_1, e_1],\ [e_1, e_2, e_1],\ [e_1, e_1, e_2],\ [e_1, e_2, e_3]\Big\}.$$

where $e_1 = [1 : 0 : 0], e_2 = [0 : 1 : 0], e_3 = [0 : 0 : 1] \in \mathbb{C}P^2$. Figure 5.20 is an example of one of these polytopes: the vertices corresponding to each image of the momentum map of each fixed point set are labelled **a**, **b**, $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ as shown. There are nine different distinct polytopes for this action as shown in Figure 1.2b which are distinguished by the magnitudes and respective ratios between Γ_1, Γ_2 and Γ_3. In section 5.6 we prove the following theorem:

Theorem. *The momentum polytopes of the* $SU(3)$ *action on* $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ *with weighted symplectic form* $\Gamma_1\omega_{FS} \oplus \Gamma_2\omega_{FS} \oplus \Gamma_3\omega_{FS}$ *fall into nine different categories for which* $\Gamma_i - \Gamma_j - \Gamma_k \neq 0$, $\Gamma_i + \Gamma_j \neq 0$, $\Gamma_i \neq 0$ *where* $i, j, k = 1, 2, 3$.



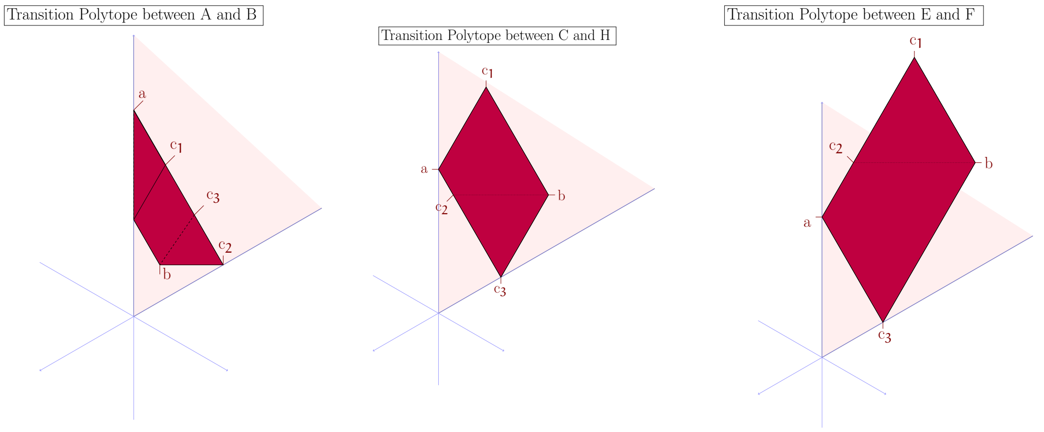
(a) As an example we have here one of the polytopes where the points
a=Spectrum$\Big(J([e_1, e_1, e_1])\Big)$,
b=Spectrum$\Big(J([e_1, e_2, e_3])\Big)$,
c$_1$=Spectrum$\Big(J([e_2, e_1, e_1])\Big)$,
c$_2$=Spectrum$\Big(J([e_1, e_2, e_1])\Big)$,
c$_3$=Spectrum$\Big(J([e_1, e_1, e_2])\Big)$.



(b) All of the nine momentum polytopes for the $SU(3)$ action on three points $(z_1, z_2, z_3) \in \mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$.

Figure 1.2: The momentum polytopes of $SU(3) \curvearrowright \mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$.

These different polytopes are separated by *transitional polytopes*. If the spectrum of a coadjoint orbit has a repeated element then the orbit is diffeomorphic to $\mathbb{C}P^2$, while if it has 3 distinct elements it is diffeomorphic to $F(2, 1)$. The first type of transitional polytope consists of two rather than one singular momentum values as shown in Figure 1.3. And the second type of transitional polytope corresponds to the 5 fixed point sets 'collapsing' into only three (one singular and two



(a) The transition polytope between Polytopes A and B.

(b) The transition polytope between Polytopes C and H.

(c) The transition polytope between Polytopes E and F.

Figure 1.3: Three examples of the *transition polytopes* of $SU(3) \curvearrowright \mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$.

regular) sets of momentum values: they look exactly like the polytopes for the $SU(3)$ action on two points in $\mathbb{C}P^2 \times \mathbb{C}P^2$. In section 5.4 we prove the following theorem:

Theorem. *The* transitional *momentum polytopes of the* $SU(3)$ *action on* $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ *fall into three different categories for which* $\Gamma_i - \Gamma_j - \Gamma_k = 0$*,* $\Gamma_i + \Gamma_j = 0$*,* $\Gamma_i = 0$ *where* $i, j, k = 1, 2, 3$*.*

Similarly for the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2$, in section 5.7 we prove the following theorem:

Theorem. *The different polytopes of theorem 1 are separated by* transitional *momentum polytopes of the* $SU(3)$ *action on* $\mathbb{C}P^2 \times \mathbb{C}P^2$ *fall into three different categories for which* $\Gamma_1 - \Gamma_2 = 0$*,* $\Gamma_1 + \Gamma_2 = 0$*,* $\Gamma_i = 0$ *where* $i, j = 1, 2$*.*

For the third category of transition polytope in which the momentum polytope of the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ resembles that of the momenutm polytopes of the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2$ we composed and proved a theorem to describe the restricted momentum map that outlines this transformation:

Theorem. *For Lie group* G *acting on manifold* (M, ω) *with* $J : M \to \mathfrak{g}^*$*. Let* X *be an invariant symplectic submanifold of* M*. The momentum map restricted to this submanifold is* $J|_X = J_X : X \to \mathfrak{g}^*$ *where for* $x \in X$ $J_X(x) = J(x)$*. Therefore* J_X *is the momentum map for* G *acting on* X*.*

In chapter 6 we find a more rigorous derivation for the shape and edges of the momentum polytope. We prove that the direction of the edges from each vertex is dictated much more locally. Therefore according to the quadratic momentum map on the symplectic slice we can distinguish vertices into definite and indefinite vertices. This distinction then prescribes the possible directions of the edges leading from that vertex. At the definite vertices the reduced space is isomorphic to the sphere.

Globally, the gradient of the edges of the polytope are determined by the bifurcation lemma. The bifucation lemma provides a link between the rank of a momentum map at a point p of a Poisson Manifold $(P, \{\cdot,\cdot\})$ and the symmetry of the manifold at p. The rank of T_pJ is the same as the dimension of the annihilator of the isotropy algebra at p providing a rank-nullity correspondence.The Weyl group reflections at the walls of the Weyl chamber also dictate the resulting shape of the polytope.
However the direction of the edge of a polytope from its vertex can be described more locally. For a vertex, the direction of any of the edges that lead away from the vertex $(a, b, c_1, c_2$ or $c_3)$ travel along the weight vectors dictated by the momentum map on the symplectic slice, N_1.
The reduced spaces are 2-dimensional symplectic manifolds for certain values of the momentum map and degenerate for others. On the momentum polytope the topologies of the reduced spaces can be separated into three categories: those that correspond to the momentum values along the edge of the momentum polytope, those that correspond to the momentum values at the interior of the momentum polytope and those that correspond to the momentum values along the dotted lines, the 'interior edges', between vertices in the interior of the polytope (for example the dotted lines between vertices b and c_2 shown in Figure 5.20). In chapter 6 we prove the following theorem:

Theorem. *A* definite vertex *of the momentum polytope corresponds to specific values for* $\mu = J(x)$ *and provides definite values for the quadratic momentum map on the symplectic slice, and the reduced space is isomorphic to the 2-sphere,*

$$J_{N_1}^{-1}(\mu)/\mathbb{T}^2 \simeq S^2.$$

for those values of μ*. An indefinite vertex does not produce definite values for the momentum map on the symplectic slice.*

A point vortex is a point of concentrated and isolated vorticity on irrotational fluid. In more recent cases the point vortices may be modelled on a backdrop of constant vorticity. Helmholtz showed that vortex interaction can be described solely by their strength and relative positions in 1858. Kirchhoff wrote the equations in Hamiltonian form, these Hamiltonian equations didn't rely on knowing the potential or kinetic energies of the system. Novikov with others, including Aref, rediscovered the three vortex problem bringing such research into the limelight in 1975 leading to some great developments. In fact Aref stated that point vortex systems research provides mathematical playgrounds - for the way they bring different strands of classical maths together. These playgrounds include those through which Kidambi and Newton showed that the motion of three vortices on a sphere is completely integrable in 1998. Lim showed that for lattice vortex systems there exist quasi-periodic orbits on the corresponding invariant tori in 1990. There have been other relevant results. Some latter-day developments include the study of different point vortex systems on a sphere, and other surfaces that are less symmetric.

Let's consider the Hamiltonian structure: for symplectic manifold with Riemannian metric (S, ω_0), consider the manifold $M = S \times S \times \cdots \times S \setminus \Pi$ which has configurations of an ordered set of n distinct points in S that removes collisions - Π is the large diagonal and therefore the subset that includes all possible collisions. Usually collisions are removed from models of vortex systems. If x_i is the coordinate of each vortex point, then Γ_i is its corresponding *vortex strength*. And the symplectic form on M is $\Omega = \Gamma_1\omega_0 \oplus \Gamma_2\omega_0 \oplus \cdots \oplus \Gamma_n\omega_0 = \pi_1^*(\Gamma_1\omega_0) + \pi_2^*(\Gamma_2\omega_0) + \cdots + \pi_n^*(\Gamma_n\omega_0)$ where $\pi_j : M \to S$ is the Cartesian projection to the j^{th} component of M. Choosing a smooth function that describes the pairwise interaction of the vortices $h_0 : S \times S \setminus \Pi \to \mathbb{R}$, which, depending on the configuration space, can be taken to be the negative of the Green's function of the Laplacian relative to the Riemannian metric. The Hamiltonian of this action is the function $H(x_1, \ldots, x_n) = \sum_{i<j} \Gamma_i\Gamma_j h_0(x_i, x_j)$ whose evolution, for vector field X_H is defined by Hamilton's equation $i_{X_H}\Omega = -dH$, is given by $\dot{x} = X_H(x)$.

If the action preserves the symplectic form and $h_0(g \cdot x, g \cdot y) = h_0(x, y)$ for all $g \in G,\ x, y \in S$, then there exists G symmetry and the vector field is equivariant. If G action on S is Hamiltonian with momentum map $J : S \to \mathfrak{g}^*$, then $\mathbf{J} : M \to \mathfrak{g}^*$ is the momentum map for the action of G on M which is in turn Hamiltonian where in particular, $\mathbf{J}(x_1, \ldots, x_n) = \sum_j \Gamma_j J(x_j)$. The different components of the momentum map are preserved by the dynamics according to Noether which is in accordance with this.

[66] shows that the actions of $\mathsf{SE}(2)$, $\mathsf{SO}(3)$ and $\mathsf{SL}(2)$ on S if it is the plane, the sphere or the hyperbolic plane are indeed Hamiltonian. [71] showed that this is not true for symplectic action on point vortices on a torus or cylinder.

We initiated research in this direction by investigating vortices on the complex projective plane. In Chapter 7 we first introduce widely-known results for systems of vortices on the 2-sphere including their Hamiltonians. The relative equilibria is a trajectory that lies in the group orbit or, almost equivalently, an invariant group orbit. This thesis provides a further extension of the playground by application of geometric methods to investigate the relative equilibria of vortices on $\mathbb{CP}^2$. Here the vortex interaction isn't modeled to be pairwise except that we use the assumption that $h_0(x, y)$ only depends on the distance between x and y (coordinates of vortices). It needs to be clarified that we are working with a generalised mathematical structure and these results don't apply actual fluid vortex dynamics.

For two vortices we find that the reduced space is just a single point, for every configuration, whereas for the 3 point vortex system we find that they are typically diffeomorphic to a sphere, sometimes a pinched sphere or just a single point (usually for points on the boundary of the polytope). At the end we consider the relative equilibria of their stability. For the two generalised vortices we have that every configuration is a relative equilibria, but this is not true for the three vortex case except for when the reduced space is a point. The main outputs from this chapter however are the relative equilibria results. We introduce our definition of the space of allowed velocity vectors and find that it is either isomorphic to a single point or the real line depending on the vortex configurations on the complex projective plane.

In section 7.3 we formulated the following definition and theorem along with the proof,

Definition. At $x \in \mathcal{P}$,

$$\mathsf{R}_0 \subset \mathsf{T}_x\mathsf{G}$$

is the subset of tangent space of the allowed velocity vectors, ξ, of x and is an RE for which $d\mathsf{H}_x = [\xi] \in \mathsf{R}_0$. Therefore

$$\mathsf{R}_0 \simeq (\mathfrak{g}_\mu/\mathfrak{g}_x)^{\mathsf{G}_x}$$

where $\mathsf{N}_0^* \simeq (\mathfrak{g}_\mu/\mathfrak{g}_x)$.

Using this definition we also show results concerning the dynamics of the configuration of a vortex system including:

Theorem. *For the* $\mathsf{SU}(3)$ *action on two vortices on* $\mathbb{C}\mathsf{P}^2$*, at a generic configuration (when the vortices are neither parallel or perpendicular),*

$$\mathsf{R}_0 \simeq \mathbb{R}$$

and this is true for every respective vortex strength.

In chapter 8 we explore areas for further research and our progress, where it exists, in each.

The results of this thesis have been taken further and are being written up as research papers [69], [70].

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My utmost thanks to my supervisor, James Montaldi, for the perfect amount of inspiration, commitment and guidance throughout. He has been a wonderful supervisor.

This thesis is dedicated to my late grandmother, my mother and my sister.

The Manchester University maths department provides the perfect atmosphere for research: the architecture of the building, the weekly seminars and the social events help foster a collaborative, supportive and easy-going village-like atmosphere.

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My experience at Manchester would not have been the same without all of the excellent people I have met here; all of which I sincerely hope to maintain contact and friendship with for years to come.

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Chapter 2

Symplectic Geometry

2.1 Weyl Chambers

Here we provide a run through of the standard theorems, propositions, lemmas, corollaries and definitions that lead up to a full mathematical description of Weyl groups, Weyl chambers and weights. We mainly state these in order and have omitted proofs but the relevant proofs along with detailed narratives and overview can be found in the following textbooks [23], [31], [42] [83] [90].

2.1.1 Rudimentary Principles

2.1.1.1 The notion of Lie algebra, linear Lie algebras and Lie algebra derivations

We signify an arbitrary (commutative) field with $\mathbb{F}$.

Definition 2.1.1. L is a vector space over a field $\mathbb{F}$, we denote by $(x, y) \mapsto [xy]$ the operation $L \times L \to L$, and call it the *commutator* or *bracket* of x and y: if the following axioms hold, it is called a *Lie algebra* over $\mathbb{F}$:
(L1) The bracket operation is bilinear.
(L2) $[xx] = 0$ for all x in L.
(L3) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ $(x, y, z \in L)$ (Jacob Identity).

If a vector space isomorphism $\phi : L \to L'$ satisfying $\phi([xy]) = [\phi(x)\phi(y)]$ exits for all x, y in L, then the two Lie algebras L, L' over $\mathbb{F}$ are said to be *isomorphic*, where ϕ would be the *isomorphism* of Lie algebras. For K a subspace of L, if $[xy] \in K$ whenever $x, y \in K$ then K is called a subalgebra. According to the operations assumed, K is a Lie algebra in its own right.

End V the set of linear transformations $V \to V$ where V is a finite dimensional vector space over $\mathbb{F}$. End V is a vector space over $\mathbb{F}$, is a ring corresponding to the fixed product operation, and has dimension n^2 (n=dim V). End V becomes a Lie algebra over $\mathbb{F}$ when we interpose a new bracket operation called the *bracket* of x and y, that's defined: $[x, y] = xy - yx$. This is an algebra structure that is different from the associative one, which we denote $\mathfrak{gl}(V)$ (the *general linear algebra*) for End V. It is the algebra of the *general linear group* $\mathsf{GL}(V)$ which is the group of all invertible endomorphisms of V.

Any subalgebra of $\mathfrak{gl}(V)$ is a *linear Lie algebra.* The *classical algebras* are distinguished according to certain long-established linear Lie groups into A_ℓ, B_ℓ, C_ℓ, D_ℓ ($\ell \geq 1$):

$\underline{A_\ell}$: We identify the set of endomorphisms of V having trace zero with $\mathfrak{sl}(V)$ or $\mathfrak{sl}(\ell+1, \mathbb{F})$. It is a subalgebra of $\mathfrak{gl}(V)$ and is called the *special linear algebra* and is coupled with the *special linear group* $\mathsf{SL}(V)$ of endomorphisms of det 1. For A_ℓ dim $V = \ell + 1$.

$\underline{B_\ell}$: If f is the nondegenerate symmetric bilinear form on V with matrices $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}$. $\mathfrak{o}(V)$ or $\mathfrak{o}(2\ell+1, \mathbb{F})$ (the *orthogonal algebra*) is made up of all endomorphisms of V with $f(x(v), w) = -f(v, x(w))$. Here dim $V = 2\ell + 1$ is odd.

$\underline{C_\ell}$: Here f on V is defined by the matrix $s = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$. The *symplectic algebra* denoted $\sim\!\!\mathfrak{l}(V)$ or $\sim\!\!\mathfrak{l}(2\ell, \mathbb{F})$ is comprised of all endomorphisms x of V satisfying $f(x(v), w) = -f(v, x(w))$. Here dim $V = 2\ell$ with basis $(v_1, \ldots, v_{2\ell})$.

$\underline{D_\ell}$: The construction of D_ℓ is exactly the same as B_ℓ's except $s = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}$ and dim $V = 2\ell$ is even. Therefore we acquire another *orthogonal algebra.*

Let's denote the set of *upper triangular matrices* with $\mathfrak{UT}(n, \mathbb{F})$, the *strictly upper triangular matrices* with $\mathfrak{SUT}(n, \mathbb{F})$ and the set of all *diagonal matrices* with $\mathfrak{d}(n, \mathbb{F})$.

A vector space $\mathfrak{U}$ over $\mathbb{F}$ that has a bilinear operation $\mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$ usually denoted by a collocation, is called a $\mathbb{F}$-*algebra* (not necessarily associative). But if $\mathfrak{U}$ is a Lie algebra, then instead of the collocation we always use the bracket. A *derivation* of $\mathfrak{U}$ a linear map $\triangle : \mathfrak{U} \to \mathfrak{U}$ satisfying the known product rule $\triangle(ab) = a \triangle (b) + \triangle(a)b$. The collection of all derivations of $\mathfrak{U}$ is a vector subspace of End $\mathfrak{U}$ denoted Der $\mathfrak{U}$, and it is a subalgebra of $\mathfrak{gl}(\mathfrak{U})$. Der L is defined because Lie algebra L is an $\mathbb{F}$-algebra. ad x denotes the $x \in L$ where $y \mapsto [xy]$ is an endomorphism of L and because we can rewrite the Jacobi identity as $[x[yz]] = [[xy]z] + [y[xz]]$ it follows ad $x \in$ Der L. These types of derivations are called *inner* derivations where every other type of derivation is called an *outer* derivation. And the adjoint representation of L is the map $L \to$DerL sendind x to ad x.

We will use $\text{ad}_L x$ to communicate x is acting on L, and $\text{ad}_K x$ for x is acting on K, to distinguish when $x \in L$ but not in K subalgebra of L.

When L is an arbitrary finite dimensional vector space over $\mathbb{F}$, it is a Lie algebra if we set $[xy] = 0$ for all $x, y \in L$. This algebra is *abelian* if it has trivial Lie multiplication. Let L denote any Lie algebra, with basis $x_1, \ldots x_n$, then the *structure constants* a_{ij}^k that arise in $[x_i x_j] = \sum_{k=1}^n a_{ij}^k x_k$ retrieve the whole multiplication table of L.

2.1.1.2 Ideals, homomorphisms, representations, automorphisms, solvability and nilpotency

An *ideal* of L is subspace I of L for which both $x \in L$, $y \in I \Rightarrow [xy] \in I$. For example 0 and L itself are ideals of L. The *derived algebra* of L, written $[LL]$, which is comparable to the commutator subgroup of a group is an ideal of L. Similarly the *center* $Z(L) = \{z \in L \mid [xz] = 0 \text{ for all } x \in L\}$ is also an ideal of L. L is *simple* if it has no ideal except itself and 0 and more importantly $[LL] \neq 0$. For I an ideal of L the *quotient algebra* L/I is formulated much in the same way as that of a quotient ring.

$N_L(K) = \{x \in L \mid [xK] \subset K\}$ defines the *normaliser* of a subalgebra K of L. K is then said to be *self-normalising* when $K = N_L(K)$. And $C_L(X) = \{x \in L \mid [xX] = 0\}$ is the *centraliser* of a subset X of L.

$\phi : L \to L'$ becomes an *homomorphism* if $\phi([xy]) = [\phi(x)\phi(y)]$, for all $x, y \in L$. When Ker $\phi = 0$, ϕ is a *monomorphism* and an *epimorphism* if $\text{Im}\phi = L'$. It is an isomorphism again when it is both an epimorphism and a monomorphism. Just as to ϕ there is $\text{Ker}\phi$, there is to an ideal I the associated *canonical map* $x \mapsto x + I$ of L onto L/I.

Proposition 2.1.2. *(1)* $L/\text{Ker}\phi \simeq \text{Im}\phi$ *if* $\phi : L \to L'$ *is an homomorphism of Lie algebras. A unique homomorphism* $\psi : L/I \to L'$ *exists if* I *is any ideal of* L

that is in Ker ϕ*, for which*

$$\begin{array}{ccc} L & \xrightarrow{\phi} & L' \\ & \searrow^{\pi} & \uparrow \psi \\ & & L/I \end{array}$$

where π*=canonical map. (2) If* $I \subset J$ *where* I *and* J *are both ideals of* L*, then* J/I *is an ideal of* L/I *and it follows that* L/J *is naturally isomorphic to* $(L/I)/(J/I)$*. (3) There exists a natural isomorphism between* $(I+J)/J$ *and* $I/(I \cap J)$ *if* I*,* J *are ideals of* L*.*

An homomorphism $\phi : L \to \mathfrak{gl}(V)$ is a *representation* of a Lie algebra L. It concludes from all of the above that *any simple Lie algebra is isomorphic to a linear Lie algebra.*

An isomorphism of L onto itself is an *automorphism* of L. If we are restricted to cases of char $\mathbb{F}$=0, if (ad $x)^k = 0$ for some $k > 0$, then $x \subset L$ is an element for which it is said that ad x is *nilpotent.*An *inner* automorphism is that of the form exp (ad x) where ad x nilpotent.

The *derived series* is a sequence of ideals of L defined by $L^{(0)} = L$, $L^{(1)} = [LL]$, $L^{(2)} = [L^{(1)}L^{(1)}], \ldots, L^{(i)} = [L^{(i-1)}L^{(i-1)}]$. If $L^{(n)} = 0$ for some n then L is *solvable.*

Proposition 2.1.3. *For* L *a Lie algebra. (1) When* L *is solvable, then all subalgebras and homomorphic images of* L *are also solvable. (2) When* I *is a solvable ideal of* L *where* L/I *is solvable, then* L *itself is solvable. (3)* $I + J$ *is a solvable ideal if both* I*,* J *are.*

For S a maximal solvable ideal where L is an arbitrary Lie algebra, and I is any other solvable ideal of L, then there exists a unique maximal solvable ideal, called the *radical* of L an denoted Rad L, since (3) of the above proposition would imply $S + I = S$ accordin to maximality, or $I \subset S$. L is called *semisimple* when Rad $L = 0$.

Write the *descending central series* (or *lower central series*) as a sequence of ideals of L as $L^0 = L$, $L^1 = [LL]$ $(= L^{(1)})$, $L^2 = [LL^1], \ldots, L^i = [LL^{i-1}]$. L is *nilpotent* if $L^n = 0$ for some n.

Proposition 2.1.4. *For Lie algebra* L*: (1) all subalgebras and homomorphic images of* L *are nilpotent if* L *is nilpotent. (2)* L *is nilpotent if* $L/Z(L)$ *is nilpotent. (3)* $Z(L) \neq 0$ *if* L *is nilpotent and nonzero.*

x *ad-nilpotent* if ad x is a nilpotent endomorphism.

2.1.1.3 Engel's Theorem

Theorem 2.1.5. *(Engel)* L *is nilpotent if all elements of* L *are ad-nilpotent.*

Lemma 2.1.6. *ad* x *is also nilpotent if* $x \in \mathfrak{gl}(V)$ *is a nilpotent endomorphism.*

Theorem 2.1.7. *For* L *a subalgebra of* $\mathfrak{gl}(V)$ *and* V *finite dimensional, if* $V \neq 0$ *and* L *is made up of nilpotent endomorphisms then there exists nonzero* $v \in V$ *for which* $L.v = 0$.

Corollary 2.1.8. *There exists a basis of* V *relative to which the matrices of* L *are all in* $\mathfrak{SUT}(n\mathbb{F})$*, meaning that according to the theorem above, there exists a flag* (V_i) *in* V *stable under* L*, with* $x.V_i \subset V_{i-1}$ *for all* i*.*

Lemma 2.1.9. *If* $K \neq 0$ *then* $K \cap Z(L) \neq 0$ *if* L *be nilpotent and* K *is an ideal of* L*. Specifically* $Z(L) \neq 0$.

2.1.2 Semisimple Lie Algebras

2.1.2.1 Lie's theorem, Jordan-Chevalley decomposition, Cartan's criterion, criterion for simplicity, simple ideals of L

Theorem 2.1.10. *Given that* $V \neq 0$ *then for all endomorphisms in* L *there is a common eigenvector contained in* V*. This is provided* L *is a solvable subalgebra of* $\mathfrak{g}(V)$ *and* V *is finite dimensional.*

Corollary 2.1.11. *The matrices of* L *relative to a suitable basis of* V *(where* L *is a solvable subalgebra of* $\mathfrak{gl}(V)$ *and dim* $V = n < \infty$*) are upper triangular. Otherwise put: some flag in* V *is stabilised by* L*.*

Corollary 2.1.12. $x \in [LL] \Rightarrow ad_L x$ *is nilpotent for* L *solvable. Specifically,* $[LL]$ *is nilpotent.*

If the roots of the minimal polynomial of $x \in$End V (V finite dimensional) over $\mathbb{F}$ are all distinct, then x is *semisimple*.

Proposition 2.1.13. *For* V *be a finite dimensional vector space over* $\mathbb{F}$ *and* $x \in$ *End* V*: (1) There exist unique* $x_s, x_n \in$ *End* V *that fulfil:* x_n *is nilpotent,* x_s x_n *commute and* $x = x_s + x_n$ *and* x_s *are semisimple. (2)* $x_s = p(x)$*,* $x_n = q(x)$ *for polynomials* $p(T)$*,* $q(T)$ *in one indeterminate, without constant term. Specifically, any endomorphism commuting with* x *commutes with* x_s *and* x_n*. (3) For subspaces* $A \subset B \subset V$*,* x *maps* B *into* A *and* x_s *and* x_n *also map* B *into* A*.*

The (additive) *Jordan-Chevalley decomposition* (or simply Jordan decomposition) of x is the decomposition $x = x_s + x_n$, where x_n is the nilpotent part of x and x_s is its semisimple part.

Lemma 2.1.14. *Let $x \in$ End V, where $x = x_s + x_n$ is its Jordan decomposition and dim $V < \infty$.Then the Jordan decomposition of ad$x$ is ad $x =$adx_s+adx_n (in End (End V)).*

Lemma 2.1.15. *Let $\mathfrak{U}$ be a finite dimenstional $\mathbb{F}$-algebra, then the derivative Der $\mathfrak{U}$ contains the nilpotent and semisimple parts (in End $\mathfrak{U}$) of all its elements.*

Lemma 2.1.16. *Let $A \subset B$ be two subspaces of $\mathfrak{gl}(V)$ where V is finite dimensional. And let $M = \{x \in \mathfrak{gl}(V) \mid [x, B] \subset A\}$. x is nilpotent if $x \in M$ satisfies Tr$(xy) = 0$ for all $y \in M$.*

Theorem 2.1.17. *For $\mathfrak{gl}(V)$, with dim $V < \infty$, let L be a subalgebra. If for all $x \in [LL]$, $y \in L$, Tr$(xy) = 0$, then L is solvable. This is known as Cartan's criterion.*

Corollary 2.1.18. *L is solvable if Tr(ad$x$ad$y$)=0 for all $x \in [LL]$, $y \in L$ for L a Lie algebra.*

For x, y in L a Lie algebra, we define the *Killing form*, κ, a *associative* ($\kappa([xy], z) = \kappa(x, [yz])$) symmetric bilinear form on L, as $\kappa(x, y) =$Tr(ad$x$ad$y$).

Lemma 2.1.19. *Let κ be the Killing form of L, if I is an ideal of L, and κ_I is the Killing form of I. It is viewed as a Lie algebra and $\kappa_I = \kappa\Big|_{I\times I}$.*

Define the *radical* of a symmetric bilinear form, $\beta(x, y)$, as $S = \{x \in L \mid \beta(x, y) = 0$ for all $y \in L\}$. $\beta(x, y)$ is *nondegenerate* if S is 0.

Theorem 2.1.20. *For L a Lie algebra. Its Killing form is nondegenerate if and only if L is semisimple.*

For the set of ideals $I_1, \ldots, I_t$, $L = I_1 + \ldots + I_t$ L is said to be a *direct sum* of the set if $L = I_1 + \ldots + I_t$ i.e. is a direct sum of subspaces.

Theorem 2.1.21. *Let L be a semisimple Lie algebra. Simple ideals, $L_1, \ldots, L_t$, of L exist such that $L = L_1 \oplus \ldots \oplus L_t$. The restriction of κ to $L_i \times L_i$ is the Killing form of L, and every simple ideal of L coincides with one of the L_i.*

Corollary 2.1.22. *For L a semisimple Lie algebra, $L = [LL]$, and each of its ideals is a sum of specific simple ideals of L.Also all homomorphic images and ideals of L are semisimple.*

2.1.2.2 Inner derivations, abstract Jordan decomposition, modules, Casimir elements of representation, Weyl's theorem, preservation of Jordan decomposition

Theorem 2.1.23. *Every derivation of* L *is inner if* L *is semisimple and ad* $L =$ *Der* L.

An abstract Jordan decomposition in a non-specific semisimple Lie algebra can be formed using the above theorem. Recall Lemma 2.1.15, and in particular Theorem 2.1.23, since $L \rightarrow$ad L is 1-1, unique elements $s, n \in L$ are determined by each $x \in L$ where ad $x =$ ad $s+$ ad n is the usual Jordan decomposition of ad x in End L. Therefore $x = s + n$, with $[sn] = 0$, s is *ad-semisimple* (i.e. ads semisimple) and n is ad-nilpotent. $n = x_n$ is the *nilpotent part* of x, and $s = x_s$ is the *semisimple part* of x.

A vector space V that has an operation $L \times V \rightarrow V$, more specifically, $(x, v) \mapsto x.v$ or just xv, satisfies the following conditions: (M1) $(ax + by).v = a(x.v) + b(y.v)$,
(M2) $x.(av + bw) = a(x.v) + b(x.w)$,
(M3) $[xy].v = x.y.v - y.x.v$. $(x, y \in L;\ v, w \in V;\ a, b \in \mathbb{F})$. It is called an *L-module.*

A linear map $\phi : V \rightarrow W$ where $\phi(x.v) = x.\phi(v)$ is called a *homomorphism of* L*-modules.* When the two modules have *equivalent* representations of L, then for ϕ an isomorphism of vector spaces, it is called an *isomorphism* of L-modules. If an L-module V has exactly two L-submodules, then is it called *irreducible. A zero dimensional vector space is not considered as an irreducible* L*-module.* A one dimensional space on which L acts is called irreducible is permitted. And V is *completely reducible* if V is a direct sum of irreducible L-submodules.

Lemma 2.1.24. *For* $\phi : L \rightarrow \mathfrak{gl}(V)$ *irreducible, the only endomorphisms of* V *that commute with all* $\phi(x)$ *where* x *is a subset of* L *are scalars.*

This is known as Schur's lemma. If V is an L-module, if we specify that for f in the dual vector space V^*, $v \in V$, $x \in L$: $(x.f)(v) = -f(x.v)$, then V^* becomes an L-module called the *contragredient* or the *dual.*

We call a representation of L *faithful* if it is 1-1. If L is semisimple and $\phi : L \rightarrow \mathfrak{gl}(V)$ is faithful with non-degenerate trace form $\beta(x, y) =$Tr$(\phi(x)\phi(y))$, then fixing basis $(x_1, \dots, x_n)$ of L, we have that c_ϕ for $c_\phi(\beta)$ and name this the *Casimir element of* ϕ.

Lemma 2.1.25. *Let* L *be a semisimple Lie algebra with representation* $\phi : L \to \mathfrak{gl}(V)$*, then* L *acts trivially on any one dimensional* L*-module and* $\phi(L) \subset \mathfrak{sl}(V)$.

Theorem 2.1.26. $\phi : L \to \mathfrak{gl}(V)$ *is completely reducible if it is a (finite dimensional) representation of a semisimple Lie algebra.*

This is Weyl's theorem

Theorem 2.1.27. *Let* V *be finite dimensional and* $L \subset \mathfrak{gl}(V)$ *a semisimple linear Lie algebra. Then the abstract and usual Jordan decompositions in* L *coincide since* L *would contain the semisimple and nilpotent parts in* $\mathfrak{gl}(V)$ *of all its elements.*

Corollary 2.1.28. *Let* $\phi : L \to \mathfrak{gl}(V)$ *a finite dimensional representation of semisimple Lie algebra* L *then the abstract Jordan decomposition of* $\phi(x)$ *is* $x = s + n$.

2.1.2.3 Weights, maximal vectors, irreducible module classification

In this section we will assume that all modules are finite dimensional over $\mathbb{F}$. L will denote $\mathfrak{sl}(2, \mathbb{F})$, which has standard basis

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

h acts diagonally on V as it's semisimple and for V an arbitrary L-module, a decomposition of V would be a direct sum of eigenspaces $V_\lambda = \{v \in V \mid h.v = \lambda v\}$, $\lambda \in \mathbb{F}$. If $V_\lambda \neq 0$ then λ is a *weight* of h in V and V_λ is a *weight space*.

Lemma 2.1.29. *If* v *is in* V_λ*, then* $x.v$ *is in* $V_{\lambda+2}$ *and* $y.v$ *is in* $V_{\lambda-2}$.

Because the sum $V = \coprod_{\lambda \in \mathbb{F}} V_\lambda$ is direct for $\dim V < \infty$ then $V_\lambda = 0$ must be true so $V_{\lambda+2} = 0$. For the λ just described, a *maximal vector* of weight λ is the name of any nonzero vector in V_λ. Let's now suggest that V is an irreducible L-module. If we pick a maximal vector such as $v_0 \in V_\lambda$ and set $v_i = (1/i!)y^i.v_0$ $(i \geq 0)$ where $v_{-1} = 0$ then,

Lemma 2.1.30. *(1)* $h.v_i = (\lambda - 2i)v_i$*, (2)* $y.v_i = (i+1)v_{i+1}$*, (3)* $x.v_i = (\lambda - i + 1)v_{i-1}$ $(i \geq 0)$.

Theorem 2.1.31. *Let* $L = \mathfrak{sl}(2, \mathbb{F})$ *with irreducible module* V. *(1) Parallel with* h, V *is the direct sum of weight spaces* V_μ, $\mu = m, m-2, \ldots, -(m-2), -m$, *where* $m+1$ *=dim* V *and dim* $V_\mu = 1$ *for each* μ. *(2) Up to nonzero scalar multiples* V *has a unique maximal vector, whose weight is* m *and is called the highest weight of* V. *(3) The above formulas are exactly the action of* L *on* V, *if the basis is chosen in the same way. Specifically, there exists no more than one irreducible* L*-module (up to isomorphism) of each possible dimensions* $m+1$, $m \geq 0$.

Corollary 2.1.32. *Let* V *be any finite dimensional* L*-module where* $L - \mathfrak{sl}(2, \mathbb{F})$ *then the eigenvalues of* h *on* V *are all integers, and each occurs along with its negative (an equal number of times). Specifically, the number of summands in any decomposition of* V *into direct sum of irreducible submodules is exactly dim*V_0+*dim*V_1.

2.1.2.4 Maximal toral subalgebras, roots, centraliser of H, orthogonality, integrality and rationality properties

If L is not nilpotent in the way established in Engel's theorem then we can locate a x in L whose abstract Jordan decomposition has nonzero semisimple part x_s. Then for example the span of such x_s is a nonzero subalgebra of L made up of semisimple elements. This subalgebra is referred to as *toral*

Lemma 2.1.33. *A toral subalgebra of* L *is abelian.*

A toral subalgebra that is not properly included in any other is a *maximal toral subalgebra* H of L. For example, H is just the trace=0 set of diagonal matrices if $L = \mathfrak{sl}(n, \mathbb{F})$.

It follows that $\mathrm{ad}_L H$ is a commuting family of semisimple endomorphisms of L and is *simultaneously diagonalisable.* For all $h \in H\}$, where α ranges over H^*, L is the direct sum of the subspaces $L_\alpha = \{x \in L \mid [hx] = \alpha(h)x$. The centraliser of H, $C_L(H)$, is exactly L_0 which, of course, includes H. Let Φ denote the set of all nonzero $\alpha \in H^*$ for which $L_\alpha \neq 0$. The elements of phi are finite and are called the *roots* of L with respect to H. The *Cartan decomposition* $L = C_L(H) \oplus \coprod_{\alpha \in \Phi} L_\alpha$ is the *root space decomposition.*

Proposition 2.1.34. $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ *for all* $\alpha, \beta \in H^*$. *ad* x *is nilpotent if* $x \in L_\alpha$ *and* $\alpha \neq 0$. L_α *is orthogonal to* L_β *relative to the Killing form* κ *of* L *if* $\alpha, \beta \in H^*$, *and* $\alpha + \beta \neq 0$.

Corollary 2.1.35. *The restriction of the Killing form to* $L_0 = C_L(H)$ *is nondegenerate.*

Lemma 2.1.36. *Let* x, y *be commuting endomorphisms of a finite dimensional vector space, Tr*$(xy) = 0$*, and* y *nilpotent then* xy *is nilpotent.*

Proposition 2.1.37. $H = C_L(H)$ *if* H *is a maximal toral subalgebra of* L.

Corollary 2.1.38. κ *restricted to* H *is nondegenerate.*

For $\phi \in H^*$ there's a corresponding (unique) element $t_\phi \in H$ for which $\phi(h) = \kappa(t_\phi, h)$ for all $h \in H$. Specifically, Φ corresponds to the subset $\{t_\alpha; \alpha \in \Phi\}$ of H. So by the above corollary we can identify H with H^*.

Proposition 2.1.39. *(1)* Φ *spans* H^**. (2) If* $\alpha \in \Phi$ *then* $-\alpha \in \Phi$*. (3)* $[xy] = \kappa(x, y)t_\alpha$ *(*t_α *as above) if* $\alpha \in \Phi$*,* $x \in L_\alpha$*,* $y \in L_{-\alpha}$*. (4)* $[L_\alpha L_{-\alpha}]$ *is one dimensional, with basis* t_α *when* $\alpha \in \Phi$ *(5)* $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$*, for* $\alpha \in \Phi$*. (6) There exists* $y_\alpha \in L_{-\alpha}$ *such that* $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ *span a three dimensional simple subalgebra of* L *isomorphic to* $\mathfrak{sl}(2, \mathbb{F})$ *via* $x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$*,* $y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$*,* $h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *if* $\alpha \in \Phi$ *and* x_α *is any nonzero element of* L_α*. (7)* $h_\alpha = -h_{-\alpha}$ *for* $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$

If like specified in (Proposition 2.1.39(b)) for each pair of roots α, $-\alpha$ $S_\alpha \simeq \mathfrak{sl}(2, \mathbb{F})$ be a subalgebra of L made up as in Proposition 2.1.39(f), we can describe $\mathrm{ad}_L S_\alpha$ so there exists a complete description of all finite dimensional S_α-modules.

If we choose a particular $\alpha \in \Phi$, then it can be shown that twice a root is never a root (i.e. 2α is not a root) and the only multiples of a root α which are roots are $\pm\alpha$.

Proposition 2.1.40. *(1) For* $H_\alpha = [L_\alpha L_{-\alpha}]$*,* $S_\alpha = L_\alpha + L_{-\alpha} + H_\alpha$ *and there exists a unique* $y_\alpha \in L_{-\alpha}$ *satisfying* $[x_\alpha y_\alpha] = h_\alpha$ *for a particular nonzero* $x_\alpha \in L_\alpha$*. Simply put,* $\alpha \in \Phi$ *implies dim* $L_\alpha = 1$*. (2) The only scalar multimples of* α *which are roots are* α *and* $-\alpha$ *if* $\alpha \in \Phi$*. (3) The numbers* $\beta(h_\alpha)$ *are called* Cartan integers*, for which if* $\alpha, \beta \in \Phi$*, then* $\beta - \beta(h_\alpha)\alpha \in \Phi$ *where* $\beta(h_\alpha) \in \mathbb{Z}$*. (4)* $[L_\alpha L_\beta] = L_{\alpha+\beta}$ *for* $\alpha, \beta, \alpha + \beta \in \Phi$*. (5) Let* $\alpha, \beta \in \Phi$*,* $\beta \neq \pm\alpha$*. If* q *is the largest integer such that* $\beta + q\alpha$ *is a root, and* r *is the largest integer such that* $\beta - r\alpha$ *is a root, then* $\beta + i\alpha \in \Phi$ *where* $-r \leq i \leq q$*, and* $\beta(h_\alpha) = r - q$*. (6) The root spaces* L_α *generate* L*.*

Let $\mathbb{E}$ be the real vector space derived as $\mathbb{E} = \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}$ i.e. extending the base field from $\mathbb{Q}$ to $\mathbb{R}$. Then Φ satisfies the following

Theorem 2.1.41. *For* L, H, Φ, $\mathbb{E}$ *as above, (1)* 0 *does not belong to* Φ *since* Φ *spans* $\mathbb{E}$*. (2) No other scalar multiple of* α *is a root, only* $-\alpha \in \Phi$ *since* $\alpha \in \Phi$*. (3)* $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\ \alpha \in \Phi$ *if* $\alpha, \beta \in \Phi$*. (4)* $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \subset \mathbb{Z}$ *then* $\alpha, \beta \subset \Phi$*.*

This theorem essentially states that in the real euclidean space $\mathbb{E}$, Φ is a *root system*, and we have the correspondence $(\mathsf{L}, \mathsf{H}) \mapsto (\Phi\mathbb{E})$.

2.1.3 Root Systems

2.1.3.1 Reflections, root systems, root pairs

In this section we are focused on a finite dimensional vector space over $\mathbb{R}$ endowed with a positive definite symmetric bilinear form (α, β), in other words a fixed Euclidean space $\mathbb{E}$. an invertible linear transformation that leaves a *hyperplane* pointwise fixed therefore sending any vector orthogonal to that hyperplane into its negative is called a *reflection* in $\mathbb{E}$.

$$\mathsf{P}_\alpha = \{\beta \in \mathbb{E} \mid (\beta, \alpha) = 0\} \tag{2.1}$$

is a *reflecting hyperplane* determined by a reflection σ_α of any nonzero vector α. A reflection preserves the inner product on $\mathbb{E}$ showing that it is orthogonal.

Lemma 2.1.42. *If* Φ *is a finite set that spans* $\mathbb{E}$*. If all the reflections* σ_α *leave* Φ *invariant then* $\sigma = \sigma_\alpha$ *where* $\sigma \in \mathsf{GL}(\mathbb{E})$ *leaves* Φ *invariant and pointwise fixes a hyperplane* P *of* $\mathbb{E}$*, sending some nonzero* $\alpha \in \Phi$ *to its negative.*

According to the following axioms, $\Phi \subset \mathbb{E}$ is a *root system*: $(\mathsf{R1})$ Φ, spans $\mathbb{E}$, doesn't contain 0 and is finite.
$(\mathsf{R2})$ The only multiples of α in Φ are $\pm\alpha$.
$(\mathsf{R3})$ The reflection σ_α leaves Φ invariant.
$(\mathsf{R4})$ $\langle \beta, \alpha \rangle \in \mathbb{Z}$ for $\alpha, \beta \in \Phi$.

Let's denote the finite subgroup of $\mathsf{GL}(\mathbb{E})$ generated by the reflections σ_α $(\alpha \in \Phi)$, $\mathcal{W}$, and call it the *Weyl group* of Φ (root system in $\mathbb{E}$).

Lemma 2.1.43. *Let* $\mathcal{W}$ *be the Weyl group of the root system* Φ *in* $\mathbb{E}$*.* $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ *for all* $\alpha \in \Phi$ *if* $\sigma \in \mathsf{GL}(\mathbb{E})$ *leaves* Φ *invariant. And* $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ *for all* $\alpha, \beta \in \Phi$*.*

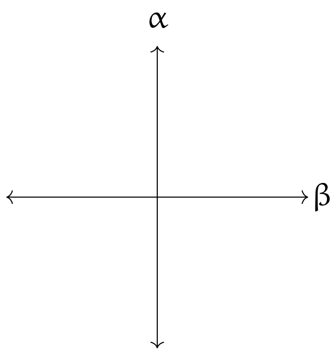
For root systems Φ, Φ' in respectivve euclidean spaces $\mathbb{E}$, $\mathbb{E}'$, $(\Phi, \mathbb{E})$ and $(\Phi', \mathbb{E}')$ *isomorphic* if there exists a vector space isomorphism ϕ. This is not necessarily an isometry.

Let's introduce$\alpha^{\nu} = \frac{2\alpha}{(\alpha,\alpha)}$ and $\Phi^{\nu} = \{\alpha^{\nu} \mid \alpha \in \Phi\}$ the *dual* or *inverse* of Φ. THis is also a root system in $\mathbb{E}$ and its Weyl group is canonically isomorphic to $\mathcal{W}$.

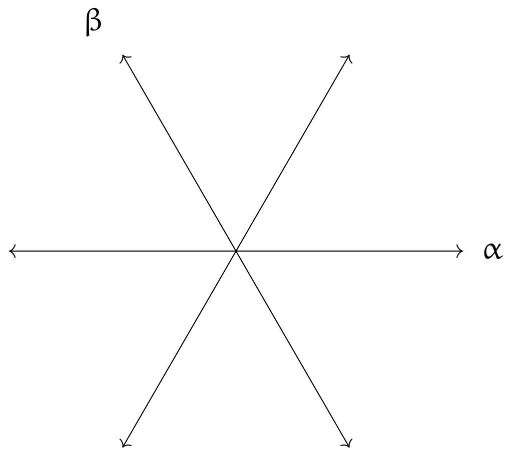
The *rank* of the root system Φ is written $\mathfrak{l}$ =dim$\mathbb{E}$. We can draw pictures for Φ for $\mathfrak{l} \leq 2$.

This is a root system that belongs to $\mathfrak{sl}(2, \mathbb{F})$ with Weyl group of order 2. Rank 2 offers more possibilities, here are three examples:

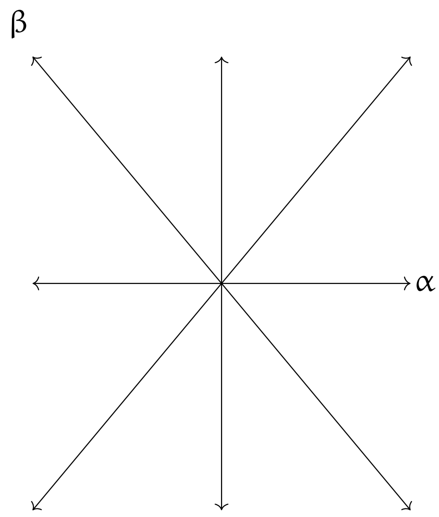
$A_1 \times A_1$



A_2



B_2



The possible angles between pairs of roots is seriously limited by axiom (R4). we have that

$$\langle\beta,\alpha\rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 2\frac{\|\beta\|}{\|\alpha\|}\cos\theta.$$

Lemma 2.1.44. *For nonproportional roots* α *and* β*, if* $(\alpha,\beta) < 0$ *then* $\alpha+\beta$ *is a root, and if* $(\alpha,\beta) > 0$ *then* $\alpha-\beta$ *is a root.*

For nonproportional roots, we call all roots of type $\beta + i\alpha$, where $i \in \mathbb{Z}$, the α-*string through* β. We have that root strings are of length no greater than 4. And from $\beta - r\alpha$ to $\beta + q\alpha$, the α-string through β is unbroken.

2.1.3.2 Weyl chambers, simple roots, the Weyl group and irreducible root systems

In this section, Φ is a root system of rank l with Weyl group $\mathcal{W}$ in a euclidean space $\mathbb{E}$.

A *base* is a name given to a subset $\triangle$ of Φ when, (B1) $\triangle$ is a basis of $\mathbb{E}$, (B2) $\beta = \sum k_\alpha\alpha$ $(\alpha \in \triangle)$, with integral coefficients k_α all nonnegative or all nonpositive, defines each root β.

We call the roots in $\triangle$ *simple*. Card $\triangle = l$ and β defined in (B2) is unique according to (B1). The *height* of a root relative to $\triangle$, may be defined ht$\beta = \sum_{\alpha\in\triangle} k_\alpha$. We call β *positive* if all $k_\alpha \geq 0$ and write $\beta \succ 0$, similarly β is *negative*, $\beta \prec 0$, if all $k_\alpha \leq 0$. The assemblage of positive and negative roots is written as Φ^+ and Φ^- respectively. $\alpha + \beta$ is positive when α and β are positive roots such that $\alpha + \beta$ is also a root. A partial order on $\mathbb{E}$ consistent with $\alpha \succ 0$ is defined by $\triangle$. In fact if we define $\beta \prec \alpha$ if and only if $\beta = \alpha$ or $\alpha - \beta$ is a sum between positive or simple roots.

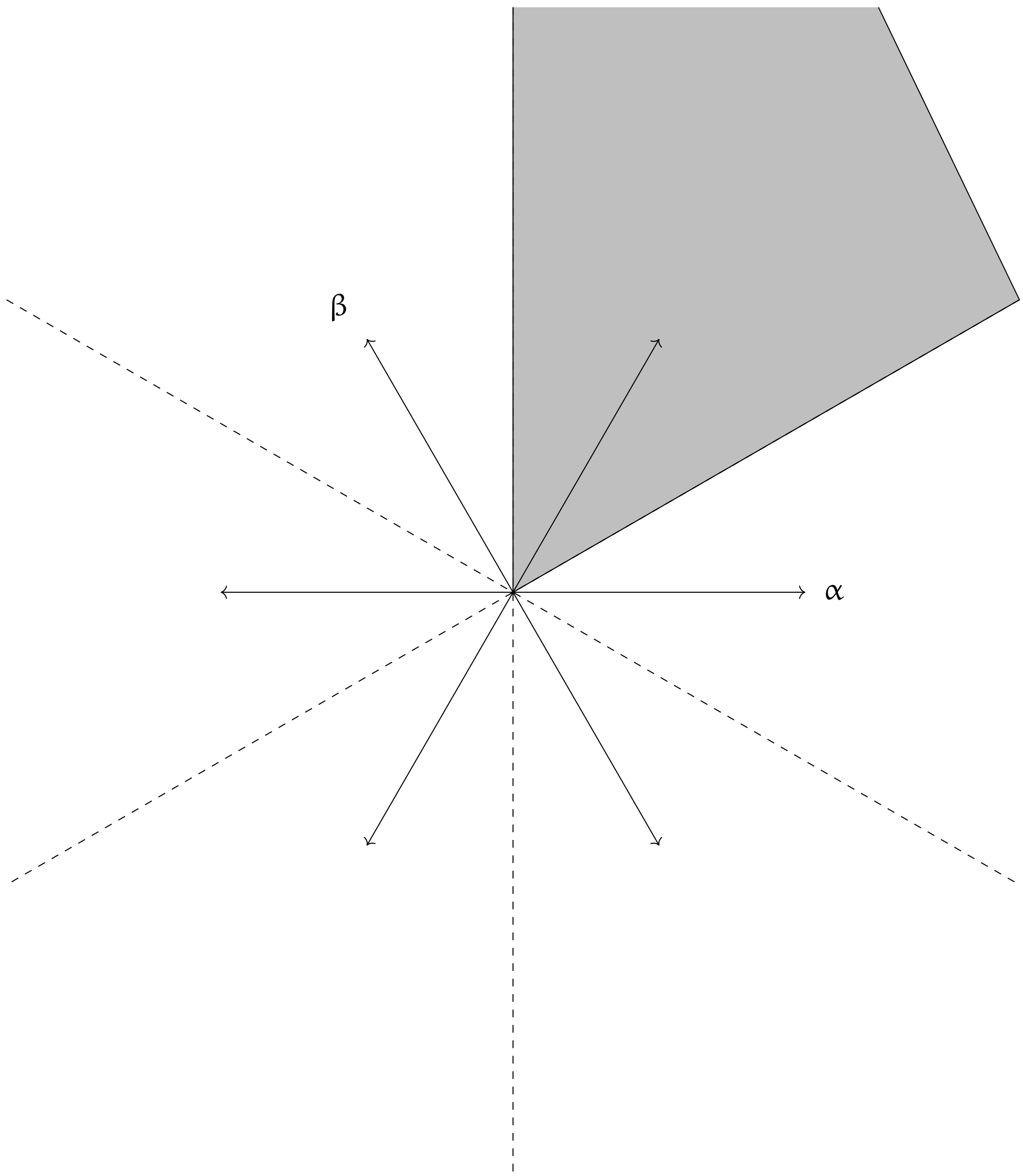
Lemma 2.1.45. $(\alpha, \beta) \leq 0$ *for* $\alpha \neq \beta$ *in* $\triangle$ *if* $\triangle$ *is a base of* Φ*, and* $\alpha - \beta$ *is not a root.*

Theorem 2.1.46. Φ *has a base.*

Define $\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\} =$ for each vector $\gamma \in \mathbb{E}$ as the set of roots that are on the "positive" side of the hyperplane orthogonal to γ. $\mathbb{E} \neq$ the union of finitely many hyperplanes P_α. If $\gamma \in \mathbb{E} - \cup_{\alpha\in\Phi} P_\alpha$ then $\gamma \in \mathbb{E}$ is called *regular*, it's called *singular* otherwise. If $\alpha = \beta_1 + \beta$, for some $\beta_i \in \Phi^+(\gamma)$ then we call $\alpha \in \Phi'(\gamma)$ *decomposable*, it is *indecomposable* otherwise.

