Routes to Chaos

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2005

MIMS EPrint: **2006.102**

ISSN 1749-9097

NL3157 **Routes to Chaos**

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If a nonlinear systems has chaotic dynamics then it is natural to ask how this complexity develops as parameters vary. For example, in the logistic map

$$x_{n+1} = rx_n(1 - x_n) \tag{1}$$

it is easy to show that if $r = \frac{1}{2}$ then there is a fixed point at $x = 0$ which attracts all solutions with initial values x_0 between 0 and 1, while if $r = 4$ the system is chaotic. How, then, does the transition to chaos occur as the parameter r varies? Indeed, is there a clean transition to chaos in any well-defined sense? The identification and description of routes to chaos has had important consequences for the interpretation of experimental and numerical observations of nonlinear systems. If an experimental system appears chaotic then it can be very difficult to determine whether the experimental data comes from a truly chaotic system, or if the results of the experiment are unreliable because there is too much external noise. Chaotic time series analysis provides one approach to this problem, but an understanding of routes to chaos provides another. In many experiments there are parameters (ambient temperature, Raleigh number, etc.) which are fixed in any realization of the experiment, but which can be changed. If recognizable routes to chaos are observed when the experiment is repeated at different values of the parameter, then there is a sense in which the presence of chaotic motion has been explained.

By the early 1980s three "scenarios" or "routes to chaos" had been identified (e.g. Eckmann (1981)): Ruelle–Takens–Newhouse, period doubling, and intermittency (which has several variants). As we shall see, in their standard forms each of these transitions uses the term "route to chaos" in a different way, so care needs to be taken over the interpretation of experimental or numerical observations of these transitions.

Ruelle–Takens–Newhouse

In 1971 Ruelle and Takens published a mathematical paper with the provocative title 'On the nature of turbulence'. In this paper and a subsequent improvement with Newhouse, they discuss the Landau scenario for the creation of turbulence by the successive addition of new frequencies to the dynamics of the fluid. They show that if the attractor of a system has three independent frequencies (four in the 1971 paper) then a small perturbation of this system has a hyperbolic strange attractor—a Plykin attractor (a solenoid in the 1971 paper). The result became known colloquially as "three frequencies implies chaos", a serious misinterpretation of the mathematical result which has been the cause of a number of misleading statements. First, the result proves the existence of chaos in systems arbitrarily close to the three frequency system in an infinite dimensional function space, but gives no indication of the probability of finding chaos in any given example. Second, I know of no experimental situation where a Plykin

Figure 1. The attractor of the logistic equation as a function of the parameter r. The horizontal axis is r, and the vertical axis x.

attractor has been shown to exist even when chaotic behavior has been observed close to systems with three frequency attractors. Numerical experiments suggest that it is much more likely that the system evolves by frequency locking.

The Ruelle–Takens–Newhouse route to chaos remains something of an enigma, and more work needs to be done to understand precisely how and when the strange attractors predicted by the theory come into being.

Period doubling

Figure 1 shows the attractor of the logistic map (1) as a function of the parameter, r, for $3.5 < r < 4$. Thus the set of points plotted on any vertical line of constant r represents the attractor of the map for that value of r and if the set is finite then the (numerically computed) attractor is a periodic orbit which cycles through the finite collection of points. Fig. 1 suggests that for small r, the attractor is always periodic and has period 2^n, with n increasing as r increases. Beyond some critical value $r = r_c$, with $r_c \approx 3.569946$, the attractor may be more complicated. There are clearly intervals of r for which the attractor is periodic, and the attractor seems to be contained in 2^n bands which merge as r increases (the final band merging from 2 bands to one band with r just below 3.68 is particularly clear).

As r increases, the periodic orbit of period 2^n is created from the orbit of period 2^{n-1} by a period-doubling bifurcation. If this bifurcation occurs with $r = r_n$ then $r_n \to r_c$ geometrically as $n \to \infty$, with

$$\lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \delta \approx 4.66920 \qquad \text{i.e.} \qquad r_n \sim r_c - \kappa\delta^{-n} \tag{2}$$

The really surprising feature of this period-doubling cascade, as shown in Feigenbaum (1978), is that the cascade can be observed in many maps and the accumulation rate δ of the period-doubling cascade (2) is the same although the constants r_c and κ depend on the map. In fact, the universal value of δ depends on the nature of the maximum of the map: $\delta \approx 4.66920$ for maps with quadratic maximum. A complete cascade of

Figure 2. (a) The quadratic map $f(x) = 1 - mx^2$ restricted to the interval $[-1, 1]$ and the second iterate, $f(f(x))$ for $m = 1.40115$ which is just below the critical value m_c. (b) The original map, f, and the rescaled map, $F_2(x) = \mathcal{T}f(x) = -a^{-1}f(f(-ax))$ on $[-1, 1]$ with $a = -f(1) = m - 1$. Note that $a = 0.40115$ is close to the universal value of $\alpha \approx 0.3995$.

band merging from 2^n bands to 2^{n-1} bands occurs at parameter values $\tilde{r}_n$ above r_c, and $\tilde{r}_n \to r_c$ as $n \to \infty$ at the same universal geometric rate δ.