Theorem 2.1.47. *The set* $\triangle(\gamma)$ *of all indecomposable roots in* $\Phi^+(\gamma)$ *is a base of* Φ *for regular* $\gamma \in \mathbb{E}$*. Also every base can be acquired this way.*

$\mathbb{E}$ is separated into finitely many regions, separated by the hyperplanes P_α. The (open) *Weyl chambers* of $\mathbb{E}$ are the connected components of $\mathbb{E} - \cup_\alpha P_\alpha$. Each regular $\gamma \in \mathbb{E}$ is contained in strictly one Weyl chamber, denoted $\mathfrak{C}(\gamma)$. If γ and γ' lie on the same side of each hyperplane P_α $(\alpha \in \Phi)$ $(\triangle(\gamma) = \triangle(\gamma')$ or $\Phi^+(\gamma) = \Phi^+(\gamma'))$ we notify this with $\mathfrak{C}(\gamma) = \mathfrak{C}(\gamma')$. In other words, bases are in 1-1 correspondence with Weyl chambers. The *fundamental Weyl chamber relative to* $\triangle$ is $\mathfrak{C}(\triangle) = \mathfrak{C}(\gamma)$ for $\triangle = \triangle(\gamma)$. The open convex set, otherwise known as the intersection of opem half-spaces is denoted $\mathfrak{C}(\triangle)$ and is made up of all γ in $\mathbb{E}$ for which $(\gamma, \alpha) > 0$ where $(\alpha \in \triangle)$.For example take type A_2 which is depicted in the above figure: it has six chambers, one of which is shaded - this is the fundamental region corresponding to the base $\{\alpha, \beta\}$. The Weyl group sends one Weyl chamber onto another, or for regular γ and $\sigma \in \mathcal{W}$, we have that $\sigma(\mathfrak{C}(\gamma)) = \mathfrak{C}(\sigma\gamma)$.



Lemma 2.1.48. *Let $\triangle$ be a fixed base of Φ. $\alpha - \beta$ is a root if α is positive but not simple and for some $\beta \in \triangle$. (Clearly $\alpha - \beta$ is a positive root.)*

Corollary 2.1.49. *Let $\triangle$ be a fixed base of Φ. For $\alpha_i \in \triangle$ where each α_i is not necessarily distinct, each $\beta \in \Phi^*$ can be written as a sum of α_i in the form: $\alpha_1 + \ldots + \alpha_k$. Where each partial sum $\alpha_1 + \ldots + \alpha_i$ is a root.*

Lemma 2.1.50. *Let $\triangle$ be a fixed base of Φ. The positive roots that are not α are permuted by σ_α if α is simple.*

Corollary 2.1.51. *Let $\triangle$ be a fixed base of Φ. $\sigma_\alpha(\triangle) = \triangle - \alpha$ for all $\alpha \in \triangle$ if we set $\triangle = \frac{1}{2}\sum_{\beta \succ 0} \beta$.*

Lemma 2.1.52. *Let $\triangle$ be a fixed base of Φ. If we write $\alpha_1, \ldots, \alpha_t \in \triangle$ where each is not necessarily distinct and introduce the notation $\sigma_i = \sigma_{\alpha i}$ then for some index $1 \leq s < t$, $\sigma_1 \ldots \sigma_t = \sigma_1 \ldots \sigma_{s-1}\sigma_{s+1} \ldots \sigma_{t-1}$ if $\sigma_1 \ldots \sigma_{t-1}(\alpha_t)$ is negative.*

Corollary 2.1.53. *Let $\triangle$ be a fixed base of Φ. $\sigma(\alpha_t) \prec 0$ if $\sigma \in \mathcal{W}$ can be written as $\sigma = \sigma_1 \ldots \sigma_t$ with regard to reflections in relation to the simple roots, where t is as small as possible.*

Theorem 2.1.54. *Let $\triangle$ be a base of Φ. Then (1) $\mathcal{W}$ acts transitively on Weyl chambers, and what this means is that there is a $\sigma \in \mathcal{W}$ for which $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \triangle$ if $\gamma \in \mathbb{E}$ andγ regular. (2) $\mathcal{W}$ acts transitively on bases which means that $\sigma(\triangle') = \triangle$, for some $\sigma \in \mathcal{W}$ if $\triangle'$ is another base of Φ. (3) There exists a $\sigma \in \mathcal{W}$ for which $\sigma(\alpha) \in \triangle$ for α any root. (4) σ_α, where $\alpha \in \triangle$, generate $\mathcal{W}$. (5) $\mathcal{W}$ acts simply transitively on bases which means that $\sigma = 1$ if $\sigma(\triangle) = \triangle$ where $\sigma \in \mathcal{W}$.*

Lemma 2.1.55. $l(\sigma) = n(\sigma)$ *for all* $\sigma \in \mathcal{W}$

Let $\lambda, \mu \in \bar{\mathfrak{C}}(\triangle)$.

Lemma 2.1.56. σ *is product of simple reflections which fix* λ *if* $\sigma\lambda = \mu$ *for some* $\sigma \in \mathcal{W}$. *Specifically,* $\lambda = \mu$.

What the above lemma defines a *fundamental domain* for the action of $\mathcal{W}$ on $\mathbb{E}$ which is the closure $\mathfrak{C}(\triangle)$ of the fundamental Weyl chamber relative to $\triangle$. Each point of this set is $\mathcal{W}$-conjugate to each vector in $\mathbb{E}$.

If a particular Φ cannot be sectioned into the union of two proper subsets where each root in one set is orthogonal to each root in the other set then it is *irreducible*. For example A_1, A_2, B_2, G_2 are irreducible, but $A_1 \times A_1$ is not. If $\triangle$ is a base of Φ, Φ is irreducible if $\triangle$ cannot be partitioned in the way just specified.

Lemma 2.1.57. *(1) For Φ irreducible, there is a unique maximal root β according to the partial ordering $\succ$. After all, $\alpha \neq \beta$ implies $\mathrm{ht}\alpha < \mathrm{ht}\beta$ and $(\beta, \alpha) \geq 0$ for all $\alpha \in \triangle$. All $K_\alpha > 0$ if $\beta - \sum k_\alpha \alpha$ where $\alpha \in \triangle$. (2) For Φ irreducible, $\mathcal{W}$ acts irreducibly on $\mathbb{E}$. Specifically, the $\mathcal{W}$-orbit of a root α spans $\mathbb{E}$. (3) For Φ*

irreducible, then no more than two root lengths can exist in Φ where each root of a particular length is conjugate to another of the same length under $\mathcal{W}$. (For Φ irreducible, the two different root lengths are distinguished into a long *root and a* short *root. When they are all of the same length then they are all referred to as long roots.) (4) For Φ irreducible, with two distinct roots, then the maximal root β is long.*

2.1.3.3 Cartan matrices and their irreducible parts, weights

Let Φ be a root system of rank l, $\mathcal{W}$ its Weyl group and $\triangle$ a base of Φ. Fix $(\alpha_1, \ldots, \alpha_l)$ to be an ordering of the simple roots then the *Cartan matrix* of Φ is the matrix $(\langle\alpha_i, \alpha_j\rangle)$ whose entries are the *Cartan integers*.

Example 2.1.58.

$$A_1 \times A_1 \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \; A_2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \; B_2 \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

The Cartan matrix is independent of the choice of $\triangle$ but does depend on the chosen ordering.

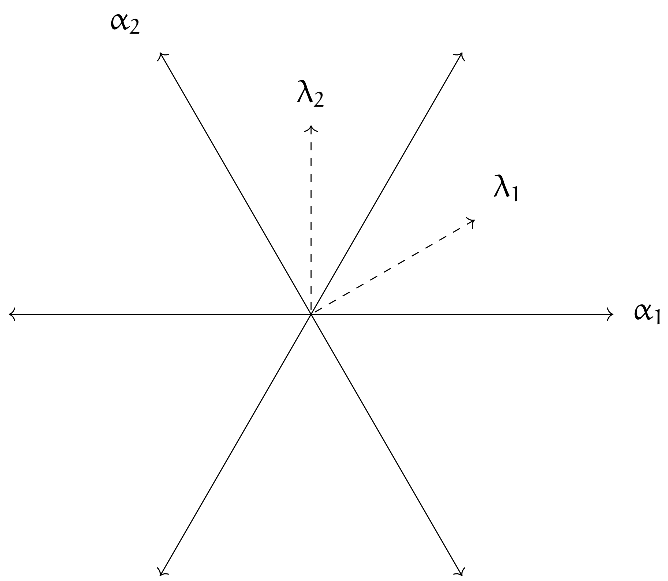
According to the following proposition, we can retrieve Φ from the Cartan integers. One can therefore write down all the roots.

Proposition 2.1.59. *Φ is determined by its Cartan matrix up to isomorphism: Let Φ be a root system of rank l, $\mathcal{W}$ its Weyl group and $\triangle$ a base of Φ. Choose $\Phi' \subset \mathbb{E}'$ as another root system with base $\triangle' = \{\alpha'_1, \ldots, \alpha'_l\}$. $\alpha_i \mapsto \alpha'_i$ uniquely extends to an isomorphism $\phi : \mathbb{E} \longrightarrow \mathbb{E}'$ therefore mapping Φ onto Φ' and fulfills $\langle\phi(\alpha), \phi(\beta)\rangle = \langle\alpha, \beta\rangle$ for all $\alpha, \beta \in \Phi$, if for $1 \leq i, j \leq l$ we have that $\langle\alpha_i, \alpha_j\rangle = \langle\alpha'_i, \alpha'_j\rangle$.*

Proposition 2.1.60. *If we section $\triangle$ into mutually orthogonal subsets: $\triangle = \triangle_1 \cup \ldots \triangle_t$. If $\mathbb{E}_i$ spans $\triangle_i$ then $\mathbb{E}$ can be written as an orthogonal direct sum: $\mathbb{E} = \mathbb{E}_1 \oplus \ldots \oplus \mathbb{E}_t$. Φ can be written as a unique union of irreducible root systems Φ_i in $\mathbb{E}_i$ so that $\mathbb{E} = \mathbb{E}_1 \oplus \ldots \oplus \mathbb{E}_t$.*

Call *weights* the elements of the set of all $\lambda \in \mathbb{E}$ for which $\langle\lambda, \alpha\rangle \in \mathbb{Z}$, we will call this set Γ. Γ is a subgroup of $\mathbb{E}$ including Φ because $\langle\lambda, \alpha\rangle = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$ linearly depends on λ. We have that $\langle\lambda.\alpha\rangle \in \mathbb{Z}$ for all $\alpha \in \triangle$ if and only if $\lambda \in \Gamma$. The *root lattice* is the subgroup of Γ generated by Φ which we denote Γ_r. Specifically a root lattice in $\mathbb{E}$ in that it is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis of $\mathbb{E}$. $\lambda \in \Gamma$ is *dominant* for a fixed base $\triangle \subset \Phi$ if all integers $\langle \lambda\alpha \rangle$ are nonnegative. If these integers are positive then it is *strongly dominant.* If we set Γ^+ as the set of all dominant weights, this means that it is the set of all weights that lie in the closure of $\mathfrak{C}(\triangle)$, the fundamental Weyl chamber. And the set of strongly dominant weights is denoted $\Gamma \cap \mathfrak{C}(\triangle)$.

The vectors $2\alpha_i/(\alpha_i, \alpha_i)$ also form a basis of $\mathbb{E}$ if $\triangle = \{\alpha_1, \dots, \alpha_l\}$. If we let $\lambda_1, \dots, \lambda_l$ be the dual basis in relation to the inner product on $\mathbb{E}$, i.e. $\frac{2(\lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij}$, then the λ_i are dominant weights because all $\langle \lambda_i, \alpha \rangle$ are nonnegative integers and are called the fundamental dominant weights (relative to $\triangle$).



the *fundamental group* of Φ is Γ/Γ_r which must be a finite group according to theory on lattices: The Cartan matrix denotes a change of basis, particularly, $m_{ij} \in \mathbb{Z}$ write $\alpha_i = \sum_j m_{ij}\lambda_j$, this leads to $\langle \alpha_i, \alpha_k \rangle = \sum_j m_{ij}$ and $\langle \lambda_j, \alpha_k \rangle = m_{ik}$. For example the Cartan matrix of type A_2 is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ therefore $\alpha_1 = 2\lambda_1 - \lambda_2$ and $\alpha_2 = -\lambda_1 + 2\lambda_2$. If we invert this matrix, $(1/3)\begin{pmatrix} (2 & 1 \\ 1 & 2 \end{pmatrix}$, we have that $\lambda_1 = (1/3)(2\alpha_1 + \alpha_2)$ and $\lambda_2 = (1/3)(\alpha_1 + 2\alpha_2)$.

2.1.4 Reduction and Isomorphism

In previous sections we showed that Φ is a root system in $\mathbb{E}$: Let L be a semisimple Lie algebra over the algebraically closed field $\mathbb{F}$ of characteristic 0, and L has maximal toral subalgebra H, and the set of roots of L relative to H is $\Phi \subset H^*$. For $l = \dim_{\mathbb{F}} H^*$ we know that the rational span of Φ in H^* has dimension l over $\mathbb{Q}$. We derive an l-dimensional real vector space $\mathbb{E}$ that is spanned by Φ if we extend the base field from $\mathbb{Q}$ to $\mathbb{R}$. $\mathbb{E}$ is a euclidean space because the symmetric bilinear form dual to the Killing form is carried along to it.

Proposition 2.1.61. *Let L, H and Φ be as above. Then Φ is an irreducible root system in the sense of 2.1.3.2.*

Therefore:

Corollary 2.1.62. *Let L, H and Φ be as above. If the decomposition of L into simple ideals is $L = L_1 \oplus \ldots \oplus L_t$ then the maximal toral subalgebra of L_i is $H_i = H \cap L_i$. The (irreducible) root system Φ that corresponds is irreducible and has canonical decomposition $\Phi = \Phi_1 \cup \ldots \cup \Phi_t$.*

Characterising simple roots by their (irreducible) root systems is easier than characterising semisimple Lie algebras by their root systems.

Proposition 2.1.63. *If we fix a small set of generators for L. Let L, H as above and Φ the root system of L relative to H. Fix $\triangle$ a base of Φ. Then the root spaces L_α, $L_{-\alpha}$ generate L as a Lie algebra. Or for $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$ arbitrary root vectors generate L.*

A *standard set of generators* for L is what we call $\{x_\alpha, y_\alpha\}$ or $\{x_\alpha, y_\alpha, h_\alpha\}$ if $0 \neq x_\alpha \in L_\alpha$ and $0 \neq y_\alpha \in L_{-\alpha}$ ($\alpha \in \triangle$), with $[x_\alpha y_\alpha] = h_\alpha$. The isomorphism $\mathbb{E} \to \mathbb{E}'$ (not necessarily an isometry) causes isomorphism $\Phi \to \Phi'$. If we multiply the inner product on $\mathbb{E}$ or $\mathbb{E}'$ by a positive scalar this doesn't affect the root system axioms.

Theorem 2.1.64. *Let L be a simple Lie algebra over $\mathbb{F}$, with maximal toral subalgebra H and root system Φ and let L' be another Lie algebra over $\mathbb{F}$, with maximal toral subalgebra H' and root system Φ'. Assume $\alpha \mapsto \alpha'$ denotes the isomorphism $\Phi \to \Phi'$ so that there exists $\pi : H \to H'$. $\triangle' = \{\alpha' \mid \alpha \in \triangle\}$ is a base of Φ' for fixed $\triangle \subset \Phi$. Pick an arbitrary Lie algebra isomorphism $\pi_\alpha : L_\alpha \to L'$ which means to pick an arbitrary non-zero $x_\alpha \in L_\alpha$, $x'_{\alpha'} \in L'_{\alpha'}$ for* *each* $\alpha \in \triangle$ *and* $\alpha' \in \triangle'$. *Then there exists a unique isomorphism* $\pi : L \to L'$ *extending* $\pi : H \to H'$ *and all* π_α *(*$\alpha \in \triangle$*).*

2.2 Momentum Map

Again, here we provide the relevant theorems, propositions, lemmas, corollaries and definitions for symplectic manifolds,almost complex structures, symplectic and Hamiltonian actions of $\mathbb{R}$, Lie groups, orbit spaces and momentum maps with some examples. Proofs and more detail can be found in [61] and [77].

2.2.1 Symplectic Manifolds

Definition 2.2.1. For M a smooth manifold that hasa closed, nondegenerate, skew-symmetric 2-form ω on it, a *symplectic manifold* is a pair (M, ω). If the 2-form ω is comprehended then we will just say that M is a symplectic manifold.

To say that ω is closed means that $d\omega = 0$ where d is the exterior derivative. Nondegenerate ω means that at any point $p \in M$, if we have that $X \in T_pM$ then if $\omega_p(X, Y) = 0$ for all $Y \in T_xM$ then this must mean that $X = 0$. And for ω to be skew-symmetric this means that $\omega_p(X, Y) = -\omega_p(Y, X)$ for all $X, Y \in T_pM$ at any $p \in M$. If we focus on the symplectic linear geometry properties of ω_p on T_pM, the nondegeneracy and skew-symmetric conditions of ω_p mean that T_pM must have even dimensions and in turn M must have even dimension.

Proposition 2.2.2. *If* M *is a symplectic manifold then it is necessarily even dimensional.*

Definition 2.2.3. A diffeomorphism from a symplectic manifold to itself that preserves the symplectic form is called a *symplectomorphism.* In particular, ψ $\in$Diff(M) is a symplectomorphism if $\psi^*\omega = \omega$ for M a symplectic manifold. Therefore at points $p \in M$ and with vectors $X, Y \in T_pM$, by the definition of the pullback we have that,

$$(\psi^*\omega)_p(X, Y) = \omega_{\psi(p)}(d\psi_p(X), d\psi_p(Y)) = \omega_p(X, Y)$$

The group, under composition, of symplectomorphisms of a symplectic manifold to itself is written as *Symp*(M, ω).

Definition 2.2.4. A submanifold Y of a symplectic manifold (M, ω) where at each point $p \in Y$, the restriction of ω_p to T_pY is symplectic is called a *symplectic submanifold.* In other words $\omega_p\Big|_{T_pY\times T_pY}$ is nondegenerate and because ω is closed and skew-symmetric this immediately means that the restriction is also closed and skew symmetric.

2.2.2 Almost Complex Structures

Definition 2.2.5. Let V be a vector space. A linear map $J : V \to V$ such that $J^2 = -\text{Id}$ is a *complex structure* on V.

Definition 2.2.6. Let (V, ω) be a symplectic vector space. J is called a *compatible* complex structure if the map $g_J : V\times V \to \mathbb{R}$ where $g_J(X, Y) = \omega(X, JY)$ for all $X, Y \in V$ is a positive inner product on V.

Proposition 2.2.7. *A compatible complex structure on* V *exists if* (V, ω) *is a symplectic vector space.*

Definition 2.2.8. Let M be a smooth manifold. A smooth field of complex structures on the vector spaces of the tangent spaces is an *almost complex structure* on M. In other words, there is a linear map $J_x : T_xM \to T_xM$ at each point $x \in M$ such that $J_x^2 = -\text{Id}$.

Definition 2.2.9. Let $(M\omega)$ be a symplectic manifold. J is a *compatible* almost complex structure on M if g the two form on TM that satisfies:

$$\begin{aligned} g_x : T_xM &\times T_xM \to \mathbb{R} \\ g_x(X, Y) &= \omega_x(X, J_x) \text{ for all } X, Y \in T_xM \end{aligned}$$

is a Riemannian metric om M. A *compatible triple* (ω, g, J) where ω is a symplectic form g is a Riemannian metric, and J is an almost complex structure if $g_x(\cdot, \cdot) = \omega_x(\cdot, J_x\cdot)$ for all $x \in M$.

Proposition 2.2.10. *Let* (M, ω) *be a symplectic manifold and* g *a Riemannian metric on* M*. Then a compatible almost complex structure* J *on* M *exists.*

Proposition 2.2.11. *Any symplectic manifold has compatible almost complex structures.*

Proposition 2.2.12. *Suppose* (V, ω) *is a symplectic vector space, and* (ω, g, J) *is a compatible triple on* V *then a linear map* $A : V \to V$ *which preserves both the symplectic and the complex structures must be unitary (*$A \in U(V)$*).*

2.2.3 Symplectic and Hamiltonian Actions of $\mathbb{R}$

Definition 2.2.13. Suppose (M, ω) is a symplectic manifold then a group homomorphism $\psi : \mathbb{R} \to$Symp(M, ω) where the evaluation map $ev_\psi : M \times \mathbb{R} \to M$ given by $ev_\psi(p, t) = \psi_t(p)$ is smooth is called a *smooth symplectic action of* $\mathbb{R}$ *on* M.

Definition 2.2.14. If X is a vector field on a symplectic manifold $(M\omega)$ then if the 1-form $i_X\omega$ is closed, as in, $di_X\omega = 0$ we call X a *symplectic vector field.*

Recall that for a smooth vector field X and a given tensor field τ, the flow of X can be written ψ_t, certainly $\psi_0 =$Id and $\frac{d}{dt}\psi_t(p) = X(\psi_t(p))$. And

$$\mathcal{L}_X\tau = \frac{d}{dt}\Big|_{t=0}\psi_t^*\tau.$$

is the Lie derivative of τ with respect to X. And the following Lie derivative identities hold: (A) The Cartan Magic Formula: $\mathcal{L}_X\tau = i_X d\tau + di_X\tau$
(B) $\frac{d}{dt}\psi_t^*\tau = \psi_t^*\mathcal{L}_X\tau$.

Proposition 2.2.15. *For* (M, ω) *be a compact, symplectic manifold and* $\psi : \mathbb{R} \to \mathsf{Symp}(M, \omega)$ *a smooth symplectic action of* $\mathbb{R}$*. Then a particular collection of vector fields* $\{X_t\}$ *are generated by* ψ *defined:*

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t.$$

This means that X_t *is a symplectic vector field for every* $t \in \mathbb{R}$*. However, a smooth family of diffeomorphisms* $\{\psi_t\}$ *satisfying:*

$$\psi_0 = \mathrm{Id}$$

and

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t.$$

is determined by the flow of X_t *if* $\{X_t\}$ *is a time-dependent family of symplectic vector fields. And* $\{\psi_t\}$ *is a smooth symplectic action* $\psi : \mathbb{R} \to \mathsf{Symp}(M\omega)$*. Thus* *there exists* {*symplectic actions of* $\mathbb{R}$ *on* M} $\leftrightarrow$ {*time-dependent symplectic vector fields on* M} *a 1-1 correspondence.*

In other words, the above proposition shows that for a given complete vector field X, its flow $\{\exp tX : M \to M \mid t \in \mathbb{R}\}$ is defined as the unique collection of diffeomorphisms is a smooth symplectic action if it satisfies

$$\exp tX\Big|_{t=0} = \mathrm{Id}$$

$$\frac{d}{dt}\exp tX = X \circ \exp tX.$$

Definition 2.2.16. Let (M, ω) be a symplectic manifold. Then for any given smooth function $H : M \to \mathbb{R}$ a vector field X_H on M can be defined by

$$i_{X_H}\omega = dH$$

by the nondegeneracy of ω

H is called an *Hamiltonian function* and X_H is called an *Hamiltonian vector field*. And X_H is tangent to the level sets of H:

$$dH(X_H) = i_{X_H}\omega(X_H) = \omega(X_H, X_H) = 0.$$

Definition 2.2.17. X_H is a symplectic vector field because $di_{X_H}\omega = ddH = 0$, so the flow ψ of X_H is a smooth symplectic action if M is compact. We call ψ is a *Hamiltonian action of* $\mathbb{R}$.

2.2.4 Lie Groups

Recall that a *Lie group* is a group G that is a smooth manifold where multiplication and inversion operations are smooth maps.

Definition 2.2.18. Let G be a Lie group. For $g \in G$, we can define left multilplication by g as $L_g : G \to G$ given by $a \mapsto g \cdot a$. A vector field X on G is *left-invariant* if for every $g \in G$ $(L_g)_*X = X$.

Proposition 2.2.19. *Let's introduce a new notation for the Lie algebra. Specifically, the* Lie algebra of the Lie group G *is the set* $\mathfrak{g}$ *of all left-invariant vector fields on* G*, together with the Lie bracket* $[\cdot, \cdot]$.

Proposition 2.2.20. *An isomorphism of vector spaces is defined by the map* $\mathfrak{g} \to T_eG$ *sends a left invariant vector field to its value at the identity* e *of* G*, given by* $X \mapsto X_e$*. Therefore, we can identify the vector space* T_eG *with* $\mathfrak{g}$.

Definition 2.2.21. Taking the identity of the map

$$\begin{aligned} \psi_g : G &\to G \\ g &\mapsto g \cdot a \cdot g^{-1} \end{aligned}$$

provides an invertible linear map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ ($\mathfrak{g} = T_eG$). The *adjoint action* is an action of G on $\mathfrak{g}$ obtained by varying g given by:

$$\begin{aligned} \mathrm{Ad} : G &\to GL(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}_g. \end{aligned}$$

Definition 2.2.22. Introduce $\mathfrak{g}^*$ as the dual vector space of $\mathfrak{g}$. The pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$, $\langle \cdot, \cdot \rangle$, is defined:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} &\to \mathbb{R} \\ (\xi, X) &\mapsto \langle \xi, X \rangle = \xi(X). \end{aligned}$$

Given $\xi \in \mathfrak{g}^*$ we define $\mathrm{Ad}^*_g \xi$ by $\langle \mathrm{Ad}^*_g \xi, X \rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X \rangle$ for any $X \in \mathfrak{g}$ therefore defining the map $\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$. The *coadjoint action* is the action of G on $\mathfrak{g}^*$ given by varying g:

$$\begin{aligned} \mathrm{Ad}^* : G &\to GL(\mathfrak{g}^*) \\ g &\mapsto \mathrm{Ad}^*_g. \end{aligned}$$

$\mathrm{Ad}_g = \mathrm{Id}$ on $\mathfrak{g}$ and $\mathrm{Ad}^*_g = \mathrm{Id}$ on $\mathfrak{g}^*$ for all $g \in G$ if G is abelian. Certainly the Lie group $\mathbb{T}^n$ is abelian.

2.2.5 Orbit Space

Definition 2.2.23. Let $\psi : G \to \mathrm{Diff}(M)$ be any action. $\{\psi_g(p) \mid g \in G\}$ is the *orbit* of G through $p \in M$.

Definition 2.2.24. The subgroup $G_p := \{g \in G \mid \psi_g(p) = p\}$ is the *isotropy subgroup* (or *stabiliser*) of $p \in M$.

For $p, q \in M$, let $\sim$ be the orbit equivalence relation between them. $p \sim q \Leftrightarrow p$ *and* q *are on the same orbit.* The *orbit space* is the space of orbits $M/\sim = M/G$. And the *point-orbit projection* is

$$\pi : M \to M/G$$
$$p \mapsto \text{orbit through } p.$$

2.2.6 Momentum Maps

Definition 2.2.25. For (M, ω) a symplectic manifold. A group homomorphism $\psi : G \to \text{Symp}(M, \omega)$ where the evaluation map $ev_\psi : M \times G \to M$ given by $ev_\psi(p, g) = \psi_g(p)$ is smooth is a *smooth symplectic action of a Lie group* G.

Definition 2.2.26. The *infinitesimal action of* ξ, for a vector $\xi \in \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of G, is the vector field X_ξ on M that satisfies:

$$X_\xi = \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(t\xi)}.$$

That X_ξ is a symplectic vector field immediately follows from $\mathbb{R} \to \text{Symp}(M, \omega) : t \mapsto \psi_{\exp(t\xi)}$.

Definition 2.2.27. Let (M, ω) be a symplectic manifold, G be a Lie group, $\mathfrak{g}$ be the lie algebra of G, $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$, and $\psi : G \to \text{Symp}(M, \omega)$ be a symplectic action. Then if there exists a map,

$$J : M \to \mathfrak{g}^*$$

we call ψ a *Hamiltonian action* which we call the *momentum map* that satisfies:
(A) We define $H_\theta : M \to \mathbb{R}$ by $H_\theta(p) = \langle \mu(p), \theta \rangle$ for each $\theta \in \mathfrak{g}$. The Hamiltonian function for the vector field X_θ is H_θ such that:

$$dH_\theta = i_{X_\theta} \omega.$$

(B) The action of ψ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$ are equivariant to J for all $g \in G$:

$$J \circ \psi_g = \text{Ad}_g^* \circ \mu$$

$\mathrm{Ad}_g^* = \mathrm{Id}$ for all $g \in G$ for G abelian, therefore for G abelian, condition (B) becomes:

$$J \circ \psi_g = J.$$

2.2.6.1 Examples

Example 2.2.28. Let (V, ω_V) be a symplectic vector space and let K be a compact connected Lie group with a fixed maximal torus T that acts on V by linear symplectic transformations. This is a Hamiltonian action with quadratic momentum map

$$J_V^\xi(v) = \frac{1}{2}\omega_V(\xi v, v)$$

where ξ_V is the image of $v \in V$ under $\xi \in \mathfrak{g}$, and is considered a linear operator on V. Pick $\mathbb{J}$, a K-invariant ω_V-compatible complex structure, on V and suppose that $\langle \cdot, \cdot \rangle$ is the Hermitian inner product whose imaginary part is equal to $-\omega_V$. Then rewrite the momentum map:

$$J_V^\xi(v) = \frac{\sqrt{-1}}{2}\langle \xi v, v\rangle. \tag{2.2}$$

If $K = T$ is a torus then V is an orthogonal direct sum of weight spaces, $V = \oplus_{\nu \in \Lambda^*} V_\nu$. ν is called a weight of symplectic action of T on V if $V_\nu \neq 0$. The weights do not depend on the choice of the complex structure, but the weight space decomposition does. This is because any two K-invariant compatible complex structures on V are conjugate by a K-equivariant linear symplectic map. And $\xi v = 2\pi\sqrt{-1}\nu(\xi)v$, so $J(v) = -\pi\|v\|^2\nu$ if v is a vector of weight ν by (2.2) (so that,

$$\Delta(V) = -\mathrm{cone}\{\nu_1, \ldots, \nu_l\},$$

where $\nu_1, \ldots, \nu_l$ are the (real) weights of V as will be clear by the end of this chapter).

Example 2.2.29. Take the $n \times n$ dimensional Unitary group $U(n)$, the Lie algebra, and its dual, is the space of Hermitian $n \times n$-matrices according to the standard invariant form (and multiplication by i). These Hermitian matrices are conjugate by an element of $U(n)$ to a diagonal matrix. A chain of subspaces $V_1 \subset V_2 \ldots \subset V_{n-1} \subset V_n = \mathbb{C}^n$ with $\dim V_i = i$ is a *maximal flag* in $\mathbb{C}^n$. If we denote the set of maximal flags by $\mathbf{F}$ then $\mathbf{F} = U(n)/T$ where T is the n-dimensional torus made up of diagonal unitary matrices. Given a maximal flag

$\{V_i\}$ let U_i be the orthogonal complement of V_{i-1} in V_i. Fix real $\lambda_1, \dots, \lambda_n$ with $\lambda_1 > \lambda_2 \dots > \lambda_n$. Then there is a unique hermitian matrix with eigenspaces U_i and corresponding eigenvalues λ_i, and $\mathbb{C}^n = \bigoplus U_i$, specifically this gives a map,

$$f_\lambda : \mathbf{F} \to \mathfrak{g}^*$$

the image of which is the (generic) coadjoint orbit. Flag manifolds with different sequences of dimensions are the other coadjoint orbits, such as the Grassmanians and the projective space $\mathbb{C}P^{n-1}$.

The coadjoint orbit $f_\lambda(\mathbf{F})$ of $U(n)$ is an integral orbit if and only if the λ_i are integers: for an acceptable choice of normalisation of the invariant quadratic form on the Lie algebra.

The standard action of $U(n+1)$ on a projective space $\mathbb{C}P^n$ has momentum map,

$$J(z) = \frac{i}{|z|^2} z z^*,$$

where z is a column vector, representing a point in $\mathbb{C}P^n$.

For another group G the transpose of $\mathfrak{g} \to \mathfrak{u}(n)$ begets $\mathfrak{u}(n)^* \to \mathfrak{g}^*$. If we take $\xi \in \mathfrak{g}$ to act on $\mathbb{C}^{n+1}$ by iA for a self-adjoint matrix A then we can assume that A is diagonal with eigenvalues λ_a, ordered so that $\lambda_1 \geq \lambda_2 \geq \dots$ and H is

$$H(\mathbf{z}) = \frac{1}{|\mathbf{z}|^2} \sum \lambda_a |z_a|^2$$

where taking the eigenvector of A with largest eigenvalue corresponds to taking the point p where H is maximal. And the weight of the action on the fibre of H over p is simply what the eigenvalue is.

$U(n)$ has rank n: the maximal torus in $U(n)$ is given by the diagonal matrices

$$\mathrm{diag}(e^{i\lambda_1}, \dots, e^{i\lambda_n}),$$

The symmetric group on n objects, acting by permutations of the eigenvalues $e^{i\lambda_n}$ is isomorphic to the Weyl group of $U(n)$. Therefore the weights of the standard representation $\mathbb{C}^n$ are λ_i with usual coordinates $(\lambda_1, \dots, \lambda_n)$ on Lie (T) for the maximal torus T of $U(n)$. The roots are $\lambda_i - \lambda_j$ for $i \neq j$ and there are $n(n-1)$ of these and $\dim G = n^2$, $\mathrm{rank}(G) = n$.

2.3 Coadjoint Orbit

$I_g : G \to G$ is the inner automorphism $I_g(h) = ghg^{-1}$, recall that the *adjoint representation* of a Lie group G is defined

$$\mathrm{Ad}_g = T_e I_g : \mathfrak{g} \to \mathfrak{g}.$$

The *coadjoint action* is given by

$$\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*$$

where, for $\mu \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$, and $\langle , \rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$,

$$\langle \mathrm{Ad}^*_{g^{-1}}(\mu), \xi \rangle = \langle \mu, \mathrm{Ad}_{g^{-1}}(\xi) \rangle$$

i.e. $\mathrm{Ad}^*_{g^{-1}}$ is the dual of the linear map $\mathrm{Ad}_{g^{-1}}$. $\mathcal{O}_\mu$ is the subset of $\mathfrak{g}^*$ and is the *coadjoint orbit* through $\mu \in \mathfrak{g}^*$ defined,

$$\mathcal{O}_\mu := \{\mathrm{Ad}^*_{g^{-1}}(\mu) \mid g \in G\} := G \cdot \mu$$

$\mathcal{O}_\mu$ is an *immersed submanifold* like the orbit of any group action. If G is compact, $\mathcal{O}_\mu$ is a closed embedded submanifold.

Example 2.3.1. The dual of the Lie algebra of $SO(3)$, $\mathfrak{so}(3)^*$, is the group of antisymmetric matrices which is isomorphic to $\mathbb{R}^3$. A coadjoint orbit of $SO(3)$ is,

$$\mathcal{O}_\Pi := \{\mathrm{Ad}^*_{\mathbf{A}^{-1}}(\Pi) \mid \mathbf{A} \in SO(3), \Pi \in \mathbb{R}^3\} := \{\mathbf{A}\Pi \mid \mathbf{A} \in SO(3)\} \tag{2.3}$$

which is the sphere in $\mathbb{R}^3$ of radius $||\Pi||$ [61].

The coadjoint orbit K_μ with its Kirillov-Kostant symplecti form ω_μ is a Hamiltonian K-manifold for every μ in $mathfrakt^*_+$. The momentum map is simply the inclusion $i_\mu : K_\mu \to \mathfrak{l}^*$.

2.3.1 Coadjoint Equivariant Momentum Map

the *Hamiltonian vector field* associated to the *Hamiltonian function* H is a vector field on M induced from the natural isomorphism between dervations on $C^\infty(M)$

$$X_H = \{\cdot, H\}.$$

Hamilton's equations $\dot{z} = X_h(z)$ can be rewritten in the Poisson bracket form for any $f \in C^\infty(M)$,

$$\dot{f} = \{f, h\}.$$

For $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $\mathfrak{g}$ a Lie algebra acting on it canonically with a momentum map $J : M \to \mathfrak{g}^*$. One can see that the map $(\mathfrak{g}, [\cdot,\cdot]) \to (C^\infty(M), \{\cdot,\cdot\})$ defined by $\xi \mapsto J^\xi$ becomes a Lie algebra homomorphism i.e.

$$J^{[\xi,\eta]} = \{J^\xi, J^\eta\}, \quad \xi, \eta \in \mathfrak{g}.$$

if and only if for any $\xi \in \mathfrak{g}$ and any $z \in M$,

$$T_z J \cdot \xi_M(z) = -\mathrm{ad}^*_\xi J(z). \tag{2.4}$$

This is then called an *infinitesimally equivariant* momentum map, which is the infinitesimal version of the *global* or *coadjoint equivariance* when the Lie algebra action corresponds to the Lie group action. J is G-equivariant when for all $g \in G$

$$\mathrm{Ad}^*_{g^{-1}} \circ J = J \circ \Psi_g \tag{2.5}$$

otherwise put, $J^{\mathrm{Ad}_g \xi}(g \cdot z) = J^\xi(z)$, for all $g \in G$, $\xi \in \mathfrak{g}$, and $z \in M$ and smooth left action of G on M, $\Psi : G \times M \to M$. The derivative of (2.5) with respect to g at $g = e$ in the direction ξ is (2.4).

Lie algebra actions that permit infinitesimally equivariant momentum maps are usually referred to as *Hamiltonian actions*; and Lie group actions with coadjoint equivariant momentum maps are called *globally Hamiltonian actions*. Global and infinitesimal equivariance of the momentum map are only equivalent if G being a connected symmetric group is proveable. The result below shows that if the momentum map corresponds to the action of a compact Lie group then the momentum map can always be chosen to be equivariant:

Proposition 2.3.2. *(Montaldi) Let* G *be a compact Lie group that acts canonically on the Poisson manifold* $(M, \{\cdot, \cdot\})$ *with an associated momentum map* $J : M \to \mathfrak{g}^*$*. Then an equivariant momentum map exists.*

A similar result for choosing an equivariant momentum map: the canonical action of a semisimple Lie algebra on a symplectic manifold admits an infinitesimally equivariant momentum map.

Lemma 2.3.3. *Let* G *be a Lie group acting on the manifold made up of the* N*-product of coadjoint orbits,* $M = \mathcal{O}_1 \times \ldots \times \ldots \times \mathcal{O}_N$*, where* $\mathcal{O}_i$ *denotes the* i*th coadjoint orbit. Let* $\omega = \Gamma_1\omega_1 \times \ldots \times \Gamma_N\omega_N$ *be the symplectic structure on this manifold, where each* ω_i *corresponds to the Kostant-Kirillov-Souriau symplectic form on the* $\mathcal{O}_i$ *coadjoint orbit, and* Γ_i *are the corresponding vortex strengths. If* $J(X_i)$ *is the momentum map* $J_i : \mathcal{O}_i \to \mathfrak{g}^*$ *defined on* $(\mathcal{O}_i, \Gamma_i, \omega_i, G)$ *then the momentum map for the symplectic manifold* M*, defined on* (M, ω, G) *is*

$$J(X_1, \ldots, X_N) = J_1(X_1) + \ldots + J_N(X_N) \tag{2.6}$$

with map $J : M \to \mathfrak{g}^*$

Proof. See Proposition 10.7.1 in [61] □

2.3.2 Symplectic Reduction

Let's introduce a more geometric description of a canonical transformation which is usually defined as a phase space transformation that takes one canonical transformation to another. A smooth map of a symplectic manifold to itself that preserves the symplectic form (or Poisson bracket) is a *symplectic map* and it is invariant. The reduction process for Hamiltonian systems with symmetry is highly reliant on the geometry of symplectic manifolds.

Let G act on symplectic manifold (S, ω) by symplectic maps (this is called a *symplectic action*). Let H be a G-invariant Hamiltonian on P and J be an equivariant momentum map. The isotropy subgroup (symmetry subgroup) at $\mu \in \mathfrak{g}^*$ is $G_\mu = \{g \in G \mid g \cdot \mu = \mu\}$ where G_μ leaves $J^{-1}(\mu)$ invariant due to equivariance. $J^{-1}(\mu)$ is a smooth manifold and that G_μ acts freely and properly on $J^{-1}(\mu)$ if μ is a regular value of J, and therefore $J^{-1}(\mu)/G_\mu =: P_\mu$ is a smooth manifold. $\pi_\mu : J^{-1}(\mu) \to P_\mu$ is the projection map and $i_\mu : J^{-1}(\mu) \to P$ is the inclusion. This means,

$$\dim P_\mu = \dim P - \dim G - \dim G_\mu$$

Marsden and Weinstein developed results by Arnold, Jacobi, Liouville, and Smale to reach the following result:

Theorem 2.3.4. *A unique symplectic structure* ω_μ *exists on* P_μ *that satisfies*

$$i_\mu^* \omega = \pi_\mu^* \omega_\mu$$

This is known as the Reduction Theorem.

Define the reduced Hamiltonian $H_\mu : P_\mu \to \mathbb{R}$ by $H = H_\mu \circ \pi_\mu$ for a given G-invariant Hamiltonian H on P. This means that the trajectories of X_H project to the trajectories of X_{H_μ}, however it would be beneficial to know how to rebuild the X_H trajectories from those of X_{H_μ}. Therefore a particular version of the symplectic reduction theorem is below, where $\mathcal{O}_\mu$ is the coadjoint orbit through μ:

Corollary 2.3.5.

$$(T^*G)_\mu \simeq \mathcal{O}_\mu \tag{2.7}$$

the *Kostant-Kirillov-Souriau orbit symplectic structure* is the symplectic structure induced on $\mathcal{O}_\mu$. It is consistent with the Lie-Poisson structure on $\mathfrak{g}^*$ because the bracket of two functions on $\mathcal{O}_\mu$ is exactly that obtained from extending them arbitrarily to $\mathfrak{g}^*$, taking the Lie-Poisson bracket on $\mathfrak{g}^*$ and then restricting to $\mathcal{O}_\mu$.

2.3.2.1 The orbit momentum map

$P_\mu = J^{-1}(\mu) = G = J^{-1}(\mathcal{O}_\mu)/G$ therefore it is unsurprising to use the *orbit momentum map*: $j : P/G \to \mathfrak{g}^*/G$ because $J^{-1}(\mathcal{O}_\mu)/G \subset P/G$. We have,

$$\begin{array}{ccc} P & \xrightarrow{J} & \mathfrak{g}^* \\ \downarrow & & \downarrow \\ P/G & \xrightarrow{j} & \mathfrak{g}^*/G \end{array}$$

the vertical arrows are quotient maps and therefore $P_\mu = j^{-1}(\mathcal{O}_\mu)$. Since the momentum value changes this is useful for investigating bifurcations. However if G_μ is not compact then the orbit space $\}^*/G$ is not necessarily a practicable space.

2.4 Morse-Bott Functions

Definition 2.4.1. If we choose M to be any compact Riemannian manifold, then a *Morse-Bott function* is a smooth function $f : M \to \mathbb{R}$ if the critical set Crit$(f) = \{x \in M \mid df(x) = 0\}$ breaks down to a finite number of connected submanifolds of M called the *critical manifolds*, and ker$\nabla^2 f$ is equal to the tangent space of the critical set. What this means is that,

$$T_x\mathrm{Crit}(f) = \ker\nabla^2 f(x)$$

for every $x \in$Crit(f).

A specific version of the Morse-Bott function for which the critical manifolds are all of dimension zero, is the Morse function. Therefore for each $x \in$Crit(f), ker$\nabla^2 f(x) = 0$ so the Hessian is nondegenerate.

To help make the above definition more instinctual, let's consider the following definition,

Definition 2.4.2. Let M be a compact Riemannian manifold, let $f : M \to M$ be a diffeomorphism, and let L be a f invariant subset of M. If for any point $x \in L$ the tangent space T_xM partitions into a direct sum of the three subbundles then L is said to be a *normally hyperbolic invariant manifold*:

$$T_xM = T_xL \oplus E_x^+ \oplus E_x^-$$

so that, (A) the *stable bundle* is the restriction of df to E^+ and is a contraction,
(B) the *unstable bundle* is the restriction of df to E^- and is an expansion,
(C) the restriction of df to TL is relatively neutral,
all with respect to some Riemannian metric on M. This means that there must exist constants $0 < c$ and $0 < \kappa < \delta^{-1} < 1$ for which:
(A) $df_xE_x^+ = E_{f(x)}^+$ and $df_xE_x^- = E_{f(x)}^-$ for all $x \in L$
(B) $\|f^n v\| \leq c\kappa^n\|v\|$ for all $v \in E^+$ and $n > 0$
(C) $\|f^{-n} v\| \leq c\kappa^n\|v\|$ for all $v \in E^-$ and $n > 0$
(D) $\|f^{-n} v\| \leq c\delta^n\|v\|$ for all $v \in TL$ and $n > 0$.

It follows that the critical manifolds of f are all normally hyperbolic invariant manifolds with respect to the negative gradient flow if f is a Morse-Bott function. Specifically, the specific collection of diffeomorphisms $\phi_t : M \to M$ defined by $\phi_0 = $id for $t \in \mathbb{R}$ and $\frac{d}{dt}\phi_t = -\nabla f \circ \phi_t$ is the negative gradient flow. For whichever critical manifold C, and for any point x in it, T_xM is the tangent space that can be partitioned into the direct sum $T_xM = T_xC \oplus E_x^+ \oplus E_x^-$ where the positive and negative eigenspaces of $\nabla^2 f(x)$ span E_x^+ and E_x^- respectively. $d\phi_t(x)$ is relatively neutral on T_xC because ker$\nabla^2 f(x) = T_xC$ and it is a contraction on E_x^+ and an expansion on E_x^-.

Definition 2.4.3. Let $W^s(C)$ be the notation for a *stable manifold* which is a manifold constructed from the trajectories, $\phi_t(x)$, of the set of points $x \in M$ that converge to some point C as $t \to \infty$. And $T_xW^s(C) = T_xC \oplus E_x^+$ for any x in C. Likewise, let $W^u(C)$ be the notation for an *unstable manifold* which is a manifold constructed from the trajectories, $\phi_t(x)$, of the set of points $x \in M$ that converge to some point C as $t \to -\infty$ and $T_xW^u(C) = T_xC \oplus E_x^-$ for any x in C.

The image $f(M) \subset \mathbb{R}$ can only be compact with a minimum and maximum because M is compact. THis mean that for any $x \in M$ the trajectory $\phi_t(x)$ has to converge to some critical manifold C as $t \to \infty$ because f decreases along the trajectory. This means,

$$M = \bigcup_C W^u(C).$$

Definition 2.4.4.

$$n^-(C) = \text{dim}W^u(C) - \text{dim}C = \text{codim}W^s(C)$$

is the *index* of a critical manifold C, and,

$$n^+(C) = \text{dim}W^s(C) - \text{dim}C = \text{codim}W^u(C).$$

is the *coindex* of a critical manifold C.

Any compact hypersurface in $\mathbb{R}^n$ can be separated into an 'inside' and an 'outside' according to the Jordan-Brouwer Separation Theorem. If M is a compact manifold embedded in $\mathbb{R}^n$, and codim$(M) \neq 1$, then $\mathbb{R}^n - M$ is connected. For embedded submanifolds of codimension$\neq 1$, the Jordan-Brouwer Separation Theorem doesn't apply. The complement $M - N$ between a compact manifold

M and its submanifold N of codimension≥ 1 has to be connected. Consider the following lemma for codimension$\neq 1$

Lemma 2.4.5. *Let* M *be a compact connected manifold and* $f : M \to \mathbb{R}$ *a Morse-Bott function such that for any of the critical manifolds* C *of* f, $n^{\pm}(C) \neq 1$. *Then the level set* $f^{-1}(c)$ *is connected for every* $c \in \mathbb{R}$.

The proof of this involves these points:
(1) There exists exactly just one connected critical manifold of coindex zero and exactly one connected critical manifold of index zero.
(2) For every regular value $c \in \mathbb{R}$, $f^{-1}(c)$ is connected.
(3) For the remaining critical values $0 < j < N$, $f^{-1}(c_j)$ is connected.

2.5 Convexity Properties of Momentum Mapping

2.5.1 Introduction

Schur announced the first relation between coadjoint orbits and convexity in a 1923 paper. This paper demonstrated that the set of diagonals of an isospectral set of $n \times n$ Hermitian matrices, which is a subset of $\mathbb{R}^n$, reside within the convex hull, the vertices of which are the vectors comprised from the $n!$ permutations of its eigenvalues.

Horn proved an archetype to the convexity theorem in the 1950s that focused on Hermitian matrices. Denote the set of $n \times n$ Hermitian matrices with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ by $\mathcal{H}^n_\lambda$, and let $a_{11}, a_{22}, \ldots, a_{nn}$ be the diagonal entries for each $A \in \mathcal{H}^n_\lambda$. The map $J : \mathcal{H}^n_\lambda \to \mathbb{R}$ is defined by $A \mapsto (a_{11}, a_{22}, \ldots, a_{nn})$, and the image of the J map is a convex polytope which, for σ that ranges over the set of permutations of $\{1, 2, \ldots, n\}$, is the convex hull of the vectors $\lambda_\sigma = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(n)})$.

A generalised version of this result that relates to coadjoint orbits was introduced by Kostant in the 1970s. If T is the Cartan subgroup of Lie group G, with lie algebra $\mathfrak{t}$. The dual of the adjoint action of G on its Lie algebra $\mathfrak{g}$ is the action of G on $\mathfrak{g}^*$ whose orbits are the coadjoint orbit. For inclusion map $\pi : \mathfrak{t} \to \mathfrak{g}$, take the transpose, $\pi^T : \mathfrak{g}^* \to \mathfrak{t}^*$. WE have the map

$$J : \mathcal{O} \to \mathfrak{t}^*$$

by restricting π to $\mathcal{O} \subset \mathfrak{g}^*$ a coadjoint orbit. And according to Kostant's theorem the image of this map is the convex polytope. Denote the set of T-fixed points on $\mathcal{O}$ as $\mathcal{O}^{\mathsf{T}}$, J maps it bijectively onto an orbit of the Weyl group $\mathsf{N}(\mathsf{T})/\mathsf{T}$ and the image of J is the convex hull of the points on this orbit.

These results are reached through application of symplectic geometry including Kirillov, Kostant and Souriau's theorem that G and its coadjoint orbits are symplectic manifolds and the action of G on $\mathcal{O}$ preserves the symplectic form. And these coadjoint orbits would be the only symplectic G-manifolds on which G acts transitively if G is compact and connected. This action is certainly Hamiltonian because $\xi_{\mathcal{O}}$ is a vector field on $\mathcal{O}$ that corresponds to $\xi \in \mathfrak{g}$ for the action of G on $\mathcal{O}$, and it is a Hamiltonian vector field. Introduce inclusion map $i : \mathcal{O} \to \mathfrak{g}^*$ and if the linear functional on $\mathfrak{g}^*$, l^{ξ}, derived from the pairing on $\xi \in \mathfrak{g}$ with elements of $\mathfrak{g}^*$, then the inner product of the symplectic form on $\mathcal{O}$, $\omega_{\mathcal{O}}$, with $\xi_{\mathcal{O}}$ is $di^* l^{\xi}$. So G acting on $\mathcal{O}$ is an Hamiltonian action with momentum map $i : \mathcal{O} \to \mathfrak{g}^*$. J above is the momentum mapping of this action restricted to the action of T whose image is a convex polytope according to Kostant. Specifically,

$$\mathsf{J}(\mathcal{O}) = \text{ conv } \mathsf{J}(\mathcal{O}^{\mathsf{T}})$$

where for A a subset of real vector space, conv A is the convex hull of it. Atiyah and Guillemin-Sternberg generalised this further so that action of G on $\mathcal{O}$ need not be transitive and symplectic, or in fact be considered at all. What their result states is that for T an n-torus that acts on compact symplectic manifold M by Hamiltonian action $\mathsf{T} \times M \to M$ with corresponding momentum map $\mathsf{J} : M \to \mathfrak{t}^*$ then (A) $\mathsf{J}(M^{\mathsf{T}})$ is finite and $\mathsf{J}(M^{\mathsf{T}}) \subset \mathfrak{t}^*$, and (B) $\mathsf{J}(M)$ is the convex hull of $\mathsf{J}(M^{\mathsf{T}})$ (simply put, $\mathsf{J}(M) = \text{conv } \mathsf{J}(M^{\mathsf{T}})$). Specifically, $\mathsf{J}(M)$ is a convex polytope. And this is the *abelian* convexity theorem.

The more intense non-abelian convexity theorem was proven only three years later by Frances Kirwan. For G not necessarily abelian but still a compact connected Lie group that acts on a compact symplectic manifold in a Hamiltonian way then the momentum map

$$\mathsf{J} : M \to \mathfrak{g}^* \tag{2.8}$$

applies, however the image is not necessarily convex, but a less obvious convexity result is approachable. Denote the orbit of the cadjoint action of G on $\mathfrak{g}^*$ by $\mathfrak{g}^*/\mathsf{G}$. The action of the Weyl group $W := \mathsf{N}(\mathsf{T})/\mathsf{T}$ on $\mathfrak{t}^*$ is the dual of the action of W on $\mathfrak{t}$, and the orbit space $\mathfrak{t}^*/W$ is isomorphic to the orbit space $\mathfrak{g}^*/G$. Select a (closed) Weyl chamber $\mathfrak{t}^*$, called the positive Weyl chamber, and denote it $\mathfrak{t}^*_+$ where $\mathfrak{g}^*/G \simeq \mathfrak{t}^*/W \simeq \mathfrak{t}^*_+$ as it is the fundamental domain of the action of the Weyl group on $\mathfrak{t}^*$ therefore (2.8) becomes,

$$J_+ : M \to \mathfrak{t}^*_+ \tag{2.9}$$

which Kirwan's theorem proves the image of is the convex polytope.

However this theorem doesn't come up with a definitive description of the image of the image of this map in the positive Weyl chamber. But some examples have been found including, for example, the equivariant Darboux theorem which states that for X an orbit of G in M which is a Hamiltonian G-manifold, then there is a structural convexity theorem for M^X_{can}, the canonical model one gets for the action. And Sjamaar [88] developed theorem for "local" construction of a convex polytope that couples the convexity theorem for M^X_{can} coupled with a Morse theory result that Kirwan used in her proof.

There are two versions of convexity theorem that apply the isospectral sets of Hermitian matrices. For example the application of Kirwan's theorem to the action of $U(n-1)$ on a generic coadjoint orbit of $U(n)$ which some have intertwined with Gelfand-Cetlin and confocal quadrics theories. If we project $\mathcal{H}^n$ onto $\mathcal{H}^{n-1}$ this associates to each $n \times n$ Hermitian matrix its $(n-1) \times (n-1)$ minor mapping $\mathcal{H}^n_\lambda$ onto the $\bigcup_\mu \mathcal{H}^{n-1}_\mu$ set that is the union over all $n-1$-tuples μ that intertwine with λ_is in the way $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for $i = 1, 2, \ldots, n-1$.

Another version concerns $U(n)$ acting diagonally on the product of two coadjoint orbits of $U(n)$: the set of λs satisfying

$$\mathcal{H}^n_\lambda \subset \mathcal{H}^n_\mu + \mathcal{H}^n_\Upsilon \tag{2.10}$$

for μ and ν n-tuples of real numbers, is a convex polytope.

Klyacho finalised a total description of the momentum polytope around two decades ago. He devised that it is sufficient and necessary for λ to satisfy equation (2.10) to prove convexity through mini-max and Morse theories.

Let's consider the momentum polytope of the $U(n)$ on $\mathcal{O}$, where $U(n) \to U(n) \times U(n)$ is the diagonal imbedding and $\mathcal{O}$ is a coadjoint orbit of $U(n) \times U(n)$. In 2000 Berenstein and Sjamaar found the momentum polytope of the compact lie group H acting on $\mathcal{O}$ which is a coadjoint orbit of compact Lie group G, with existing embedding $i : H \to G$. The polytope, according to them, is distinguished by a set of inequalities.

And yet another example of Kirwan's theorem comes about if M is a compact Kähler manifold. Let T act on M, $T \times M \to M$, in a Hamiltonian way with momentum map J, we can consider the holomorphic - non-Hamiltonian - action of complex torus $T^{\mathbb{C}} = (\mathbb{C}^{\times})^n$ on M, where $T \simeq (S^1)^n$ is the n-torus and $\mathbb{C}^{\times}$ is the multiplicative group of complex numbers. Denote the orbit of $T^{\mathbb{C}}$ through $p \in M$ as $T_p^{\mathbb{C}}$, this provides a more local assembly of this convexity theorem that Atiyah proved in 1982 such that the closure of $T_p^{\mathbb{C}}$ is a convex polytope. WHich in turn is the convex hull of the image of the momentum map of the set $\bar{T}_p^{\mathbb{C}} \cap M^T$.

For G a compact connected Lie group that has Kählerian action on a compact Kähler manifold, M we have a non-abelian generalisation to Atiyah's 1982 result that was found by Brion in 1987. WIth respect to the image of the momentum map J_+ of an orbit of the holomorphic action of complex Lie group, $G^{\mathbb{C}}$, through a choice of point of M is convex. With respect to the image of the momentum map $J(Bp)$ of the Borel subgroup, B, of $G^{\mathbb{C}}$, -orbit through a p in M associated to the opposite chamber $-t_+^*$ intersects the interior of the positive Weyl chamber in a convex set. The intersection is contained in the intersection:

$$\text{Int}\mathfrak{t}_+^* \cap \bigcap_{b \in B} J(T^{\mathbb{C}} bp). \tag{2.11}$$

$J(Bp) \cap \text{Int}\mathfrak{t}_+^*$ is a convex polytope and equal to the intersection equation (2.11) if M is a projective variety. Another further edition of Kirwan's theorem for Kähler manifolds involves a lower semicontinuous function of p in the form of equation (2.11).

The convexity theorem has continued to be generalised for a number of different useful ways, such as for the Hamiltonian actions of non-compact Lie groups, for the Hamiltonian actions of compact Lie groups on non-compact symplectic manifolds, for the quasi-Hamiltonian actions of Lie groups, for poisson actions of Lie groups on Poisson manifolds, and the action of infinite dimensional groups such as groups of gauge transformations and loop groups on infinite-dimensional symplectic manifolds.

A very important development is Delzant's theorem [27] that focuses on the faithful action of T on M, a compact Hamiltonian T-manifold where $\dim M \geq 2\dim T$ (The T-action is called a "toric" action or "completely integrable" if dim$M$ = 2dim$T$). According to Delzant the momentum polytope for this system is established up to a T-equivariant symplectomorphism. The non-abelian version of this result is fascinating. Let G be a compact Lie group and M a connected G-manifold, all points in dense open subset, U, of M have the same orbit type for U such that for all p and q in U, the stabiliser groups, G_p and G_q, are conjugate in G. If the principle isotropy group of the action, G_p, which is unique up to conjugacy is discrete then $\dim M \geq \dim G + \operatorname{rank} G$. Delzant's conjecture states that M can be extrapolated up to isomorphism from its momentum polytope and its principle isotrpoy group. If $\dim M = \dim G + \operatorname{rank} G$ then the G action is called a *multiplicity-free* action. This is still only a conjecture.

The Białynicki-Birula theorem [14] showed that a torus action on a nonsingular complex projective variety, that is birationally equivalent to projective space, with finitely many fixed points allows a partition into affine spaces. This therefore provided plausability for a conjecture of the convexity result that allows for Kählerisability. It states that if the fixed point set, M^T, of M a compact Hamiltonian T-manifold, is finite then a T-invariant complex structure which is compatible with its symplectic structure exists for M. And the Białynicki-Birula theorem hinted that the the birational classiification wouldn't be complicated if the Kähler assumption was removed. But Sue Tolman disproved this conjecture via a counterexample using the fact that the shape of the momentum map is dictated by the $T^{\mathbb{C}}$-orbits according to a corollary of Atiyah's convexity theorem.

Duistermaat-Heckman theory provided greater scaffolding to the momentum polytope geometry. They showed that the momentum polytope is a disjoint union of "action chambers" - these are open convex subpolytopes - each with a corresponding polynomial called the Duistermaat-Heckman polynomial. And $\omega^n/n!$ is the symplectic form on symplectic manifold (M^{2n}, ω).

2.5.2 The Atiyah-Guillemin-Sternberg Convexity Theorem

2.5.2.1 Preliminaries

Lemma 2.5.1. *Let* (M, ω) *be a compact connected symplectic manifold, and let a compact group* G *act symplectically on it:* $G \to \mathrm{Symp}(M, \omega) : \tau \to \psi_\tau$. *Therefore an almost complex structure* $\mathbb{J}$ *exists on* M *that is compatible with the symplectic form and for which* $\psi_\tau^* \mathbb{J} = \mathbb{J}$ *for every* $\tau \in G$.

Recall that for some unique geodesic γ (determined by Reimannian metric g)

satisfying $\gamma(0) = x$ with initial velocity $\gamma'(0) = \xi$ for a given point $x \in M$, and a vector $\xi \in T_x M$, the exponential map is $\exp_x : T_x M \to M$ by $\exp_x(\xi) = \gamma(1)$

Proposition 2.5.2. *Let the set of points of* M *that are fixed by each symplectomorphism in Im*(H) $\subset$*Symp*(M, ω) *for* H *a subgroup of* G*, be denoted Fix*(H) *i.e. Fix*$(H) = \bigcap_{h \in H}$*Fix*(ψ_h)*: then Fix*$(H) \subset M$ *is a symplectic submanifold of* M.

Lemma 2.5.3. *Let* M *be a compact connected symplectic manifold with symplectic form* ω*. Let* $\mathbb{T}^m \to$*Symp*$(M, \omega) : \theta \mapsto \psi_\theta$ *describe the Hamiltonian torus action with momentum map* $\mu : M \to \mathbb{R}^m$*. Let* $H_\theta = \langle J, \theta \rangle : M \to \mathbb{R}$ *be the Hamiltonian function for every* $\theta \in \mathfrak{g}^* = \mathbb{R}^m$*. Then* $\mathtt{Crit}(H_\theta) = \bigcap_{\tau \in T_\theta} \mathtt{Fix}(\psi_\tau)$ *where* $T_\theta = \mathtt{cl}(\{t\theta + k \mid t \in \mathbb{R}, k \in \mathbb{Z}^m\}/\mathbb{Z}^m)$ *for Im*(T_θ) $\subset$*Symp*(M, ω)*.* $\mathtt{Crit}(H_\theta)$ *is the critical set and symplectic submanifold of the Morse-Bott function* H_θ *which has critical manifolds, also of even dimension, and of even index and coindex.*

Definition 2.5.4. A momentum map $J : M \to \mathbb{R}^m$ is *irreducible* if its components $J = (J_1, \ldots, J_m)$ have linearly independent 1-forms: $dJ_1, \ldots, dJ_m$. In particular, $\alpha_1 dJ_1(x)(\xi) + \ldots + \alpha_m dJ_m(x)(\xi) = 0$ for scalar $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ for all $x \in M$ and all vectors $\xi \in T_x M$ if and only if $\alpha_1 = \ldots = \alpha_m = 0$. If not then J is *reducible.*

Definition 2.5.5. If $\frac{\theta_i}{\theta_j}$ for $\{\theta_i \mid 1 \leq i \leq s, \theta_i \in \mathbb{R}\}$ is rational for all $\theta_{i,j} \neq 0$ with $1 \leq i, j \leq s$, then the set θ_i is said to be *rationally dependent.*

Proposition 2.5.6. $\mathbb{T}^{m-1} \to$*Symp*$(M, \omega) : \tau \mapsto \psi'_\tau$ *is action reduced to the* $(m-1)$*-torus action since* J *is reducible. It has momentum map* $J' : M \to \mathbb{R}^{m-1}$*. And* $\psi_\theta = \psi'_{A\theta}$ *and* $\mu(x) = A^T \mu'(x)$ *for* $\theta \in \mathbb{T}^m$*,* $x \in M$ *and* $A \in \mathbb{Z}^{(m-1)\times m}$.

2.5.2.2 Convex Properties

Theorem 2.5.7. *(The Atiyah-Guillemin-Sternberg Convexity Theorem) Hamiltonian torus action* $\mathbb{T}^m \to$*Symp*$(M, \omega) : \theta \mapsto \psi_\theta$ *on compact connected symplectic manifold* (M, ω) *has momentum map* $\mu : M \to \mathbb{R}^m$ *whose image is convex, in particular a convex subset of* $\mathbb{R}^m$*. The levels of* J *are connected and* $\bigcap_{\theta \in \mathbb{T}^m} \mathtt{Fix}(\psi_\theta) = \bigcup_{j=1}^N C_j$ *where* $C_1, \ldots, C_N$ *denote the finite union of connected symplectic submanifolds that are the decomposition of the points of* M *fixed by every symplectictomorphism in Im*$(\mathbb{T}^m)$ $\subset$*Symp*(M, ω)*, whose image is constant* $J(Cj) = \eta_j \in \mathbb{R}^m$*. And* $J(M) = K(\eta_1, \ldots, \eta_N)$*, is the convex hull of these points which are the images of the fixed points of the action.*

The proof of this is can be broken down into the proofs of the following steps using only the maths that has been introduced here so far:
(A) For every regular value $\eta \in \mathbb{R}^m$ it can be shown that the preimage $J^{-1}(\eta) \subset M$ is connected by induction over the dimension m of the torus.
(B) Use induction over the dimension m of the torus to show that the image $J(M) \subset \mathbb{R}^m$ is convex.
(C) $C_1, \ldots, C_N$ denote the finite union of connected symplectic submanifolds that are the decomposition of the points of M fixed by every symplectictomorphism in Im($\mathbb{T}^m$) $\subset$Symp(M, ω), whose momentum map image, $J(Cj)$, is constant.
(D) For the points $\eta_j = J(C_j) \in \mathbb{R}^m$, $1 \le j \le N$ the image of J is the convex hull of them.

If G acts on a manifold M in a way such that $\bigcap_{p \in M} G_p = \{e\}$ G_p is the stabiliser of p i.e. each $g \in G$, $g \neq e$ moves at least one $p \in M$, then this is an *effective* action.

Corollary 2.5.8. *If the* $\mathbb{T}^m$*-action on* $(M, \omega, \mathbb{T}^m, \mu)$*, a Hamiltonian* $\mathbb{T}^m$*-space, is effective then dim* $M \ge 2m$*.*

Definition 2.5.9. Let (M, ω) be a compact connected symplectic manifold with effective hamiltonian action of a torus $\mathbb{T}$ with associated momentum map J. And let dim$\mathbb{T} = \frac{1}{2}$dim M, then M is a *(symplectic) toric manifold.*

2.5.3 The non-abelian convexity theorem

Let τ denote an Hamiltonian action of compact connected Lie group G on compact symplectic manifold M, with corresponding momentum map $J : M \to \mathfrak{g}^*$. Consider the map $\mathfrak{g}^* \to \mathfrak{g}^*/G = \mathfrak{t}^*_+$ and compose it with J to get a map $M \to \mathfrak{t}^*_+$. The image of this is a convex polytope called the Kirwan polytope, $\triangle$, according to the non-abelian convexity theorem.

The local convexity theorem is as follows

Theorem 2.5.10. *Let* U *and* U' *be the neighbourhoods of* p *in* M *and* μ*, the momentum map image of point* p*, in* $\mathfrak{t}^*_+$*, respectively. Let* U *and* U' *satisfy* $J(U) \cap \mathfrak{t}^*_+ = C_p \cap U'$ *where* C_p *is a convex conic polytope.* $C_p = C_q$ *for* p *and* q *points near enough to each other on the level set,* $J^{-1}(\mu)$*.*

The level set, J^{-1}, is connected according to Atiyah's convexity theorem (the proof of this involves Morse theory and topology). And this therefore means that

$C_p = C_q$ for any p and q on the $J^{-1}(\mu)$ level set. Therefore for sufficiently small U', $\triangle \cap U' = C_p \cap U'$. Therefore $\triangle$ satisfies local convexity in a neighbourhood of every point. And every closed set which is locally convex is convex.

The Marle, and Guilllemin-Sternberg equivariant Darboux theorem which describes by simple canonical model what, for a G-invariant neighbourhood of p, the G action looks like.

2.5.4 Reyer Sjamaar's re-examination

The Atiyah-Guillemin-Sternberg abelian convexity theorem provided a much more quantitative information on the shape of the momenum polytope than Kirwan's general convexity theorem. For example, due to the application of Morse theory and the equivariant Darboux Theorem to the momentum map components, the abelian version provides full information for the vertices of the polytope as given by images of fixed points in M. It also provides that one can read off the shape of the polytope from the isotropy action on the tangent space at a corresponding fixed point.

The main result of Sjamaar's paper [88] is to provide such detailed information for the non-abelian version, therefore providing a "sharpened" adaptation of Kirwan's convexity theorem that is inspired by Brion's application of Kirwan's theorem for projective varieties. He gives a description of the polytope's shape in the local neighbourhood of $\mu = J(m)$ $(m \in M)$ in the momentum polytope through the action of G_m on polynomials on $T_m M$ and the conditions a point μ in the polytope must satisfy to be a vertex.

"In the language of the orbit method, the momentum polytope of M is the "classical" analogue of the set of highest weights of the unitary irreducible representations occuring in the "quantisation" of M." [88]

Sjamaar's main theorem is,

Theorem 2.5.11. *Let $J : M \to \mathfrak{g}^*$ be a proper momentum map. Then*
A. $\triangle(M) = \bigcap_{m \in J^{-1}(\mathfrak{t}^*_+)} \triangle_m$ and this intersection is a closed convex polyhedral subset of $\mathfrak{t}^*_+$ because it is locally finite.
Let the commutator subgroup of G be denoted $[G, G]$: and since $[G, G]$ is normal, then for every subgroup F of G, $[G, G]F$ is a subgroup of G. And $[G, G]F$ is a closed reductive subgroup if F is.
B. Consider the fibre $J^{-1}(\mu)$ and let m be any point on it: if μ is a vertex of

$\triangle(M)$ then $\mathcal{O}_\mu = [\mathcal{O}_\mu, \mathcal{O}_\mu]\mathcal{O}_m$ for coadjoint orbit $\mathcal{O}_\mu$ through μ in $\mathfrak{t}^*_+$ and has Kirillov-Kostant-Souriau symplectic form ω_μ, or equivalently $\mathfrak{g}_\mu = [\mathfrak{g}_\mu, \mathfrak{g}_\mu] + \mathfrak{g}_m$. Specifically, for m fixed by T it follows that $\mu = J(m)$ is a vertex of the convex polytope within $\mathfrak{t}^*_+$.

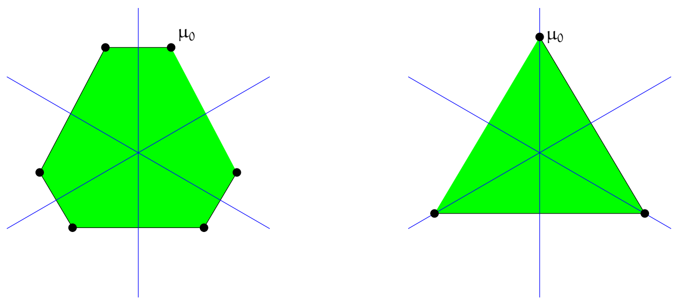
C. There exist $m \in M$ for which $\mu \in \mathfrak{t}^*_+$ and $\mathfrak{g}_\mu = [\mathfrak{g}_\mu, \mathfrak{g}_\mu] + \mathfrak{g}_m$ for $\mu = J(m)$. Let E denote the subset of M that is made up of all such m. The momentum map image of this subset, $J(E)$ is a discrete subset of $\mathfrak{t}^*_+$, and $\triangle(M)$ is the convex hull of this image if M is compact.

2.5.5 Simple examples of projections of some coadjoint orbits to $\mathfrak{t}^*$

The momentum map for the T action on $\mathcal{O}_\mu$ is the restricition of the projection $\pi : \mathfrak{g}^* \to \mathfrak{t}^*$ to $\mathcal{O}_\mu$. The fixed points of this action provide the vertices which correspond to instances for which the map $\pi|_{\mathcal{O}_\mu}$ has rank 0, and these are the only instances for this. Introduce the map $\xi : \mathfrak{g}^* \to \mathbb{R}$ whose restriction to $\mathcal{O}_\mu$ is $\xi|_{\mathcal{O}_\mu} = \xi_\mu$ which will also have rank 0 at these points. These points are critical points. And the points of the intersection $\mathcal{O}_\mu \cap \mathfrak{t}^*$ that make up the Weyl group orbit coincide with the vertices. Therefore the convex polytope is invariant under the Weyl group.

Definition 2.5.12. A *regular point* of the (modified) coadjoint action is a $\mu \in \mathfrak{g}$ for which all the isotropy subgroups in a neighbourhood of it are conjugate.

The only ciritcal points for $\xi \circ \pi|_{\mathcal{O}_\mu}$ are the points $\xi \in \mathfrak{t}$ that are regular, which are exactly the points that are not orthogonal to any of the faces of the polytope.



(a) Generic Orbit (b) Degenerate Orbit

The two diagrams here are those for $G = SU(3)$ where the Weyl group is the symmetric group on 3 letters S_3. The 'semiregular' hexagon corresponds to the polytope for generic/regular initial point μ_0 and $\dim(\mathcal{O}_{\mu_0}) = 6$. The equilateral triangle corresponds to μ_0 on a wall of the Weyl chambers or line of reflexion and $\dim(\mathcal{O}_{\mu_0}) = 4$. The polytope is a single point for $\mathcal{O}_{\mu_0} = 0$ which is true for $\mu_0 = 0$. Any general linear function on $\mathfrak{t} = \mathbb{R}^2$'s critical points correspond to each of the vertices.

$G = SO(4)$ has Weyl group is $W = \mathbb{Z}_2 \times \mathbb{Z}_2$.

2.5.6 A by-no-means-exhaustive bibliography of modern developments

2.5.6.1 Infinite dimensional Lie theory

By providing a convexity theorem for loop groups of simply connected and compact connected Lie groups through incorporating Kostant's convexity result [49], Atiyah and Pressley [10] developed the first infinite dimensional convexity result. Kac and Peterson [44] further elaborated a theory for any coadjoint orbit (except some degenerate orbits) of any Kac-Moody Lie groups that correspond to Cartan matrices that are generalisable and symmetrisable. Neumann provided the first convexity theorem for the Banach Lie group of unitary operators on a divisible Hilbert space, by straight-forward generalisation of the Schur Horn version [74], [75]. Schatten classes, operator topology, and infinite dimensional orthogonal symplectic groups are incorporated to find a topology to close the convex hull. Birtea, Ortega and Ratiu showed that the momentum map provies a convexity result so long as it is closed therefore providing the foundation for further developments of infinite dimensional convexity theories [15]. They also provide theory for cylinder-valued momentum maps [16].

2.5.6.2 Linear and nonlinear symplectic actions

Due to Kostant's stratification of Kirwan's convexity theory for projective algebraic varieties Brion [21] introduced a more precise clarification of the polytope. And as stated above Sjamaar extended this to provide a local description of the polytope shape. Pflaum has recently published a book on the stratification [79].

The first result for momentum polytope of Lie group on non-compact symplectic manifold is thanks to Hilgert, Neeb and Planck [40]. The first result for momentum map of action of linear compact lie group on symplectic vector spaces, that may not be proper, is thanks to Lerman, it adopts methods of symplectic cutting, [53]. He along with Meinrenken, Tolman and Woodward [54] developed a theory for symplectic orbifolds with local description of the momentum polytope and proved that the map from the orbit space to the polytope is an open one. A slice theorem for Hamiltonian actions using the Marle-Guillemin-Sternberg Normal form provides local description of polyhedral cones that the generally convex, unbounded, but locally finite intersection of which, make the momentum polytope (that has discrete vertices) of non-compact manifolds [59], [36].

A version of Kirwan's convexity theory for the action of the complexification of the compact Lie group on an irreducible complex Kähler, not necessarily compact, manifold was introduced by Heinzner and Huckleberry [39]. And the convex Hamiltonian manifold was introduced by Knop [48].

Duistermaat generalised the convexity result so it not only applied to flag manifolds (as shown by Kostant), or symplectic action (as shown by Atiyah) but to provide the polytope for the momentum map of toral Hamiltonian action on a compact connected symplectic manifold from the image of the fixed point set of antisymplectic involution [28], [29].

There are even convexity theorems for actions that don't admit momentum maps. For example, Benoist and Giacobbe derive the momentum map associated to a separately appropriate coverings of the symplectic manifold [12], [32], [33]. Giacobbe showed that for the torus action, that the smallest covering possible that admits a momentum map whose image is the product of a compact convex polytope with a vector space; and despite small perturbations of the symplectic form, this remains stable. Giacobbe extended Kirwan's convexity theorem for compact group actions as well as the fact that the action of an n-dimensional torus on a $2n$-dimensional closed connected symplectic manifold is Hamiltonian. Ratiu and Zung proved a convexity theorem for the torus action on presymplectic manifolds. Lin and Sjamaar proved the convexity theorem for the general compact lie group action on a presymplectic manifold therefore providing a presymplectic convexity theorem [57].

The Lu-Weinstein Poisson Lie structure is one of the most important Poisson Lie group structures on compact lie groups that have all been fully classified, as well as the associated dual groups [58]. Flaschka and Ratiu formed a convexity theorem named after them for which the compact Lie group is a Poisson Lie group [30]. Alekseev connects the usual Kirwan Convexity theorem by reducing the convexity theory for Poisson actions of compact Poisson Lie groups on symplectic manifolds that comes about from using the Poisson Lie group structure to adjust the symplectic structure of the manifold [3]. Along with Malkin and Meinrenken he provided a convexity theory for quasi-Hamiltonian action that is non-linear [4].

2.5.6.3 The Local-Global Convexity Principle

Morse theory cannot always provide an easy advancement from the local to the global version of this theory. Tietze and Nakajima showed that a connected closed subset $S \subset \mathbb{R}^n$ is convex if it is locally convex therefore initiating the development of the aptly named Local-Global Convexity Theory [91], [73]. This was furthered by Schoenberg, Klee, Sacksteder, Straus, and Valentine, Blumenthal and Freese, Kay, Cel etc in their relevant papers. The first people to apply the local-global convexity principle were Condevaux, Dazord, and Molino who used it to provide simpler proofs to both the Atiyah-Guillemin-Sternberg and Kirwan Convexity Theorems [25]. Hilgert, Neeb and Planck provided a version of the local-global convexity theory that can be used as a tool for symplectic convexity [40]. And since Morse theory doesn't work to globalise the theory for Poisson Lie groups other than those with Lu-Weinstein structure: this tool was crucial for the Flaschka-Ratiu convexity theorem proof for all Poisson Lie groups.

Prato provided a crucial result of the failure of convexity for non-compact manifolds [81]. She showed this was true even for torus actions, and she also showed that if a momentum map of Hamiltonian action of the symmetry torus can be provided for an integral element in the Lie algebra which is proper where the unique ciritcal value is its minimum then the image of it is the convex hull of affine rays that start at the fixed point of the action and of which there is a finite number [82]. A map from a connected Hausdorff topological space to a convex set in Euclidean space that is both proper and continuous was whown by Bjorndahl and Karshon therefore showing a further version of the Local-Global Convexity Principle [17]. Zung [92] supplied a simplification of the Local-Global Convexity theory that he along with Ratiu and Wacheux applied to toric-focus Hamiltonian systems to provide a global convexity result published in 2017 [84].

2.6 The Bifurcation Lemma

Lemma 2.6.1. *(*Bifurcation Lemma*) Let $\mathcal{L}$ denote the symplectic leaf of the Poisson manifold* $(P, \{\cdot,\cdot\})$ *containing* $p \in P$*. Let* $J : P \to \mathfrak{g}^*$ *be the momentum* *map associated to the Lie group acting on* P*. Then,*

$$T_pJ(T_p\mathcal{L}) = (\mathfrak{g}_p)^\circ \tag{2.12}$$

where $(\mathfrak{g}_p)^\circ$ *represents the annihilator of the isotropy subalgebra* $\mathfrak{g}_p$ *of* p*, in* $\mathfrak{g}^*$*. If* P *is a symplectic manifold then* (2.12) *can be rewritten as*

$$\mathrm{Range}(T_pJ) = (\mathfrak{g}_p)^\circ \tag{2.13}$$

The bifurcation lemma provides a link between the rank of the momentum map at a point p and the symmetry of the manifold at p, and we will see that the rank of T_pJ is the same dimension of $(\mathfrak{g}_p)^\circ$, providing a rank-nullity correspondence.

Background before proof of the Bifurcation Lemma

Proposition 2.6.2. *For a momentum map* $J : M \to \mathfrak{g}^*$ *associated to the action of the Lie group* G *on a symplectic manifold* M*, the image of the momentum map at a point* m *is a subset of the dual of the Lie algebra and* $(\mathfrak{g}^*_m)^\circ = \mathfrak{g}_m$ *i.e. the annihilator of the dual of the Lie isotropy subalgebra at* m *is equal to the Lie isotropy subalgebra at* m*.*

Proof. Recall $dJ^\xi + \llcorner_{\xi_M}\omega = 0$, the transpose of the tangent momentum map is $T_mJ^\top : \mathfrak{g} \to T^*_mM$ and $\mathrm{Im}(dJ_m)^\circ = \mathrm{Ker}(dJ^\top_m)$; the kernel of the map is:

$$\mathrm{Ker}T_mJ^\top = \{\xi \in \mathfrak{g} \mid (\llcorner_{\xi_M}\omega)_m = 0\}. \tag{2.14}$$

Meaning that the kernel of the map is the elements ξ for which $(\xi_M)_m = 0$ i.e. the set of Lie algebra of the isotropy subgroup G_m. Since the image of the momentum map is the annihilator of the above kernel definition this is enough to prove that $(\mathfrak{g}^*_m)^\circ$ is a subspace of $\mathrm{Im}T_mJ$ and is $\mathfrak{g}_m$. □

Proposition 2.6.3. *The stabiliser (isotropy subgroup)* G_m *is discrete if and only if the momentum map* $J : M \to \mathfrak{g}^*$ *associated to the canonical action of* G *onto a symplectic manifold* (M, ω) *is a submersion.*

Proof. The preceding proposition showed that the dimension of $(\mathfrak{g}^*_m)^\circ$ is equal to the rank of the tangent momentum map:

$$\mathrm{rank}(T_m J) = \dim(\mathrm{Ker}(dJ_m)^\circ) = \dim(\mathfrak{g}/\mathfrak{g}_m) \tag{2.15}$$

As $\mathrm{Ker}(dJ_m)^\circ$ $=\mathrm{Ker}dJ_m^\top$. J_m is a submersion if the differential map dJ_m has constant rank which is only true if the dimension of the isotropy subalgebra, and in turn the dimension of the isotropy subgroup, is discrete. □

Proposition 2.6.4. *Ker*dJ_m *is symplectically orthogonal to the space tangent to the orbit through* m.

Proof.

$$\begin{aligned}\langle dJ_m(\nu), \xi\rangle = 0 \quad \forall \xi \;&\Leftrightarrow\; \omega_m(\nu, \xi_M(m)) = 0 \quad \forall \xi \\ &\Leftrightarrow\; \nu \text{ orthogonal to subspace generated by infinitesimal generators}\end{aligned}$$

□

We also need to revisit and prove Noether's theorem:

Theorem 2.6.5. *The momentum map satisfies Noether's condition if the level set* $J^{-1}(\mu)$ *of the momentum map is tangent and constant to the Hamiltonian vector field* X_h *where* $h \in C^\infty(M)$ *is a Hamiltonian function that is invariant under the Hamiltonian action of the Lie group* G.

Proof. To clarify, for map $J : P \to S$ where S is a set, J is said to satisfy Noether's condition and is called a Noether momentum map associated to the Lie group pr algebra's canonical action on Poisson manifold $(P, \{\cdot,\cdot\})$ when the flow determined by any Hamiltonian vector field of this G-invariant or $\mathfrak{g}$-invariant Hamiltonian function preserves the fibers of J:

$$J \circ F_t = J|_{\mathrm{Dom}(F_t)} \tag{2.16}$$

[77]. For $h \in C^\infty(M)^{\mathfrak{g}}$ Hamiltonian function, G-invariance implies $\xi_M[h] = 0$,

$$\begin{aligned}\{J^\xi, h\}(p) &= dJ^\xi(p) \cdot X_h(p) \\ &= -dh(p) \cdot X_{J^\xi}(p) \\ &= -dh(p) \cdot \xi_P(p) \\ &= -\xi_P[h](p) = 0\end{aligned}$$

□

Proof of the Bifurcation Lemma

Proof. This requires the proof of 1) $T_pJ(T_p\mathcal{L}) \subset (\mathfrak{g}_p)^\circ$ and 2)$(\mathfrak{g}_p)^\circ \subset T_pJ(T_p\mathcal{L})$ or $[T_pJ(T_p\mathcal{L})]^\circ \subset \mathfrak{g}_p$:

1. For $\nu_p \in T_p\mathcal{L}$ an infinitesimal generator of Hamiltonian function $h \in C^\infty(M)$, with $\nu_p = X_h(m)$ then

$$\begin{aligned}\langle T_pJ \cdot \nu_p, \xi\rangle &= \langle T_pJ \cdot X_h(p), \xi\rangle \\ &= dJ^\xi(p) \cdot X_h(p) \quad [\text{recall}\langle J(m), \xi\rangle = J^\xi(m)] \\ &= -\xi_p[h](p) = 0 \quad [\text{see proof of Noether's condition}]\end{aligned}$$

2. To show $[T_pJ(T_p\mathcal{L})]^\circ \subset \mathfrak{g}_p$, choose $\eta \in \mathfrak{g}$ i.e. $\eta \in (T_pJ(T_p\mathcal{L}))^\circ$: this implies that:

$$\begin{aligned}\langle T_pJ \cdot T_p\mathcal{L}, \eta\rangle &= dJ^\eta(p) \cdot X_h(p) \\ &= 0\end{aligned}$$

Both since $\xi_P[h](p) = \eta_P[h](p) = 0$ as h arbitrary Hamiltonian function.

Since P is a symplectic manifold then $T_p\mathcal{L} = T_pP$ therefore

$$T_pJ \cdot T_p\mathcal{L} = T_pJ \cdot T_pP = \text{Range}(T_pJ) \tag{2.17}$$

□

Corollary 2.6.6. *If a Lie group acts canonically on the locally free Poisson manifold associated with $J : P \to \mathfrak{g}^*$ momentum map then this momentum map is a submersion.*

Proof. A group action on a manifold is free if the isotropy subgroup of every element in the manifold is only the identity element. This means the isotropy subalgebra can only be the trivial element. This means $\mathfrak{g}_p = \{0\}\ \forall p \in P$.
$T_pJ : T_pP \to \mathfrak{g}$ is the tangent momentum map which is surjective since by the bifurcation lemma equation (2.17), therefore the momentum map is a submersion and so an open map where the image of the momentum map is an open subset of $\mathfrak{g}^*$. □

2.7 Dynamics

2.7.1 Relative Equilibrium

An equilibrium point p in phase space $\mathcal{P}$ for which $X_H(p) = 0$, or equivalently $dH_p = 0$, is a point in the phase space that is invariant under the dynamics. A group orbit is a relative equilibria in a symmetric dynamical system if it is invariant under the dynamics. Otherwise put: a relative equilibrium is an equilibrium point of the dynamics instigated on the orbit space. Relative equilibria are simply the equilibria or group orbits of the equilibria for finite groups. Simply put it is the group orbit that is invariant under the dynamics.

Definition 2.7.1. For each $t \in \mathbb{R}$ a symmetry transformation $g_t \in G$ exists for which trajectory $\gamma(t)$ in $\mathcal{P}$ satisfies $\gamma(t) = g_t \cdot \gamma(0)$. This trajectory is a *relative equilibrium.*

If the trajectory through p is $\gamma_(t)$ then the trajectory through $g \cdot p$ is $g \cdot \gamma(t)$ which, according to the definition above, means that the entire group orbit is invariant. Conversely: since the trajectory remains within a single group orbit, which if this orbit is invariant under the dynamics means that all the trajectories within it are relative equilibria.

Example 2.7.2. Consider the N-body problem in space: rigid rotations about an axis i.e. motions for which the shape of the bodies doesn't change, are the relative equilibria of this system.

Proposition 2.7.3. *[66] The following statements are equivalent for H a G-invariant Hamiltonian for G-action on $\mathcal{P}$ with momentum map J where $\mu = J(p)$ for $p \in \mathcal{P}$:*

- $\gamma(t)$ *is the trajectory through* p *that is a relative equilibrium,*

- $G \cdot p$ *is the group orbit that is invariant under the dynamics,*

- *There exists* $\xi \in \mathfrak{g}$ *for which* $\gamma(t) =$*exp*$(t\xi) \cdot x$, $\forall t \in \mathbb{R}$,

- *There exists* $\xi \in \mathfrak{g}$ *for which* $H_\xi = H - \phi_\xi$ *has critical point* p,

- H *restricted to the level set* $\Phi^{-1}(\mu)$ *has critical point* p.

Proof. The equivalence of the third and fourth statements: the third statement is equivalent to $X_H(p) = \xi_{\mathcal{P}}(x)$. Using the symplectic form, this is in turn equivalent to $dH(x) = d\phi_\xi(x)$. For the rest of the proof and further discussion of relative equilibria see [66] □

The Riemann ellipsoid problem is the name of Riemann's classification of all the possible relative equilibria in an affine fluid flow - this was the first full classification of relative equilibria, and is otherwise known as the affine rigid body problem or pseudo-rigid problem. The symmetry group of this system is $SO(3) \times SO(3)$ and the set of all 3×3 invertible matrices is the configuration. Since the momentum is conserved, Riemann found the geometric limitations on the variety of feasible versions of relative equilibria along with the 6 conserved quantities that are of the form given in the third statement of proposition 2.7.3. Riemann used the geometric condition expressed in the corollary below for general action of groups.

Corollary 2.7.4. *$coad^\theta_\xi J(p) = 0$ for $p \in P$ is a point of a relative equilibrium with angular velocity ξ and θ is the cocyle associated to the momentum map.*

The angular velocity and momentum of a relative equilibrium *commute* for suitable choice of equivariant momentum map for compact G. This means that if a system has $SO(3)$ symmetry then the angular velocity and the value of the momentum are parallel at any relative equilibrium.

If there exists a non-degenerate relative equilibrium for a given value of the momentum map then Arnold observed in 1978 that if that value satisfies a particular condition of regularity then a unique non-degenerate relative equilibrium exists on each level set of the momentum map close to the level set that contains the value in the neighbourhood of it. In other words the relative equilibrium persists. And it is a regular point of the coadjoint group of the Lie group according to the regularity hypothesis. The relative equilibrium points have to have trivial isotropy and the Hamiltonian reduced to the given value must have a non-degenerate critical point at the relative equilibrium so that the relative equilibrium is non-degenerate. To prove this apply the implicit function theorem on the orbit space which is smooth. The relative equilibrium is Lyapunov stable (even for perturbations that alter the momentum value) if the quadratic part of the reduced Hamiltonian is positive definite [8] [55]. This method of proving the relative equilibrium stability is known as the Energy Casimir method, which is different from the Energy-Momentum method Marsden developed with colleagues. In [65] Montaldi gave persistence and stability results for G compact and extremal relative equilibria (the relative equilibrium i a local extremum for the reduced Hamiltonian, without application of the regularity hypothesis.

A smooth submanifold of the phase space with dimension dim$G$+rank$G$ is made up of the set of relative equilibria in a neighbourhood of a relative equilibrium for locally free action with regular velocity according to George Patrick in 1995. And this means that the relative equilibrium doesn't have to be extremal and the given momentum map value doesn't have to be regular in $\mathfrak{g}^*$ either.

2.8 The Witt-Artin Decomposition

The reduction to compact group actions is based on the Marle-Guillemin-Sternberg normal form for symplectic actions and momentum maps. Let again a connected Lie group G act in a Hamiltonian manner on a symplectic manifold M with a momentum map $\Phi : M \to \mathfrak{g}^*$ and the corresponding cocycle $\phi : G \to \mathfrak{g}^*$. The tangent space at $m \in M$can be decomposed into

$$T_m M = T \oplus N \tag{2.18}$$

where T is the tangent space to the orbit $G \cdot m$ and N is the normal to the action. And this can be further decomposed into

$$T_m M = T_0 \oplus T_1 \oplus N_1 \oplus N_0. \tag{2.19}$$

At $x \in M$, consider the four spaces:

$$\begin{aligned}
T_0 &= T_x(G \cdot x) \cap \ker d\Phi(x) = T_x(G_\mu \cdot x), \\
T_1 &= T_x(G \cdot x)/T_0, \\
N_1 &= \ker d\Phi(x)/T_0, \\
N_0 &= T_x M/(T_x(G \cdot x) + \ker d\Phi(x)).
\end{aligned}$$

Since $\ker d\Phi_x$ is the symplectic complement to $T_x(G \cdot x)$, these spaces depend only on the G-action and not on the choice of Φ. Using the compactness of

$G_x \subset G_\mu$, we can realise the quotients T_1, N_1, N_0 as G_x-invariant subspaces of $T_x M$ satisfying:

$$T_0 \oplus T_1 = T_x(G \cdot x),$$
$$T_0 \oplus N_1 = \ker d\Phi(x),$$
$$T_0 \oplus T_1 \oplus N_1 \oplus N_0 = T_x M$$

If H is a G-invariant Hamiltonian on P, then the differential dH_x at x annihilates $T_0 \oplus T_1$. If x is a relative equilibrium of H, then dH_x annihilates $T_0 \oplus N_1$ also. Therefore dH_x is naturally an element of N_0 which since G is compact then

$$N_0^* \simeq \mathfrak{g}_\mu / \mathfrak{g}_p.$$

Chapter 3

$SU(3)$ action on products of $\mathbb{C}P^2$

After introducing the 3×3 special unitary matrix group and its properties, we introduce its coadjoint orbits. We obtain the isotropy subgroups and the corresponding fixed point sets of the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2$ and $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$. We go further and establish the isotropy subgroups and associated fixed point sets of the $SU(n+1)$ action on $\mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n$.

3.1 Some General Properties of $SU(3)$ and $\mathbb{C}P^2$

The $SU(n)$ matrix group is a connected Lie group. It is the set of $n \times n$ complex matrices whose elements $U \in SU(n)$ satisfy the properties

$$U^\dagger U = I, \quad \det U = 1. \tag{3.1}$$

$SU(3)$ is 8 dimensional and of rank 2. The Lie algebra of $SU(3)$, $\mathfrak{su}(3)$, is the set of 3×3 traceless anti-Hermitian matrices. By The First Conjugation Theorem, any Hermitian matrix can be put into the diagonal form.
The Cartan subalgebra of $SU(3)$, H, is the set of all diagonal Hermitian matrices with trace=0, e.g. for $D \in H$:

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{3.2}$$

therefore the Cartan subalgebra is 2 dimensional, $H \simeq \mathbb{R}^2$, and the walls of the Weyl chambers for $SU(3)$ are the hyperplanes $\lambda_1 = \lambda_2$, $\lambda_2 = \lambda_3$, $\lambda_1 = \lambda_3$ in $\mathbb{R}^2$.

Besides the single point at the origin of type $\left(\frac{SU(3)}{SU(3)}\right)$, $SU(3)$ has two types of coadjoint orbits, the 6-dimensional flag manifold and the complex projective plane:

$$\frac{SU(3)}{U(1)\times U(1)} \simeq F(2,1), \quad \frac{SU(3)}{SU(2)\times U(1)} \simeq \mathbb{C}P^2, \tag{3.3}$$

see [13]. $\mathbb{C}P^2$ is the 4-dimensional complex projective plane: complex lines through the origin in $\mathbb{C}^3$.
Each orbit of the Weyl group of $SU(3)$, $W(SU(3))$ is the intersection between H^* and a coadjoint orbit.

3.2 Isotropy subgroups and Corresponding Fixed Point Sets of $SU(3)$ acting on $\mathbb{C}P^2 \times \mathbb{C}P^2$

$SU(3)$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ both have the same dimension, however the quotient space $\mathbb{C}P^2 \times \mathbb{C}P^2/SU(3)$ is not isomorphic to a single point: $SU(3)$ does not act transitively on $\mathbb{C}P^2 \times \mathbb{C}P^2$.

Definition 3.2.1. The action of a compact Lie group on a manifold, whose quotient is one dimensional, is called a *cohomegeneity one action.*

Theorem 3.2.2. *For a cohomogeneity one action, the orbit space* M/G*, is homeomorphic to either (i) a circle, (ii) the open unit interval, (iii) the half-open interval, or (iv) the closed unit interval.*

For a proof of this theorem see [37]. This means that the orbit space of this quotient space is one dimensional, i.e. it's the space of a 1 parameter family of orbits of codimension 1, rather than one single orbit. The interior of the singular orbit space corresponds to the orbit with the smallest isotropy subgroup of the action and the boundaries of the orbit space correspond to the isotropy subgroups of greater size(s).
Let us investigate the different isotropy subgroups of the action of $SU(3)$ on $\mathbb{C}P^2 \times \mathbb{C}P^2$. Recall that $\mathbb{C}P^2$ coordinates are unique up to rescaling, therefore $[0:1:0] = [0:\upsilon:0]$ where $\upsilon \in \mathbb{C}^*$. For a point $[1:0:0] \in \mathbb{C}P^2$, the $SU(3)$

matrices that fix this point are of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix} \text{ where } A = \begin{pmatrix} b & c \\ d & e \end{pmatrix} \text{ and } 1 = a\det A \tag{3.4}$$

A is the group of 2×2 unitary matrices. In other words, we will denote the subgroup of $SU(3)$ that fixes the point $[1:0:0] \in \mathbb{C}P^2$ as E_1 and it is

$$SU(3)_{[1:0:0]} = E_1 \cong U(2). \tag{3.5}$$

The $SU(3)$ matrix

$$\begin{pmatrix} b & c & 0 \\ d & e & 0 \\ 0 & 0 & a \end{pmatrix} \text{ where } A = \begin{pmatrix} b & c \\ d & e \end{pmatrix} \text{ and } 1 = a\det A, \tag{3.6}$$

a different subgroup, E_2 also isomorphic to $U(2)$ fixes $[0:0:1] \in \mathbb{C}P^2$, in other words $SU(3)_{[0:0:1]} = E_2 \cong U(2)$. Therefore the fixed point set of this isotropy subgroup is

$$\text{Fix}(E_1, E_2 \cong U(2), \mathbb{C}P^2 \times \mathbb{C}P^2) = \{[1:0:0], [0:0:1]\} \tag{3.7}$$

For the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2$, we investigate the subgroups of $SU(3)$, $SU(3)_{(P_1,P_2)}$, that would fix two different points P_1 and P_2 in $\mathbb{C}P^2 \times \mathbb{C}P^2$. If the two points are parallel this means they are equal ($P_1 = P_2$) and E_1 and E_2 are the isotropy subgroups that fix $P_1 = P_2 = [1:0:0]$ and $P_1 = P_2 = [0:0:1]$ in $\mathbb{C}P^2 \times \mathbb{C}P^2$, respectively:

$$SU(3)_{([1:0:0],[1:0:0])} \cong U(2) \cong SU(3)_{([0:0:1],[0:0:1])}. \tag{3.8}$$

The fixed point set of which has two elements:

$$\text{Fix}(E_1, E_2 \cong U(2), \mathbb{C}P^2 \times \mathbb{C}P^2) = \{(e_i, e_i) \mid i = 1, 3\}, \tag{3.9}$$

where $e_1 = [1:0:0]$, $e_2 = [0:1:0]$ and $e_3 = [0:0:1]$ in $\mathbb{C}P^2$. For P_2 orthogonal to P_1, for example $P_1 = [1:0:0]$ and $P_2 = [0:1:0]$, the isotropy subgroup that fixes this choice of P_1 and P_2 is the maximal torus subgroup of $SU(3)$:

$$SU(3)_{([1:0:0],[0:1:0])} \cong \mathbb{T}^2 \cong U(1) \times U(1). \tag{3.10}$$

The fixed point set of which has four elements:

$$\text{Fix}(\mathbb{T}^2, \mathbb{C}P^2 \times \mathbb{C}P^2) = \{(e_i, e_j) \mid i = 1, 3, j = 1, 2, 3 \text{ and } i \neq j\}. \tag{3.11}$$

Consider when P_1 is neither equal to nor perpendicular to P_2, for example $P_1 = [1 : 0 : 0]$ and $P_2 = [a : b : 0]$ for $a \neq 0$ and $b \neq 0$, then the isotropy subgroup for the set of two generic points, i.e. two points that are not parallel nor perpendicular to each other, in $\mathbb{C}P^2 \times \mathbb{C}P^2$ is

$$SU(3)_{([1:0:0],[a:b:0])} \cong \mathbb{T}^1 \cong U(1). \tag{3.12}$$

And the fixed point set of which is

$$\text{Fix}(\mathbb{T}^1, \mathbb{C}P^2 \times \mathbb{C}P^2) = [a : b] \cong \mathbb{C}P^1. \tag{3.13}$$

The dimensions of the different isotropy subgroups of the action of $SU(3)$ on $\mathbb{C}P^2 \times \mathbb{C}P^2$ are different and distinct therefore $\mathbb{C}P^2 \times \mathbb{C}P^2/SU(3)$ is homeomorphic to either the half open interval or the closed interval. Since the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2$ has three different isotropy subgroups of three different sizes that don't coincide, i.e. not just two, this means that $\mathbb{C}P^2 \times \mathbb{C}P^2/SU(3)$ is homeomorphic to the closed interval. The boundaries of the orbit space of the action $\mathbb{C}P^2 \times \mathbb{C}P^2/SU(3)$ correspond to the isotropy subgroups $U(2)$ and $\mathbb{T}^2$ and the interior corresponds to the $\mathbb{T}^1$ isotropy subgroup.

3.3 Isotropy Subgroups and Corresponding Fixed Point Sets of $SU(3)$ acting on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$

The isotropy subgroup that fixes three parallel points in $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ is:

$$SU(3)_{([1:0:0],[1:0:0],[1:0:0])} = E_1 \cong U(2) \cong E_2 = SU(3)_{([0:0:1],[0:0:1],[0:0:1])}$$

The fixed point set of this isotropy subgroup is

$$\mathrm{Fix}(E_1, E_2 \cong U(2), \mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2) = \{(a, a, a) \mid a = e_1 \text{ or } e_3\}. \tag{3.14}$$

The maximal torus fixes three points for which two points are equal to each other and both perpendicular to the third point, for example, $P_1 = P_2 = [1 : 0 : 0]$ and $P_3 = [0 : 1 : 0]$,

$$SU(3)_{([1:0:0],[1:0:0],[0:1:0])} \cong \mathbb{T}^2,$$

The isotropy subgroup that fixes three perpendicular points, i.e. $P_1 \perp P_2 \perp P_3$, is also the maximal torus:

$$SU(3)_{([1:0:0],[0:1:0],[0:0:1])} \cong \mathbb{T}^2$$

and its fixed point set is

$$\mathrm{Fix}(\mathbb{T}^2, \mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2) = \{(a, a, b), (a, b, a), (b, a, a) \mid a = e_1 \text{or } e_3, \\ b = e_1 \text{ or } e_2 \text{ or } e_3 \text{ and } a \neq b\} \cup \{(e_1, e_2, e_3)\}. \tag{3.15}$$

3.4 Isotropy subgroups of $SU(n+1)$ acting on $\mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n$

Denote three points in $\mathbb{C}P^n$ as p_1, p_2 and p_3. For $p_1 = p_2 = p_3 = [1 : 0 : \ldots : 0]$ the isotropy subgroup of $SU(n+1)$ that fixes these is isomorphic to $U(n)$, another subgroup fixes $p_1 = p_2 = p_3 = [0 : 0 : \ldots : 1]$ that is also isomorphic to $U(n)$.
For three points for which two of the points are equal and perpendicular to the third point, or all three points are perpendicular to each other in $\mathbb{C}P^n$, the $SU(n+1)$ sub-matrices that fix such configurations of the three points are all isomorphic to $U(n-2)\times\mathbb{T}^2$. Therefore the fixed point set of these isotropy subgroups that are all isomorphic to $U(n-2) \times \mathbb{T}^2$ is the set given by the configurations $\{(a, a, b), (a, b, a), (b, a, a), (a, b, c)\}$ where either $a = e_1$ and $b, c = e_2, e_3$ or $a = e_n,\ b, c = e_{n-1}, e_{n-2}$ only.
For p_3 neither parallel nor perpendicular to points p_1 and p_2, the isotropy subgroup that fixes these points is:

$$SU(n+1)_{\{p_1,p_2,p_3 \in \mathbb{C}P^n | p_3 \not\perp p_1,p_2 \text{ and } p_3 \neq p_1,p_2\}} \cong U(n-2) \tag{3.16}$$

the fixed point set of this isotropy subgroup is isomorphic to $\mathbb{C}P^2$

$$\mathrm{Fix}(U(n-2), \mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n) = [a : b : c] \cong \mathbb{C}P^2.$$

this is also the isotropy subgroup and fixed point set for the action of $SU(n+1)$ on one point $p = [a : b : c : 0 : \ldots : 0]$ where $a, b, c \in \mathbb{C}$ and $a, b, c \neq 0$.
Without loss of generality, these results provide a novel proof to the following:

Lemma 3.4.1. *Any* k *points in* $\mathbb{C}P^n$ *lie in a common* $\mathbb{C}P^r$*,* $r \leq k-1$ *and any symmetric dynamics will preserve that* $\mathbb{C}P^n$ *by the fixed point argument.*

Chapter 4

The Momentum Map of the $SU(3)$ Action on Products of $\mathbb{C}P^2$

The symplectic form on the complex projective plane is the Fubini-Study form: we explore its properties including the fact that it is a basic symplectic form. After introducing the momentum map that maps the complex projective plane to the dual of the Lie algebra of $SU(3)$ we explore the properties of this momentum map, namely, the uniqueness of the image of this momentum map, its $SU(3)$ equivariance and show that the Kirillov-Kostant-Souriau symplectic form on the image of the momentum map pulls back to the Fubini-Study symplectic form. We then establish the momentum map on products of the complex projective plane and state the symplectic form on the products of the complex projective plane which is the direct sum of 'weighted' copies of the Fubini-Study symplectic: the weights are scalar values.