The quantitative universality in parameter space described by the scaling δ has a counterpart in phase space. If x_n denotes the point on the periodic orbit of period 2^{n-1} that is closest to the critical point (or turning point) of the map with $r = r_n$, then

$$\lim_{n\to\infty} \frac{x_{n+1} - \frac{1}{2}}{x_n - \frac{1}{2}} = -\alpha \tag{3}$$

where α is another universal constant, which, for maps with a quadratic turning point, takes the value $\alpha \approx 0.3995 = 1/2.50\ldots$.

In families of one hump (unimodal) maps this universality can be explained by a renormalization argument. Restrict attention to families of one hump maps with critical point (maximum) at $x = 0$, parametrized by μ and normalized so that $f(0) = 1$. As shown in Figure 2a, for parameter values near μ_c (the accumulation of period doubling) the second iterate of the map, $f(f(x))$, restricted to an interval about the critical point is a one hump map with a minimum. So, after a rescaling (and flipping) of the coordinates it is another one hump map with the same normalization as shown in Figure 2b. Mitchell Feigenbaum was able to show (by arguments which have been made rigorous in the past ten years) that the universal properties described above are due to the structure of the doubling operator $\mathcal{T}$, which is a map on one hump maps $f : [-1, 1] \to [-1, 1]$, with critical point at $x = 0$ and $f(0) = 1$, defined by

$$\mathcal{T}f(x) = -a^{-1}f(f(-ax)) \tag{4}$$

where $a = -f(1)$ so that the normalization $\mathcal{T}f(0) = 1$ is preserved. This operator does the rescaling and flipping referred to above. In the appropriate universality class, for example quadratic critical point together with some further technical conditions, there is a fixed point f_*, of $\mathcal{T}$, so $f_* = \mathcal{T}f_*$, and the universal scaling of phase space is given by $\alpha = -f_*(1)$. Furthermore, the universal accumulation rate δ of (2) is an unstable eigenvalue of the (functional) derivative of $\mathcal{T}$ at f_*.

We can now consider measures of chaos such as the topological entropy or the Lyapunov exponents of the map. If $\{f_\mu\}$ is a family of one hump maps which undergoes period doubling then the parameter μ can be chosen so that the period-doubling cascade is for $\mu < \mu_c$. Coullet & Tresser (1980) show that the universal structure described above implies that if $H(\mu)$ is either the topological entropy (which can be thought of as the growth rate of the number of periodic orbits) or the Lyapounov exponent of f_μ with $\mu > \mu_c$ then

$$H(\mu) \sim C(\mu - \mu_c)^{\frac{\log 2}{\log \delta}} \tag{5}$$

The Lyapunov exponent is a very poorly behaved function of the parameters and this scaling provides only an envelope for the graph of the exponent, but the topological entropy is continuous. Indeed, the proof of Sharkovsky's Theorem (see entry on One-dimensional Maps) shows that if a continuous map of the interval has a periodic orbit which is not a power of two then there is a horseshoe for some iterate of the map, and hence the map has positive topological entropy if $\mu > \mu_c$. The entropy is zero if $\mu < \mu_c$, so if by chaos we mean positive topological entropy, then the period-doubling route is a true route to chaos.

Intermittency

The first stable periodic orbit of each of the windows of periodic motion in $r > r_c$ which can be seen in Figure 1 is created in a saddlenode (or tangent) bifurcation. Throughout the parameter interval for which such orbits are stable there is a repelling strange invariant set, but most solutions tend to the stable periodic orbit. Just before the creation of the stable periodic orbit, chaotic solutions spend long periods of time near the points at which the stable periodic orbit will be created (the "laminar" phase), then move away and behave erratically before returning to the laminar phase. This behavior is called intermittency by Pomeau & Manneville (1980), who were the first to describe the scaling of the time spent in the laminar phase. They looked at the average time T_A spent by solutions in the laminar phase as a function of the parameter r close to the value r_{sn} at which the saddlenode bifurcation occurs. A simple argument based on the passage time of a trajectory of a map close to a tangency with the diagonal (the condition for the saddlenode bifurcation) establishes that the average time in the laminar phase diverges as a power law:

$$T_A \sim |r - r_{sn}|^{-\frac{1}{2}} . \tag{6}$$

Other types of intermittency (involving period-doubling bifurcations etc.) can be analyzed using the same ideas. Note that a strange invariant set exists throughout the parameter regions being considered here, so in this case the term "route to chaos" refers to the stability of the chaotic invariant set, not the creation of a chaotic set. Moreover, in any open neighborhood of r_{sn} there are parameters for which the map has other stable periodic orbits, so the full description of parameters with stable chaotic motion is much more complicated than the description above suggests.

Other routes to chaos

Since the pioneering work of the late 1970s a number of other routes to chaos have been identified. New routes to chaos are still being identified, and the list provided here is by no means complete. Arnéodo, Coullet & Tresser (1981) show that there can be cascades of homoclinic bifurcations to chaos via a mechanism closely related to period doubling. This gives the less standard convergence rates involving non-quadratic turning points immediate relevance. The bifurcation which creates the strange invariant set of the Lorenz model is another type of homoclinic bifurcation, and this strange invariant set becomes stable by a "crisis" in which the strange invariant set collides with a pair of unstable periodic orbits. Ott (2002) contains a good account of such transitions. More complicated transitions involving maps of the circle are detailed in MacKay & Tresser (1986) and Newhouse, Palis & Takens (1983) give another transition.

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See also Attractors; Bifurcations; Chaotic dynamics; Difference equations; Intermittency; Lorenz equations; Period doubling; One-dimensional maps; Time series analysis

Further Reading

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