4.1 The Fubini-Study Form

The Fubini-Study form is a symplectic form on the Complex Projective Space. For the n-dimensional complex projective space, $\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^*$, the pull back of the Fubini-Study form to $\mathbb{C}^{n+1}$ is

$$\omega_{FS} = \frac{\sqrt{(-1)}}{2}\partial\bar{\partial}\log|z|^2 \tag{4.1}$$

$$= \frac{i}{2|z|^4}\sum_{j,k=1}^{n}\left(|z_j|^2 dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k\right) \tag{4.2}$$

and it is the only $SU(3)$-invariant form on $\mathbb{C}P^2$ up to a scalar multiple[1]. In the case that $n = 1$, $z_1 = z = x + iy$,

$$\int_{\mathbb{C}P^1} \omega_{FS} = \int_{\mathbb{R}^2} \frac{dx \wedge dy}{(1 + x^2 + y^2)^2} = \pi. \tag{4.3}$$

The Fubini-Study inclusion-quotient diagram, as introduced in proposition 2.3.2, is

$$\begin{array}{ccc} & S^{2n+1} & \xrightarrow{i} \mathbb{C}^{n+1} \\ \pi & \downarrow & \\ & \mathbb{C}P^n & \end{array}$$

where $\mathbb{C}P^n = S^{2n+1}/S^1$. S^1 acts on $\mathbb{C}^{n+1}$ with momentum map $\sum_j |z_j|^2/2$ and S^{2n+1} is the level set of the momentum map

$$\pi^* \omega_{FS} = i^* \omega_{\mathbb{C}^{n+1}} \tag{4.4}$$

where $\omega_{\mathbb{C}^{n+1}} = \sum_{j=0}^n dx_j \wedge dy_j = \frac{i}{2} \sum_{j=0}^n dz_j \wedge d\bar{z}_j$ is the standard symplectic form on $\mathbb{C}^{n+1}$.

Claim: The Fubini-Study form is a basic symplectic form.

Proof. We can see that the Fubini-Study form is invariant under S^1, for $\lambda \in S^1$:

$$\begin{aligned} \omega_{FS} = & \frac{i}{2|\lambda|^4|z|^4} \sum_{j,k=1}^{n} \left(|\lambda|^2 |z_j|^2 \lambda dz_k \wedge \bar{\lambda} d\bar{z}_k - \bar{\lambda} \bar{z}_j \lambda z_k \lambda dz_j \wedge \bar{\lambda} d\bar{z}_k \right) \\ & = \frac{i}{2|z|^4} \sum_{j,k=1}^{n} \left(|z_j|^2 dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k \right). \end{aligned}$$

The Fubini-Study form annihilates vertical (or horizontal) vectors, meaning it is a horizontal (or vertical) vector. This means that $w_0(d\pi_{v_1}, \cdots, d\pi_{v_n}) = 0$ if any $d\pi_{v_i} = 0$, then the pull-back is zero. In other words $\omega(v_1, \cdots, v_p) = 0$ when atleast one of v_i are vertical ($d\pi(v_i) = 0$ for an $i \in n$). Since the Fubini-Study form is both invariant and horizontal it is therefore a *basic* symplectic form. □

[1]Without the $1/2$ coefficient in (4.2) and (4.2), $w_{FS}(n = 1) = 2\pi$.

4.2 Properties of a Momentum Map of the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ with Unweighted Symplectic Form

Let's define a momentum map of the $SU(3)$ action on products of $\mathbb{C}P^2$ and prove some of its properties. For $Z = [z_1 : z_2 : z_3]$ a coordinate point in $\mathbb{C}P^2$, the map

$$K : [z_1 : z_2 : z_3] \mapsto \frac{1}{\sum_{i,j=1,2,3} |z_j|^2}(Z \otimes \bar{Z}) - \frac{1}{3}I \tag{4.5}$$

Claim:

(1) Im(K) is a single coadjoint orbit

(2) K is $SU(3)$-equivariant

(3) Has a Kirillov-Kostant-Souriau (KKS) symplectic form on Im(K) that pulls back to ω_{FS} on $\mathbb{C}P^2$.

Proof. If the second requirement holds then it is enough to verify the third requirement at a single point in $\mathbb{C}P^2$ as both forms are invariant. For condition (1), $\mathbb{C}P^n$ can be decomposed into $\mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \ldots \sqcup \mathbb{C}^1 \sqcup 0$:

$$\begin{aligned} &\text{For } [z_0 : z_1 : \ldots : z_n] \text{ in } \mathbb{C}P^n \text{ at} \\ z_0 \text{ fixed and } z_0 \neq 0 \;&:\; [1 : z_1 : \ldots : z_n] \simeq \mathbb{C}^n \\ z_0 = 0 \;&:\; [0 : z_1 : \ldots : z_n] \simeq \mathbb{C}P^{n-1} \\ z_0 = 0, z_1 \text{ fixed and } z_1 \neq 0 \;&:\; [0 : 1 : \ldots : z_n] \simeq \mathbb{C}^{n-1} \\ z_0 = 0, z_1 = 0 \;&:\; [0 : 0 : \ldots : z_n] \simeq \mathbb{C}P^{n-2} \text{ etc.} \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{C}P^n &= \mathbb{C}^n \sqcup \mathbb{C}P^{n-1} && (4.6) \\ &= \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \mathbb{C}P^{n-2} && (4.7) \\ &= \ldots && (4.8) \end{aligned}$$

and therefore,

$$\mathbb{C}P^2 = \mathbb{C}^2 \sqcup \mathbb{C}^1 \sqcup \mathbb{C}^0. \tag{4.9}$$

On $[z_1 : z_2 : z_3]$, we have that $[0 : 0 : 1] \simeq \mathbb{C}^0$, $[0 : 1 : z_3] \simeq \mathbb{C}^1$ and $[1 : z_2 : z_3] \simeq \mathbb{C}^2$. For $\mathbb{C}^1$, without loss of generality, we will only consider the specific

point $[0:1:1]$ and for $\mathbb{C}^2$, again without loss of generality, we only consider the point $[1:1:1]$. Now in $\mathbb{C}^3$, acting on the point $[0:1:1]$ by conjugation, we rotate it to the point $[0:0:1]$ (these points are conjugate to each other) and similarly, $[1:1:1]$ is conjugate to $[0:1:1]$ and in turn conjugate to $[0:0:1]$. To belong to a single coadjoint orbit, (K([0:0:1])), (K([0:1:1])) and (K([1:1:1])) must have the same eigenvalues. As explained, by conjugation we need only consider (K([0:0:1])) whose eigenvalues are $2/3, 2/3, -1/3$. Repeated eigenvalues imply that the orbit is diffeomorphic to $\mathbb{C}P^2$ (if every eigenvalue is distinct then the orbit is diffeomorphic to the flag manifold $F(2,1)$).
For property (2) Consider the action of $A \in SU(3)$

$$K \ : \ Z \to \frac{Z \otimes \overline{Z}}{|Z|^2} - \frac{1}{3}I \tag{4.10}$$

$$K \ : \ AZ \to \frac{AZ \otimes \overline{AZ}}{|AZ|^2} - \frac{1}{3}I = \frac{Z \otimes \overline{Z}}{|Z|^2} - \frac{1}{3}I. \tag{4.11}$$

For property (3) since property (2) holds we can consider the coordinate $z_0 = [1 : 0 : 0]$, also $T_{z_0}\mathbb{C}P^2 = (0, w_1, w_2)$

$$K : [1:0:0] \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}I \tag{4.12}$$

The defining equation for the momentum map, (2.2) re-written here, is

$$\langle dJ_x(v), \xi \rangle = \omega_x(v, \xi_p(x)) \tag{4.13}$$

where $x \in P$, $v \in T_xP$ and $\xi \in \mathfrak{g}$.

The symplectic form on $\mathfrak{su}(3)^*$

$$dJ(u) = \mathrm{ad}^*_\eta \mu \quad \text{for some } \eta \tag{4.14}$$

Then for the KKS form

$$\omega_\mu(dJ(u), dJ(v)) = \langle \mu, [\xi, \eta] \rangle \tag{4.15}$$

and $\omega_{FS}(u, v)(z_0) = \langle \mu, [\xi, \eta] \rangle = \mathrm{tr}(\mu^*[\xi, \eta])$ for $\mathbf{u} = (u_2, u_3)$ and $\mathbf{v} = (v_2, v_3)$ and K is a momentum map. $\square$

4.3 The Momentum Map of the $SU(3)$ Action on Products of $\mathbb{C}P^2$ with Weighted Symplectic Form

For the action of $SU(3)$ on products of $\mathbb{C}P^2$, we will introduce the *weighted* symplectic form. The symplectic form on N-copies of $\mathbb{C}P^2$ is

$$\omega = \Gamma_1 \omega_{FS} \oplus \ldots \oplus \Gamma_N \omega_{FS} \quad \text{where} \quad \Gamma_1, \ldots \Gamma_N \in \mathbb{R} \tag{4.16}$$

so the symplectic form is the sum of scalar multiples of the Fubini-Study form on each $\mathbb{C}P^2$. The momentum map $J : \prod_{i=1}^{N} \mathbb{C}P^2 \to \mathfrak{su}(3)^*$ of the $SU(3)$ action on N-products of $\mathbb{C}P^2$ is

$$J : (Z_1, \ldots Z_N) \to \sum_{i=1}^{N} \Gamma_i K(Z_i), \quad Z_i \in \mathbb{C}P^2.$$

Chapter 5

The Momentum Polytopes of the $SU(3)$ Action on Products of $\mathbb{C}P^2$

In this chapter we classify the momentum polytopes of the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2$ with symplectic form $\omega = \Gamma_1\omega_{FS} \oplus \Gamma_2\omega_{FS}$ according to the ratios between the separate weights Γ_1 and Γ_2. We also classify the momentum polytopes of the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ according to the ratios between Γ_1, Γ_2 and Γ_3 of the associated symplectic form $\omega = \Gamma_1\omega_{FS} \oplus \Gamma_2\omega_{FS} \oplus \Gamma_3\omega_{FS}$.

The momentum map, J, of the $SU(3)$ action on products of $\mathbb{C}P^2$ with symplectic form $\omega = \bigoplus_{i=1}^{N} \Gamma_i\omega_{FS}$ has an 'Orbit Momentum Map', j:

$$\begin{array}{ccc} \prod_{i=1}^{N} \mathbb{C}P^2 & \xrightarrow{J} & \mathfrak{su}(3)^* \\ \downarrow & & \downarrow \\ \prod_{i=1}^{N} \mathbb{C}P^2/SU(3) & \xrightarrow{j} & \mathfrak{su}(3)^*/SU(3) \cong \mathfrak{t}^*_+ \end{array}$$

the vertical arrows define reductions and $\mathfrak{t}^*_+$ is the positive Weyl chamber. Moreover each coadjoint orbit intersects $\mathfrak{t}^*_+$ at precisely one point: $\mathfrak{g}^*/G \simeq \mathfrak{t}^*_+$. We identify the elements of $\mathfrak{g}^*$ with 3×3 traceless (anti-)Hermitian matrices. The maximal torus of $SU(3)$ is $\mathbb{T}^2$ and the Weyl group acts on it as described in section 2.1. The action of the Weyl group $W(G)$ on a point of $\bar{C}$ (defined in section 2.1) is the orbit of the Weyl group. The pink area in the diagram below (see figure 5.1) is the positive Weyl chamber which has two walls ε_{α_1} and ε_{α_2}. The generic and degenerate types of orbits of $SU(3)$ were stated in chapter 3. A coadjoint orbit intersects with $\mathfrak{h}^*$ denoting an orbit of $W(SU(3))$. For a generic case, an orbit of $W(SU(3))$ has 6 elements, each element intersecting with the interior of each Weyl chamber at one point, if an initial point lies in the interior of the positive Weyl chamber. And for a degenerate (non-generic) case the orbit of $W(SU(3)$ has 3 elements which intersect one and only one wall of each Weyl chamber, if the initial point is on the wall of a positive Weyl chamber [13].

Each coadjoint orbit corresponds to an isospectral submanifold (i.e. the set of matrices with given spectrum (eigenvalues). Each Weyl chamber is conjugate to its consecutive Weyl chambers by reflection e.g. (2.1). Reflection along the Weyl Chamber walls is a permutation of the given spectrum. Therefore the image of the momentum map is reflected with respect to the Weyl chamber walls onto the positive Weyl chamber.

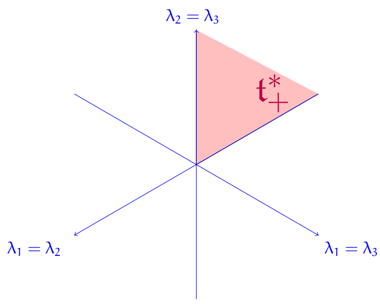


Figure 5.1: The positive Weyl chamber, $\mathfrak{t}^*_+$, as introduced in definition 2.1.56, points in which correspond to the set of eigenvalues of the diagonal elements of the image of the momentum map of the action, $(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3$, $\lambda_1 \geq 0$ and $\lambda_3 \leq 0$.

5.1 The Weighted Momentum Map of the Fixed Point Sets of the $SU(3)$ Action on $\mathbb{C}P^2 \times \mathbb{C}P^2$

The weighted momentum map for the $SU(3)$ action on the manifold $\bar{X} = \mathbb{C}P^2 \times \mathbb{C}P^2$ is

$$J : Z_1, Z_2 \to i \sum_{i=1}^{2} \Gamma_i \frac{Z_i \otimes \bar{Z}_i}{|Z_i|^2} - \frac{1}{3} I \sum_{i=1}^{2} \Gamma_i$$

as derived in section 4. The diagonal elements of the image of this map coincide with the positive Weyl chamber. The images of the momentum map action on $Z_1 = Z_2 = e_i$ all lie on the same coadjoint orbit and all happen to map to $\mathfrak{t}^*$: once a given spectrum is permuted, if the image of the momentum map of different elements of the fixed point set coincide then they lie on the same coadjoint orbit. For example, the elements of the fixed point set of the $\mathsf{U}(2)$ action (3.9) have the same spectrum. Firstly the spectrum values for $Z_1 = Z_2 = e_1$:

$$\begin{aligned} J([e_1, e_1]) &= \begin{pmatrix} \frac{2(\Gamma_1+\Gamma_2)}{3} & 0 & 0 \\ 0 & \frac{-\Gamma_1-\Gamma_2}{3} & 0 \\ 0 & 0 & \frac{-\Gamma_1-\Gamma_2}{3} \end{pmatrix} \\ \text{Spectrum}\Big(J([e_1, e_1])\Big) &= (\lambda_1, \lambda_2, \lambda_3) = \Big[\frac{2(\Gamma_1+\Gamma_2)}{3}, \frac{-\Gamma_1-\Gamma_2}{3}, \frac{-\Gamma_1-\Gamma_2}{3}\Big] \end{aligned}$$

similarly the image of the momentum map acting on $Z_1 = Z_2 = e_3$ is

$$\begin{aligned} J([e_3, e_3]) &= \begin{pmatrix} \frac{-\Gamma_1-\Gamma_2}{3} & 0 & 0 \\ 0 & \frac{-\Gamma_1-\Gamma_2}{3} & 0 \\ 0 & 0 & \frac{2(\Gamma_1+\Gamma_2)}{3} \end{pmatrix} \\ \text{Spectrum}\Big(J([e_3, e_3])\Big) &= \Big[\frac{-\Gamma_1-\Gamma_2}{3}, \frac{-\Gamma_1-\Gamma_2}{3}, \frac{2(\Gamma_1+\Gamma_2)}{3}\Big] \end{aligned}$$

Permuting this spectrum so that it intersects with the positive Weyl chamber,

$$\text{Spectrum}\Big(J([e_3, e_3])\Big) = \Big[\frac{2(\Gamma_1+\Gamma_2)}{3}, \frac{-\Gamma_1-\Gamma_2}{3}, \frac{-\Gamma_1-\Gamma_2}{3}\Big].$$

Therefore the two different spectra coincide within the positive Weyl chamber:

$$\text{Spectrum}\Big(J([e_1, e_1])\Big) = \text{Spectrum}\Big(J([e_3, e_3])\Big) = \Big[\frac{2(\Gamma_1+\Gamma_2)}{3}, \frac{-\Gamma_1-\Gamma_2}{3}, \frac{-\Gamma_1-\Gamma_2}{3}\Big]. \tag{5.1}$$

and similarly,

$$\text{Spectrum}\Big(J([e_1, e_1])\Big) = \text{Spectrum}\Big(J([e_2, e_2])\Big) = \text{Spectrum}\Big(J([e_3, e_3])\Big).$$

The spectrum specified in (5.1) satisfies the positive Weyl chamber domain variables for specific values of Γ_1 and Γ_2. For example, for $\Gamma_1, \Gamma_2 < 0$, the permutation of the spectrum that orders it accordingly gives

$$\text{Spectrum}\Big(J([[(e_i, e_i) \mid i = 1, 2, 3]])\Big) = (\lambda_1, \lambda_2, \lambda_3) = \Big[\frac{-\Gamma_1-\Gamma_2}{3}, \frac{-\Gamma_1-\Gamma_2}{3}, \frac{2(\Gamma_1+\Gamma_2)}{3}\Big].$$

The different spectra of the four different elements of the fixed point set of the maximal torus action, (3.11), all coincide in the positive Weyl chamber once permuted. Firstly,

$$\begin{aligned}\text{Spectrum}\Big(J([e_1,e_2])\Big) &= \text{Spectrum}\Big(J([e_2,e_1])\Big)\\ \text{Spectrum}\Big(J([e_1,e_3])\Big) &= \text{Spectrum}\Big(J([e_3,e_1])\Big)\\ \text{Spectrum}\Big(J([e_2,e_3])\Big) &= \text{Spectrum}\Big(J([e_3,e_2])\Big)\end{aligned}$$

up to permutation, and

$$\text{Spectrum}\Big(J([e_1,e_2])\Big) = \text{Spectrum}\Big(J([e_1,e_3])\Big) = \text{Spectrum}\Big(J([e_2,e_3])\Big)$$

again, up to permutation, where

$$\text{Spectrum}\Big(J([(e_i,e_j) \mid i,j = 1,2,3 \text{ and } i \neq j])\Big) = \Big[\frac{2\Gamma_1-\Gamma_2}{3}, \frac{2\Gamma_2-\Gamma_1}{3}, \frac{-\Gamma_1-\Gamma_2}{3}\Big]$$

but this spectrum coincides with the positive Weyl chamber for certain values of Γ_1 and Γ_2 only. Other Γ_1 to Γ_2 ratios require different permutations:

For $\Gamma_1 > \Gamma_2 > 0$ the correct permutation gives the spectrum

$$\text{Spectrum}\Big(J([(e_i,e_j) \mid i,j = 1,2,3 \text{ and } i \neq j])\Big) = \Big[\frac{2\Gamma_1-\Gamma_2}{3}, \frac{2\Gamma_2-\Gamma_1}{3}, \frac{-\Gamma_1-\Gamma_2}{3}\Big];$$

For $0 > \Gamma_1 > \Gamma_2$ the correct spectrum is

$$\text{Spectrum}\Big(J([(e_i,e_j) \mid i,j = 1,2,3 \text{ and } i \neq j])\Big) = \Big[\frac{-\Gamma_1-\Gamma_2}{3}, \frac{2\Gamma_2-\Gamma_1}{3}, \frac{2\Gamma_1-\Gamma_2}{3}\Big];$$

For $\Gamma_1 > 0 > \Gamma_2$ where $|\Gamma_2| > |\Gamma_1|$,

$$\text{Spectrum}\Big(J([(e_i,e_j) \mid i,j = 1,2,3 \text{ and } i \neq j])\Big) = \Big[\frac{2\Gamma_1-\Gamma_2}{3}, \frac{-\Gamma_1-\Gamma_2}{3}, \frac{2\Gamma_2-\Gamma_1}{3}\Big];$$

For $\Gamma_2 > 0 > \Gamma_1$ where $|\Gamma_2| > |\Gamma_1|$,

$$\text{Spectrum}\Big(J([(e_i,e_j) \mid i,j = 1,2,3 \text{ and } i \neq j])\Big) = \Big[\frac{2\Gamma_2-\Gamma_1}{3}, \frac{-\Gamma_1-\Gamma_2}{3}, \frac{2\Gamma_1-\Gamma_2}{3},\Big];$$

For $\Gamma_1 = \Gamma_2$ where Γ_1 and Γ_2 are both positive and for $\Gamma_1 = -\Gamma_2 = \Gamma$ where Γ is positive two different spectrum equate,

$$\left[\frac{2\Gamma_1-\Gamma_2}{3},\frac{2\Gamma_2-\Gamma_1}{3},\frac{-\Gamma_1-\Gamma_2}{3}\right]=\left[\frac{2\Gamma_2-\Gamma_1}{3},\frac{2\Gamma_1-\Gamma_2}{3},\frac{-\Gamma_1-\Gamma_2}{3}\right];$$

For $\Gamma_1 = \Gamma_2$ where both Γ_1 and Γ_2 are both negative and for $\Gamma_1 = -\Gamma_2 = \Gamma$ where Γ is negative two different spectrum equate,

$$\left[\frac{-\Gamma_1-\Gamma_2}{3},\frac{2\Gamma_1-\Gamma_2}{3},\frac{2\Gamma_2-\Gamma_1}{3}\right]=\left[\frac{-\Gamma_1-\Gamma_2}{3},\frac{2\Gamma_2-\Gamma_1}{3},\frac{2\Gamma_1-\Gamma_2}{3}\right].$$

5.1.1 The Γ_1 to Γ_2 Ratio

For two scalar values Γ_1 and Γ_2 there exist a few distinct ratios between them that distinguish the correct permutations of the eigenvalues so that the image of the momentum map intersects with the positive Weyl chamber. These can be distinguished on the $\Gamma_1 - \Gamma_2$ plane:

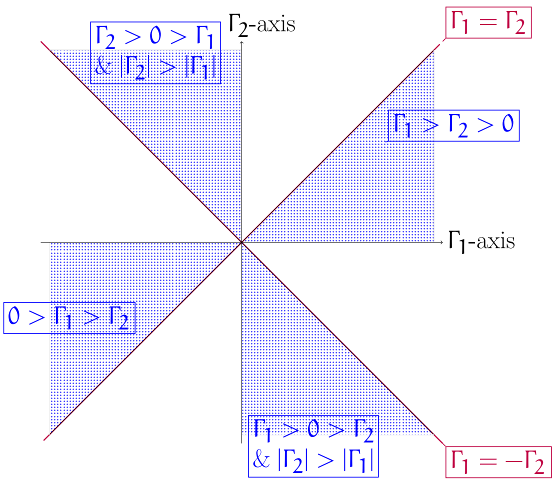


Figure 5.2: There are six different ratios shown on this plane including (going clockwise) $\Gamma_1 = \Gamma_2$, $\Gamma_1 > \Gamma_2 > 0$, $\Gamma_1 = -\Gamma_2$, $\Gamma_1 > 0 > \Gamma_2$ where $|\Gamma_2| > |\Gamma_1|$, $0 > \Gamma_1 > \Gamma_2$ and $\Gamma_2 > 0 > \Gamma_1$ where $|\Gamma_2| > |\Gamma_1|$.

Therefore the Momentum Polytopes of the $SU(3)$ Action on $\mathbb{C}P^2 \times \mathbb{C}P^2$ are classified into 6 different Polytopes: the Mometum Polytope for $\Gamma_1 = \Gamma_2$; the Momentum Polytope for $\Gamma_1 = -\Gamma_2$; the Momentum Polytope for $\Gamma_1 > \Gamma_2 > 0$; the Momentum Polytope for $0 > \Gamma_1 > \Gamma_2$; the Momentum Polytope for $\Gamma_1 > 0 > \Gamma_2$ where $|\Gamma_2| > |\Gamma_1|$; and the Momentum Polytope for $\Gamma_2 > 0 > \Gamma_1$ where $|\Gamma_2| > |\Gamma_1|$.

5.2 The Weighted Momentum Map of the Fixed Point Sets of the $SU(3)$ Action on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$

The momentum map for the $SU(3)$ action on the manifold $\bar{X} = \mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ is

$$J : Z_1, Z_2, Z_3 \to i \sum_{i=1}^{3} \Gamma_i \frac{Z_i \otimes \bar{Z}_i}{|Z_i|^2} - \frac{1}{3} I \sum_{i=1}^{3} \Gamma_i$$

As the roots can be permuted according to the constant rank theorem, to satisfy the above positive Weyl chamber root conditions then the only fixed points that need to be considered are:

$$\Big\{[e_1, e_1, e_1],\ [e_2, e_1, e_1],\ [e_1, e_2, e_1],\ [e_1, e_1, e_2],\ [e_1, e_2, e_3]\Big\} \tag{5.2}$$

whose momentum map images are:

$$\begin{aligned}
J([e_1, e_1, e_1]) &= \begin{pmatrix} \frac{2(\Gamma_1+\Gamma_2+\Gamma_3)}{3} & 0 & 0 \\ 0 & \frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3} & 0 \\ 0 & 0 & \frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3} \end{pmatrix} \\
\text{Spectrum}\Big(J([e_1, e_1, e_1])\Big) &= \Big[\frac{2(\Gamma_1+\Gamma_2+\Gamma_3)}{3}, \frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3}, \frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3}\Big] \\
J([e_2, e_1, e_1]) &= \begin{pmatrix} \frac{-\Gamma_1+2(\Gamma_2+\Gamma_3)}{3} & 0 & 0 \\ 0 & \frac{2\Gamma_1-\Gamma_2-\Gamma_3}{3} & 0 \\ 0 & 0 & \frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3} \end{pmatrix} \\
\text{Spectrum}\Big(J([e_2, e_1, e_1])\Big) &= \Big[\frac{-\Gamma_1+2(\Gamma_2+\Gamma_3)}{3}, \frac{2\Gamma_1-\Gamma_2-\Gamma_3}{3}, \frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3}\Big] \\
J([e_1, e_2, e_1]) &= \begin{pmatrix} \frac{2\Gamma_1-\Gamma_2+2\Gamma_3}{3} & 0 & 0 \\ 0 & \frac{-\Gamma_1+2\Gamma_2-\Gamma_3}{3} & 0 \\ 0 & 0 & \frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3} \end{pmatrix} \\
\text{Spectrum}\Big(J([e_1, e_2, e_1])\Big) &= \Big[\frac{2\Gamma_1-\Gamma_2+2\Gamma_3}{3}, \frac{-\Gamma_1+2\Gamma_2-\Gamma_3}{3}, \frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3}\Big] \\
J([e_1, e_1, e_2]) &= \begin{pmatrix} \frac{2(\Gamma_1+\Gamma_2)-\Gamma_3}{3} & 0 & 0 \\ 0 & \frac{-\Gamma_1-\Gamma_2+2\Gamma_3}{3} & 0 \\ 0 & 0 & \frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3} \end{pmatrix} \\
\text{Spectrum}\Big(J([e_1, e_1, e_2])\Big) &= \Big[\frac{2(\Gamma_1+\Gamma_2)-\Gamma_3}{3}, \frac{-\Gamma_1-\Gamma_2+2\Gamma_3}{3}, \frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3}\Big]
\end{aligned}$$

$$
\begin{aligned}
J([e_1,e_2,e_3]) &= \begin{pmatrix} \frac{2\Gamma_1-\Gamma_2-\Gamma_3}{3} & 0 & 0 \\ 0 & \frac{-\Gamma_1+2\Gamma_2-\Gamma_3}{3} & 0 \\ 0 & 0 & \frac{-\Gamma_1-\Gamma_2+2\Gamma_3}{3} \end{pmatrix} \\
\text{Spectrum}\Big(J([e_1,e_2,e_3])\Big) &= \Big[\frac{2\Gamma_1-\Gamma_2-\Gamma_3}{3}, \frac{-\Gamma_1+2\Gamma_2-\Gamma_3}{3}, \frac{-\Gamma_1-\Gamma_2+2\Gamma_3}{3}\Big]
\end{aligned}
$$

Again, these points and the resulting momentum polytope that is made up of these points depends on the respective magnitudes of Γ_1, Γ_2 and Γ_3 and their respective ratios i.e. their classification.

5.2.1 The Γ_1 to Γ_2 to Γ_3 Ratio

For three scalar values Γ_1, Γ_2 and Γ_3 there exist a few distinct ratios between them that distinguish the correct permutations of the eigenvalues so that the image of the momentum map intersects with the positive Weyl chamber.
We only need to consider the Momentum Polytopes for values of $\Gamma_i \neq 0$, this is because when one of the $\Gamma_i = 0$ we end up recreating and repeating the Momentum Polytopes of the action of $SU(3)$ on the $\mathbb{C}P^2$ or $\mathbb{C}P^2 \times \mathbb{C}P^2$ manifolds which is redundant and does not produce any new information for us. We will separate the momentum polytopes according to the following equalities: $\Gamma_1 = 0$, $\Gamma_2 = 0$, $\Gamma_3 = 0$, $\Gamma_2 = \Gamma_3$, $\Gamma_1 = \Gamma_3$, $\Gamma_1 = -\Gamma_2$, $\Gamma_2 = -\Gamma_3$, $\Gamma_1 = -\Gamma_3$, $\Gamma_1 = \Gamma_2 + \Gamma_3$, $\Gamma_2 = \Gamma_1 + \Gamma_3$ and $\Gamma_3 = \Gamma_1 + \Gamma_2$.
So let us distinguish the different Momentum Polytopes that aren't defined by these equations.

The hyperplane $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ for an arbitrary $\Delta \in \mathbb{R}_+$ on the $\Gamma_1 - \Gamma_2 - \Gamma_3$ axes:

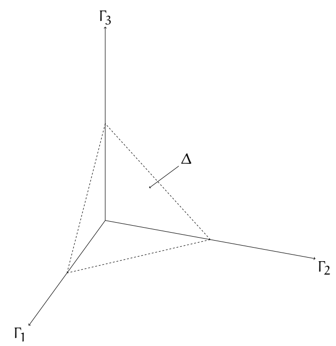


Figure 5.3: Δ marks the hyperplane $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ for an arbitrary $\Delta \in \mathbb{R}_+$

On the plane $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$,

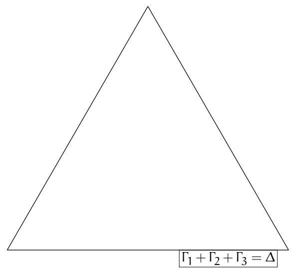


Figure 5.4: The plane $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ for an arbitrary $\Delta \in \mathbb{R}_+$ flattened onto the page and without the Γ_1, Γ_2 or Γ_3 axes shown

We mark out the lines that correspond to the different coincidences between Γ_1, Γ_2 and Γ_3:

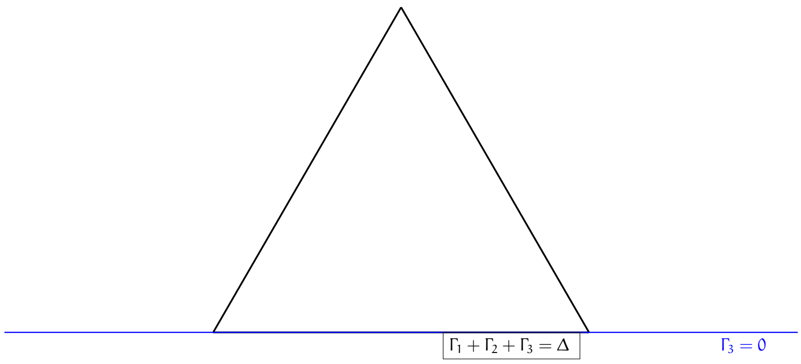


Figure 5.5: $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ and $\Gamma_3 = 0$ drawn and labelled

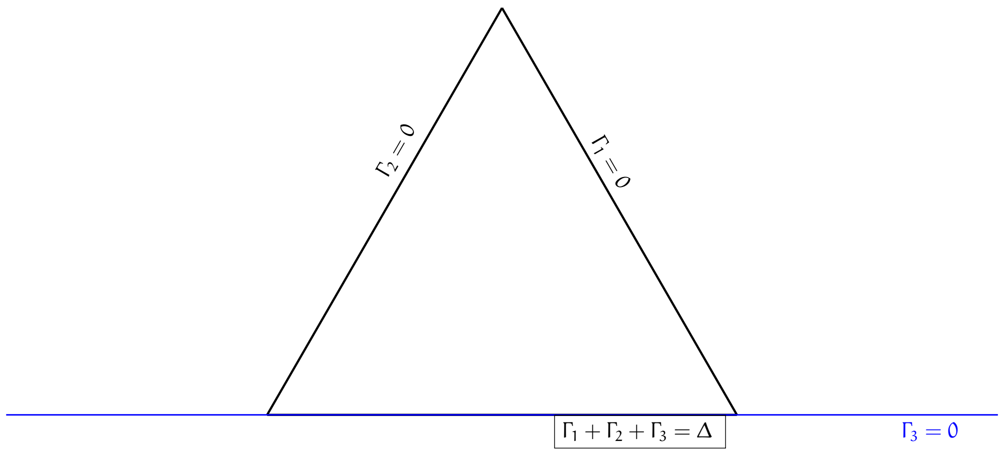


Figure 5.6: $\Gamma_1 = 0$ and $\Gamma_2 = 0$ also marked

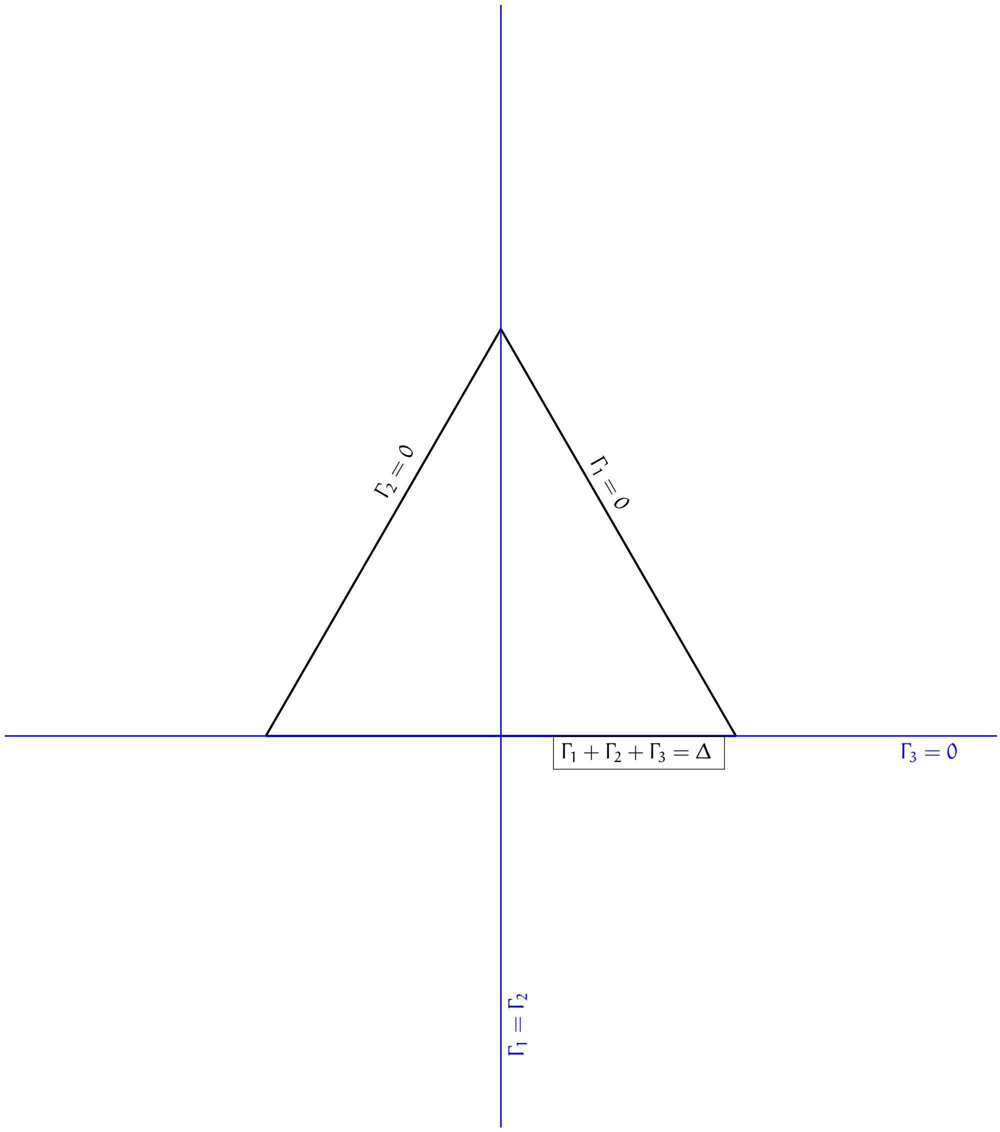


Figure 5.7: $\Gamma_1 = \Gamma_2$ also drawn and labelled

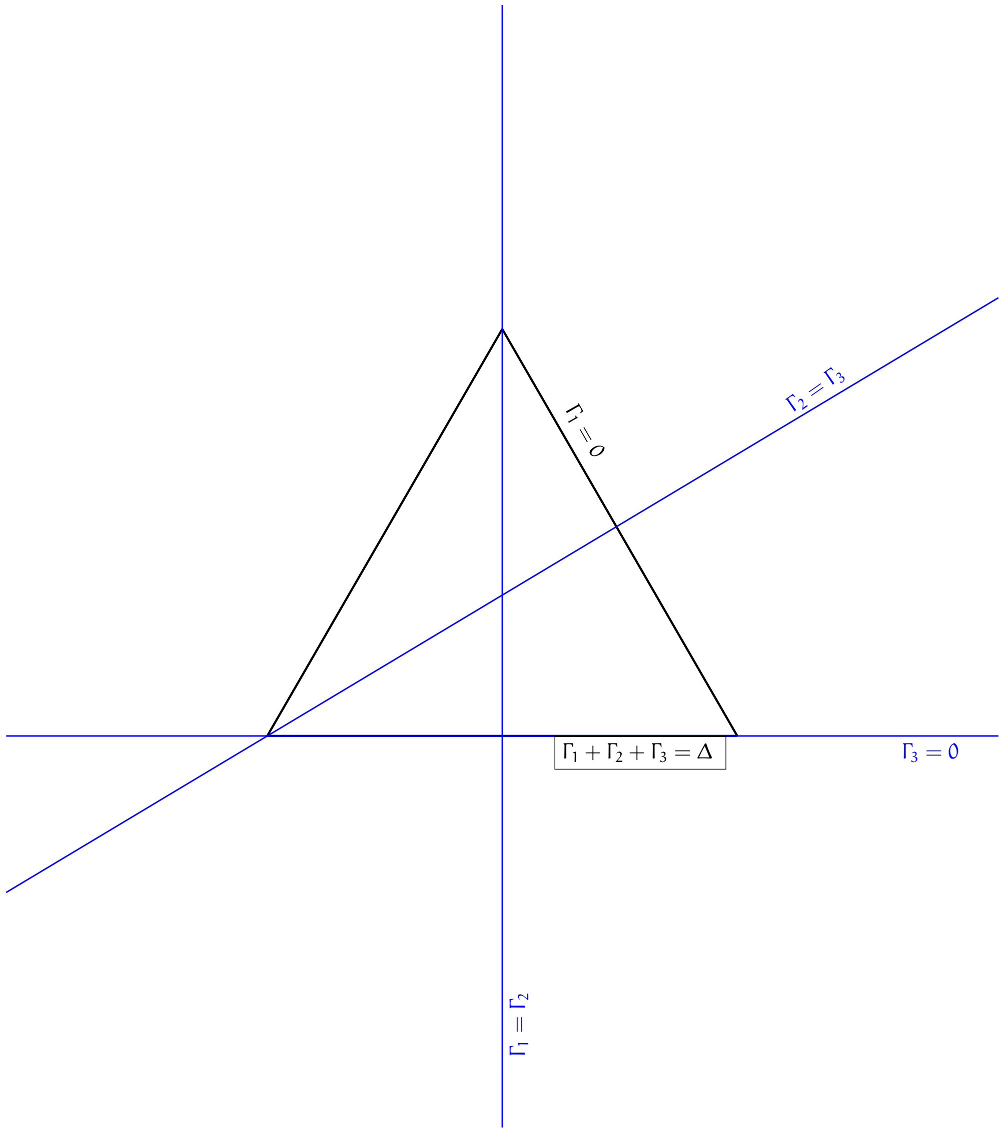


Figure 5.8: $\Gamma_2 = \Gamma_3$ added

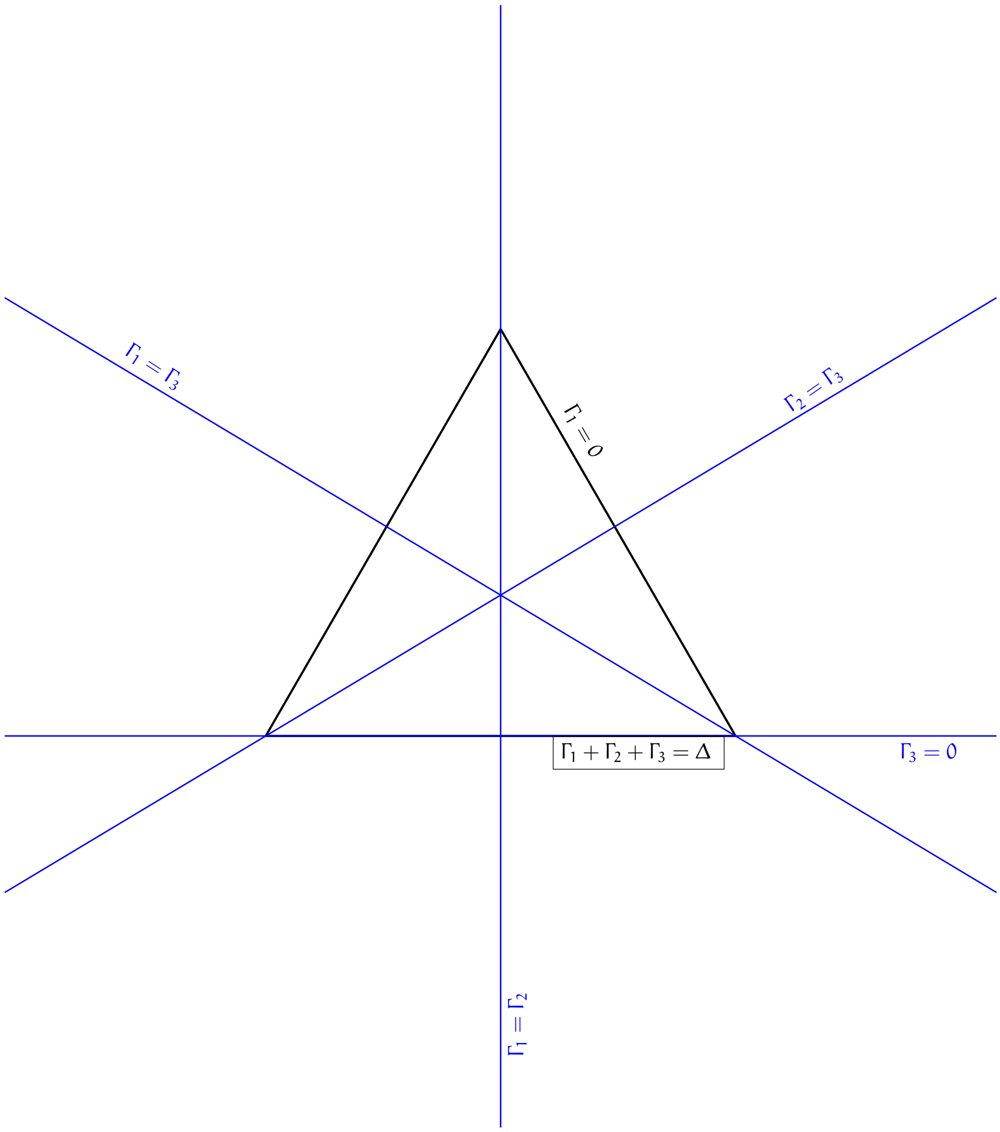


Figure 5.9: $\Gamma_1 = \Gamma_3$ added

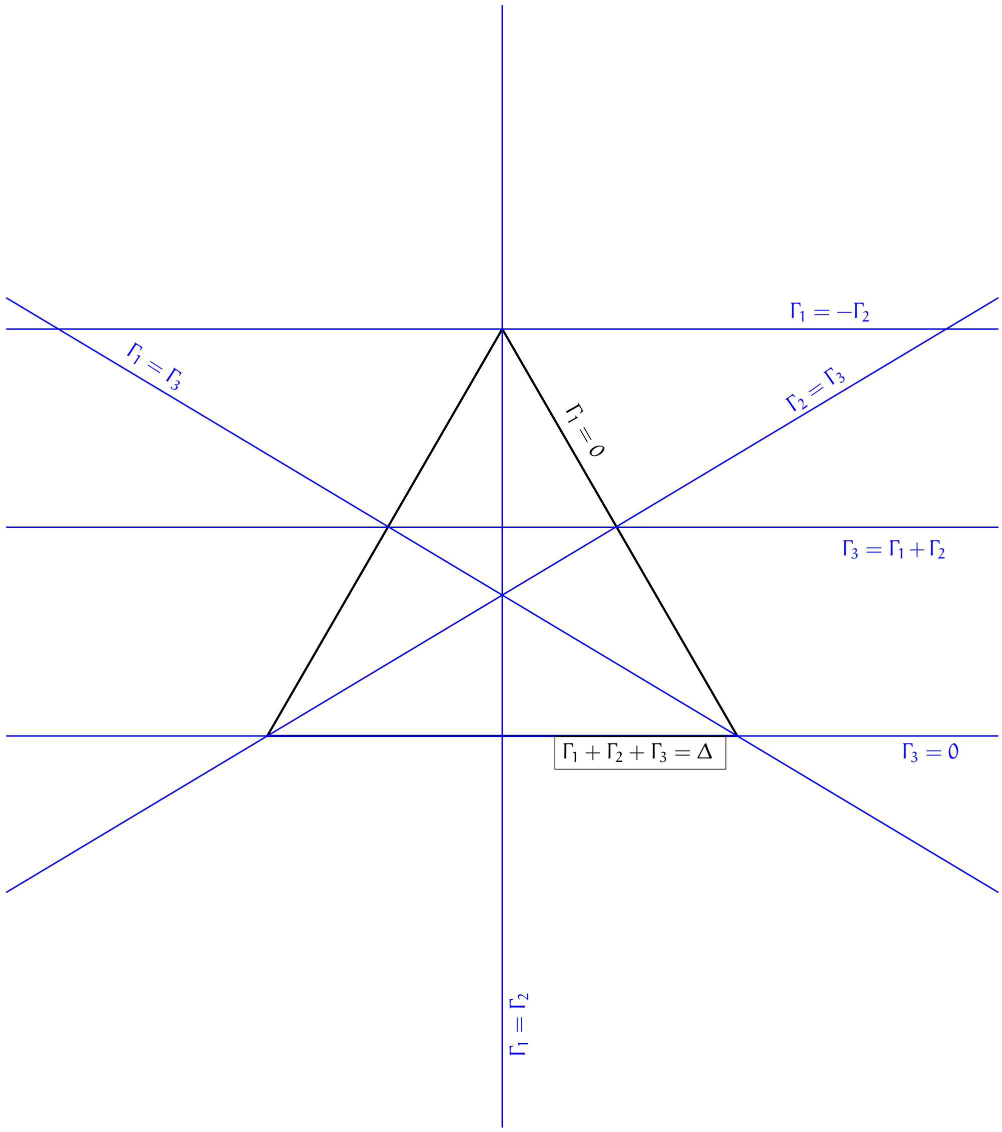


Figure 5.10: $\Gamma_1 = -\Gamma_2$ and $\Gamma_3 = \Gamma_1 + \Gamma_2$ added

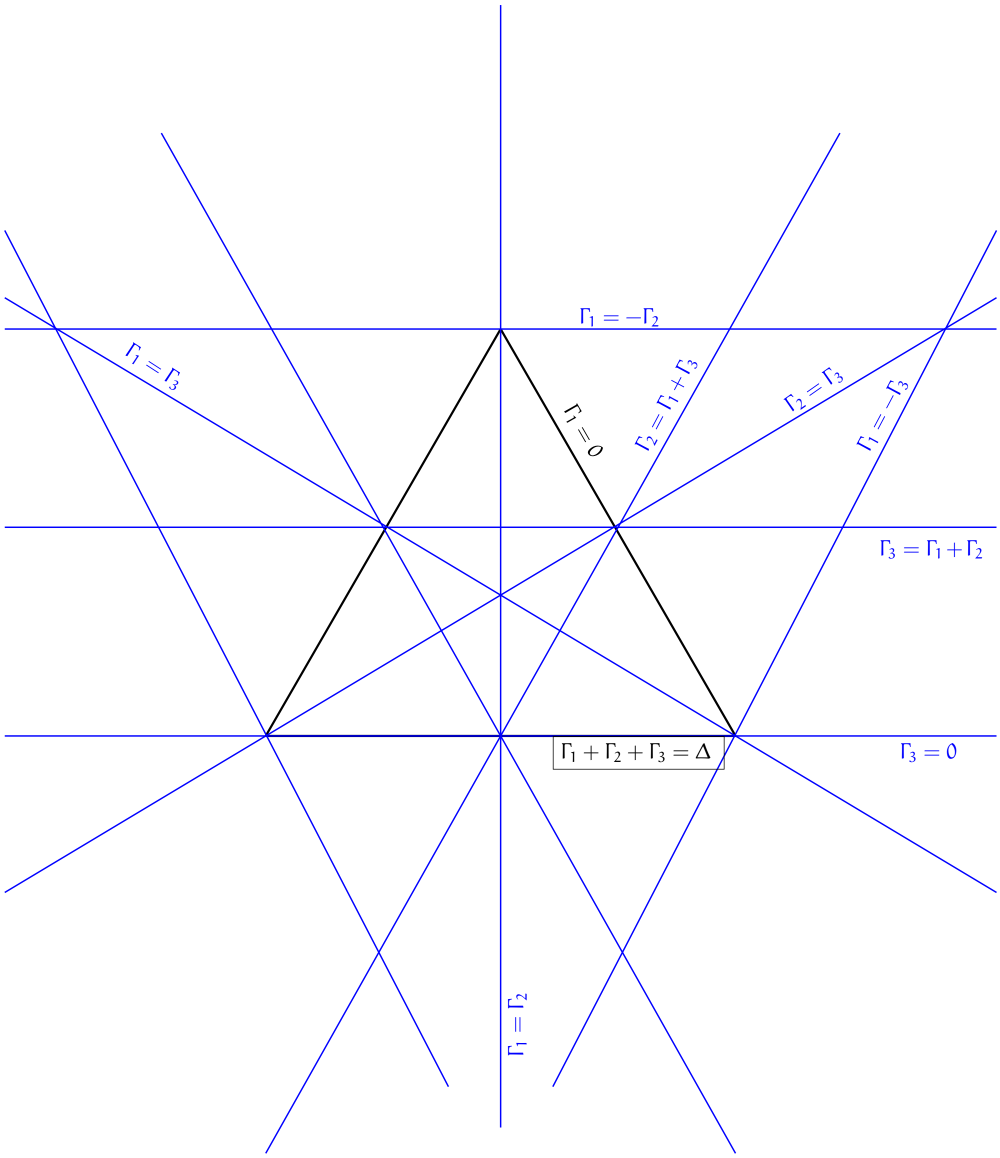


Figure 5.11: $\Gamma_1 = -\Gamma_3$ and $\Gamma_2 = \Gamma_1 + \Gamma_3$ added

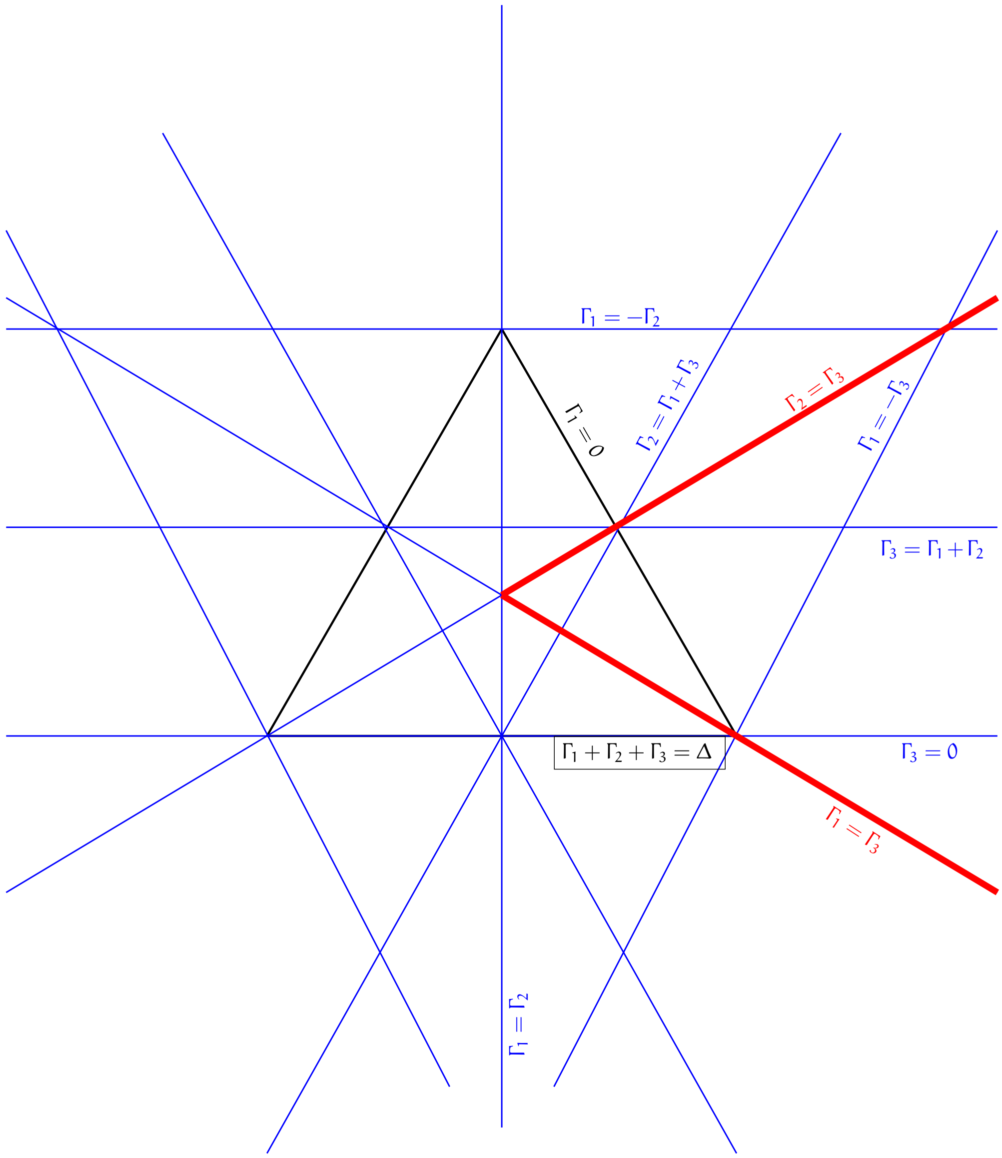


Figure 5.12: Due to symmetries of the momentum map we choose to focus on the region between the lines marked in red $\Gamma_2 = \Gamma_3$ and $\Gamma_1 = \Gamma_3$

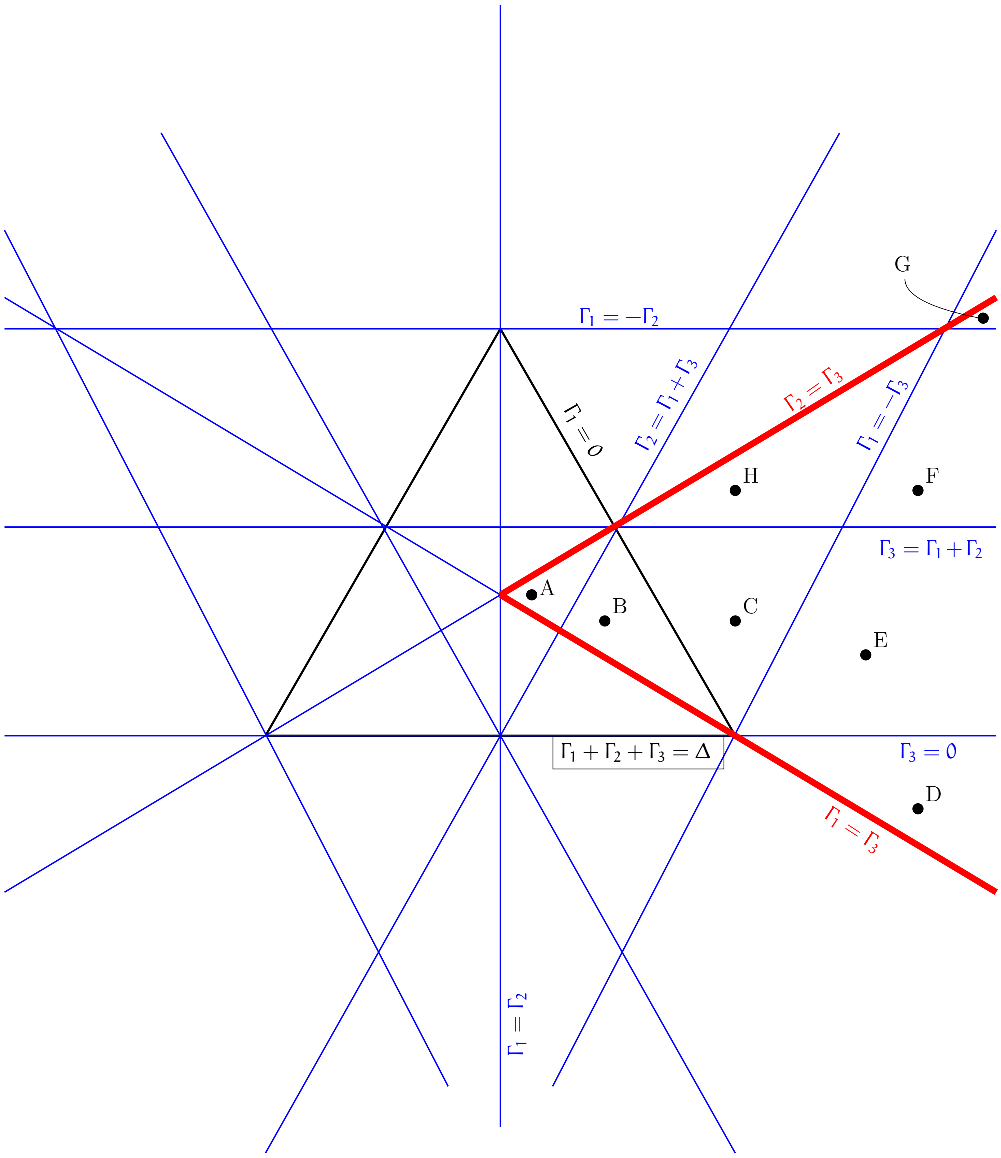


Figure 5.13: The separate sections outlined by $\Gamma_2 = \Gamma_3$, $\Gamma_1 = \Gamma_3$ and the other equalities that have been drawn and labelled are lettered separately

Figure 5.4 is a 2-dimensional representation of the plane $\Gamma_1+\Gamma_2+\Gamma_3=\Delta$ with $\Delta\in\mathbb{R}_+$ (with the emphasised triangle outlining the resulting plane for a specific $\Delta\in\mathbb{R}_+$). The lines correspond to all possible equalities between the Γ_i's and have been drawn and labelled accordingly in figures 5.5 through to 5.11: those figures show the plane $\Gamma_1+\Gamma_2+\Gamma_3=\Delta$ with the lines $\Gamma_3=0$, $\Gamma_1=0$ and $\Gamma_2=0$ marked, as well as $\Gamma_1=\Gamma_2$, $\Gamma_2=\Gamma_3$, $\Gamma_1=\Gamma_3$, $\Gamma_1=-\Gamma_2$, $\Gamma_3=\Gamma_1+\Gamma_2$, $\Gamma_1=-\Gamma_3$, $\Gamma_2=\Gamma_1+\Gamma_3$. In figure 5.11 the plane is split into six sections separated by the lines $\Gamma_1=\Gamma_2$, $\Gamma_2=\Gamma_3$ and $\Gamma_1=\Gamma_3$. These six sections are interchangeable with one another contingent on the swapping of a Γ_i for a Γ_j, therefore recognising that there is symmetry across this plane, we need only consider one of the six sections to distinguish and classify the Momentum Polytopes that are separated by the Γ_i equalities. Hence why we mark out one of these regions in figure 5.12: the region within the lines $\Gamma_2=\Gamma_3$ and $\Gamma_1=\Gamma_3$.
In figure 5.13, between each of the equalities drawn and labelled within the region outlined by $\Gamma_2=\Gamma_3$ and $\Gamma_1=\Gamma_3$, are the sections which have been lettered **A**,**B**,**C**,**D**,**E**,**F**,**G** and **H**. The sections **A** through to **H** represent 8 different classifications of the Momentum Polytopes of the $SU(3)$ action on $\mathbb{C}P^2\times\mathbb{C}P^2\times\mathbb{C}P^2$. Including the Momentum Polytope for $\Gamma_1+\Gamma_2+\Gamma_3=0$ this makes 9 different classifications. Therefore classifying the Momentum Polytopes of this action into 9 different 'types' including the Momentum Polytope for $\Gamma_1+\Gamma_2+\Gamma_3=0$.

The image of the momentum map of the action of $SU(3)$ on $\mathbb{C}P^2\times\mathbb{C}P^2\times\mathbb{C}P^2$ where all the points are parallel, $J([e_1,e_1,e_1])$, is equal to the image of the momentum map of the action of $SU(3)$ on parallel points in $\mathbb{C}P^2\times\mathbb{C}P^2$, $J([e_1,e_1])$, if we equate $\Gamma_1=\Gamma_2$ and then make the replacements $2\Gamma_1=\Gamma_a$ and $\Gamma_3=\Gamma_b$:

$$\begin{aligned}\text{Spectrum}\Big(J([e_1,e_1,e_1])\Big) &= \left[\frac{2(\Gamma_1+\Gamma_2+\Gamma_3)}{3},\frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3},\frac{-\Gamma_1-\Gamma_2-\Gamma_3}{3}\right]\\ &= \left[\frac{4\Gamma_1+2\Gamma_3}{3},\frac{-2\Gamma_1-\Gamma_3}{3},\frac{-2\Gamma_1-\Gamma_3}{3}\right]\text{ at }\Gamma_1=\Gamma_2\\ &= \left[\frac{2(\Gamma_a+\Gamma_b)}{3},\frac{-\Gamma_a-\Gamma_b}{3},\frac{-\Gamma_a-\Gamma_b}{3}\right]\\ &= \text{Spectrum}\Big(J([e_1,e_1])\Big) \end{aligned} \tag{5.3}$$

Similarly, images of the momentum map of the $SU(3)$ action on $\mathbb{C}P^2\times\mathbb{C}P^2\times\mathbb{C}P^2$ coincide with images of the momentum map of the action on $\mathbb{C}P^2\times\mathbb{C}P^2$ for $\Gamma_2=\Gamma_3$ with the replacements $2\Gamma_2=\Gamma_a$ and $\Gamma_1=\Gamma_b$, as it would for $\Gamma_1=\Gamma_3$

with replacements $2\Gamma_1 = \Gamma_a$ and $\Gamma_2 = \Gamma_b$. Likewise, a different class of polytopes correspond to the equalities $\Gamma_1 = 0$, $\Gamma_2 = 0$, $\Gamma_3 = 0$, $\Gamma_1 = \Gamma_2$, $\Gamma_2 = \Gamma_3$, $\Gamma_1 = \Gamma_3$, $\Gamma_1 = -\Gamma_2$, $\Gamma_2 = -\Gamma_3$, $\Gamma_1 = -\Gamma_3$, $\Gamma_1 = \Gamma_2 + \Gamma_3$, $\Gamma_2 = \Gamma_1 + \Gamma_3$ and $\Gamma_3 = \Gamma_1 + \Gamma_2$ and another class of momentum polytopes correspond to their inequalities: the latter are explored and fully classified in section 5.3, and the former are fully classified in section 5.7.

5.3 The Momentum Polytopes of the $SU(3)$ Action on $\mathbb{C}P^2 \times \mathbb{C}P^2$

The action $SU(3) \curvearrowright \mathbb{C}P^2 \times \mathbb{C}P^2$ is a cohomogeneity one action whose orbit is homeomorphic to the closed interval. We show that the momentum polytopes of this action fall into different categories separated by the ratios between the Γ_is.

Theorem 5.3.1. *The momentum polytopes of the* $SU(3)$ *action on* $\mathbb{C}P^2 \times \mathbb{C}P^2$ *with weighted symplectic form* $\Gamma_1\omega_{FS} \oplus \Gamma_2\omega_{FS}$ *fall into four different categories for which* $\Gamma_1 - \Gamma_2 \neq 0$, $\Gamma_1 + \Gamma_2 \neq 0$, $\Gamma_i \neq 0$ *where* $i, j = 1, 2$.

Proof. $\Gamma_1 > \Gamma_2 > 0$, $\Gamma_2 > 0 > \Gamma_1$ where $|\Gamma_2| > |\Gamma_1|$, $0 > \Gamma_1 > \Gamma_2$ and $\Gamma_1 > 0 > \Gamma_2$ where $|\Gamma_2| > |\Gamma_1|$ are the regions defined within the hyperplanes $\Gamma_1 - \Gamma_2 = 0$, $\Gamma_1 + \Gamma_2 = 0$ and $\Gamma_i = 0$ where $i, j = 1, 2$ on the $\Gamma_1 - \Gamma_2$ axes as shown in figure 5.2. Let us categorise these regions and explore the different Momentum Polytopes they make.

<u>The First Category:</u>

The spectrum of the image of the momentum maps of the $SU(3)$ action on two parallel points and two perpendicular points in $\mathbb{C}P^2 \times \mathbb{C}P^2$ with weighted symplectic form for which $\Gamma_1 > \Gamma_2 > 0$ that satisfy the positive Weyl chamber domain conditions $\lambda_1 \geq \lambda_2 \geq \lambda_3$ with $\lambda_1 \geq 0$ and $\lambda_3 \leq 0$ as shown in figure 5.1 are:

$$\text{Spectrum}\Big(J([(e_i, e_i) \mid i = 1, 2, 3])\Big) = (\lambda_1, \lambda_2, \lambda_3) = \left[\frac{2(\Gamma_1 + \Gamma_2)}{3}, \frac{-\Gamma_1 - \Gamma_2}{3}, \frac{-\Gamma_1 - \Gamma_2}{3}\right]$$

$$\text{Spectrum}\Big(J([(e_i, e_j) \mid i, j = 1, 2, 3 \text{ and } i \neq j])\Big) = \left[\frac{2\Gamma_1 - \Gamma_2}{3}, \frac{2\Gamma_2 - \Gamma_1}{3}, \frac{-\Gamma_1 - \Gamma_2}{3}\right]$$

$\text{Spectrum}\Big(J([(e_i, e_i) \mid i = 1, 2, 3])\Big)$ has a repeated element so is diffeomorphic to $\mathbb{C}P^2$ and $\text{Spectrum}\Big(J([(e_i, e_j) \mid i, j = 1, 2, 3 \text{ and } i \neq j])\Big)$ has 3 distinct elements so is diffeomorphic to $\mathsf{F}(2,1)$. The coadjoint orbits through points in the boundary of the Weyl chamber are $\mathbb{C}\mathsf{P}^2$s, while those through the interior points are the flag manifolds. Plotting these spectra on the positive Weyl chamber, the line that joins them is the convex Momentum Polytope of this action:

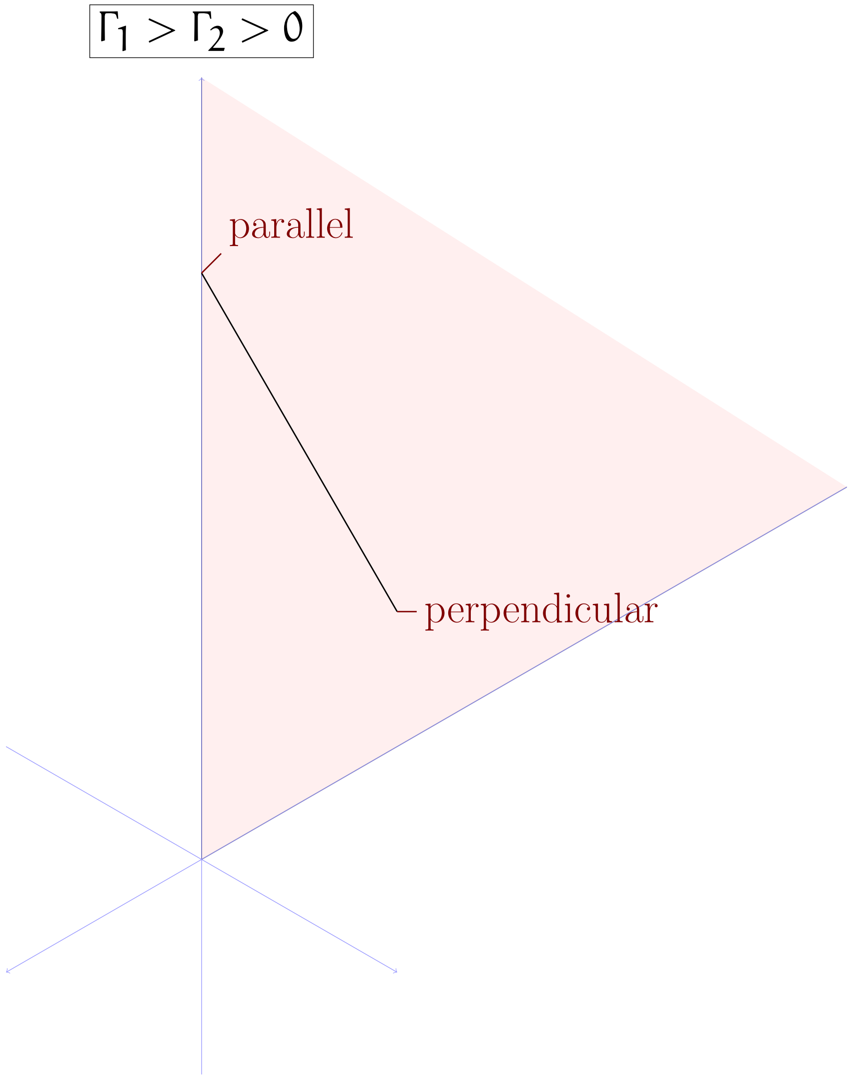


Figure 5.14: Momentum Polytope for $\Gamma_1 > \Gamma_2 > 0$ clearly indicating the regular and singular orbits in relation to parallel or perpendicular points.

The Second Category:

If the weighting of the symplectic form satisfy $\Gamma_2 > 0 > \Gamma_1$ where $|\Gamma_2| > |\Gamma_1|$, the spectra for parallel and perpendicular points must be permuted so that they coincide with the positive Weyl chamber. The resultant spectra for the action on parallel and perpendicular points in $\mathbb{C}P^2 \times \mathbb{C}P^2$ are respectively:

$$\text{Spectrum} \quad \Big(J([(e_i, e_i) \mid i = 1,2,3])\Big) = (\lambda_1, \lambda_2, \lambda_3) = \Big[\frac{2(\Gamma_1+\Gamma_2)}{3}, \frac{-\Gamma_1-\Gamma_2}{3}, \frac{-\Gamma_1-\Gamma_2}{3}\Big]$$
$$\text{Spectrum} \quad \Big(J([(e_i, e_j) \mid i,j = 1,2,3 \text{ and } i \neq j])\Big) = \Big[\frac{2\Gamma_2-\Gamma_1}{3}, \frac{-\Gamma_1-\Gamma_2}{3}, \frac{2\Gamma_1-\Gamma_2}{3}\Big]$$

the Momentum Polytope for which is generally of the form:

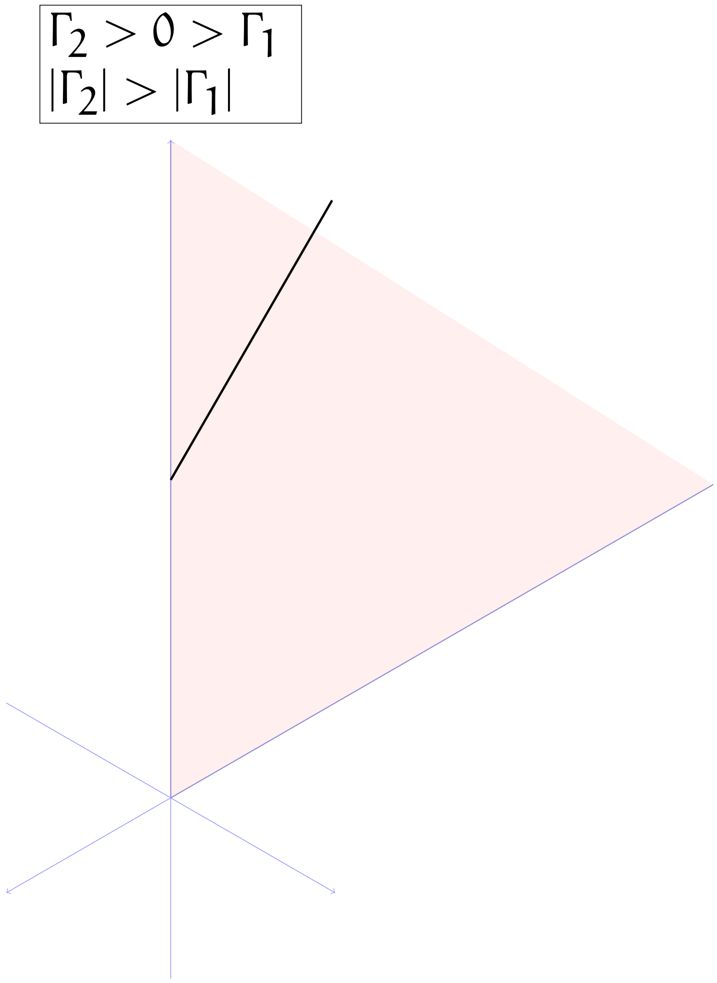


Figure 5.15: Momentum polytope for $\Gamma_2 > 0 > \Gamma_1$ where $|\Gamma_2| > |\Gamma_1|$

The Third and Fourth Category: The Momentum Polytopes of the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2$ with weighted symplectic form for which $\Gamma_1 > 0 > \Gamma_2$ where $|\Gamma_2| > |\Gamma_1|$ and $0 > \Gamma_1 > \Gamma_2$ represent the above first and second category for the Momentum Polytopes of the action but reflected along the $\lambda_2 = 0$ line in the positive Weyl chamber.

The Third Category (a reflection of the Momentum Polytopes of the first category reflected along the $\lambda_2 = 0$ line along the chosen positive Weyl chamber): For the action with weighting restriction $0 > \Gamma_1 > \Gamma_2$, the permuted spectra for the action on parallel and perpendicular points are

$$\text{Spectrum} \quad \Big(J([(e_i, e_i) \mid i = 1, 2, 3])\Big) = (\lambda_1, \lambda_2, \lambda_3) = \left[\frac{-\Gamma_1 - \Gamma_2}{3}, \frac{-\Gamma_1 - \Gamma_2}{3}, \frac{2(\Gamma_1 + \Gamma_2)}{3}\right]$$
$$\text{Spectrum} \quad \Big(J([(e_i, e_j) \mid i, j = 1, 2, 3 \text{ and } i \neq j])\Big) = \left[\frac{-\Gamma_1 - \Gamma_2}{3}, \frac{2\Gamma_1 - \Gamma_2}{3}, \frac{2\Gamma_2 - \Gamma_1}{3}\right]$$

for which the Momentum Polytope is of the form

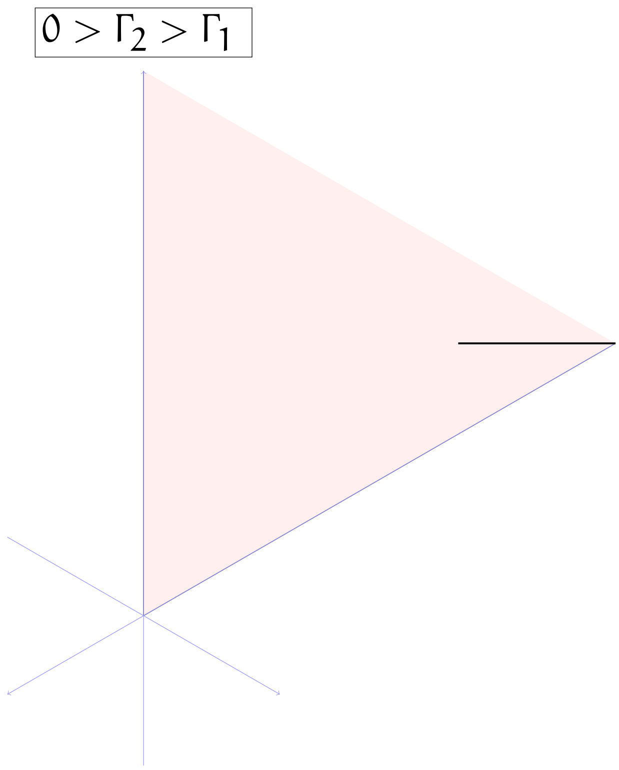


Figure 5.16: Momentum polytope for $0 > \Gamma_1 > \Gamma_2$

The Fourth Category (a reflection of the Momentum Polytopes of the second category reflected along the $\lambda_2 = 0$ line along the chosen positive Weyl chamber): The action with weighting restriction $\Gamma_1 > 0 > \Gamma_2$ where $|\Gamma_2| > |\Gamma_1|$, the permuted spectra for the action on parallel and perpendicular points are

$$\text{Spectrum}\quad \Big(J([[(e_i, e_i) \mid i = 1, 2, 3]])\Big) = (\lambda_1, \lambda_2, \lambda_3) = \Big[\frac{2(\Gamma_1 + \Gamma_2)}{3}, \frac{-\Gamma_1 - \Gamma_2}{3}, \frac{-\Gamma_1 - \Gamma_2}{3}\Big]$$
$$\text{Spectrum}\quad \Big(J([[(e_i, e_j) \mid i, j = 1, 2, 3 \text{ and } i \neq j]])\Big) = \Big[\frac{2\Gamma_1 - \Gamma_2}{3}, \frac{-\Gamma_1 - \Gamma_2}{3}, \frac{2\Gamma_2 - \Gamma_1}{3}\Big]$$

whose Momentum Polytope is of the form

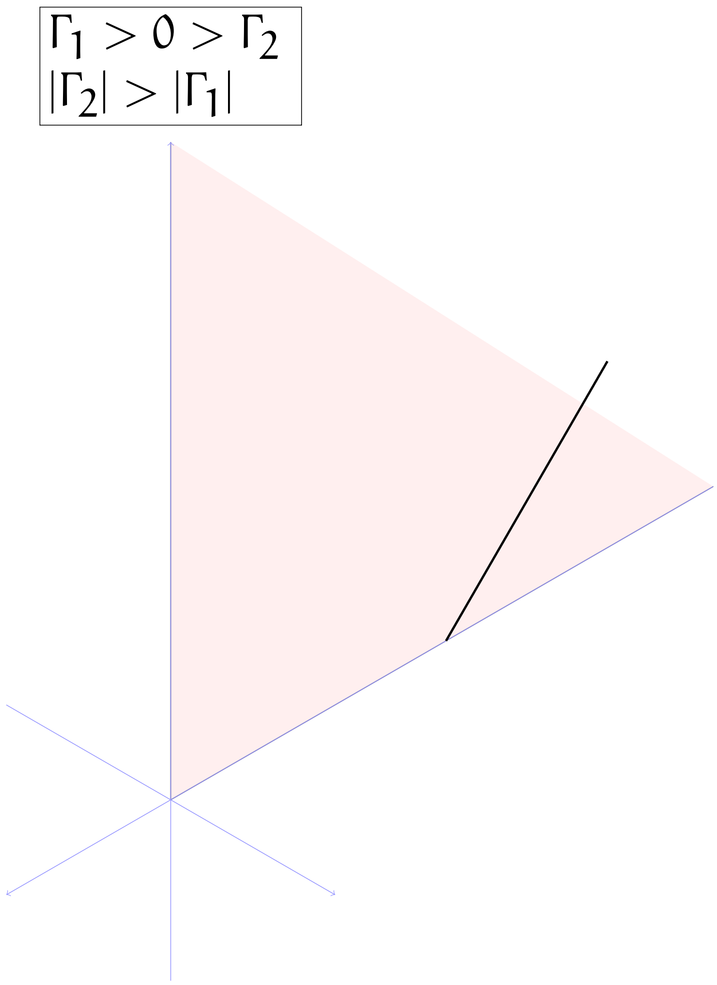


Figure 5.17: Momentum polytope for $\Gamma_1 > 0 > \Gamma_2$ where $|\Gamma_2| > |\Gamma_1|$

□

5.4 The Transitional Momentum Polytopes of the $SU(3)$ Action on $\mathbb{C}P^2 \times \mathbb{C}P^2$

Theorem 5.4.1. *The different polytopes of theorem 1 are separated by* transitional *momentum polytopes of the* $SU(3)$ *action on* $\mathbb{C}P^2 \times \mathbb{C}P^2$ *fall into three different categories for which* $\Gamma_1 - \Gamma_2 = 0$, $\Gamma_1 + \Gamma_2 = 0$, $\Gamma_i = 0$ *where* $i, j = 1, 2$.

Proof. The First Category:
The permuted spectra for the action with weighting $\Gamma_1 - \Gamma_2 = 0$ are

$$\text{Spectrum } \Big(J([(e_i, e_i) \mid i = 1, 2, 3])\Big) = (\lambda_1, \lambda_2, \lambda_3) = \left[\frac{2(\Gamma_1 + \Gamma_2)}{3}, \frac{-\Gamma_1 - \Gamma_2}{3}, \frac{-\Gamma_1 - \Gamma_2}{3}\right]$$
$$\text{Spectrum } \Big(J([(e_i, e_j) \mid i, j = 1, 2, 3 \text{ and } i \neq j])\Big) = \left[\frac{2\Gamma_1 - \Gamma_2}{3}, \frac{2\Gamma_2 - \Gamma_1}{3}, \frac{-\Gamma_1 - \Gamma_2}{3}\right]$$

whose resultant Momentum Polytope is of the general form:

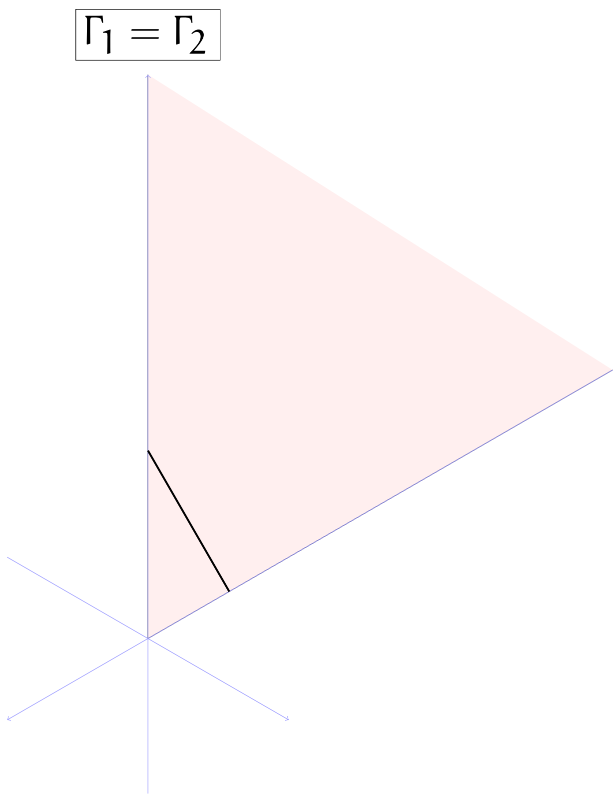


Figure 5.18: Momentum polytope for $\Gamma_1 = \Gamma_2$

The Second Category:

The permuted spectra for the action with weighting $\Gamma_1 + \Gamma_2 = 0$ are

$$\text{Spectrum}\ \Big(J([(e_i, e_i) \mid i = 1, 2, 3])\Big) = (\lambda_1, \lambda_2, \lambda_3) = \Big[0, 0, 0\Big]$$
$$\text{Spectrum}\ \Big(J([(e_i, e_j) \mid i, j = 1, 2, 3 \text{ and } i \neq j])\Big) = \left[\frac{2\Gamma_1 - \Gamma_2}{3}, \frac{-\Gamma_1 - \Gamma_2}{3}, \frac{2\Gamma_2 - \Gamma_1}{3}\right]$$

whose resultant Momentum Polytope is of the general form:

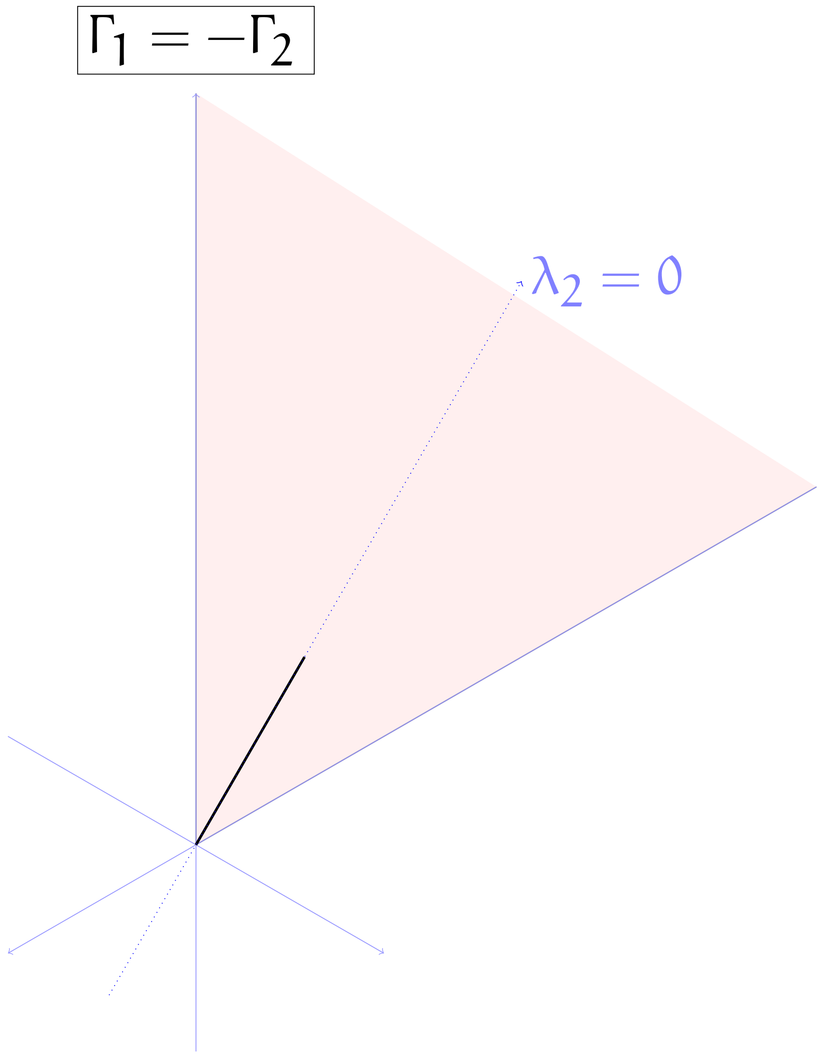


Figure 5.19: Momentum polytope for $\Gamma_1 = -\Gamma_2$

The Third Category:
For the action with weighting $\Gamma_i = 0$ the image of the momentum map is equal to that of the $SU(3)$ action on a single copy of $\mathbb{C}P^2$ and Momentum Polytope can only be a point in the boundary of the Weyl Chamber. □

5.5 The Directions of the Edges of the Momentum Polytopes of the $SU(3)$ Action on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$

Globally, the gradient of the edges of the polytope are determined by the bifurcation lemma. The bifucation lemma, whose proof is provided in section 2.6, provides a link between the rank of a momentum map at a point p of a Poisson Manifold $(P, \{\cdot, \cdot\})$ and the symmetry of the manifold at p. The rank of T_pJ is the same as the dimension of the annihilator of the isotropy algebra at p providing a rank-nullity correspondence.The Weyl group reflections at the walls of the Weyl chamber also dictate the resulting shape of the convex polytope.
For a point on the surface of a sphere, the gradient at that point will be the coordinates of the plane tangent to the sphere at that point. Applying this to points in $\mathbb{C}P^2$ we have for

$$\begin{aligned} z_i = [0:0:1] \quad \dot{z}_i = [a:b:0] \\ z_i = [0:1:0] \quad \dot{z}_i = [a:0:c] \\ z_i = [1:0:0] \quad \dot{z}_i = [0:b:c] \end{aligned}$$

Therefore, for

$$\begin{aligned} P &= [e_1, e_1, e_2] \\ T_{z_0}P &= \left(\begin{pmatrix} 0 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} 0 \\ b_2 \\ c_2 \end{pmatrix}, \begin{pmatrix} a_3 \\ 0 \\ c_3 \end{pmatrix} \right) = \nu \end{aligned}$$

The gradient of the momentum map at P is

$$\begin{aligned} dJ_v[e_1, e_1, e_2] &= \Gamma_1[e_1 v_1^\dagger + v_1 e_1^\dagger] + \Gamma_2[e_1 v_2^\dagger + v_2 e_1^\dagger] + \Gamma_3[e_2 v_3^\dagger + v_3 e_2^\dagger] \\ &= \Gamma_1 \begin{pmatrix} 0 & \bar{b}_1 & \bar{c}_1 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} + \Gamma_2 \begin{pmatrix} 0 & \bar{b}_2 & \bar{c}_2 \\ b_2 & 0 & 0 \\ c_2 & 0 & 0 \end{pmatrix} + \Gamma_3 \begin{pmatrix} 0 & a_3 & 0 \\ \bar{a}_3 & 0 & \bar{c}_3 \\ 0 & c_3 & 0 \end{pmatrix} \end{aligned}$$

The dot product of the torus action $\Xi = \begin{pmatrix} e^{i\psi} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix}$, where $\psi, \theta, \phi \in [0, 2\pi]$, with the vector v gives

$$\Xi \cdot v = \left(\begin{pmatrix} 0 \\ b_1 e^{i\theta} \\ c_1 e^{i\phi} \end{pmatrix}, \begin{pmatrix} 0 \\ b_2 e^{i\theta} \\ c_2 e^{i\phi} \end{pmatrix}, \begin{pmatrix} a_3 e^{i\psi} \\ 0 \\ c_3 e^{i\phi} \end{pmatrix} \right) \tag{5.4}$$

and

$$\begin{aligned} \xi_\psi \cdot v &= \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} ia_3 \\ 0 \\ 0 \end{pmatrix} \right) \text{ (at } \psi = 0) \\ \xi_\theta \cdot v &= \left(\begin{pmatrix} 0 \\ ib_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ ib_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) \text{ (at } \theta = 0) \\ \xi_\phi \cdot v &= \left(\begin{pmatrix} 0 \\ 0 \\ ic_1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ ic_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ ic_3 \end{pmatrix} \right) \text{ (at } \phi = 0) \end{aligned}$$

where $\xi_\theta = \frac{d}{d\theta}\Xi$, $\xi_\phi = \frac{d}{d\phi}\Xi$ and $\xi_\psi = \frac{d}{d\psi}\Xi$.
Therefore the torus action that generates θ or rather fixes θ is defined at $a_3 = 0 = c_1 = c_2 = c_3$. For symplectic form $\omega(z_1, z_2) = \Gamma\Im(z_1\bar{z}_2)$, i.e. the imaginary part of $(z_1\bar{z}_2)$ with $\Gamma \in \mathbb{R}$ and $\Gamma \neq 0$ the momentum map satisfies:

$$\langle J(v), \xi \rangle = \omega(\xi \cdot v, v) \tag{5.5}$$

Using the bifurcation lemma (section 2.6): $\mathrm{Im}\, dJ_z = \mathfrak{g}_z^\circ$.

$$dJ\left(\begin{pmatrix} 0 \\ ib_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ ib_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) = \Gamma_1 \begin{pmatrix} 0 & \bar{b}_1 & 0 \\ b_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \Gamma_2 \begin{pmatrix} 0 & \bar{b}_2 & 0 \\ b_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.6)$$

$$= \begin{pmatrix} 0 & \sum_{i=1}^2 \Gamma_i \bar{b}_i & 0 \\ \sum_{i=1}^2 \Gamma_i b_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.7)$$

the spectrum of which is $\lambda^3 - \lambda \sum_{i=1}^2 \Gamma_i \bar{b}_i \sum_{i=1}^2 \Gamma_i b_i = 0 = \lambda(\lambda^2 - \sum_{i=1}^2 \Gamma_i \bar{b}_i \sum_{i=1}^2 \Gamma_i b_i)$ rescalling $\sum_{i=1}^2 \Gamma_i \bar{b}_i \sum_{i=1}^2 \Gamma_i b_i = 1$ then

$$\mathfrak{g}_z^\circ = \begin{cases} (0,1,-1), & (0,-1,1) \\ (1,0,-1), & (-1,0,1) \\ (1,-1,0), & (-1,1,0) \end{cases}$$

Meaning that the edges of the polytope must be perpendicular to any of the walls of any of the Weyl chambers. So, locally from every fixed point of the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ the weights associated to that point travel in any of the $(\lambda_1, \lambda_2, \lambda_3)$ directions of $\mathfrak{g}_z^\circ$ shown above where $(0,1,-1) = -(0,-1,1)$, $(1,0,-1) = -(-1,0,1)$ and $(1,-1,0) = -(-1,1,0)$ therefore specifying three different directions or their negative counterparts. When the weight of a fixed point coincides with that of another fixed point then an edge to the polytope is formed. If this does not create a *convex* polytope then one of the fixed points is reflected along one of the walls of the positive Weyl chamber until a convex shape is reached whose edges coincide with any of the $\mathfrak{g}_z^\circ$ directions shown.

5.6 The Momentum Polytopes of the $SU(3)$ Action on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$

We will calculate the momentum polytopes for the $SU(3)$ action on points in the manifold $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ with weighted symplectic form $\Gamma_1 \omega_{FS} \oplus \Gamma_2 \omega_{FS} \oplus \Gamma_3 \omega_{FS}$. There are nine different distinct polytopes for this action which are distinguished by the magnitudes and respective ratios between Γ_1, Γ_2 and Γ_3.

Theorem 5.6.1. *The momentum polytopes of the* $SU(3)$ *action on* $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ *with weighted symplectic form* $\Gamma_1 \omega_{FS} \oplus \Gamma_2 \omega_{FS} \oplus \Gamma_3 \omega_{FS}$ *fall into nine different categories for which* $\Gamma_i - \Gamma_j - \Gamma_k \neq 0$, $\Gamma_i + \Gamma_j \neq 0$, $\Gamma_i \neq 0$ *where* $i, j, k = 1, 2, 3$.

Proof. The fixed point sets of the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ with weighted symplectic form have been narrowed down to just 5 fixed points as specified in equation (5.2). Once the spectrum for each of these points defined $\text{Spectrum}\Big(J([e_1, e_1, e_1])\Big)$, $\text{Spectrum}\Big(J([e_1, e_2, e_3])\Big)$, $\text{Spectrum}\Big(J([e_2, e_1, e_1])\Big)$, $\text{Spectrum}\Big(J([e_1, e_2, e_1])\Big)$ and $\text{Spectrum}\Big(J([e_1, e_1, e_2])\Big)$ have been plotted, the edges that connect these points to make the convex polytope must be perpendicular to one of the walls of the Weyl chambers as shown in section 5.5. We will label each of these spectra:

$$\begin{aligned}
\mathbf{a} &= \text{Spectrum}\Big(J([e_1, e_1, e_1])\Big), \\
\mathbf{b} &= \text{Spectrum}\Big(J([e_1, e_2, e_3])\Big), \\
\mathbf{c}_1 &= \text{Spectrum}\Big(J([e_2, e_1, e_1])\Big), \\
\mathbf{c}_2 &= \text{Spectrum}\Big(J([e_1, e_2, e_1])\Big), \\
\mathbf{c}_3 &= \text{Spectrum}\Big(J([e_1, e_1, e_2])\Big).
\end{aligned}$$

Figure 5.20 is an example of one of these polytopes: the vertices corresponding to each image of the momentum map of the 5 fixed points are labelled $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}_1$,$\mathbf{c}_2$ and $\mathbf{c}_3$ as shown. The edge connecting $\mathbf{a}$ to $\mathbf{c}_1$ is perpendicular to the wall $\lambda_1 = \lambda_3$, the edge connecting $\mathbf{a}$ to $\mathbf{c}_3$ is perpendicular to the wall $\lambda_1 = \lambda_2$, the edge connecting $\mathbf{b}$ to $\mathbf{c}_1$ is perpendicular to the wall $\lambda_1 = \lambda_2$, the edge connecting $\mathbf{b}$ to $\mathbf{c}_3$ is perpendicular to the wall $\lambda_1 = \lambda_3$, and finally the 'internal' edge connecting $\mathbf{b}$ to $\mathbf{c}_2$ is perpendicular to the wall $\lambda_2 = \lambda_3$.

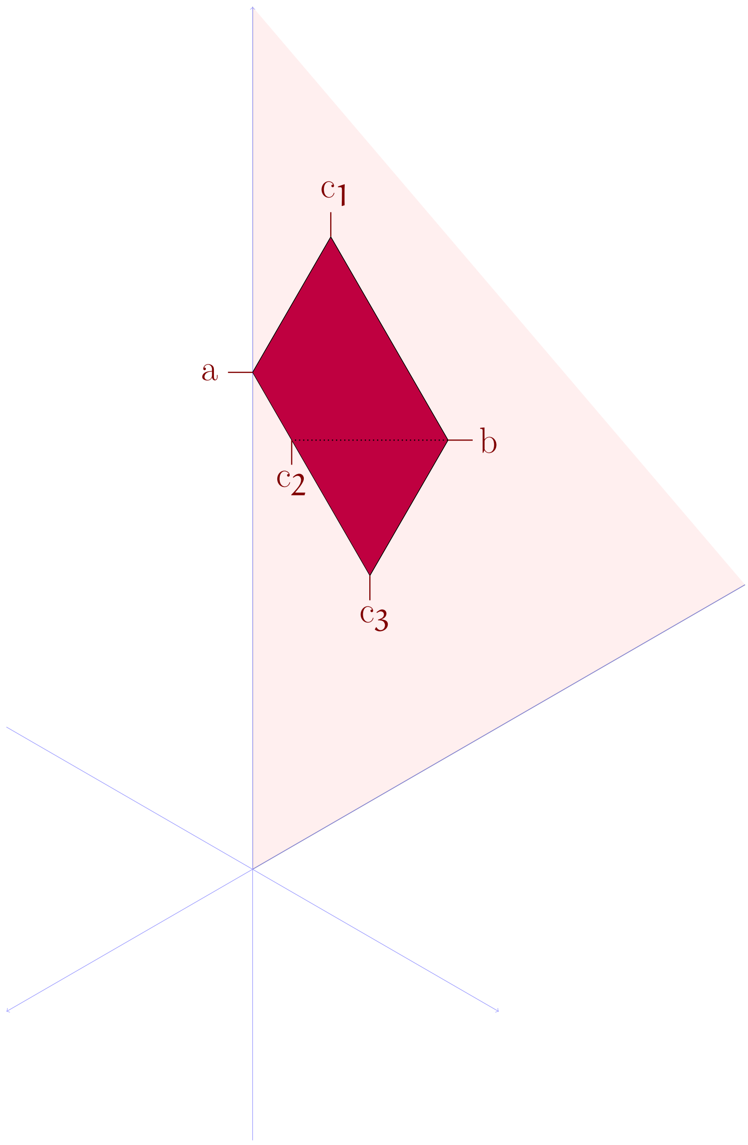


Figure 5.20: As an example we have here the polytope for region C from figure 5.13 where the labelled points $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}_1$,$\mathbf{c}_2$ and $\mathbf{c}_3$ each correspond to a specific spectrum: $\mathbf{a}$=Spectrum$\Big(J([e_1, e_1, e_1])\Big)$, $\mathbf{b}$=Spectrum$\Big(J([e_1, e_2, e_3])\Big)$, $\mathbf{c}_1$=Spectrum$\Big(J([e_2, e_1, e_1])\Big)$, $\mathbf{c}_2$=Spectrum$\Big(J([e_1, e_2, e_1])\Big)$, $\mathbf{c}_3$=Spectrum$\Big(J([e_1, e_1, e_2])\Big)$ and the edges of this convex momentum polytope are each perpendicular to a particular wall of the Weyl chambers.

Let us now show all of the different polytopes that fall into the nine different categorisations:

The **A** Polytope

The Momentum Polytope defined by the region labelled **A** in figure 5.13 between the boundaries $\Gamma_1 = \Gamma_3$, $\Gamma_2 = \Gamma_3$ and $\Gamma_2 = \Gamma_1 + \Gamma_3$ on the $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ plane has the general shape,

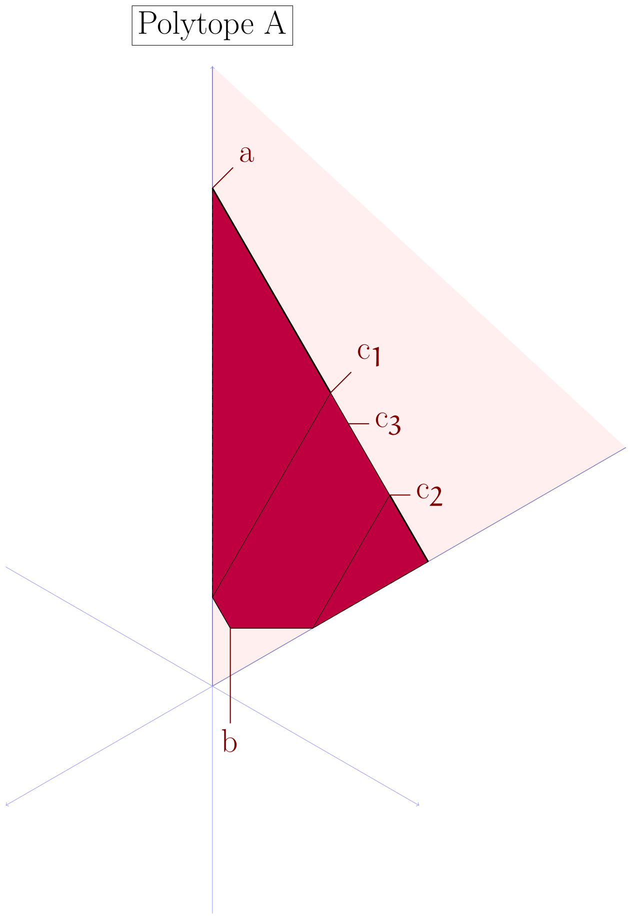


Figure 5.21: The polytope for region A from figure 5.13 where again the labelled points each correspond to a specific spectrum: $\mathbf{a}$=Spectrum$\Big(J([e_1, e_1, e_1])\Big)$, $\mathbf{b}$=Spectrum$\Big(J([e_1, e_2, e_3])\Big)$, $\mathbf{c}_1$=Spectrum$\Big(J([e_2, e_1, e_1])\Big)$, $\mathbf{c}_2$=Spectrum$\Big(J([e_1, e_2, e_1])\Big)$, $\mathbf{c}_3$=Spectrum$\Big(J([e_1, e_1, e_2])\Big)$ and the edges of this convex momentum polytope are each perpendicular to any one wall of the Weyl chambers.

This Polytope is different to the example Polytope in figure 5.20: For this polytope to be convex, the points $\mathbf{c}_1$ and $\mathbf{c}_2$ are reflected along the $\lambda_2 = \lambda_3$ and $\lambda_1 = \lambda_2$ walls of the positive Weyl chamber, respectively. Let us detail how this polytope has been formed. First we plot the 5 different spectrum values as shown in figure 5.22

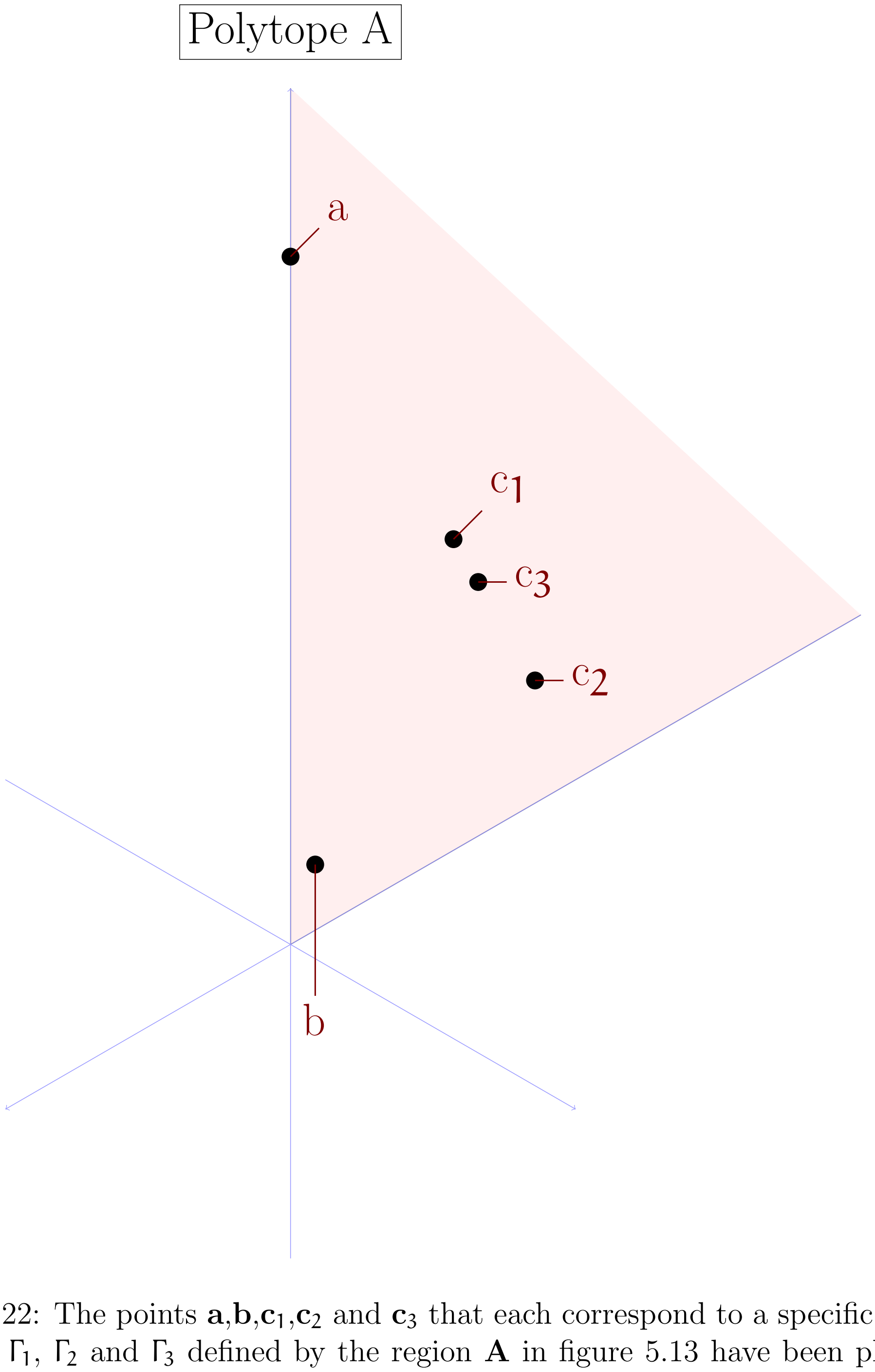


Figure 5.22: The points **a**,**b**,$\mathbf{c}_1$,$\mathbf{c}_2$ and $\mathbf{c}_3$ that each correspond to a specific spectrum for Γ_1, Γ_2 and Γ_3 defined by the region **A** in figure 5.13 have been plotted in the positive Weyl chamber. Points **a**,$\mathbf{c}_1$,$\mathbf{c}_3$ and $\mathbf{c}_2$ lie on a straight line that is perpendicular to the $\lambda_1 = \lambda_2$ wall of the positive Weyl chamber, but this line with the point **b** do not make a convex shape that is a momentum polytope.

As can be seen in figure 5.22 four points in a straight line, points $\mathbf{a}$,$\mathbf{c}_1$,$\mathbf{c}_3$ and $\mathbf{c}_2$, with a fifth point, point $\mathbf{b}$, cannot be connected to make a convex shape whose edges are each perpendicular to one of the walls of the Weyl chambers. Secondly we reflect points $\mathbf{c}_1$ and $\mathbf{c}_2$ across their closest Weyl chamber wall:

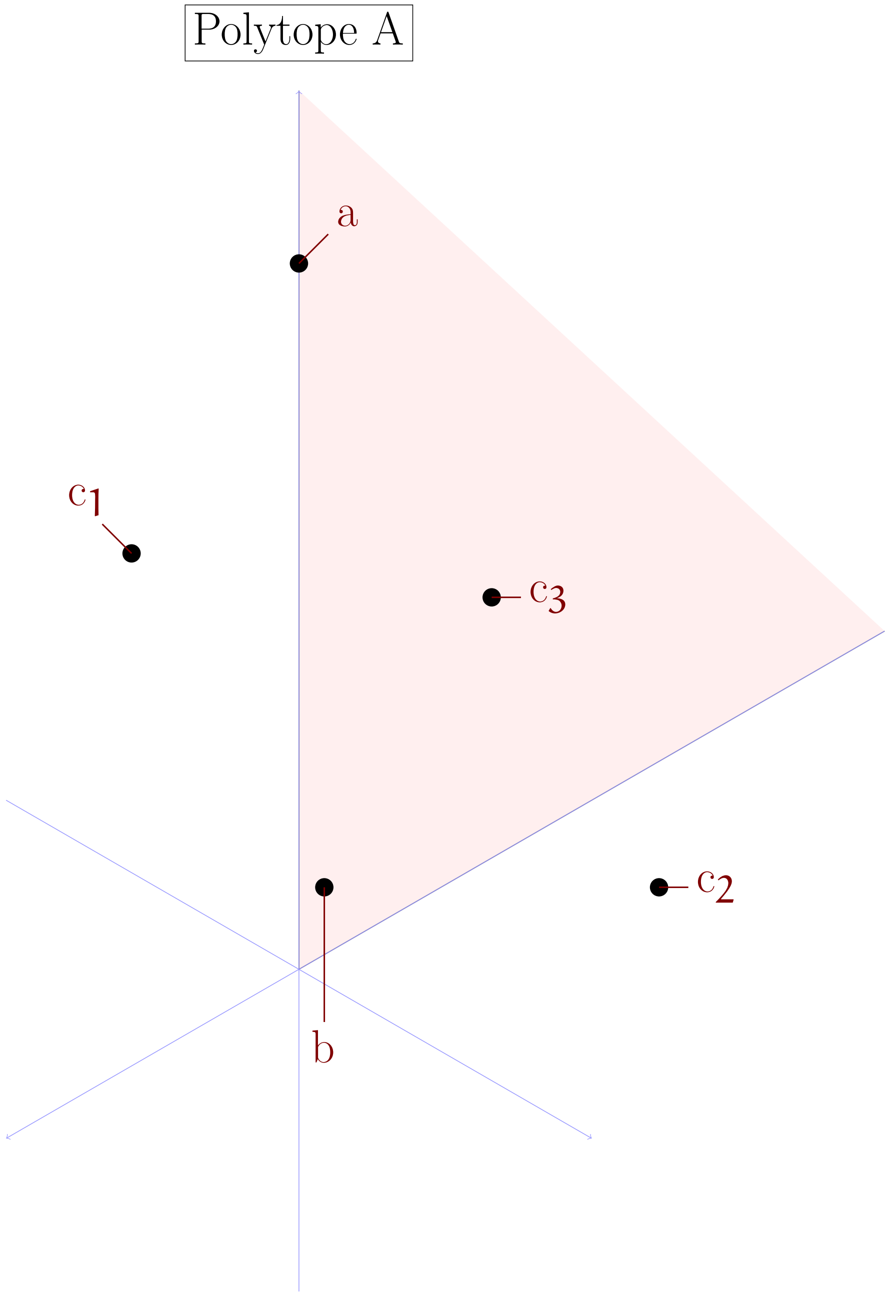


Figure 5.23: The points of figure 5.22 with point $\mathbf{c}_1$ reflected along the $\lambda_2 = \lambda_3$ wall of the positive Weyl chamber and point $\mathbf{c}_2$ reflected across the $\lambda_1 = \lambda_2$ wall of the positive Weyl chamber.

The edges that connect the new set of points make a convex shape whose edges comply with the specifications outlined by the bifurcation lemma and calculations made in section 5.5.

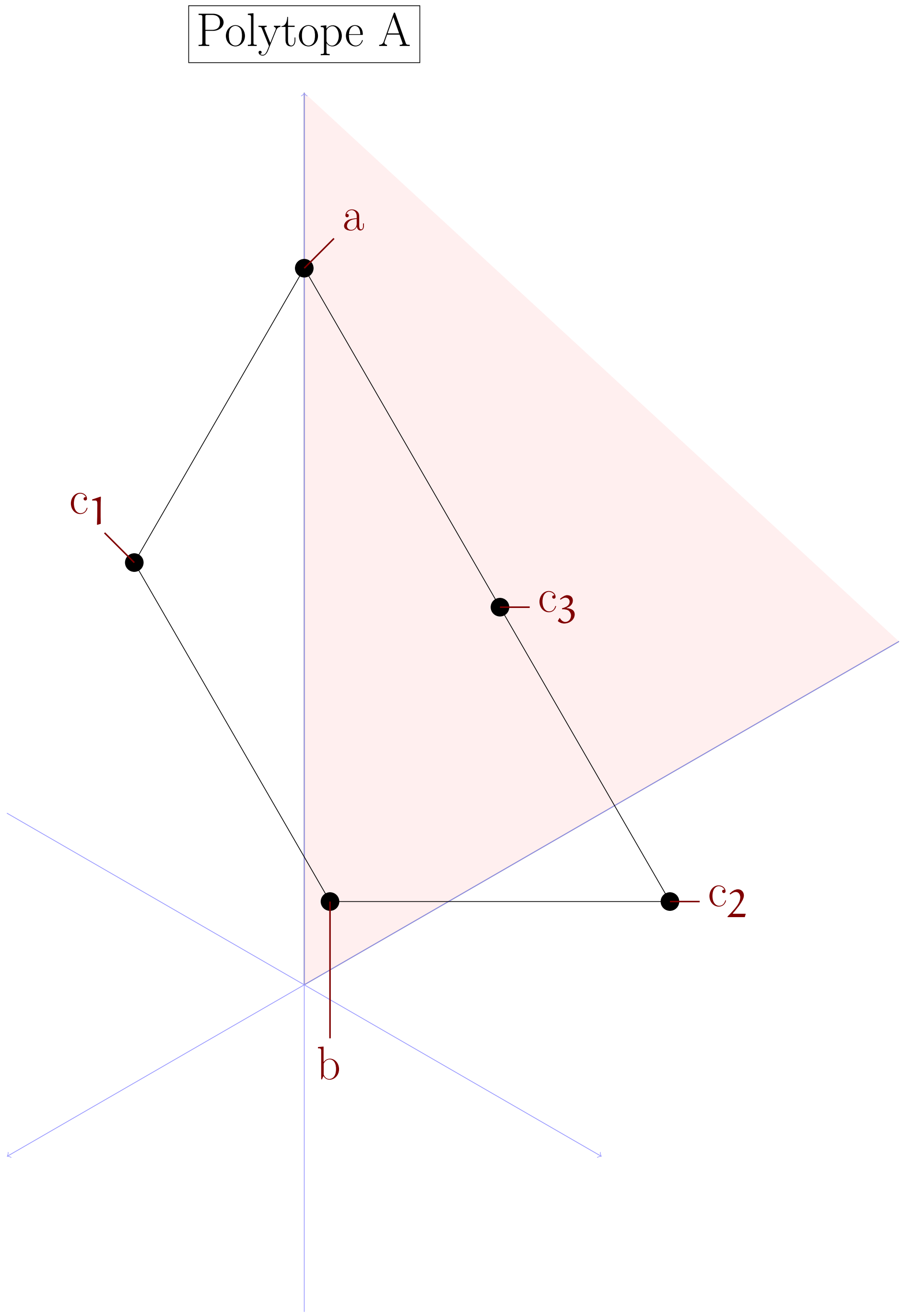


Figure 5.24: We connect the points shown in figure 5.24 to make a convex shape whose edges are each perpendicular to one of the walls of the Weyl chambers.

The edge from point **a** to $\mathbf{c}_1$ is perpendicular to the wall $\lambda_1 = \lambda_3$, the edge from point $\mathbf{c}_1$ to **b** is perpendicular to the wall $\lambda_1 = \lambda_2$, the edge from point **b** to $\mathbf{c}_2$ is perpendicular to the wall $\lambda_2 = \lambda_3$, the edge from point $\mathbf{c}_2$ to $\mathbf{c}_3$ to **a** is perpendicular to the wall $\lambda_1 = \lambda_2$. The actual Momentum Polytope has vertices $\mathbf{c}_1$ and $\mathbf{c}_2$ reflected back into the positive Weyl chamber:

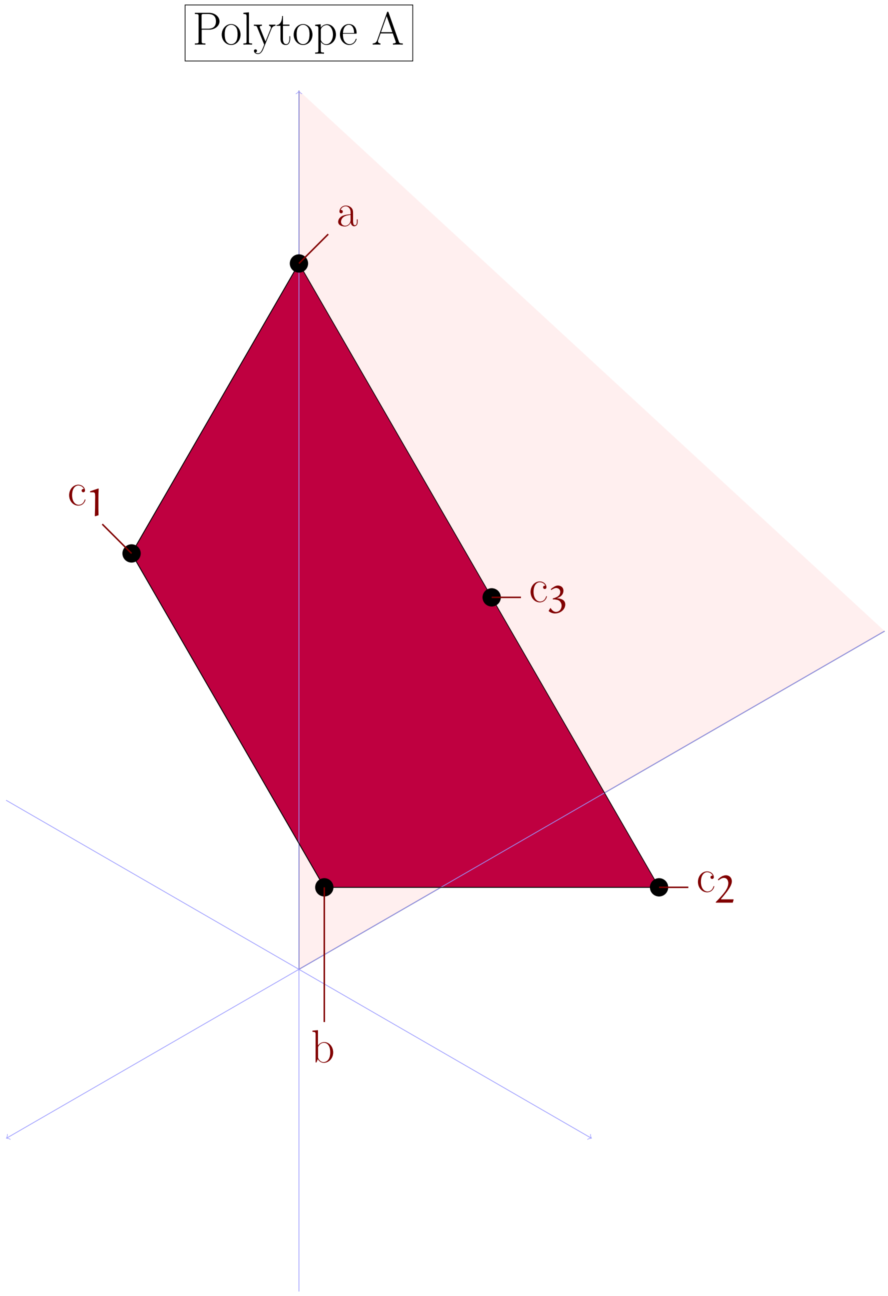


Figure 5.25: The convex shape from the refected points $\mathbf{c}_1$ and $\mathbf{c}_2$ whose edges are each perpendicular to one of the walls of the Weyl chambers.

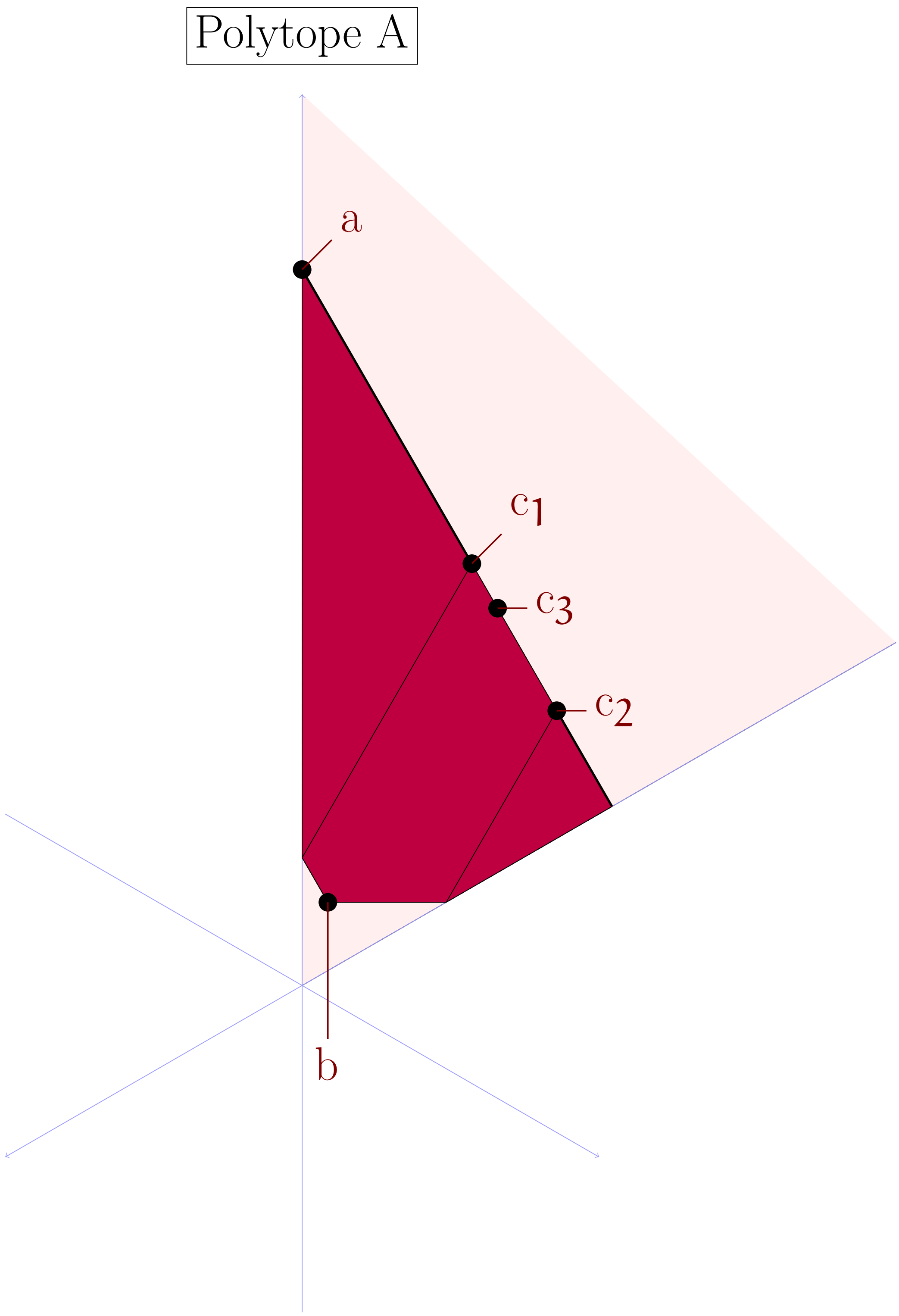


Figure 5.26: The convex shape from the reflected points $\mathbf{c}_1$ and $\mathbf{c}_2$ being reflected back into the positive Weyl chamber whose edges are each perpendicular to one of the walls of the Weyl chambers. This is the Momentum Polytope.

The points of polytope A have been reflected back into the positive Weyl chamber and the resultant Momentum Polytope is shown.

As can be seen in figures 5.25 and 5.26, the internal lines (from the re-reflected point $\mathbf{c}_1$ to the wall $\lambda_2 = \lambda_3$ and from the re-reflected point $\mathbf{c}_2$ to the wall $\lambda_1 = \lambda_2$) show the parts of the Polytope that have been reflected back into the positive Weyl chamber.

The **B** Polytope

The Momentum Polytope defined by the region labelled **B** in figure 5.13 between the boundaries $\Gamma_1 = 0$, $\Gamma_1 = \Gamma_3$ and $\Gamma_2 = \Gamma_1 + \Gamma_3$ on the $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ plane has the general shape,

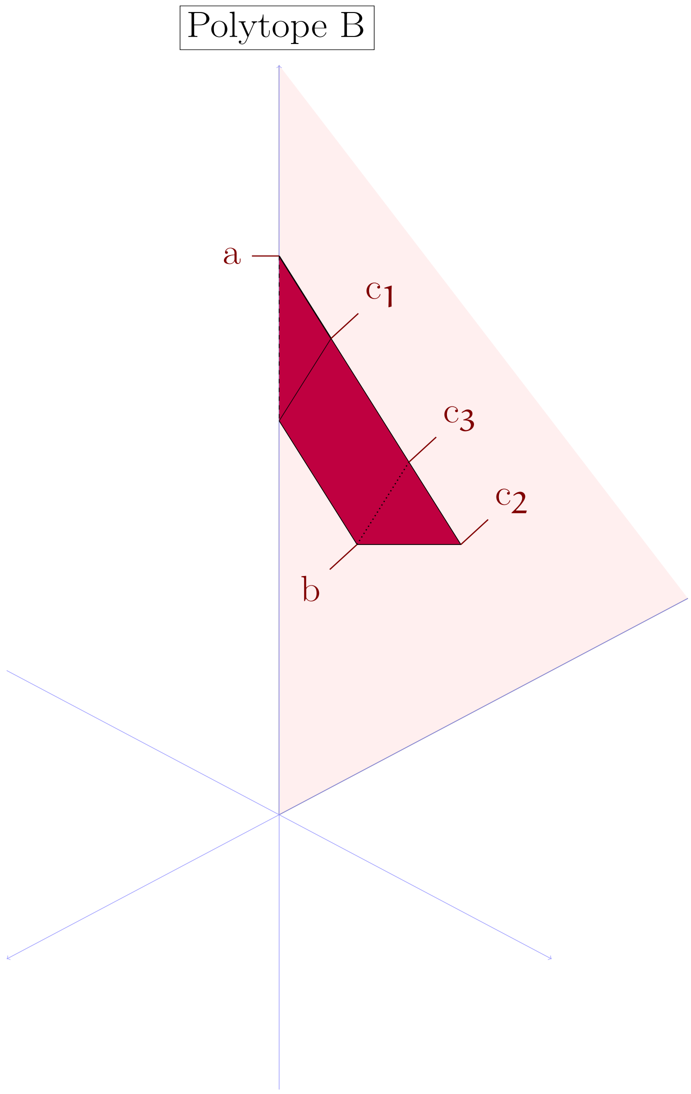


Figure 5.27: The Momentum Polytope for region B from figure 5.13 where point $\mathbf{c}_1$ is a re-reflected vertex.

This Polytope is also different to the example Polytope in figure 5.20: For this polytope to be convex, the point $\mathbf{c}_1$ has been reflected along the $\lambda_2 = \lambda_3$ wall of the positive Weyl chamber to make a convex shape whose edges are each perpendicular to a wall of the Weyl chambers before being re-reflected back in to make the Momentum Polytope for region **B** defined in figure 5.13.

The **C** Polytope

The Momentum Polytope defined by the region labelled **C** in figure 5.13 between the boundaries $\Gamma_1 = 0$, $\Gamma_1 = -\Gamma_3$ and $\Gamma_3 = \Gamma_1 + \Gamma_2$ on the $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ plane has the general shape,

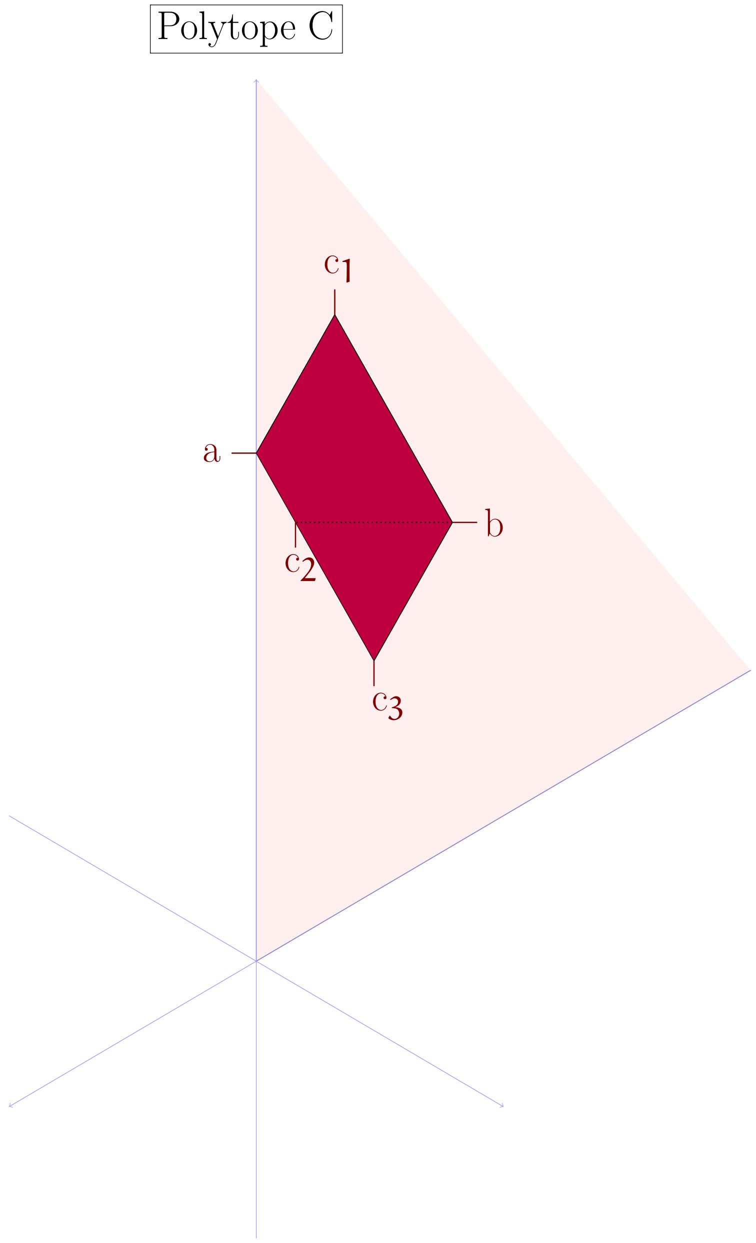


Figure 5.28: The Momentum Polytope for region **C** from figure 5.13

This Polytope is the same as the example Polytope in figure 5.20: it is the Momentum Polytope for region **C** defined in figure 5.13.

The **D** Polytope

The Momentum Polytope defined by the region labelled **D** in figure 5.13 between the boundaries $\Gamma_3 = 0$ and $\Gamma_1 = \Gamma_3$ on the $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ plane has the general shape,

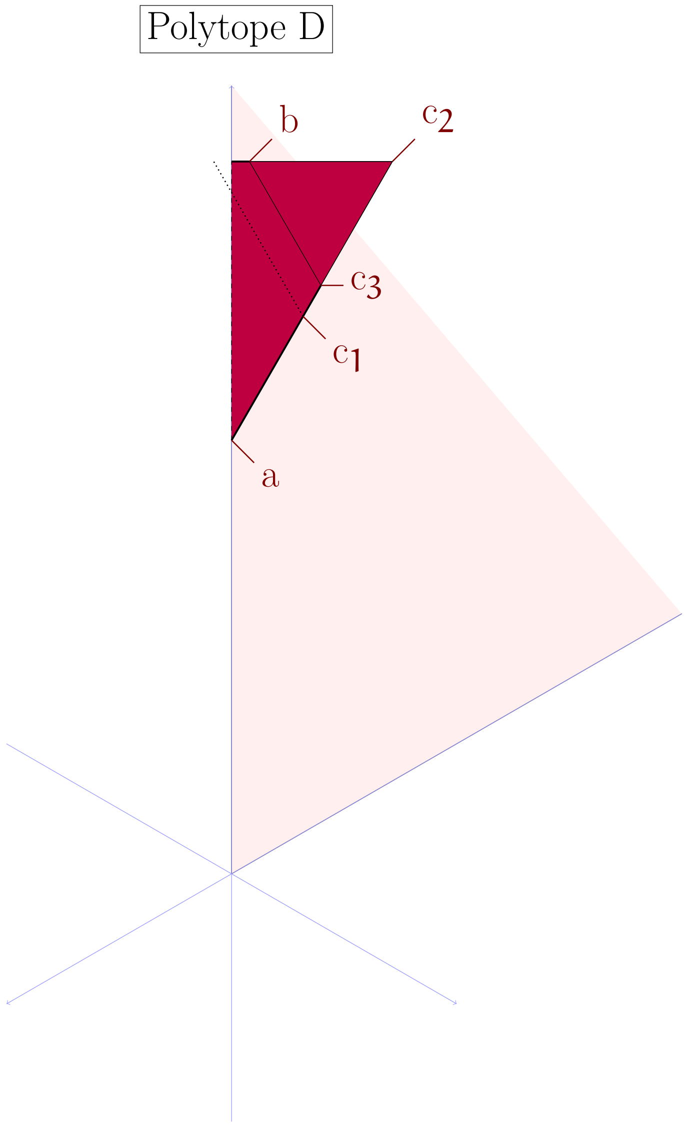


Figure 5.29: The Momentum Polytope for region **D** from figure 5.13 where point $\mathbf{c}_2$ is a re-reflected vertex and the dotted line leading from vertex $\mathbf{c}_1$ is an *internal edge*.

Let us compare the convex shape of the points including the reflected point $\mathbf{c}_2$ with the resultant Momentum Polytope:

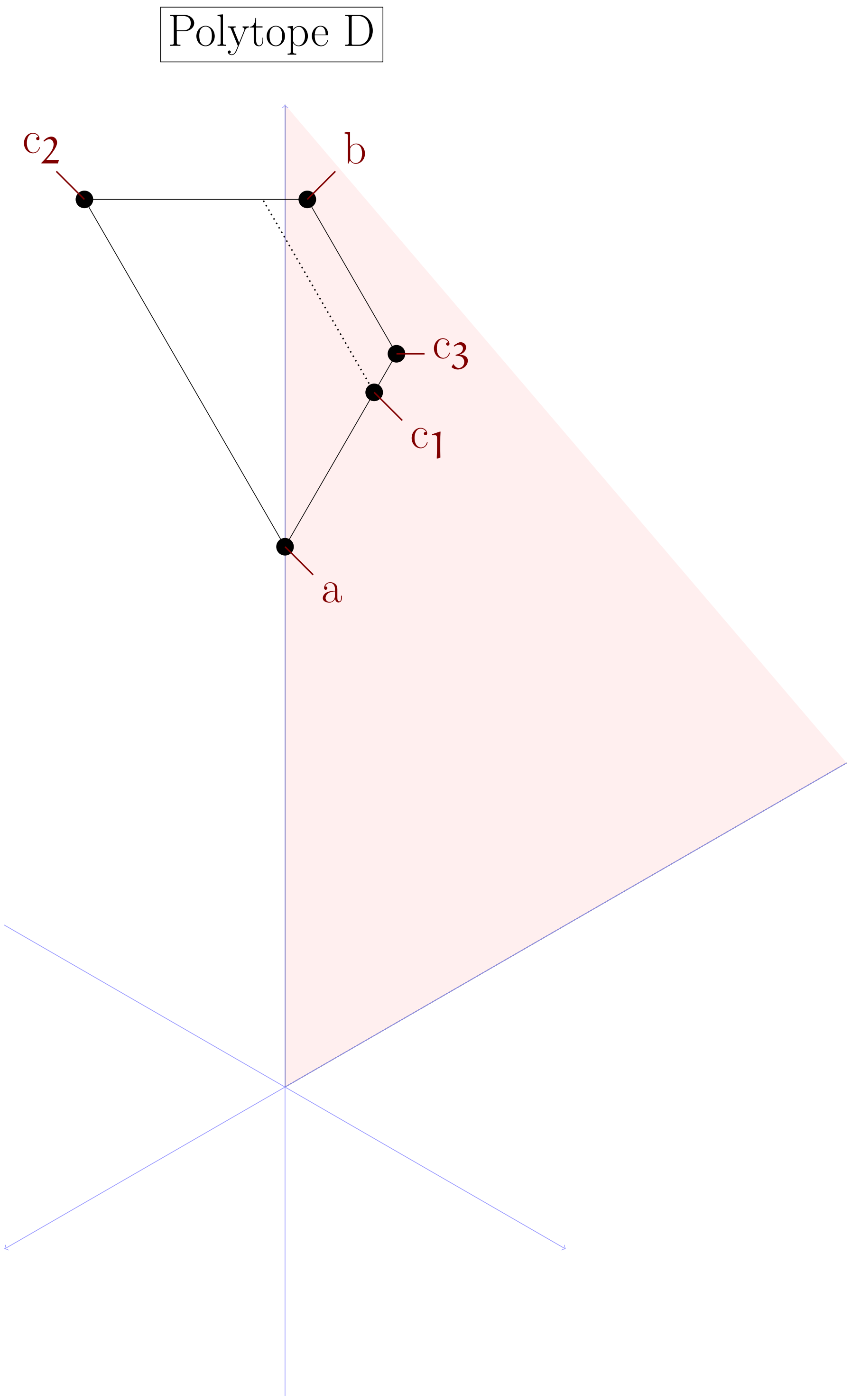


Figure 5.30: The convex shape from the plotted spectra including the refected point $\mathbf{c}_2$ whose edges are each perpendicular to one of the walls of the Weyl chambers. The dotted line from point $\mathbf{c}_1$ to the edge connecting points $\mathbf{c}_2$ and $\mathbf{b}$ is an *internal edge*

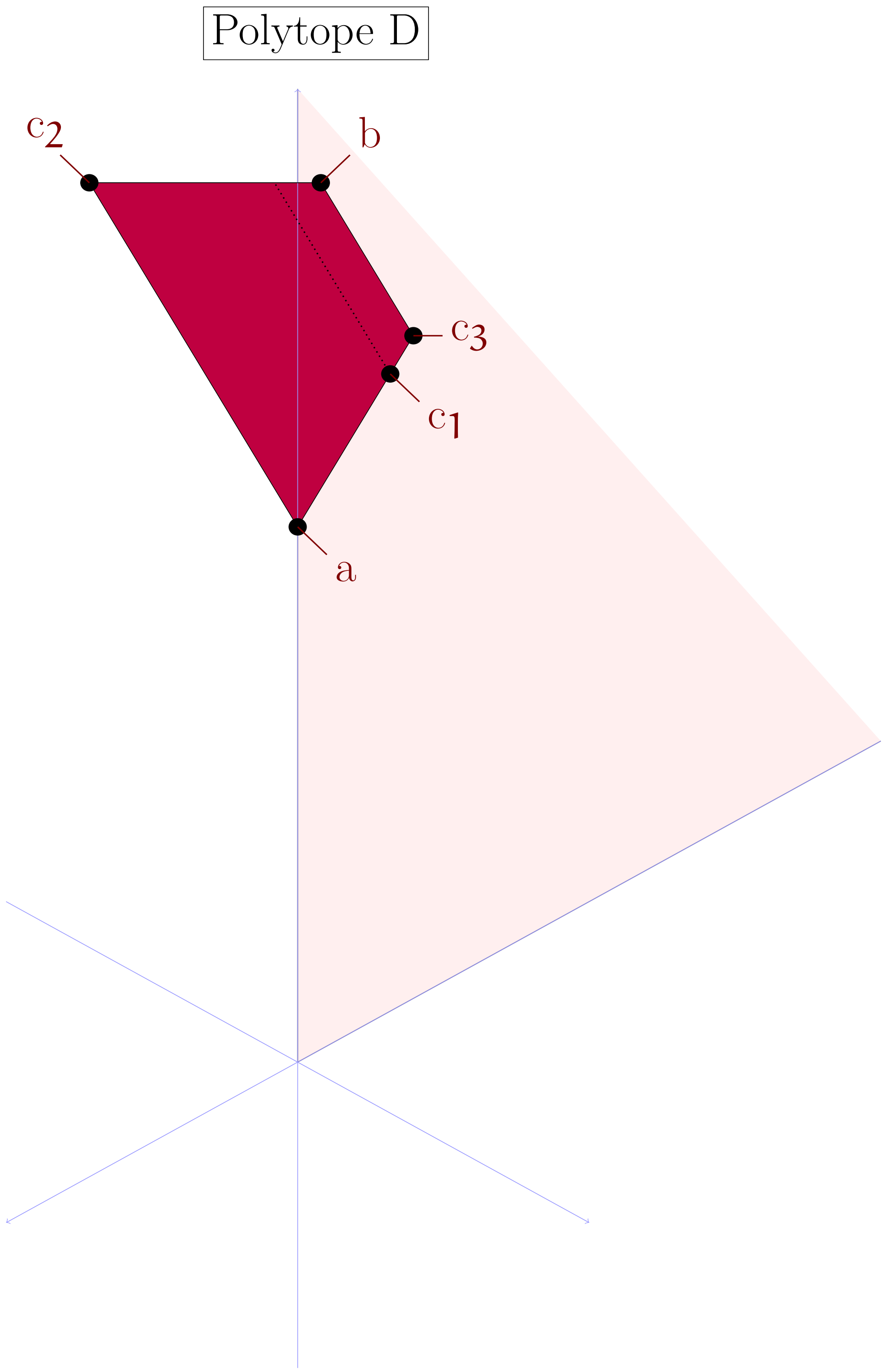


Figure 5.31: The convex shape has been filled so that it is distinguishable.

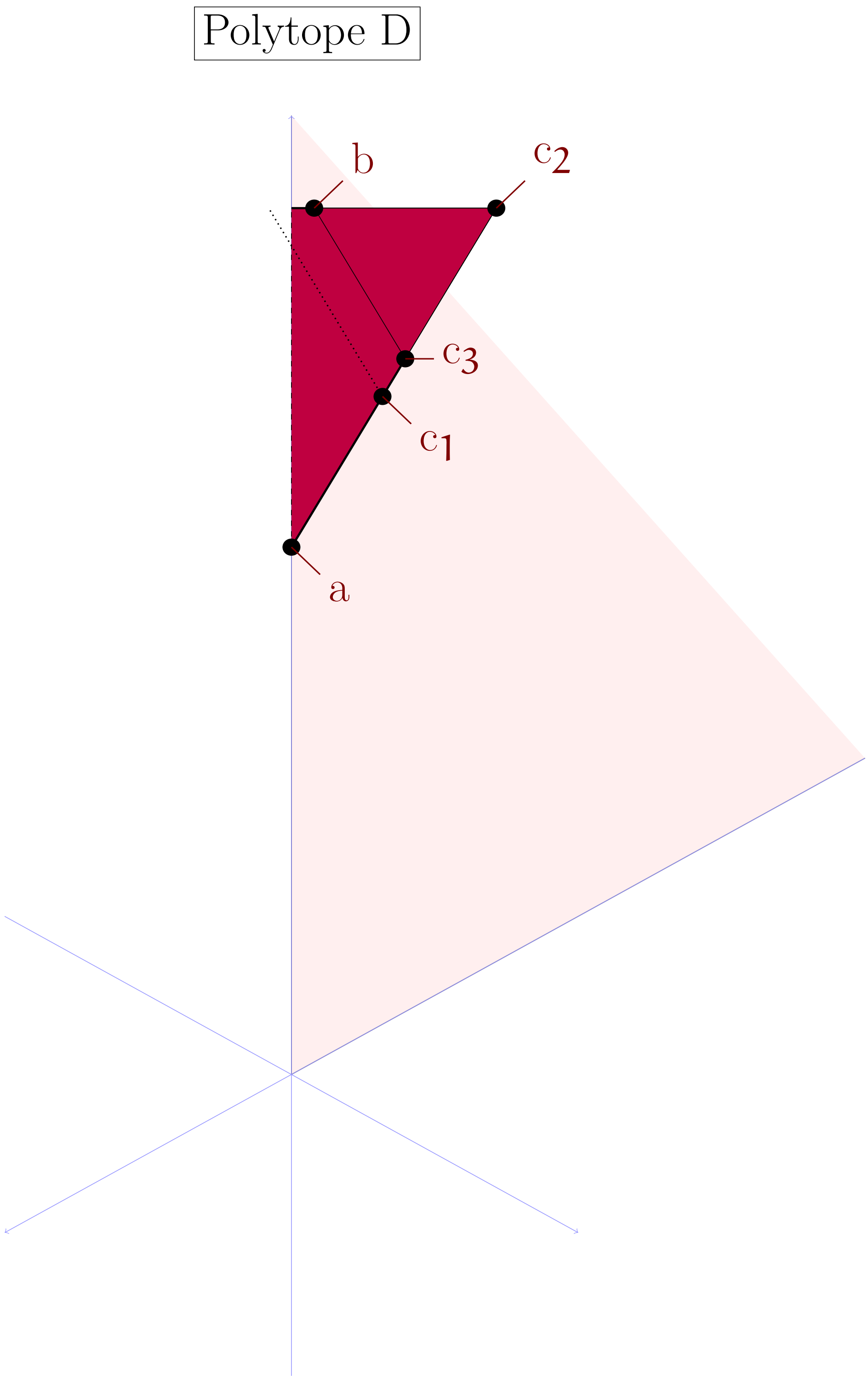


Figure 5.32: The convex shape from the reflected point $\mathbf{c}_2$ being reflected back into the positive Weyl chamber whose edges are each perpendicular to one of the walls of the Weyl chambers and is contained within the positive Weyl chamber. This is the Momentum Polytope.

The points of polytope **D** have been reflected back into the positive Weyl chamber and the resultant Momentum Polytope is shown. This Polytope includes an internal edge: the reduced space along the edge of the Momentum Polytope, the interior of the Momentum POlytope and along the internal edge are all different. This will be discussed in detail in section 7.2.2.

The **E** Polytope

The Momentum Polytope defined by the region labelled **E** in figure 5.13 between the boundaries $\Gamma_3 = 0$, $\Gamma_1 = -\Gamma_3$ and $\Gamma_3 = \Gamma_1 + \Gamma_2$ on the $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ plane has the general shape,

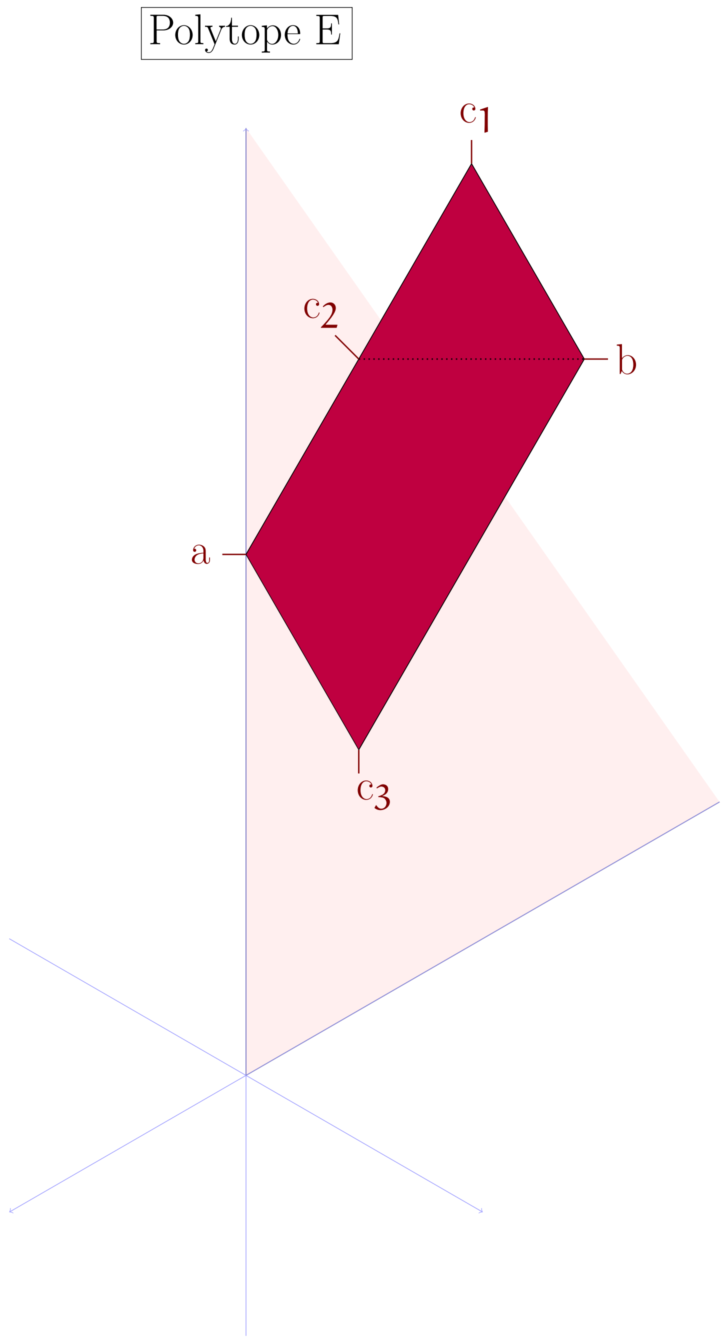


Figure 5.33: The Momentum Polytope for region **E** from figure 5.13

The **F** Polytope

The Momentum Polytope defined by the region labelled **F** in figure 5.13 between the boundaries $\Gamma_1 = -\Gamma_2$, $\Gamma_1 = -\Gamma_3$ and $\Gamma_3 = \Gamma_1 + \Gamma_2$ on the $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ plane has the general shape,

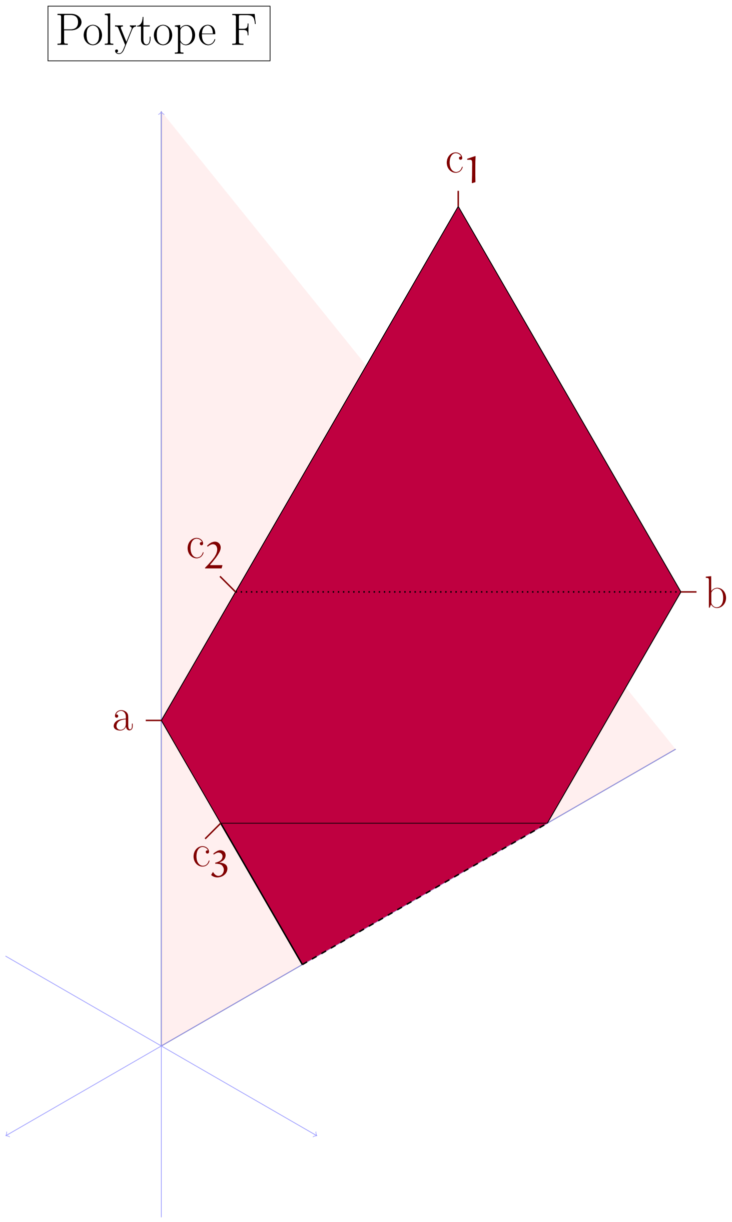


Figure 5.34: The Momentum Polytope for region **F** from figure 5.13: vertex $\mathbf{c}_3$ is reflected back into the positive Weyl chamber along the wall $\lambda_1 = \lambda_2$, and there is an internal edge between vertices $\mathbf{c}_2$ and **b**

The **G** Polytope

The Momentum Polytope defined by the region labelled **G** in figure 5.13 between the boundaries $\Gamma_2 = \Gamma_3$ and $\Gamma_1 = -\Gamma_2$ on the $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ plane has the general shape,

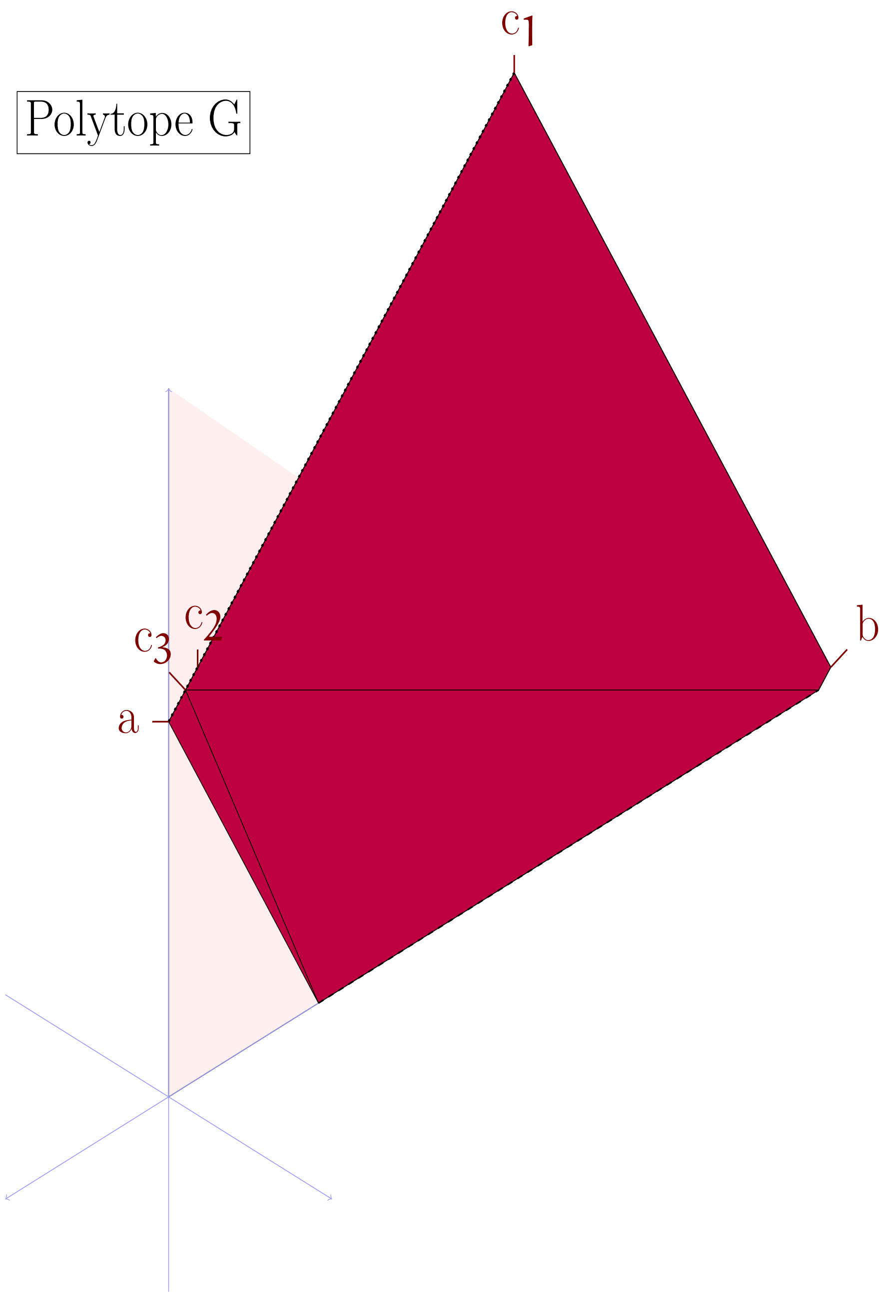


Figure 5.35: The Momentum Polytope for region **G** from figure 5.13: vertex $\mathbf{c}_3$ is reflected back into the positive Weyl chamber along the wall $\lambda_1 = \lambda_2$.

The **H** Polytope

The Momentum Polytope defined by the region labelled **H** in figure 5.13 between the boundaries $\Gamma_2 = \Gamma_3$, $\Gamma_1 = -\Gamma_3$ and $\Gamma_3 = \Gamma_1 + \Gamma_2$ on the $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Delta$ plane has the general shape,

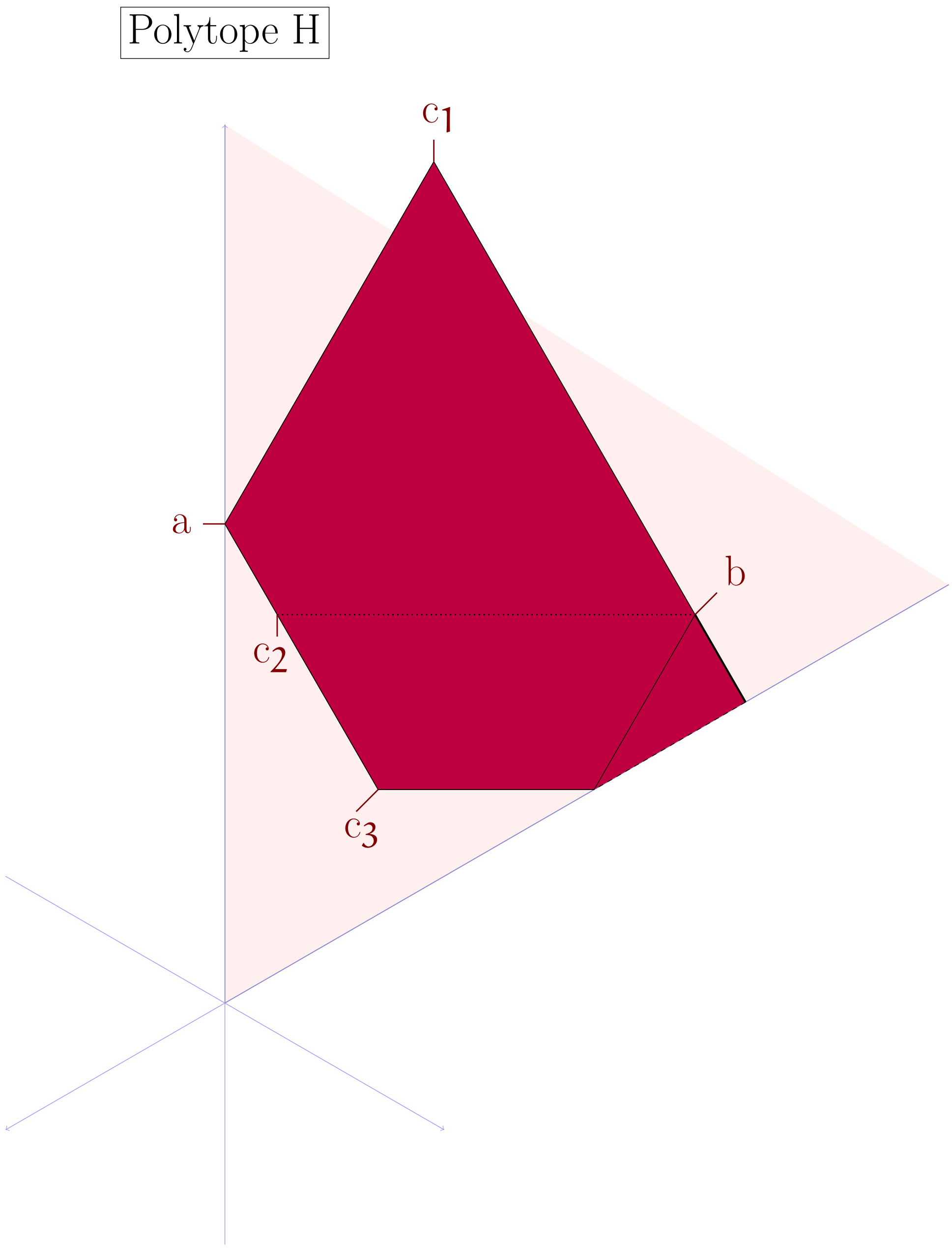


Figure 5.36: The Momentum Polytope for region **F** from figure 5.13: vertex **b** is reflected back into the positive Weyl chamber along the wall $\lambda_1 = \lambda_2$, and there is an internal edge between vertices $\mathbf{c}_2$ and **b**

The **Zero** Polytope

The Momentum Polytope defined by the Γ_i relations $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ which we will refer to as the Zero Polytope has the general shape,

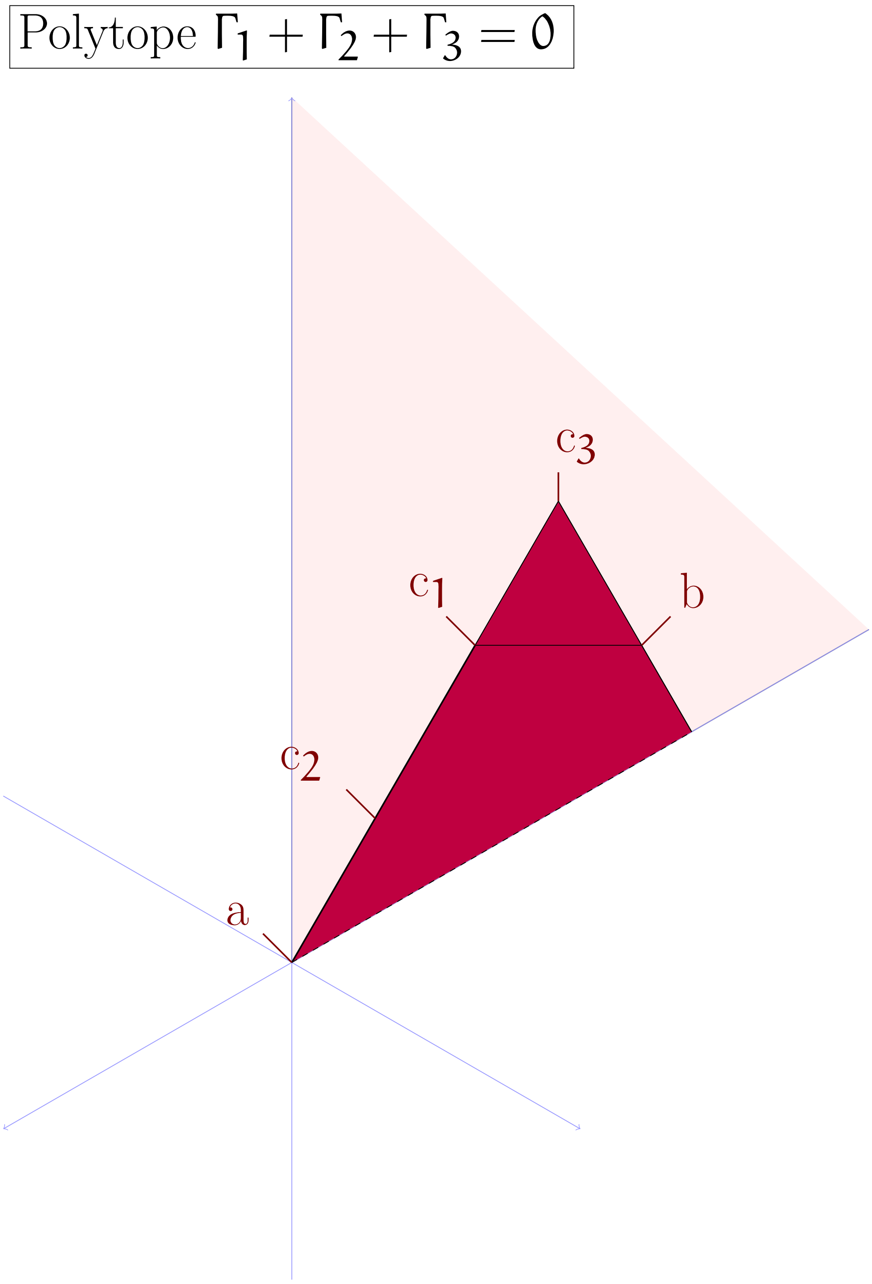


Figure 5.37: The Momentum Polytope for region $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$: vertex $\mathbf{c}_3$ is reflected back into the positive Weyl chamber along the wall $\lambda_1 = \lambda_2$

□

5.7 The Transitional Momentum Polytopes of the $SU(3)$ Action on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$

The different momentum polytopes depend on the respective magnitudes of Γ_1, Γ_2 and Γ_3. However, transitions between one type of relation between the Γ_i's to other Γ_j's take place across lines that have been drawn and labelled in figure 5.13. Therefore the different polytopes in section 5.6 are separated by *transitional polytopes*. For example, between polytope **A** and **B** is the transition polytpoe that corresponds to the relationship $\Gamma_2 = \Gamma_1 + \Gamma_3$. The different transition polytopes can be separated into three different categories. The first type of transitional polytope consists of two rather than one singular momentum value, with each singular momentum value landing on opposite walls of the positive Weyl chamber. The second type of transitional polytope also consists of two rather than one singular momentum value however both singular momentum value land on the same point: two of the vertices coincide at one point on one wall of the positive Weyl chamber. The third type of transitional polytope corresponds to the 5 fixed point sets 'collapsing' into only three (one singular and two regular) sets of momentum values: they look exactly like the polytopes for the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2$. The third transitional polyopes are described by Theorem 5.7.2.

Theorem 5.7.1. *The* transitional *momentum polytopes of the* $SU(3)$ *action on* $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ *fall into three different categories for which* $\Gamma_i - \Gamma_j - \Gamma_k = 0$, $\Gamma_i + \Gamma_j = 0$, $\Gamma_i = 0$ *where* $i, j, k = 1, 2, 3$.

Proof. The First Category of Transition Polytopes

The first category of the transitional polytopes consists of polytopes for which $\Gamma_i = \Gamma_j + \Gamma_k$. This type of transitional polytope consists of two rather than one singular momentum value. In the case for $\Gamma_2 = \Gamma_1 + \Gamma_3$ both the spectrums $\mathbf{a}$=Spectrum$\Big(J([e_1, e_1, e_1])\Big)$ and $\mathbf{c}_2$=Spectrum$\Big(J([e_1, e_2, e_1])\Big)$ have a repeated element: each of these singular momentum values are each a point on the opposite walls of the positive Weyl chamber, respectively:

$$\mathbf{a} = \text{Spectrum}\Big(J([e_1, e_1, e_1])\Big) = \Big[\frac{4(\Gamma_1 + \Gamma_3)}{3}, \frac{-2(\Gamma_1 + \Gamma_3)}{3}, \frac{-2(\Gamma_1 + \Gamma_3)}{3}\Big]$$

$$\begin{aligned}
\mathbf{c}_1 = \text{Spectrum}\Big(J([e_2, e_1, e_1])\Big) &= \Big[\frac{\Gamma_1 + 4\Gamma_3}{3}, \frac{\Gamma_1 - 2\Gamma_3}{3}, \frac{-2(\Gamma_1 + \Gamma_3)}{3}\Big] \\
\mathbf{c}_2 = \text{Spectrum}\Big(J([e_1, e_2, e_1])\Big) &= \Big[\frac{\Gamma_1 + \Gamma_3}{3}, \frac{\Gamma_1 + \Gamma_3}{3}, \frac{-2(\Gamma_1 + \Gamma_3)}{3}\Big] \\
\mathbf{c}_3 = \text{Spectrum}\Big(J([e_1, e_1, e_2])\Big) &= \Big[\frac{4\Gamma_1 + \Gamma_3}{3}, \frac{-2\Gamma_1 + \Gamma_3}{3}, \frac{-2(\Gamma_1 + \Gamma_3)}{3}\Big] \\
\mathbf{b} = \text{Spectrum}\Big(J([e_1, e_2, e_3])\Big) &= \Big[\frac{\Gamma_1 - 2\Gamma_3}{3}, \frac{\Gamma_1 + \Gamma_3}{3}, \frac{-2\Gamma_1 + \Gamma_3}{3}\Big]
\end{aligned}$$

The momentum value

$$\mathbf{a} = \Big[\frac{4(\Gamma_1 + \Gamma_3)}{3}, \frac{-2(\Gamma_1 + \Gamma_3)}{3}, \frac{-2(\Gamma_1 + \Gamma_3)}{3}\Big]$$

is a point on the $\lambda_2 = \lambda_3$ wall of the positive Weyl chamber (afterall $\Gamma_1 + \Gamma_3 > 0$). And the momentum value

$$\mathbf{c}_2 = \Big[\frac{\Gamma_1 + \Gamma_3}{3}, \frac{\Gamma_1 + \Gamma_3}{3}, \frac{-2(\Gamma_1 + \Gamma_3)}{3}\Big]$$

is a point on the $\lambda_1 = \lambda_2$ wall of the Weyl chamber. Therefore for each transition polytope that corresponds to the weight equality $\Gamma_2 = \Gamma_1 + \Gamma_3$, $\mathbf{a}$ lands on the $\lambda_2 = \lambda_3$ wall of the positive Weyl chamber and $\mathbf{c}_2$ lands on the $\lambda_1 = \lambda_2$ wall of the positive Weyl chamber.

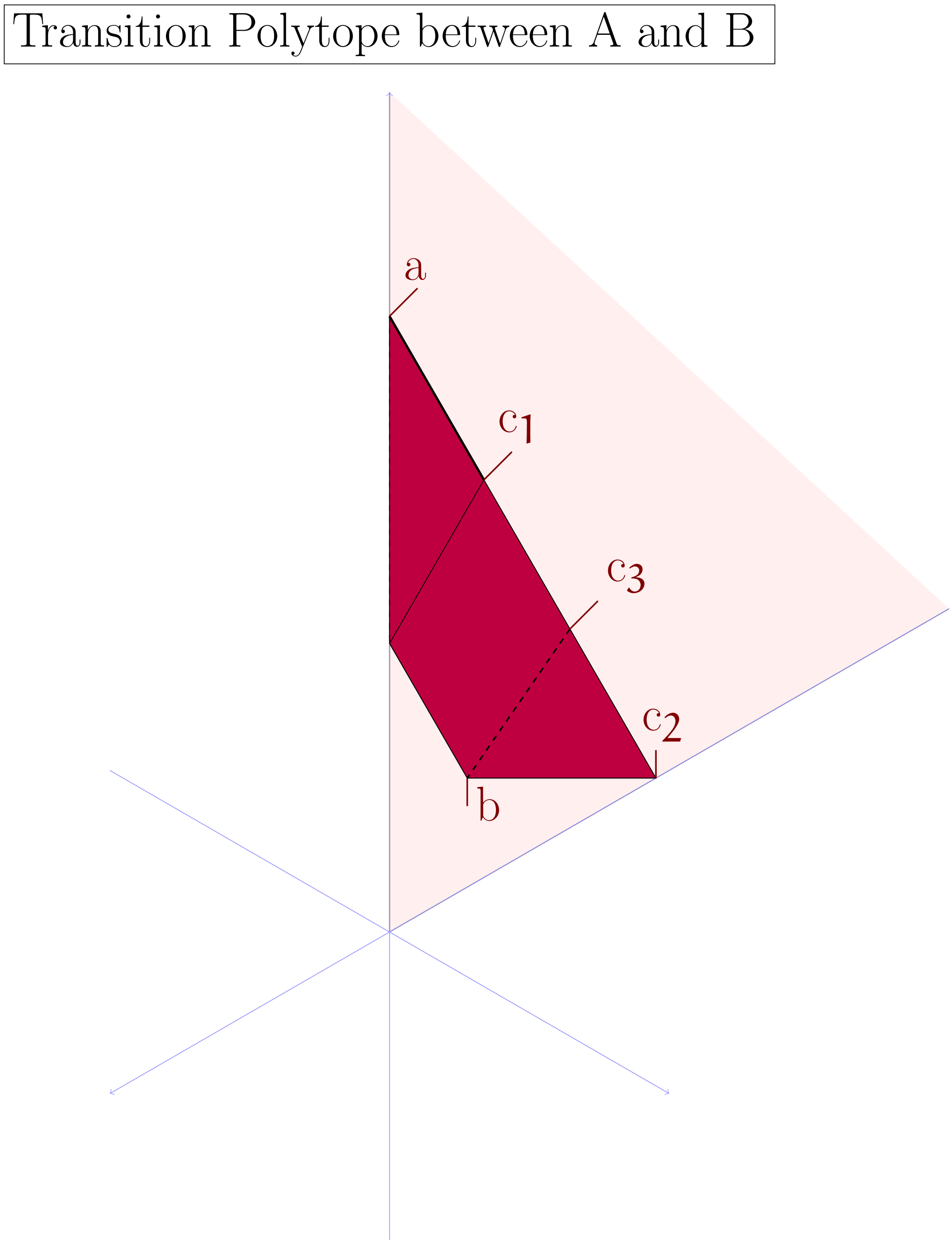


Figure 5.38: The transitional momentum polytope from polytope **A** to **B** that corresponds to $\Gamma_2 = \Gamma_1 + \Gamma_3$. Vertices **a** and $\mathbf{c}_2$ are each singular momentum values. Vertex $\mathbf{c}_1$ is reflected back into the positive Weyl chamber along the wall $\lambda_2 = \lambda_3$. Between vertices **b** and $\mathbf{c}_3$ is an internal edge.

Between regions **C** and **H** is the polytope corresponding to the $\Gamma_3 = \Gamma_1 + \Gamma_2$ that is the transition polytope between polytope **C** and polytope **H**. Between regions **E** and **F** is the polytope corresponding to the $\Gamma_3 = \Gamma_1 + \Gamma_2$, which is the transition polytope between polytope **E** and polytope **F**. In this case for $\Gamma_3 = \Gamma_1 + \Gamma_2$ both the spectrums $\mathbf{a}$=Spectrum$\Big(J([e_1, e_1, e_1])\Big)$ and $\mathbf{c}_3$=Spectrum$\Big(J([e_1, e_1, e_2])\Big)$ have a repeated element and with each singular momentum value landing on opposite walls of the positive Weyl chamber: the singular momentum value

$$\mathbf{a} = \text{Spectrum}\Big(J([e_1, e_1, e_1])\Big) = \Big[\frac{4(\Gamma_1 + \Gamma_2)}{3}, \frac{-2(\Gamma_1 + \Gamma_2)}{3}, \frac{-2(\Gamma_1 + \Gamma_2)}{3}\Big]$$

is a point on the $\lambda_2 = \lambda_3$ wall of the positive Weyl chamber and the other singular momentum value

$$\mathbf{c}_3 = \text{Spectrum}\Big(J([e_1, e_1, e_2])\Big) = \Big[\frac{\Gamma_1 + \Gamma_2}{3}, \frac{\Gamma_1 + \Gamma_2}{3}, \frac{-2(\Gamma_1 + \Gamma_2)}{3}\Big]$$

is a point on the $\lambda_1 = \lambda_2$ wall of the Weyl chamber. Therefore for each polytope that corresponds to the weight equality $\Gamma_3 = \Gamma_1 + \Gamma_2$, $\mathbf{a}$ lands on the $\lambda_2 = \lambda_3$ wall of the positive Weyl chamber and $\mathbf{c}_3$ lands on the $\lambda_1 = \lambda_2$ wall of the positive Weyl chamber.

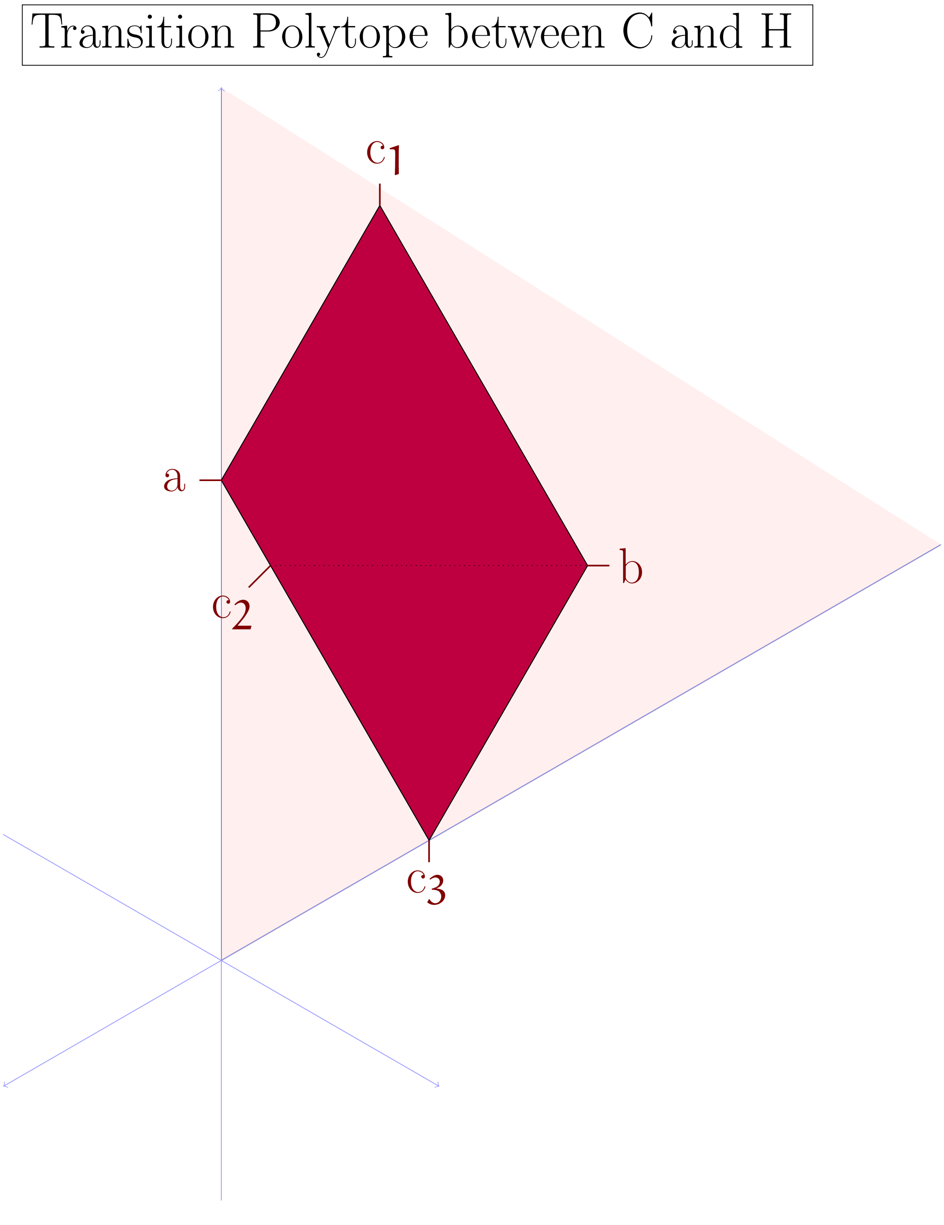


Figure 5.39: The Momentum Polytope for the polytope corresponding to the equality $\Gamma_3 = \Gamma_1 + \Gamma_2$. This is a transition polytope that depicts the transition between polytope **C** and polytope **H**. Vertices **a** and $\mathbf{c}_3$ are singular momentum values and each points on the opposite walls of the positive Weyl chamber. Between vertex **b** and $\mathbf{c}_2$ is an internal edge.

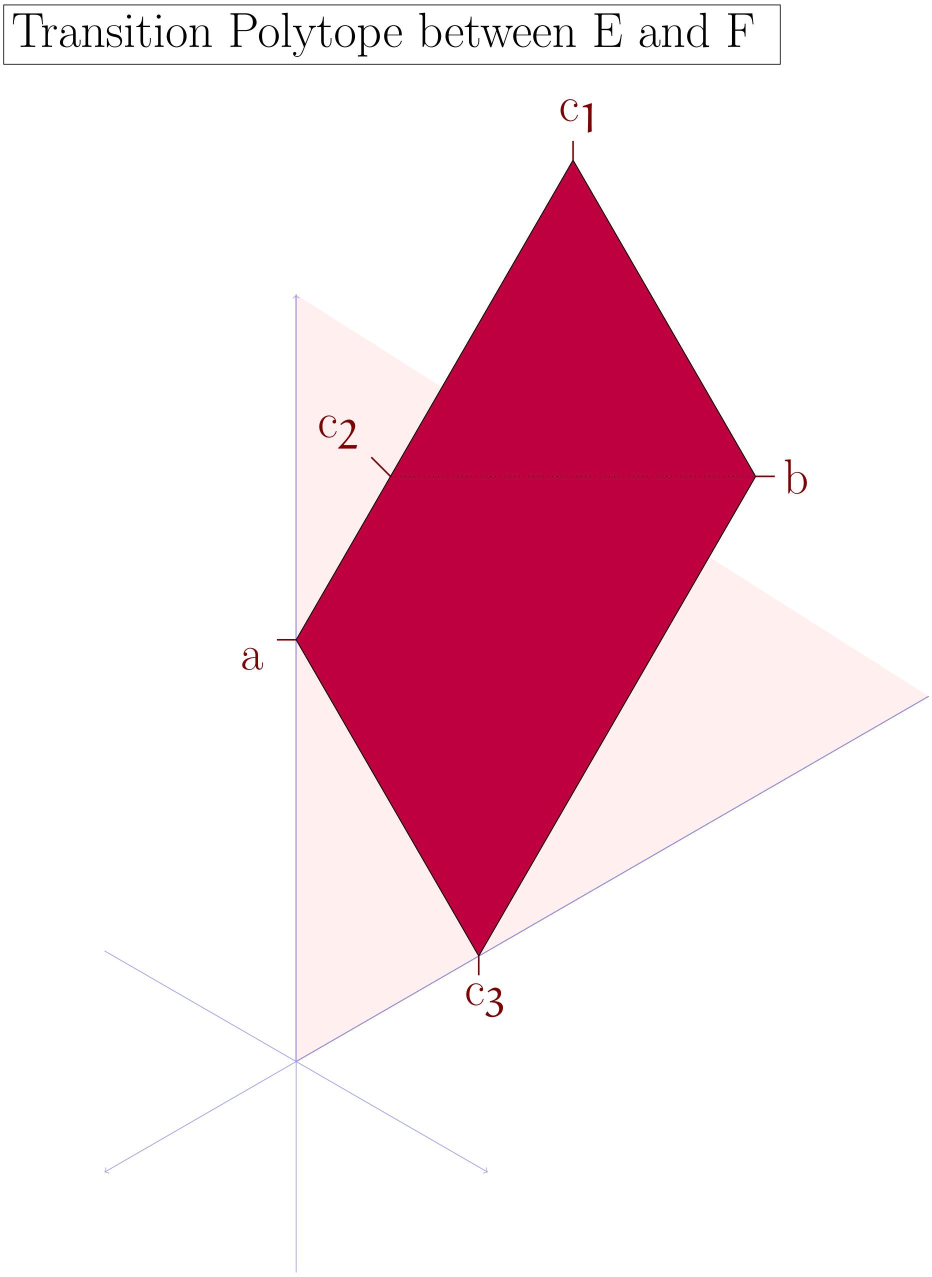


Figure 5.40: The Momentum Polytope for $\Gamma_3 = \Gamma_1 + \Gamma_2$ depicting the transition polytope that depicts the transition between polytope **E** and polytope **F**. Vertices **a** and $\mathbf{c}_3$ are the singular momentum values and each points on the opposite walls of the positive Weyl chamber. Between vertex **b** and $\mathbf{c}_2$ is an internal edge.

The Second Category of Transition Polytopes

The second category of transition polytope corresponds to the equality $\Gamma_i + \Gamma_j = 0$ for i,j=1,2,3. In the quadrant shown in figure 5.13 we have three instances of this case: there are two types of transition polytopes for which $\Gamma_1 = -\Gamma_3$ between regions of polytope **C** and polytope **E** and between the regions of polytope **F** and **H**; the third type is the transition polytope for which $\Gamma_1 = -\Gamma_2$ between regions for polytope **F** and **G**. At $\Gamma_1 = -\Gamma_3$ the momentum values

$$\begin{aligned} \mathbf{a} &= \text{Spectrum}\Big(J([e_1, e_1, e_1])\Big) = \left[\frac{2\Gamma_2}{3}, \frac{-\Gamma_2}{3}, \frac{-\Gamma_2}{3}\right] \\ \mathbf{c}_2 &= \text{Spectrum}\Big(J([e_1, e_2, e_1])\Big) = \left[\frac{-\Gamma_2}{3}, \frac{2\Gamma_2}{3}, \frac{-\Gamma_2}{3}\right] \end{aligned}$$

coincide after permutation

$$\mathbf{a} = \mathbf{c}_2 = \left[\frac{2\Gamma_2}{3}, \frac{-\Gamma_2}{3}, \frac{-\Gamma_2}{3}\right]$$

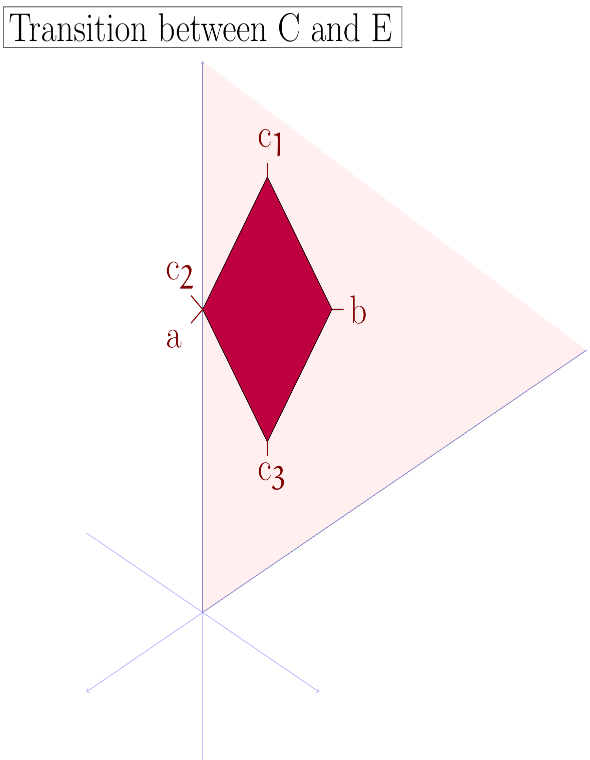


Figure 5.41: Between regions **C** and **E** in figure 5.13 is the polytope with weights $\Gamma_1 = -\Gamma_3$ that is the Transition Polytope between polytope **C** and polytope **E**. Vertices **a** and $\mathbf{c}_2$ coincide on the $\lambda_2 = \lambda_3$ wall of the Weyl chamber.

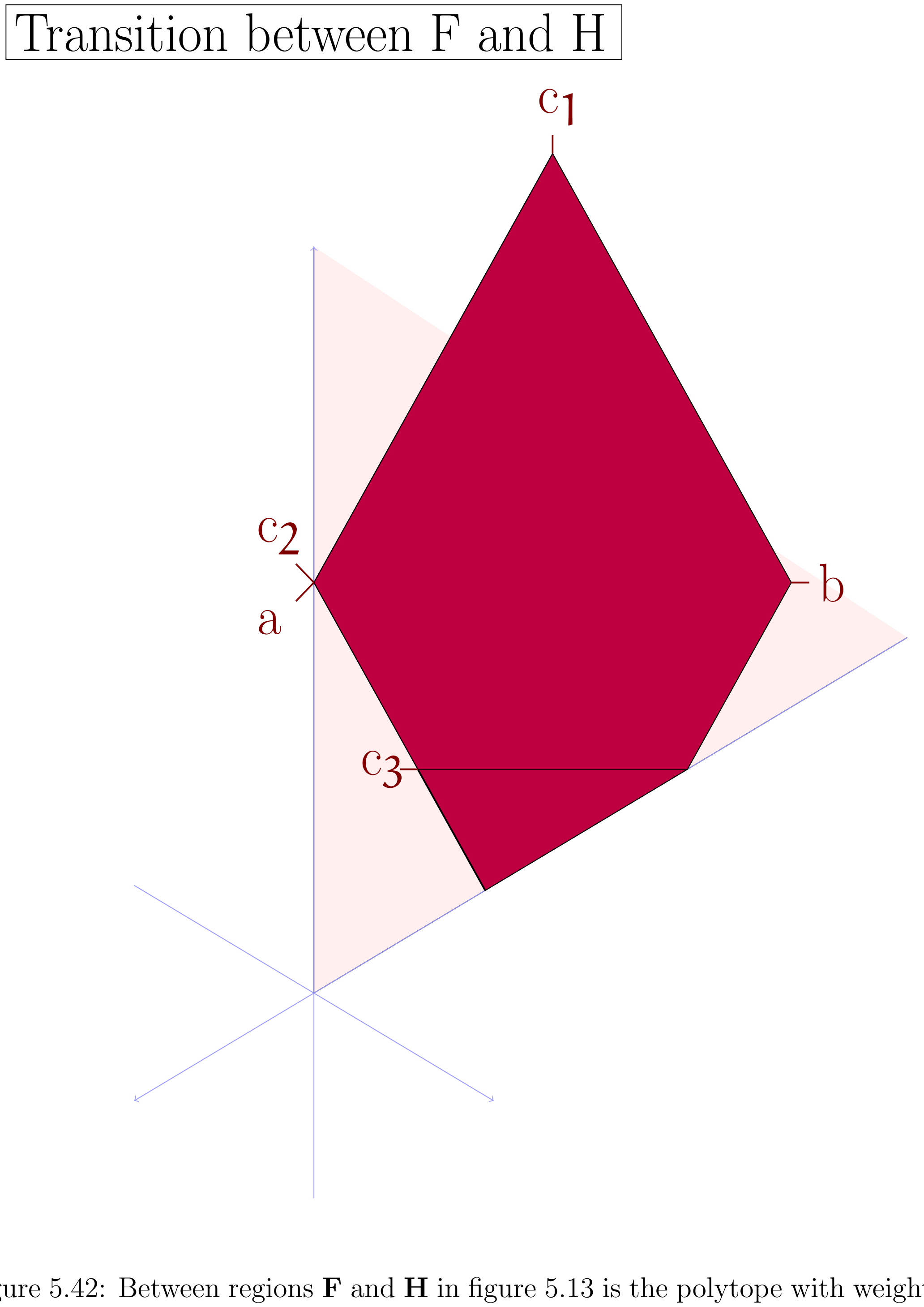


Figure 5.42: Between regions **F** and **H** in figure 5.13 is the polytope with weights $\Gamma_1 = -\Gamma_3$ that is the Transition Polytope between polytope **F** and polytope **H**. Vertices **a** and $\mathbf{c}_3$ coincide on the $\lambda_2 = \lambda_3$ wall of the Weyl chamber. Vertex $\mathbf{c}_3$ is a vertex reflected along the $\lambda_1 = \lambda_2$ wall of the positive Weyl chamber.

However at $\Gamma_1 = -\Gamma_2$ the momentum values

$$\begin{aligned} \mathbf{a} &= \text{Spectrum}\Big(J([e_1, e_1, e_1])\Big) = \left[\frac{2\Gamma_3}{3}, \frac{-\Gamma_3}{3}, \frac{-\Gamma_3}{3}\right] \\ \mathbf{c}_3 &= \text{Spectrum}\Big(J([e_1, e_2, e_1])\Big) = \left[\frac{-\Gamma_3}{3}, \frac{2\Gamma_3}{3}, \frac{-\Gamma_3}{3}\right] \end{aligned}$$

coincide after permutation

$$\mathbf{a} = \mathbf{c}_3 = \left[\frac{2\Gamma_3}{3}, \frac{-\Gamma_3}{3}, \frac{-\Gamma_3}{3}\right]$$

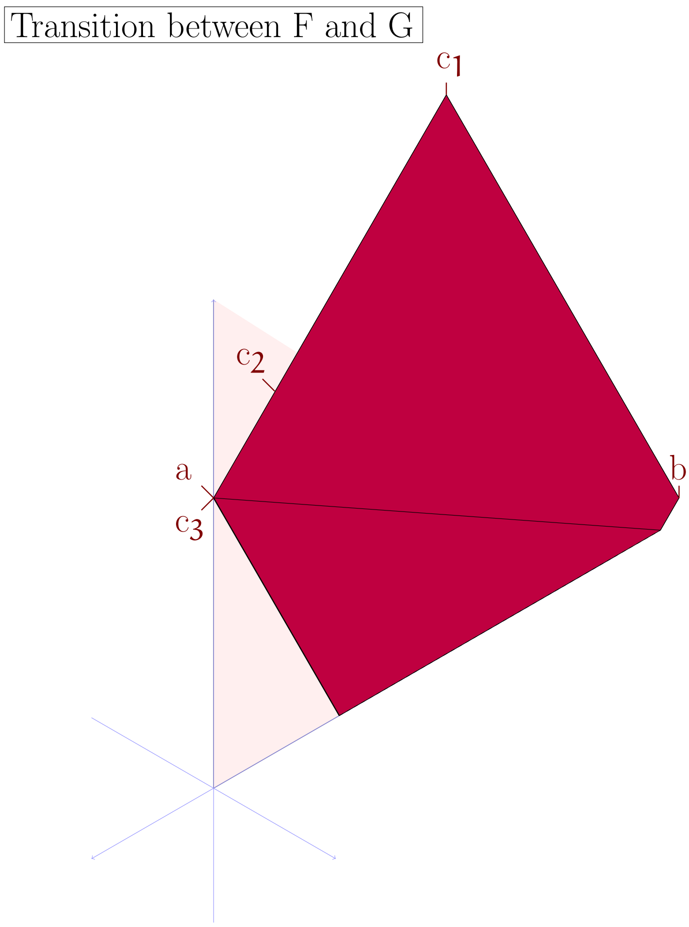


Figure 5.43: Between regions **F** and **G** in figure 5.13 is the polytope with weights $\Gamma_1 = -\Gamma_3$ that is the Transition Polytope between polytope **F** and polytope **G**. Vertices **a** and $\mathbf{c}_2$ coincide on the $\lambda_2 = \lambda_3$ wall of the Weyl chamber. Vertex $\mathbf{c}_2$ has also been reflected along the $\lambda_1 = \lambda_2$ wall of the positive Weyl chamber.

The Third Category of Transition Polytopes

For the action with weighting $\Gamma_i = 0$ the image of the momentum map is equal to that of the $SU(3)$ action on a double copy of $\mathbb{C}P^2$ and the polytopes are those described in theorem 5.3. We will construct a theorem to descibe, not the momentum map of the action with $\Gamma_i = 0$, but for which the momentum map of the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ would match that of the momentum map of the $SU(3)$ action on $\mathbb{C}P^2 \times \mathbb{C}P^2$ which we name the *restricted momentum map*:

Theorem 5.7.2. *For Lie group* G *acting on manifold* (M, ω) *with* $J : M \to \mathfrak{g}^*$*. Let* X *be an invariant symplectic submanifold of* M*. The momentum map restricted to this submanifold is* $J|_X = J_X : X \to \mathfrak{g}^*$ *where for* $x \in X$ $J_X(x) = J(x)$*. Therefore* J_X *is the momentum map for* G *acting on* X*.*

Proof. The symplectic form is:

$$\langle dJ_X(x), \xi \rangle = \omega(\xi_X(x), \nu) \tag{5.8}$$

therefore $\xi_X(x) \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$. The structure of ω is pre-symplectic. For momentum map

$$J(z_1, z_2, z_3) = \sum_j \Gamma_j z_j \otimes \bar{z}_j - \frac{1}{3}(\sum_j \Gamma_j) I \tag{5.9}$$

if we choose the invariant symplectic submanifold of $\mathbb{C}P^2$ to be $X_1 = \{(z_1, z_2, z_3) \in (\mathbb{C}P^2)^3 \mid z_i = 0\}$ which is $SU(3)$ invariant then

$$\begin{aligned} J_{X_1}(z_2, z_3) &= J(z_1, z_2, z_3) && (5.10) \\ &= \Gamma_2 z_2 \otimes \bar{z}_2 + \Gamma_3 z_3 \otimes \bar{z}_3 - \frac{1}{3}(\sum_j \Gamma_j) I && (5.11) \end{aligned}$$

So

$$(X_1, \omega) = (\mathbb{C}P^2, \Gamma_2 \omega_0) \times (\mathbb{C}P^2, \Gamma_3 \omega_0) \tag{5.12}$$

Another example is $X_{23} = \{(z_1, z_2, z_3) \in (\mathbb{C}P^2)^3 \mid z_2 = z_3\}$ which is symplectic provided $\Gamma_2 + \Gamma_3 \neq 0$ and $SU(3)$ invariant. Then

$$\begin{aligned} J_{X_{23}}(z_1, z_2) &= J(z_1, z_2, z_3) && (5.13) \\ &= \Gamma_1 z_1 \otimes \bar{z}_1 + (\Gamma_2 + \Gamma_3) z_2 \otimes \bar{z}_2 - \frac{1}{3}(\sum_j \Gamma_j) I && (5.14) \end{aligned}$$

So $(X_{23}, \omega) = (\mathbb{C}P^2, \Gamma_1\omega_0) \times (\mathbb{C}P^2, (\Gamma_2 + \Gamma_3)\omega_0)$. □

□

Chapter 6

The Definite and Indefinite Vertices of the Momentum Polytopes of the $SU(3)$ Action on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ and the Reduced Space at the Definite Vertex

Globally the gradient of the edges of the momentum polytopes are determined by the bifurcation lemma and the Weyl group reflections at the walls of the Weyl chamber, as shown in previous sections. However the direction of the edge of a polytope from its vertex can be described more locally by the quadratic momentum map on the symplectic slice, N_1.

Theorem 6.0.3. *A* definite vertex *of the momentum polytope corresponds to specific values for* $\mu = J(x)$ *and provides definite values for the quadratic momentum map on the symplectic slice, and the reduced space is isomorphic to the 2-sphere,*

$$J_{N_1}^{-1}(\mu)/\mathbb{T}^2 \simeq S^2. \tag{6.1}$$

for those values of μ. *An indefinite vertex does not produce definite values for the momentum map on the symplectic slice.*

Proof. If we take the second differential of the momentum map,

$$\begin{aligned} J(z) &= \Gamma\Big(\frac{1}{|z|^2} z \otimes \bar{z} - \frac{1}{3} I\Big) \\ \ddot{J}(z) &= \Gamma\Big(\frac{1}{|z|^2} z \otimes \bar{z} - \frac{1}{3} I\Big)^{\cdot\cdot} \\ &= \Gamma \frac{1}{|z|^2}\Big(\ddot{z} \otimes \bar{z} + 2\dot{z} \otimes \dot{\bar{z}} + z \otimes \ddot{\bar{z}}\Big) \end{aligned}$$

For normalised z, $|z| = 1$ therefore,

$$\ddot{J}(z) = \Gamma\Big(\ddot{z} \otimes \bar{z} + 2\dot{z} \otimes \dot{\bar{z}} + z \otimes \ddot{\bar{z}}\Big) \tag{6.2}$$

For example $\ddot{J}(p)$ at $p = [p^1 : p^2 : p^3] = e_1 = [1 : 0 : 0]$ and $\dot{p} = [0 : q : r]$

$$\ddot{J}_{[1:0:0]} = \Gamma \begin{pmatrix} \ddot{p}^1 + \ddot{\bar{p}}^1 & \ddot{\bar{p}}^2 & \ddot{\bar{p}}^3 \\ \ddot{p}^2 & 2|q|^2 & 2q\bar{r} \\ \ddot{p}^3 & 2\bar{q}r & 2|r|^2 \end{pmatrix} \tag{6.3}$$

Now for $J(z_1, z_2, z_3)$ where $z_1 = e_2 = [0 : 1 : 0]$ and $z_2 = z_3 = e_1$ and $z_i = [z_i^1 : z_i^2 : z_i^3]$ their differentials are $\dot{z}_1 = [u_1, 0, v_1]$ and $\dot{z}_i = [0, u_i, v_i]$ for $i = 2, 3$

$$\ddot{J}(z_1) = \Gamma_1 \begin{pmatrix} 2|u_1|^2 & \ddot{z}_1^1 & 2u_1\bar{v}_1 \\ \ddot{\bar{z}}_1^1 & \ddot{z}_1^2 + \ddot{\bar{z}}_1^2 & \ddot{\bar{z}}_1^3 \\ 2\bar{u}_1 v_1 & \ddot{z}_1^3 & 2|v_1|^2 \end{pmatrix} \tag{6.4}$$

Similarly,

$$\ddot{J}(z_i) = \Gamma_i \begin{pmatrix} \ddot{z}_i^1 + \ddot{\bar{z}}_i^1 & \ddot{\bar{z}}_i^2 & \ddot{\bar{z}}_i^3 \\ \ddot{z}_i^2 & 2|u_i|^2 & 2u_i\bar{v}_i \\ \ddot{z}_i^3 & 2\bar{u}_i v_i & 2|v_i|^2 \end{pmatrix} \quad \text{for } i = 2, 3 \tag{6.5}$$

Let's consider the projection $\pi : \mathfrak{su}(3) \to \mathfrak{t}^2$ for $\mathfrak{t}^2 \subset \mathfrak{su}(3)$, in other words a map from an $\mathfrak{su}(3)$ matrix to its diagonal elements, and apply it to the momentum map. Therefore

$$\begin{aligned} \pi \circ \ddot{J}(e_2, e_1, e_1) = \mathrm{Diag}[\ddot{J}(e_2, e_1, e_1)] = [&2\Gamma_1|u_1|^2 + \Gamma_2(\ddot{z}_2^1 + \ddot{\bar{z}}_2^1) + \Gamma_3(\ddot{z}_3^1 + \ddot{\bar{z}}_3^1), \\ &\Gamma_1(\ddot{z}_1^2 + \ddot{\bar{z}}_1^2) + 2\Gamma_2|u_2|^2 + 2\Gamma_3|u_3|^2, \\ &2\Gamma_1|v_1|^2 + 2\Gamma_2|v_2|^2 + 2\Gamma_3|v_3|^2] \end{aligned} \tag{6.6}$$

Let's recall (section 2.3.2) that the Hopf map maps a $(2n+1)$-sphere to $\mathbb{C}P^n$, therefore $\mathbb{C}P^2$ coordinates satisfy the spherical equation,

$$
\begin{aligned}
|z_1^1|^2 + |z_1^2|^2 + |z_1^3|^2 &= 1 \\
|z_2^1|^2 + |z_2^2|^2 + |z_2^3|^2 &= 1 \\
|z_3^1|^2 + |z_3^2|^2 + |z_3^3|^2 &= 1
\end{aligned}
$$

which when solved for $z_1 = e_2 = [0 : 1 : 0]$ and $z_2 = z_3 = e_1$ and $\dot{z}_1 = [u_1, 0, v_1]$ and $\dot{z}_i = [0, u_i, v_i]$ for $i = 2, 3$ give,

$$
\begin{aligned}
\ddot{z}_1^2 + \ddot{\bar{z}}_1^2 &= -2|u_1|^2 - 2|v_1|^2 \\
\ddot{z}_2^1 + \ddot{\bar{z}}_2^1 &= -2|u_2|^2 - 2|v_2|^2 \\
\ddot{z}_3^1 + \ddot{\bar{z}}_3^1 &= -2|u_3|^2 - 2|v_3|^2
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathrm{Diag}[\ddot{J}(e_2, e_1, e_1)] = 2[&\Gamma_1|u_1|^2 - \Gamma_2(|u_2|^2 + |v_2|^2) - \Gamma_3(|u_3|^2 + |v_3|^2), \\
&-\Gamma_1(|u_1|^2 + |v_1|^2) + \Gamma_2|u_2|^2 + \Gamma_3|u_3|^2, \\
&\Gamma_1|v_1|^2 + \Gamma_2|v_2|^2 + \Gamma_3|v_3|^2]. \qquad (6.7)
\end{aligned}
$$

Lemma 6.0.4. *The restriction of the quadratic form* $d^2h_\xi(p)$*, where* p *is a relative equilibria and* $\xi \in \mathfrak{g}$ *is a group velocity, to the kernel equations of the first derivative of the momentum map is a well-defined quadratic form on the symplectic slice* N_1*.*

Now the momentum map on the symplectic slice N_1, J_{N_1}, is the second differential of the momentum map restricted to Kernel(DJ).

To find Kernel(DJ) we must first calculate DJ:

$$\dot{J}(z_i) = \Gamma_i(\dot{z}_i \otimes \bar{z}_i + z_i \otimes \dot{\bar{z}}_i) \qquad (6.8)$$

and

$$\dot{J}(z_1, z_2, z_3) = \begin{pmatrix} 0 & \Gamma_1 u_1 + \Gamma_2 \bar{u}_2 + \Gamma_3 \bar{u}_3 & \Gamma_2 \bar{v}_2 + \Gamma_3 \bar{v}_3 \\ \Gamma_1 \bar{u}_1 + \Gamma_2 u_2 + \Gamma_3 u_3 & 0 & \Gamma_1 \bar{v}_1 \\ \Gamma_2 v_2 + \Gamma_3 v_3 & \Gamma_1 v_1 & 0 \end{pmatrix} \qquad (6.9)$$

therefore

$$\text{the Kernel of } dJ(z_1, z_2, z_3) = \begin{cases} \Gamma_1\bar{u}_1 + \Gamma_2 u_2 + \Gamma_3 u_3 = 0 \\ \Gamma_2 v_2 + \Gamma_3 v_3 = 0 \\ \Gamma_1 v_1 = 0 \end{cases}$$

so that

$$\begin{aligned} |u_1|^2 &= \left(\frac{\Gamma_2}{\Gamma_1}\right)^2 |u_2|^2 + \frac{\Gamma_2\Gamma_3}{\Gamma_1^2}(\bar{u}_2 u_3 + u_2\bar{u}_3) + \left(\frac{\Gamma_3}{\Gamma_1}\right)^2 |u_3|^2 \\ |v_3|^2 &= \left(\frac{\Gamma_2}{\Gamma_3}\right)^2 |v_2|^2 \\ v_1 &= 0 \end{aligned}$$

If we substitute this into (6.7) and $u_2 = x_2 + iy_2$ and $u_3 = x_3 + iy_3$ we have that Diag $[\ddot{J}(e_2, e_1, e_1)]\Big|_{\text{Ker } DJ} = J_{N_1} =$

$$\begin{aligned} = \Big[(x_2^2 + y_2^2)\Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big) + (x_3^2 + y_3^2)\Big(\frac{\Gamma_3^2}{\Gamma_1} - \Gamma_3\Big) + \frac{\Gamma_2\Gamma_3}{\Gamma_1}(2x_2x_3 + 2y_2y_3) - |v_2|^2\Big(\frac{\Gamma_2^2}{\Gamma_3} + \Gamma_2\Big), \\ -(x_2^2 + y_2^2)\Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big) - (x_3^2 + y_3^2)\Big(\frac{\Gamma_3^2}{\Gamma_1} - \Gamma_3\Big) - \frac{\Gamma_2\Gamma_3}{\Gamma_1}(2x_2x_3 + 2y_2y_3), \\ |v_2|^2\Big(\frac{\Gamma_2^2}{\Gamma_3} + \Gamma_2\Big)\Big] \end{aligned}$$

If we complete the square in the fashion,

$$\Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big)x_2^2 + 2\frac{\Gamma_2\Gamma_3}{\Gamma_1}x_2x_3 + \Big(\frac{\Gamma_3^2}{\Gamma_1} - \Gamma_3\Big)x_3^2 = \Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big)(x_2 + \frac{\Gamma_3}{\Gamma_2 - \Gamma_1}x_3)^2 + \frac{\Gamma_2\Gamma_3}{\Gamma_1 - \Gamma_2}x_3^2$$

similarly

$$\Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big)y_2^2 + 2\frac{\Gamma_2\Gamma_3}{\Gamma_1}y_2y_3 + \Big(\frac{\Gamma_3^2}{\Gamma_1} - \Gamma_3\Big)y_3^2 = \Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big)(y_2 + \frac{\Gamma_3}{\Gamma_2 - \Gamma_1}y_3)^2 + \frac{\Gamma_2\Gamma_3}{\Gamma_1 - \Gamma_2}y_3^2$$

Let's rewrite $x_2' = x_2 + \frac{\Gamma_3}{\Gamma_2 - \Gamma_1}x_3$ and $y_2' = y_2 + \frac{\Gamma_3}{\Gamma_2 - \Gamma_1}y_3$ therefore

$$\begin{aligned} J_{N_1} = \Big[\Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big)(x_2'^2 + y_2'^2) + \frac{\Gamma_2\Gamma_3}{\Gamma_1 - \Gamma_2}(x_3^2 + y_3^2) - |v_2|^2\Big(\frac{\Gamma_2^2}{\Gamma_3} + \Gamma_2\Big), \\ -\Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big)(x_2'^2 + y_2'^2) - \frac{\Gamma_2\Gamma_3}{\Gamma_1 - \Gamma_2}(x_3^2 + y_3^2), \\ -|v_2|^2\Big(-\frac{\Gamma_2^2}{\Gamma_3} - \Gamma_2\Big)\Big] \end{aligned}$$

Therefore the first term of the diagonal is

$$\Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big)(x_2'^2 + y_2'^2) + \frac{\Gamma_2\Gamma_3}{\Gamma_1 - \Gamma_2}(x_3^2 + y_3^2) - |\nu_2|^2\Big(\frac{\Gamma_2^2}{\Gamma_3} + \Gamma_2\Big), \tag{6.10}$$

the second term of the diagonal is

$$-\Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big)(x_2'^2 + y_2'^2) - \frac{\Gamma_2\Gamma_3}{\Gamma_1 - \Gamma_2}(x_3^2 + y_3^2), \tag{6.11}$$

and the third term of the diagonal is

$$-|\nu_2|^2\Big(-\frac{\Gamma_2^2}{\Gamma_3} - \Gamma_2\Big). \tag{6.12}$$

For the second term of the diagonal, equation (6.11), the coefficients of the quadratic terms $\Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big)$ and $\frac{\Gamma_2\Gamma_3}{\Gamma_1-\Gamma_2}$ as shown in (6.11) must either all be positive definite or all be negative definite for this to describe the 3-sphere. The third term of the diagonal, (6.12), describes the 1-sphere.
Collectively the second and third terms of the diagonal describe $S^3 \times S^1$ if and only if all of the coefficients in (6.11) and (6.12) are all positive definite or all negative definite. In other words the coefficients $\Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big)$ and $\frac{\Gamma_2\Gamma_3}{\Gamma_1-\Gamma_2}$ as shown in (6.11) and the coefficient $-\Big(\frac{\Gamma_2^2}{\Gamma_3} + \Gamma_2\Big)$ as shown in (6.12) must either all be positive definite for the particular Γ_i or must all be negative definite for the particular Γ_i for this diagonal to describe $S^3 \times S^1$. Therefore,

$$J_{N_1}^{-1}(\mu)/\mathbb{T}^2 \simeq S^3 \times S^1/S^1 \times S^1 \simeq S^2 \tag{6.13}$$

where the values of μ correspond to the definite vertices of the momentum polytopes. Otherwise if these coefficients don't satisfy these conditions i.e. they are neither all positive definite nor negative definite, then they are collectively indefinite.
Therefore one must compare these coefficients $\Big(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\Big)$, $\frac{\Gamma_2\Gamma_3}{\Gamma_1-\Gamma_2}$ and $-\Big(\frac{\Gamma_2^2}{\Gamma_3} + \Gamma_2\Big)$ for each choice of Γ_i as distinguished for each polytope. □

Here we will demonstrate the above conclusions concerning the definiteness of each vertex and what that means in the formation of the momentum polytopes using an example momentum polytope.

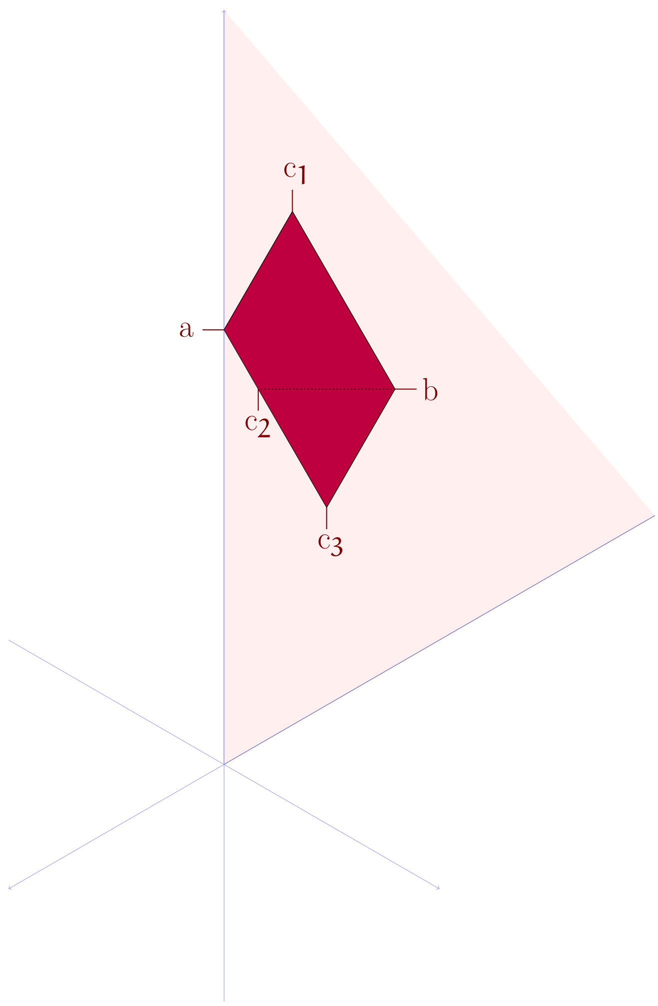


Figure 6.1: An example from the nine different categories of mometum polytopes for which $\Gamma_1 = -5.5$, $\Gamma_2 = 27.5$ and $\Gamma_3 = 8.25$.

The polytope in figure 6.1 has been constructed with $\Gamma_1 = -5.5$, $\Gamma_2 = 27.5$ and $\Gamma_3 = 8.25$

Equation (6.10) is the hessian for the $\mathbf{c}_1$ vertex. And indeed for these values of Γ_i:

$$\left(\frac{\Gamma_2^2}{\Gamma_1} - \Gamma_2\right) = -165, \quad \frac{\Gamma_2\Gamma_3}{\Gamma_1 - \Gamma_2} = -6.875 \quad \text{and} \quad -\left(\frac{\Gamma_2^2}{\Gamma_3} + \Gamma_2\right) = -119.1667 \tag{6.14}$$

these are all negative, so it is negative definite. Therefore as can be seen in figure 6.1 $\mathbf{c}_1$ is a negative *definite vertex*. The weights leading from this vertex are only in the negative root directions: this is in reference to the α_1, α_2, α_3, $-\alpha_1$, $-\alpha_2$

and $-\alpha_3$ root directions as introduced in section 2.1:

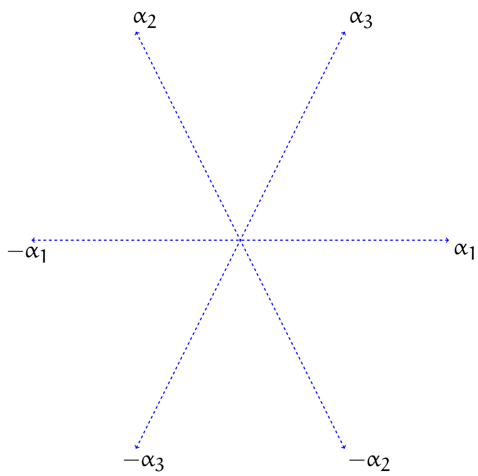


Figure 6.2: The α_1, α_2, α_3, $-\alpha_1$, $-\alpha_2$ and $-\alpha_3$ root directions as introduced in section 2.1.

This means that for vertex $\mathbf{c}_1$ the weights leading from this vertex travel along the $-\alpha_1$, $-\alpha_2$ and $-\alpha_3$ root directions.

If we repeat all of the above calculations however for vertex $\mathbf{c}_2$, the coefficients are indefinite: they are neither all positive or all negative but a mix of both for the Γ_i specified for this example polytope. Indeed in figure 6.1 the vertex $\mathbf{c}_2$ lies along the edge that joins vertex $\mathbf{a}$ to vertex $\mathbf{c}_3$ and indeed it is not a definite vertex but what we will name an *indefinite vertex*. The weights leading from the $\mathbf{c}_2$ vertex point travel along the α_2, $-\alpha_2$ and α_1 root directions.

Again if we repeat all of the above calculations however for vertex $\mathbf{c}_3$, the coefficients are again different: they are positive definite. Therefore in figure 6.1 the vertex $\mathbf{c}_3$ is a positive *definite vertex*. The weights leading from this vertex travel in the positive root directions. In figure 6.1 the weights leading from the $\mathbf{c}_3$ vertex travel along the α_2 and α_3 root directions.

For example, let us plot the five different spectrum values of the polytope in figure 6.1:

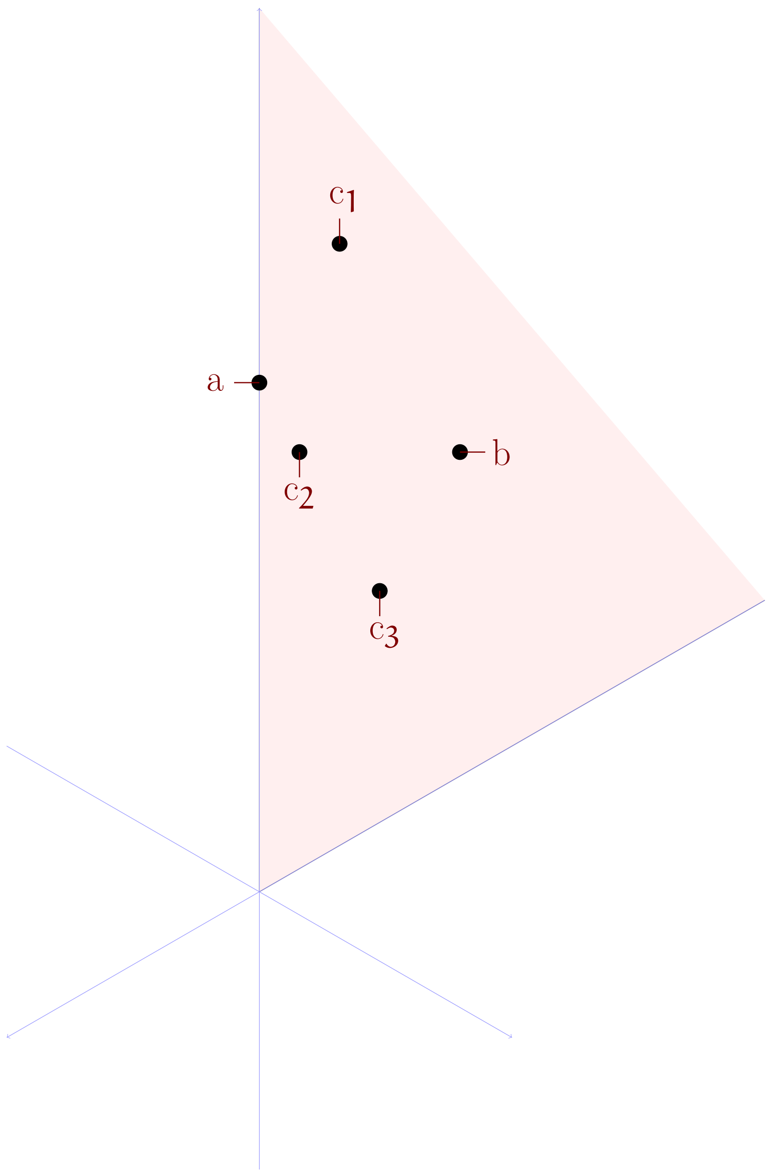


Figure 6.3: The spectrum values **a**, $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, **b** for the momentum polytope in figure 6.1 with $\Gamma_1 = -5.5$, $\Gamma_2 = 27.5$ and $\Gamma_3 = 8.25$.

As we've established, points **a**, **b** and $\mathbf{c}_2$ are indefinite(the weights leading from these points are only in any of the root directions α_i and $-\alpha_i$), however $\mathbf{c}_1$ is negative definite (the weights leading from $\mathbf{c}_1$ are only in the negative root directions $-\alpha_1$, $-\alpha_2$ and $-\alpha_3$) and $\mathbf{c}_3$ is positive definite (the weights leading from $\mathbf{c}_3$ are only in the positive root directions α_1, α_2 and α_3). Let us include these root directions as calculated from each of the spectrum points:

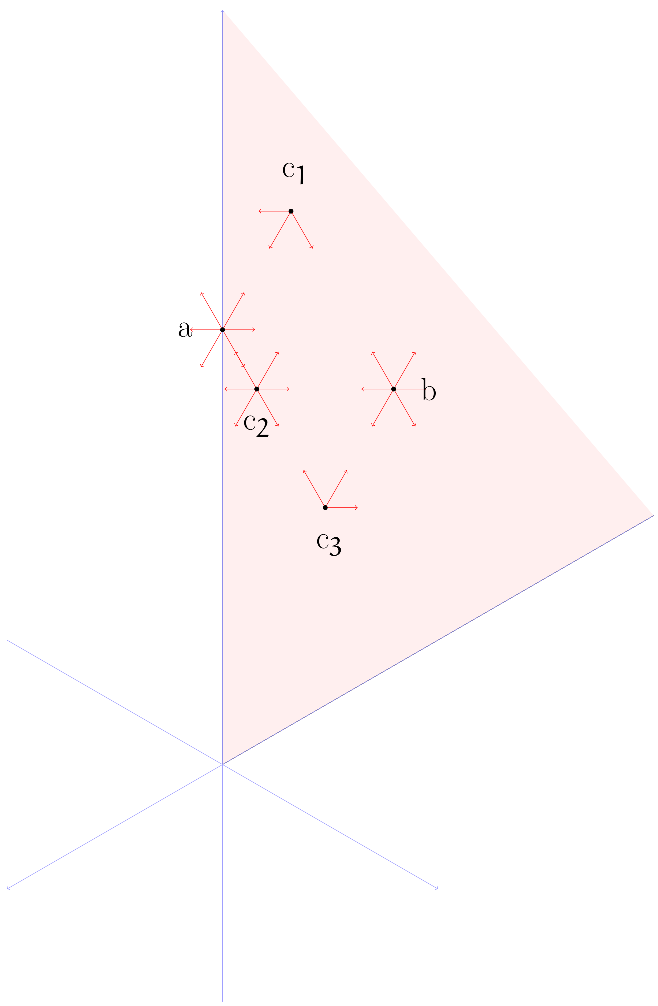


Figure 6.4: The positive and negative definite and indefinite certices $\mathbf{a}$, $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, $\mathbf{b}$ for the momentum polytope in figure 6.1 with $\Gamma_1 = -5.5$, $\Gamma_2 = 27.5$ and $\Gamma_3 = 8.25$ and their corresponding root directions according to calculations.

Following these root directions from each spectrum point, this is the construction of potential edges leading from the vertices of the polytope. When two potential edges leading from seperate vertices, but in the same and opposite directions, meet this is when a potential edge becomes an actual edge of the polytope. For example, let us consider vertices $\mathbf{c}_1$ and $\mathbf{b}$. As explained above, vertex $\mathbf{c}_1$ is a negative definite vertex, therefore the roots travel from it in the $-\alpha_1$, $-\alpha_2$ and $-\alpha_3$ directions. This means that the edges of the momentum polytope connecting vertex $\mathbf{c}_1$ to the rest of the spectrum points will be in the directions $-\alpha_1$, $-\alpha_2$ or $-\alpha_3$. The indefinite vertex $\mathbf{b}$ has roots travelling from it in all the directions pertained to an A2 type Lie algebra. And this means that the edges of the momentum polytope connecting vertex **b** to the rest of the spectrum points will be in any of the α_i and $-\alpha_i$ root directions.

As we can see from figure 6.4 the potential edge leading from vertex $\mathbf{c}_1$ in the $-\alpha_2$ direction will coincide with only one other potential edge: the potential edge leading from vertex **b** in the α_2 direction. These potential edges will meet and merge creating a clear-cut edge to the polytope leading from definite vertex $\mathbf{c}_1$ to indefinite vertex **b**. As shown in figure 6.5

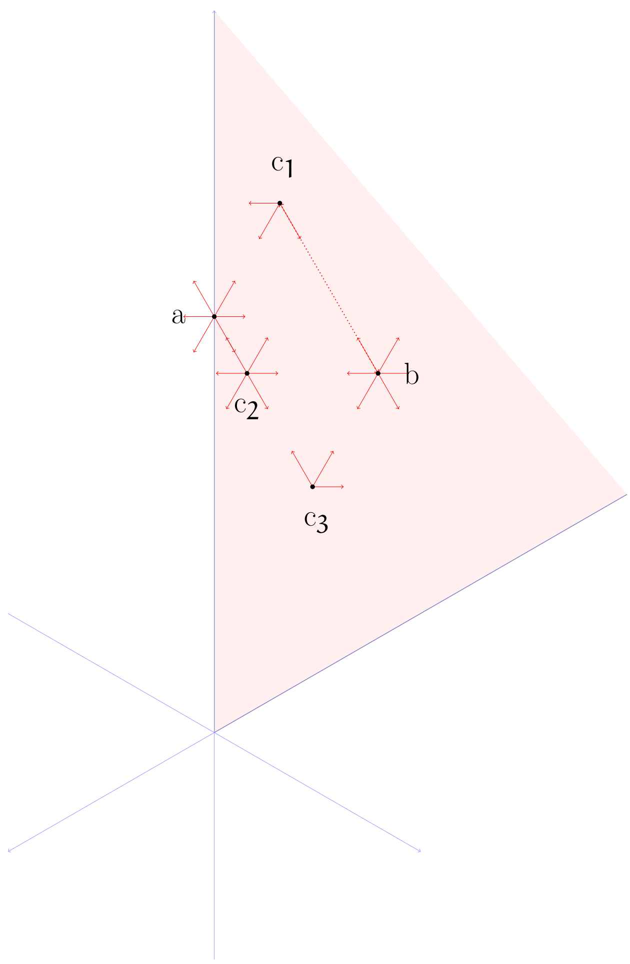


Figure 6.5: The potential edge leading from vertex $\mathbf{c}_1$ in the $-\alpha_2$ direction coincides with the potential edge leading from vertex **b** in the α_2 direction. These potential edges meet and merge creating a clear-cut edge to the polytope leading from definite vertex $\mathbf{c}_1$ to indefinite vertex **b**. This is the only edge that exists connecting vertices $\mathbf{c}_1$ and **b** satisfying the convexity property of momentum polytopes, the bifurcation lemma and information from the quadratic mometum map on the symplectic slice.

Let us follow suit for the rest of the vertices: Between vertex $\mathbf{a}$ and $\mathbf{c}_1$, the only potential edges that will coincide, merge and unite to form an actual edge of the polytope connecting these two vertices, as required according to the convexity properties of the mometum polytope, will be the potential edge travelling from indefinite vertex $\mathbf{a}$ in the α_3 direction and the potential edge travelling from definite vertex $\mathbf{c}_1$ in the $-\alpha_3$ direction. Between indefinite vertices $\mathbf{a}$ and $\mathbf{c}_2$ the only potential edges that will coincide, merge and unite to form an actual edge of the polytope connecting these two vertices will be the potential edge travelling from indefinite vertex $\mathbf{a}$ in the $-\alpha_2$ direction and the potential edges travelling from indefinite vertex $\mathbf{c}_2$ in the α_2 direction. Between indefinite vertex $\mathbf{c}_2$ and positive definite vertex $\mathbf{c}_3$ the potential edges that coincide to form an actual edge are the potential edge travelling from the indefinite vertex $\mathbf{c}_2$ in the $-\alpha_2$ direction and the potential edge travelling from the positive definite vertex $\mathbf{c}_3$ in the equal and opposite direction, namely in the α_2 weight direction. The edge connecting postive definite vertex $\mathbf{c}_3$ to indefinite vertex $\mathbf{b}$ is that travelling in the α_3 and $-\alpha_3$ from each vertex, respectively. These edges formed from potential edges are marked out in figure 6.6

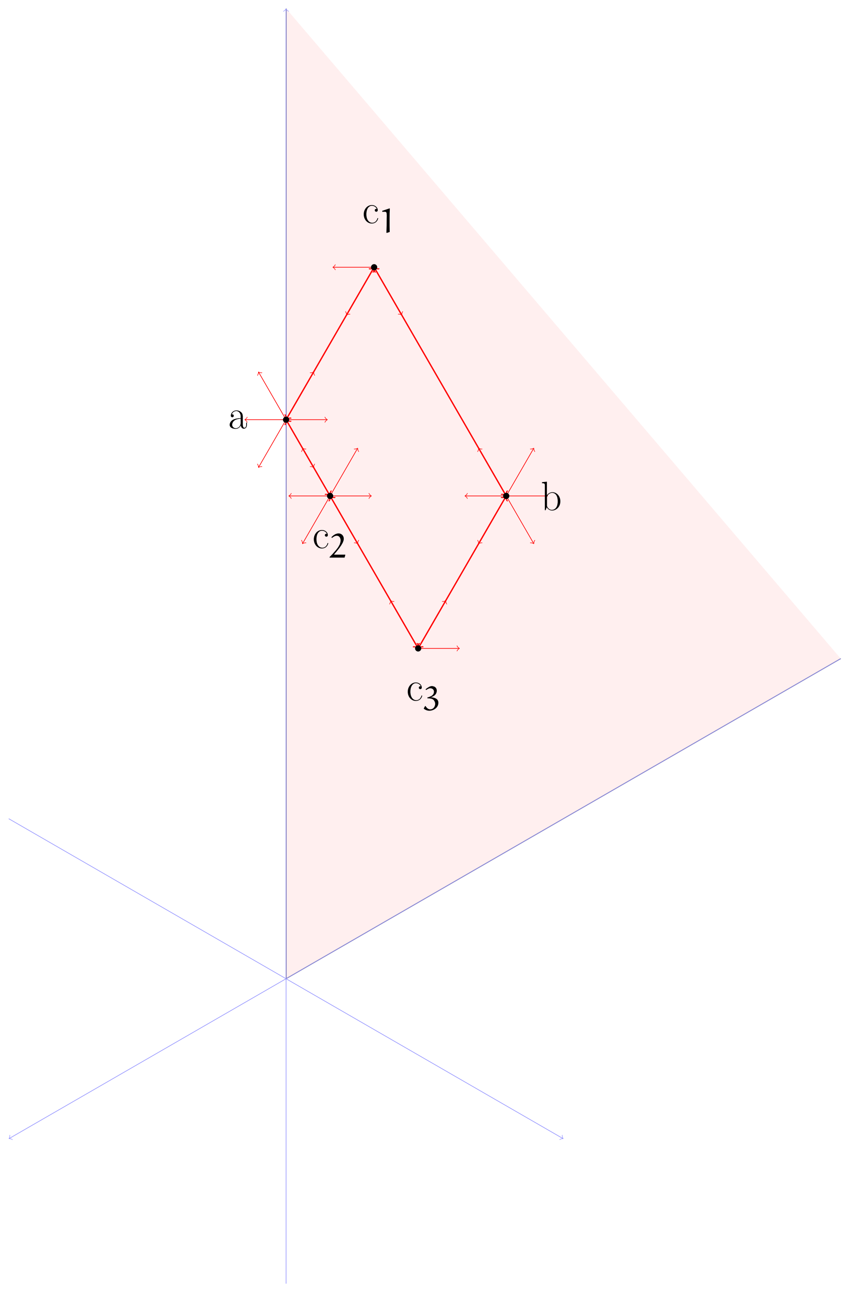


Figure 6.6: Here we can see all the edges connecting vertices $\mathbf{a}$,$\mathbf{c}_1$,$\mathbf{c}_2$,$\mathbf{c}_3$,$\mathbf{b}$ to each other according to the opposite and equal weight directions travelling from each vertex that meet to form a polytope fulfilling convexity requirements.

There is an internal edge between indefinite vertices $\mathbf{c}_2$ and $\mathbf{b}$. This is formed from the potential edges leading from these vertices in the α_1 and $-\alpha_1$ directions, respectively.

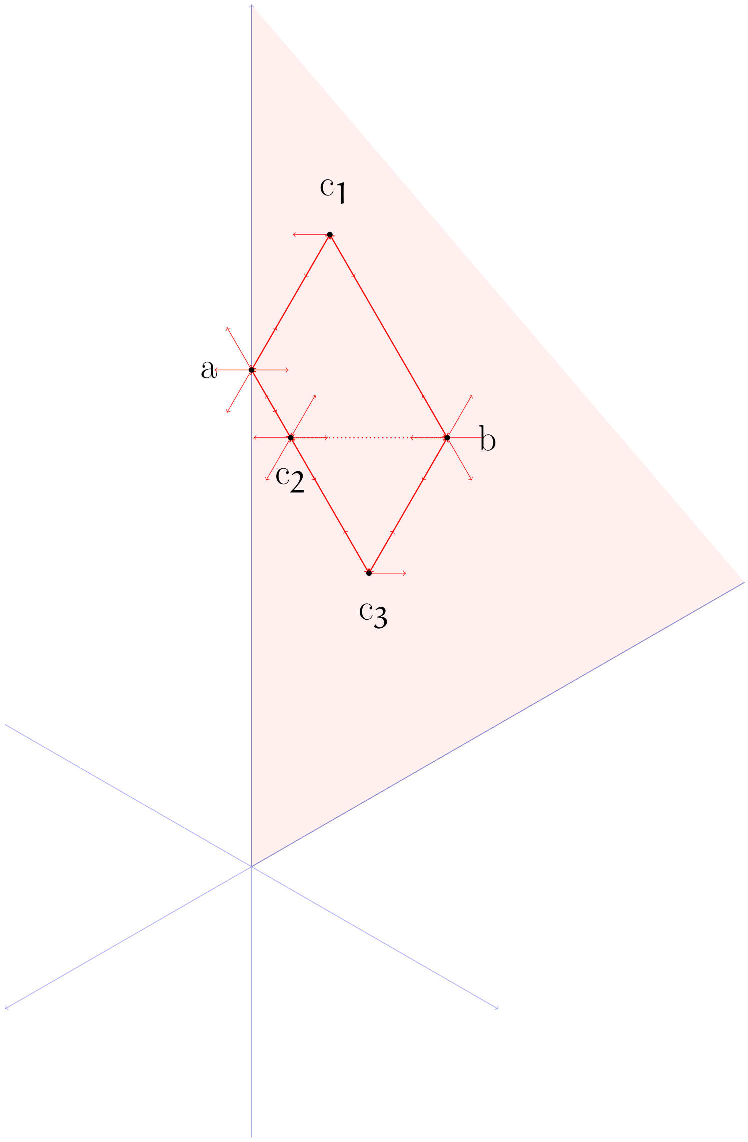


Figure 6.7: Between indefinite vertices $\mathbf{c}_2$ and $\mathbf{b}$, the only potential edges that will coincide, merge and unite to form the internal edge (depicted here by the dotted line) of the polytope connecting these two vertices, as required according to the convexity properties of the mometum polytope, will be the potential edge travelling from indefinite vertex $\mathbf{c}_2$ in the α_1 direction and the potential edge travelling from indefinite vertex $\mathbf{b}$ in the $-\alpha_1$ direction.

Finally, we have the convex structure for the momentum polytope for $\Gamma_1 = -5.5$, $\Gamma_2 = 27.5$ and $\Gamma_3 = 8.25$,

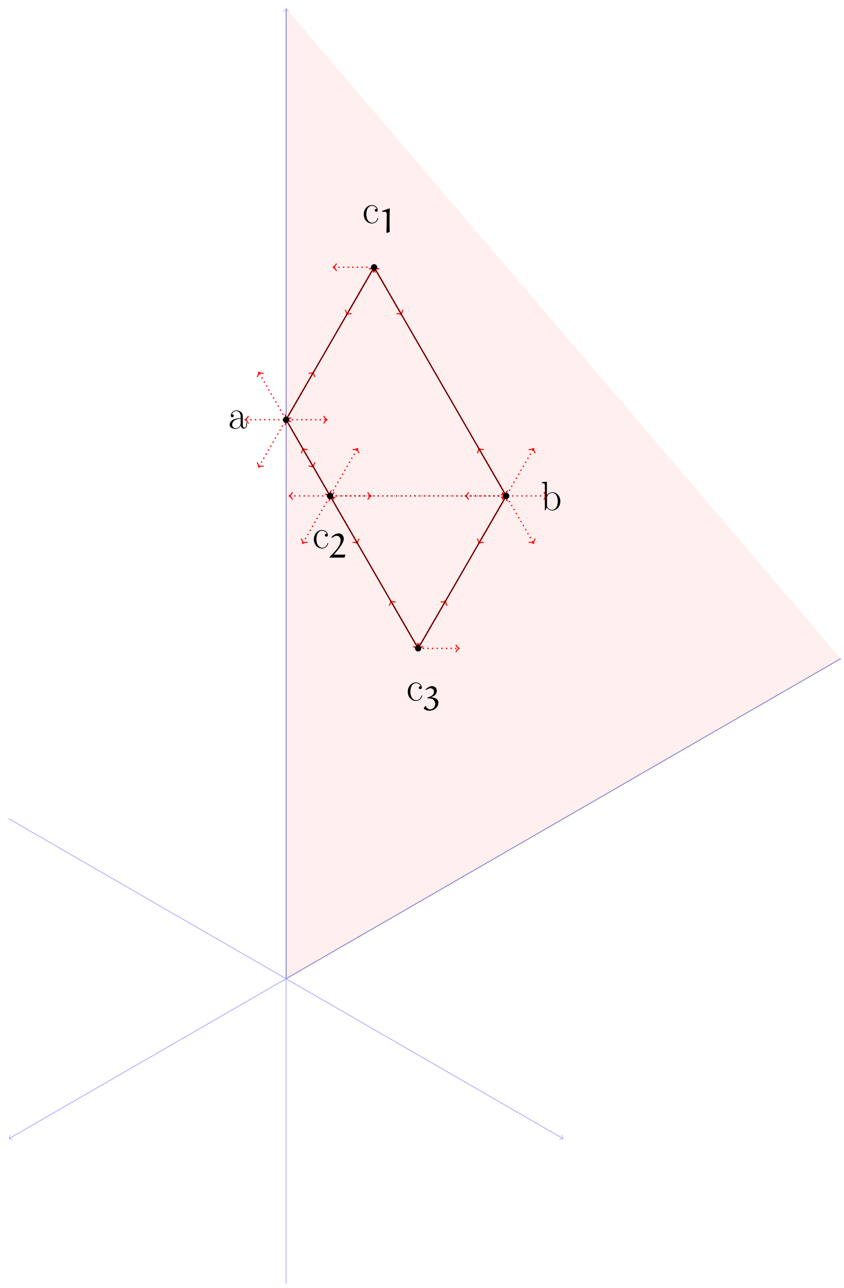


Figure 6.8: Here we have the final convex skeleton (black lines) for the unique momentum polytope for $\Gamma_1 = -5.5$, $\Gamma_2 = 27.5$ and $\Gamma_3 = 8.25$ according to all its external (solid black lines) and internal edge (dotted black line). The final convex momentum polytope for these weights is shown in figure 6.1.

Chapter 7

The Dynamics of the $SU(3)$ Action on Products of $\mathbb{C}P^2$

7.1 Generalised Vortex Dynamics

Hermann Ludwig Ferdinand Helmholtz (1821-1894) wrote a seminal paper nearly 150 years ago in which he "initiated" vortex dynamics theory. This is especially peculiar because Helmholtz was a professor of physiology and anatomy at the time. He established three laws of vortex motion which one will find in textbooks in usually practically exactly the same form.

For N point vortices $\alpha = 1, \ldots N$ with strength Γ_α (constant scalars) interacting on unbounded xy-plane with respective positions (x_α, y_α) we have,

$$\begin{aligned}\frac{dx_\alpha}{dt} &= -\frac{1}{2\pi}\sum_{\beta=1}^{N}{}' \Gamma_\beta \frac{y_\alpha - y_\beta}{l_{ab}^2}, \\ \frac{dy_\alpha}{dt} &= -\frac{1}{2\pi}\sum_{\beta=1}^{N}{}' \Gamma_\beta \frac{x_\alpha - x_\beta}{l_{ab}^2},\end{aligned} \tag{7.1}$$

are the $2N$ first order, nonlinear ODEs where $\alpha = 1, 2, \ldots N$, and

$$l_{\alpha\beta} = \sqrt{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2}$$

is the distance between vortices α and β and the omission of the singular term $\beta = \alpha$ is indicated by the prime on the summation. and so according to Helmholtz, the point vortices interact and their interaction develops in a way that is only dependent on their positions and strengths.

We can rewrite (7.1) as an ODE system for N complex coordinates $z_\alpha = x_\alpha + iy_\alpha$

$$\frac{dz_\alpha^*}{dt} = \frac{1}{2\pi i}\sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}, \qquad \alpha = 1, 2, \dots, N$$

where the asterisk denotes complex conjugation.

Kirchoff showed that the system (7.1) may be rewritten in Hamiltonian form,

$$\Gamma_\alpha \frac{dx_\alpha}{dt} = \frac{\partial H}{\partial y_\alpha}, \quad \Gamma_\alpha \frac{dy_\alpha}{dt} = -\frac{\partial H}{\partial x_\alpha}, \qquad \alpha = 1, 2, \dots, N$$

that is conserved during the motion of the Hamiltonian.

$$H = -\frac{1}{4\pi}\sum_{\alpha,\beta=1}^{N}{}' \Gamma_\alpha \Gamma_\beta \log l_{\alpha\beta},$$

And this Hamiltonian isn't governed by potential and kinetic energies.

The system (7.1) has three independent first integrals: $Q = \sum_{\alpha=1}^{N} \Gamma_\alpha x_\alpha$, $P = \sum_{\alpha=1}^{N} \Gamma_\alpha y_\alpha$ and $I = \sum_{\alpha=1}^{N} \Gamma_\alpha (x_\alpha^2 + y_\alpha^2)$ where H, I and $P^2 + Q^2$ are *in involution* for $N = 2$ irregardless of the vortex strength values. And for $N = 3$ this system is integrable irregardless of vortex strength values according to Louiville's analytical dynamics theorem.

Gröbli's 1877 is an extensive study of the integrability of different three-vortex systems. Novikov and Aref revived interest in Gröbli and Synge's early work on the three-vortex problem again in the 1970s. Vortices were catapulted into mainstream modern science research when chaos theory was shown to model $N = 4$ vortex systems.

Here we reiterate the fact that we implement generalised point vortex system dynamics and not fluid vortex dynamics systems as demonstrated in the sections below.

7.1.1 Spherical Vortices

First, vortex dynamics on S^2 with $G = SO(3)$. S^2 is a coadjoint orbit of G, and we consider $\bar{X} = S^2 \times \dots \times S^2$ (N copies), with symplectic form

$$\omega = \Gamma_1 \omega_0 \oplus \dots \oplus \Gamma_N \omega_o,$$

where ω_0 is some fixed choice of invariant symplectic form on S^2, and the γ_j are non-zero real numbers (called vortex strengths). The Hamiltonian on $\bar{X}$ is

$$H(x_1, \ldots, x_N) = -\sum_{j<k} \Gamma_j \Gamma_k \log(\|x_j - x_k\|^2).$$

This is not defined on the large diagonal (where some $x_j = x_k$), so the phase space is $X = \bar{X} \setminus \Delta$ where Δ is the large diagonal. (The log term is (essentially) the Green's function for the Laplacian on the sphere. Note the argument of log is (the square of) the Euclidean distance between the points, rather than the spherical distance. This is just what the Green's function for the Laplacian happens to be.)

The dynamics is then determined by these geometric ingredients: in particular by the choice of the Γ_j (equivalently, by the choice of the symplectic form on each copy of S^2).

7.1.2 Our Generalisation

Repeat above with any symplectic manifold. Let (S, ω_0) be a symplectic manifold with an action of a Lie group G. Let $\Gamma_1, \ldots, \Gamma_N$ be non-zero real numbers. The complete phase space for N point vortices on S is $\bar{X} = S \times \ldots \times S$ (N copies), with symplectic form

$$\omega = \Gamma_1 \omega_0 \oplus \cdots \Gamma_N \omega_0.$$

The Hamiltonian will be a function of the form

$$H(x_1, \ldots, x_N) = -\sum_{j<k} \Gamma_j \Gamma_k h(r_{jk})$$

where r_{jk} is a distance between x_j and x_k, and $h(r)$ is a smooth function of r, possibly not defined when $r = 0$.

If G is compact and acts transitively on S then S is necessarily a coadjoint orbit of G (Theorem of Guillemin-Sternberg).

Since the action is Hamiltonian, then there exists a momentum map. We have established and investigated the momentum map for $G = SU(3)$ with coadjoint orbit $\mathbb{C}P^2$. Noether's theorem states that the components of the momentum map are preserved by the dynamics.

7.2 Using Symplectic Geometry to Model Systems of Vortices on $\mathbb{C}P^2$

Recall that $SU(3)$ has three types of coadjoint orbits: the Flag manifold $F(2,1)$, the complex projective plane $\mathbb{C}P^2$ and the single point. We are interested in $SU(3)$ actions on vortices on the complex projective plane. Therefore the results in the previous chapters can be used to investigate vortex systems on $\mathbb{C}P^2$ and the $SU(3)$ actions on them, with particular attention to 2- and 3-vortex systems. Let $\bar{X} = \mathbb{C}P^2 \times \ldots \times \mathbb{C}P^2$ be the product of N copies of the complex projective plane in $\mathbb{C}^3$.

7.2.1 The $SU(3)$ action on two vortices on phase space $\bar{X} = \mathbb{C}P^2 \times \mathbb{C}P^2$

For the 2-vortex system Γ_1 is the vortex strength of one of the point vortices and Γ_2 is the vortex strength of the second point vortex, each with coordinates z_1 and z_2 respectively. The matrix subgroups of $SU(3)$ derived in chapter 3 that fix two points in $\mathbb{C}P^2 \times \mathbb{C}P^2$ (isotropy subgroups) are the matrix subgroups that fix two vortices when their coordinates are perpendicular, parallel and on generic coordinates (neither parallel nor perpendicular to each other) with respect to one another on respective complex projective planes. This defines the range of distances between the coordinates of 2 vortices on the complex projective plane. The action of $SU(3)$ on $\mathbb{C}P^2 \times \mathbb{C}P^2$ is a cohomogeneity 1 action therefore there are three isotropy subgroups fixing three distinct fixed point sets for the vortex system. The matrix group $U(2) \subset SU(3)$ fixes two vortices that are parallel to each other; the matrix group $\mathbb{T}^2 \subset SU(3)$ is the istoropy subgroup that fixes two vortices perpendicular to each other; the matrix group $\mathbb{T}^1 \subset SU(3)$ is the isotropy subgroup that fixes two generic point vortices that are neither parallel or perpendicular to each other. Section 3.2 provides the calculations for the isotropy subgroups of $SU(3)$ acting on $\mathbb{C}P^2 \times \mathbb{C}P^2$.

For vortices on the orbits, conservation of momentum means each vortex cannot move from one orbit to another, and the momentum map will only move them in particular directions on that orbit.

Without deriving a Hamiltonian, the dynamics and geometry of an action can be investigated via its momentum map (see section 2.3.2.1 and sections 4 and

5): the momentum map, J, for the $SU(3)$ action on two vortices on phase space $\bar{X} = \mathbb{C}P^2 \times \mathbb{C}P^2$:

$$J : \mathbb{C}P^2 \times \mathbb{C}P^2 \to \mathfrak{su}(3)^* \tag{7.2}$$

where $\mathfrak{su}(3)$ is the dual to the Lie algebra of $SU(3)$ (the set of traceless (anti-) Hermitian 3×3 matrices). By lemma 2.3.1 considering several vortices at once is simply the addition of the momentum map of the action on each vortex. The intersection $J \cap \mathfrak{t}_+^*$ of the image of the momentum map with the positive Weyl chamber $\mathfrak{t}_+^*$ in $\mathfrak{t}^*$ is the convex polytope also known as the *Moment Polytope* or *Momentum Polytope*. Section 2.1 introduced Weyl Chambers with particular attention to that for $SU(3)$, information for this is derived mainly from Frank Adams [2] and Raoul Bott's [20] respective lecture series. Weyl chambers are essential when considering vortices on $SU(n)$ as the trajectory of their momentum map depends on their initial position and scope of movement on the root diagram as explained via the moment polytope. Sections 4 and 5 include the definitions of momentum maps and moment polytopes which are a type of convex polyhedron that describe the shape of the trajectory of the momentum map as well as the direction.

The momentum polytope coincides with the image of the orbit momentum map: section 2.3.1 provides a narrow description of the Orbit-Reduction Method and complete integrability.

The momentum polytopes as shown in section 5.3 classify the different configurations of the 2 vortex system according to the ratios between the vortex strengths.

7.2.2 The $SU(3)$ action on three vortices on phase space $\bar{X} = \mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$

The 3-vortex system follows the same methodology as that for the 2-vortex system, however as shown in section 3 and 5 there are 5 different fixed point sets here. In this case different moment polytopes arise according to the different fixed vortex strengths Γ_1, Γ_2 and Γ_3 assigned to each vortex. Section 2.6 gives a proof of the bifurcation lemma which provides a rank-nullity correspondence that shows how the edges of the polytopes that connect the fixed points of subgroups of $SU(3)$ acting on $\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2$ are constructed as shown in section 5.5.

The nine different polytopes that result from the different nontrivial Γ_1, Γ_2 and Γ_3 vortex strength combinations are shown in section 2.5 as well as the transitions

between the nine different polytopes. Chapters 2 outlined mathematical conditions such as the nondegeneracy of the momentum map.
Chapter 4 explained some properties of the momentum map explained via the weighted roots of $\mathfrak{su}(3)$ how the edges of the polytopes are constructed locally from a fixed point. And theorem 5.7.2 showed some of the transitions from a 2-vortex polytope to a 3-vortex polytope.

We can apply the results in section to write the relative equilibria of the 2-vortex system and further steps beyond this point.

7.3 The Space of Allowed Velocity Vectors R_0

For $\xi \in \mathfrak{g}$, the associated vector field on P is denoted by ξ_P and, given a Hamiltonian function $H : P \to \mathbb{R}$, the associated vector field is denoted by X_H. Equivariant momentum map is defined by the differential condition,

$$\langle d\Phi_x(v), \xi\rangle = \omega(X_\xi, v) \quad \forall \xi \in \mathfrak{g}, \ \forall v \in T_x P \tag{7.3}$$

If $H : P \to \mathbb{R}$ is an invariant Hamiltonian function, it passes down to a function $\bar{H} : P/G \to \mathbb{R}$, whose restriction to the reduced space P_μ is denoted H_μ. If P_μ is smooth , a relative equilibrium $x_\mu \in P_\mu$ in *non-degenerate* if $d^2H_\mu(x_\mu)$ is a non-degenerate quadratic form; it is *regular* if the velocity ξ of the relative equilibrium is a regular element of $\mathfrak{g}$. (The velocity is defined by $X_H(x) = X_\xi(x)$ for some point p of the relative equilibrium; this velocity is well-defined modulo the adjoint action, since by equivariance $X_H(g \cdot x) = X_{\mathrm{Ad}_g(\xi)}(g \cdot x)$.) And for simplicity we assume throughout that the Hamiltonian H is such that the associated vector field X_H is complete.

Proposition 7.3.1. *Let* $x \in \mathcal{P}$ *be a relative equilibrium for the Hamiltonian system* H*, and suppose that* $X_H(x) = X_\xi(x)$ *(so* ξ *is well-defined modulo* $\mathfrak{g}_x$*). Then, viewing* dH_x *as an element of* $N_0^* \simeq (\mathfrak{g}_\mu/\mathfrak{g}_x)$ *as described above, we have*

$$dH_x = [\xi] \in (\mathfrak{g}_\mu/\mathfrak{g}_x)^{G_x}. \tag{7.4}$$

Proof. The isomorphism is defined via the symplectic form:

$$[\xi] \in \mathfrak{g}_\mu/\mathfrak{g}_x \mapsto [z \mapsto \omega(X_\xi(x), z)] \in N_0^*.$$

Since $X_H(x) = X_\xi(x)$ is follows that for $n \in N_0$, $dH_x(z) = \omega(X_H(x), z) = \omega(X_\xi(x), z)$ as required. □

We write $K < G$ to mean K is a closed subgroup of G. The fixed point set of a subgroup $K < G$ is

$$\mathrm{Fix}(K, \mathcal{P}) = \{x \in \mathcal{P} \mid g \cdot x = x, \forall g \in K\}$$

it is a closed submanifold of $\mathcal{P}$. If K is compact and formed of symplectic symmetries, then $\mathrm{Fix}(K, \mathcal{P})$ is invariant under the flow of the dynamical system. If in addition K is compact, then $\mathrm{Fix}(K, \mathcal{P})$ is a Hamiltonian subsystem with Hamiltonian given by the restriction of H to $\mathrm{Fix}(K, \mathcal{P})$.

How does one locate relative equilibria?

From above, relative equilibria are critical points of the Hamiltonian restricted to the level sets of the momentum map, so results on critical point of G-invariant functions are of particular interest. Let G be a Lie group and $H : \mathcal{P} \to \mathbb{R}$ a G-invariant function. Assume that G acts properly on $\mathcal{P}$. The *Principle of Symmetric Criticality* states that if the directional derivatives $dH_x(u)$ vanish for all directions u at x tangent to $\mathrm{Fix}(K, \mathcal{P})$, then directional derivatives in directions transverse to $\mathrm{Fix}(K, \mathcal{P})$ also vanish. In particular any isolated point of $\mathrm{Fix}(K, \mathcal{P})$ is a critical point of H. Therefore

$$\xi_{\mathcal{P}}(x) = X_H(x) \in (\mathfrak{g}_\mu/\mathfrak{g}_x)^{G_x} = \mathrm{Fix}(G_x, \mathfrak{g}_\mu/\mathfrak{g}_x)$$

since by equivariance

$$\phi_t(g \cdot X) = g \cdot \phi_t(X)$$
$$\text{i.e.} \quad x \in \mathrm{Fix}(\kappa) \Rightarrow \phi_t(x) \in \mathrm{Fix}(\kappa)$$

In other words,

Theorem 7.3.2. *The flow ϕ_t leaves Fix$(K, \mathcal{P})$ invariant, meaning that if $x \in$Fix$(K, \mathcal{P})$ then $\phi_t(x) \in$Fix$(K, \mathcal{P})$. This is known as the conservation of symmetry.*

Therefore we will introduce the following definition:

Definition 7.3.3. At $x \in \mathcal{P}$,

$$R_0 \subset T_x G \tag{7.5}$$

is the subset of tangent space of the allowed velocity vectors, ξ, of x and is an RE for which $dH_x = [\xi] \in R_0$. Therefore

$$R_0 \simeq (\mathfrak{g}_\mu/\mathfrak{g}_x)^{G_x} \tag{7.6}$$

where $N_0^* \simeq (\mathfrak{g}_\mu/\mathfrak{g}_x)$.

Using this definition we will show that

Theorem 7.3.4. *For the $SU(3)$ action on two vortices on $\mathbb{C}P^2$, at a generic configuration (when the vortices are neither parallel or perpendicular),*

$$R_0 \simeq \mathbb{R} \tag{7.7}$$

and this is true for every respective vortex strength.

Proof. We look at the Relative Equilibrium for every configuration, generic and otherwise, on $\mathbb{C}P^2 \times \mathbb{C}P^2$, specifically the velocity space R_0 of their relative motion on $\mathbb{C}P^2 \times \mathbb{C}P^2$ for the sake of being meticulous:

$\underline{\Gamma_1 = \Gamma_2 = \Gamma}$

When parallel, $x = (z_1, z_2) = ([1:0:0],[1:0:0])$ (for example), $G_x = U(2)$ and $G_\mu = U(2)$

$$J(x) = \mu = \frac{2\Gamma}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

When the vortices are perpendicular, $x = (z_1, z_2) = ([1:0:0],[0:1:0])$ (for example), $G_x = \mathbb{T}^2$ and $G_\mu = U(2)$:

$$J(x) = \mu = \frac{\Gamma}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

At niether parallel or perpendicular, $x = (z_1, z_2) = ([c:d:0],[c:-d:0])$ (for example) , $G_x = \mathbb{T}^1$ and $G_\mu = \mathbb{T}^2$:

$$J(x) = \mu = 2\Gamma \begin{pmatrix} (\frac{|c|^2}{|c|^2+|d|^2} - 1/3) & 0 & 0 \\ 0 & (\frac{|d|^2}{|c|^2+|d|^2} - 1/3) & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$$

$\Gamma_1 = -\Gamma_2 = \Gamma$

At parallel, $G_x = U(2)$ and $G_\mu = SU(3)$:

$$J(x) = \mu = \frac{2\Gamma}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

However, identically to $\Gamma_1 = \Gamma_2 = \Gamma$ when perpendicular, $G_x = \mathbb{T}^2$, $G_\mu = \mathbb{T}^2$ and at neither parallel or perpendicular $G_x = \mathbb{T}^1$, $G_\mu = \mathbb{T}^2$.

For other ratios between Γ_1 and Γ_2

For $\Gamma_1 > \Gamma_2 > 0$, $\Gamma_1 > 0 > \Gamma_2$ where $|\Gamma_2| > |\Gamma_1|$, $0 > \Gamma_1 > \Gamma_2$ and $\Gamma_2 > 0 > \Gamma_1$ where $|\Gamma_2| > |\Gamma_1|$: at parallel, $G_x = U(2)$, $G_\mu = U(2)$. At perpendicular $G_x = \mathbb{T}^2$, $G_\mu = \mathbb{T}^2$. And at neither parallel or perpendicular $G_x = \mathbb{T}^1$, $G_\mu = \mathbb{T}^2$.

$\Gamma_1 = \Gamma_2$	parallel: $G_x = U(2) = G_\mu$	$R_0 \simeq (\mathfrak{u}(2)/\mathfrak{u}(2))^{U(2)} = \{0\}$
	perpendicular: $G_x = \mathbb{T}^2$, $G_\mu = U(2)$	$R_0 \simeq (\mathfrak{u}(2)/\mathfrak{t}^2)^{\mathbb{T}^2} = \{0\}$
	inbetween: $G_x = \mathbb{T}^1$, $G_\mu = \mathbb{T}^2$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{t}^1)^{\mathbb{T}^1} = \mathbb{R}$
$\Gamma_1 = -\Gamma_2 = \Gamma$	parallel: $G_x = U(2)$, $G_\mu = SU(3)$	$R_0 \simeq (\mathfrak{su}(3)/\mathfrak{u}(2))^{U(2)} = \{0\}$
	perpendicular: $G_x = \mathbb{T}^2 = G_\mu = U(2)$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{t}^2)^{\mathbb{T}^2} = \{0\}$
	inbetween: $G_x = \mathbb{T}^1$, $G_\mu = SU(3)$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{t}^1)^{\mathbb{T}^1} = \mathbb{R}$
Otherwise	parallel&perpendicular: $G_x = G_\mu$	$R_0 \simeq (\mathfrak{g}_\mu/\mathfrak{g}_x)^{G_x} = \{0\}$
	inbetween: $G_x = \mathbb{T}^1$, $G_\mu = \mathbb{T}^2$	$R_0 \simeq (\mathfrak{t}^2/\mathfrak{t}^1)^{\mathbb{T}^1} = \mathbb{R}$

- $(\mathfrak{su}(3)/\mathfrak{t}^2)^{\mathbb{T}^2} = \{0\}$
- $(\mathfrak{su}(3)/\mathfrak{u}(2))^{U(2)} \subset (\mathfrak{su}(3)/\mathfrak{t}^2)^{\mathbb{T}^2} = \{0\}$
- $(\mathfrak{t}^2/\mathfrak{t}^1)^{\mathbb{T}^1} \simeq (S^1)^{S^1} \simeq \mathbb{R}$

□

Chapter 8

Further Proposed Research

8.1 The Littlewood-Richardson Polytope

This question arose from conversations with Professor Gert Heckman:
The vertices of the momentum polytope may be split up into the *Cartan vertex* which is the highest vertex of the polytope, and the *Parthasarathy-Ranga Rao-Varadarajan vertex* which is the lowest vertex of the polytope. By this notation we can endeavour to describe the finite algorithm to describe the *Littlewood-Richardson polytope* (or *Horn* polytope) which provides an explicit description of a polytope. For our case of momentum polytopes which provide non-trivial solutions, this will help to further understand the general problem and even further to understand the Littlewood-Richardson polytopes for the actions of groups of rank greater than 2. A useful paper with good references for this problem is [38]. We have already confirmed that the highest vertex of each of our momentum polytopes and the lowest vertex of each of our momentum polytopes satisfy the definitions of the Cartan vertex and Parthasarathy-Ranga Rao-Varadarajan vertex, respectively. And we have made some first steps for a proof for this.

8.2 Hamiltonian

The interaction of vortices on the sphere has been researched extensively by many including Paul Newton and James Montaldi with some extensions including, for example, point vortices on the hyperbolic plane for its non-compact symmetry properties. The dynamics of a system of vortices is usually determined by the Hamiltonian: in particular by the choice of the Γ_i. The Hamiltonian of a system of vortices on the sphere is described by the Green's function for the Laplacian of the Euclidean coordinates on the sphere.

Current work on the symmetric square of quaternionic and complex projective spaces [19] sheds some light on a distance metric that can be used to formulate a Hamiltonian for the system of vortices on $\Pi_{i=1}^{n}\mathbb{C}P^2$. Once a possible Hamiltonian has been established we may assess the critical points and stability. We are establishing a possible Hamiltonian using the chordal distance between two points in projective spaces that defines a metric that matches the Fubini-Study metric form from [24].

8.3 Further Generalisation

The dynamics of a system of vortices can also be determined by the choice of the symplectic form on each copy of the manifold it is on. Therefore instead of using scalar multiples of a given one symplectic form, we can investigate the system of vortices that is governed by different symplectic forms with guidance from Marsden and Weinstein's paper [60].

8.4 Semi-Toric Systems

Recently at the Finite Dimensional Integrable Systems Conference in Barcelona on July 2017 I attended a talk given by Joseph Palmer from Rutgers University on New Constructions of Semi-Toric Systems and the results of his collaboration with Yohann le Floch and Alvaro Pelayo [51]. This included constructing *coupled angular momentum systems* according to some parameter that shifts the semi-toric polytope and from identifying the corresponding singularity points. Therefore this method can be used to construct a Hamiltonian from the Momentum Map according to deformations of the polytope rather than independently from the Momentum Map.

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