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Rowley, Peter and Ward, David

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Fusion in 2-cores of Maximal Parabolic Subgroups of the Baby Monster

Peter Rowley and David Ward

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Abstract

In this paper the fusion of the 2-cores of the four maximal parabolic subgroups of the Baby Monster, namely those of shape $2^{9+16}Sp_8(2)$, $2^{3+32}(L_3(2) \times Sym(5))$, $2^{2+10+20}(Sym(3) \times M_{22} : 2)$ and $2^{1+22}Co_2$, are determined.

1 Introduction

Fischer's Baby Monster group $\mathbb{B}$, also sometimes denoted by F_2 (see [2]), is the second largest sporadic simple group and has order

$$2^{41}.3^{13}.5^6.7^2.11.13.17.19.23.31.47.$$

This group was uncovered by B. Fischer [8] during his investigations into $\{3,4\}$-transposition groups. We recall that such a group G is generated by a conjugacy class X of involutions for which the product of any two elements of X has order $1, 2, 3$ or 4. In [8] Fischer described many of its basic properties and subsequently Hunt [11] computed the Baby Monster's character table. Building on the information in [8], Leon and Sims [12] gave the first construction of this group as a permutation group of degree $13,571,955,000$. Later Wilson [17] constructed $\mathbb{B}$ as a 4370×4370 matrix group over $GF(2)$, and this was followed by Parker and Wilson [14] who additionally gave matrix representations over $GF(3)$ and $GF(5)$. These matrix representations of $\mathbb{B}$ were vital in the ensuing work which determined the maximal subgroups of $\mathbb{B}$ (see Wilson [18], [19], [20]) and detailed information about its conjugacy classes [21]. These endeavours relied heavily on computer calculations, however, see Meierfrankenfeld and Shpectorov [13] where there is a computer–free analysis of the 2-local subgroups of $\mathbb{B}$.

In [16] the authors generalize the notion of a cuspidal character, a well esstablished idea in the representation theory of finite Lie-type groups, to any finite group. Further, in [16], all cuspidal characters of the sporadic simple groups are determined. We give a brief summary of the main ideas in [16]. Suppose that G is a finite group, and p is a prime. Let $S \in Syl_p(G)$, and set $B = N_G(S)$. A *p-minimal subgroup P of G (with respect to B)* is a subgroup of G properly containing B and such that B is contained in a unique maximal subgroup of P. Set

$$\mathscr{M}(G,B) = \{P | P \text{ is a } p\text{-minimal subgroup of } G \text{ (with respect to } B)\},$$

Now a subset of $\mathscr{M}(G,B)$,

$$\mathscr{M}_0 = \{P_i | P_i \in \mathscr{M}(G,B), i \in I\},$$

is called a *minimal parabolic system of characteristic p for G* or a *p-minimal parabolic system of G*, if $G = \langle P_i | i \in I \rangle$ and $G \neq \langle P_j | j \in I \setminus \{i\} \rangle$ for any $i \in I$. The *rank* of $\mathscr{M}_0$ is $|I|$. Provided $B \neq G$, we always have $G = \langle \mathscr{M}(G,B) \rangle$ and so there is always at least one minimal parabolic system of characteristic p for G.

We recall that for H a finite group, $O_p(H)$ is the largest normal p-subgroup of H, and we shall refer to $O_p(H)$ as the p-core of H. Now suppose that $O_p(G) = 1$. For a minimal parabolic system $\mathscr{M}_0 = \{P_i | P_i \in \mathscr{M}(G,B), i \in I\}$ of G we define a B-parabolic system $\mathfrak{X} = \{(P_J, Q_J) | J \subseteq I\}$ by

$$P_J = \begin{cases} \langle P_j | j \in J \rangle & \text{if } \emptyset \neq J \subseteq I; \text{ and} \\ B & \text{if } J = \emptyset. \end{cases}$$

and $Q_J = O_p(P_J)$ for all $J \subseteq I$. (See Definition 1.1 of [16] for a precise description of a B-parabolic system.) Let χ be an irreducible (complex) character of G. Then we say that χ is $\mathfrak{X}$-*cuspidal* if for all $(P_J, Q_J) \in \mathfrak{X}$ with $Q_J \neq 1$ we have

$$\sum_{g \in Q_J} \chi(g) = 0. \tag{1}$$

The condition (1) will be known as the *cuspidal condition on* Q_J and is equivalent to $(\chi_{Q_J}, 1_{Q_J}) = 0$. Should G be a simple group of Lie type of characteristic p, then B would be the Borel subgroup of G and $\{P_J | J \subseteq I\}$ the parabolic subgroups of G (containing B). Further, the Q_J would be the unipotent radical of P_J for $J \subseteq I$. Turning to the sporadic simple groups, the p-minimal parabolic systems were catalogued by Ronan and Stroth [15] for groups whose Sylow p-subgroups are non-cyclic, and there we see that, up to conjugacy, $\mathbb{B}$ has a unique minimal parabolic system for $p = 2$.

In studying cuspidal characters of $\mathbb{B}$ in [16] to check whether or not (1) holds we need to know the $\mathbb{B}$-fusion in the 2-cores Q_J. Such data may be of independent interest and this is what sparked this short note.

2 The Baby Monster

Let $G = \mathbb{B}$, the Baby Monster. There are five conjugacy classes of 2-minimal parabolic subgroups having representatives $P_i \sim [2^{40}].\,\mathrm{Sym}(3)$ for $i = 1, \ldots, 4$ and $P_5 \sim [2^{38}].\,\mathrm{Sym}(5)$. These give rise to a unique 2-minimal parabolic system $\{P_1, P_2, P_3, P_5\}$. The maximal parabolic subgroups of this system are given by $P_{123} \sim 2^{9+16+6+4}.L_4(2)$, $P_{125} \sim 2^{3+32}(L_3(2) \times \mathrm{Sym}(5))$, $P_{135} \sim 2^{2+10+20}(\mathrm{Sym}(3) \times M_{22}2)$ and $P_{235} \sim 2_+^{1+22}.Co_2$. All of these maximal parabolic subgroups are 2-radical. Indeed, from [22] we observe that all 2-parabolic subgroups generated by $P_1, \ldots, P_5$ are 2-radical with the exception of P_3, P_4 and P_{34}. The reader can find further information regarding the structure of the 2-radical parabolic subgroups in [22]. We will consider the 2-maximal parabolic subgroups P_{125}, P_{135}, P_{235} and $P_{1234} \sim 2^{9+16}.Sp_8(2)$ (which does not appear as a parabolic subgroup of the given minimal parabolic system).

We note that for each 2-maximal parabolic subgroup, P, of $\mathbb{B}$, the $O_2(P)$-classes and their associated $\mathbb{B}$ classes may be obtained in the mathematical algebra system GAP [9] using the command `FusionConjugacyClasses()`. This information together with the available representations of $\mathbb{B}$ over $GF(2)$, $GF(3)$ and $GF(5)$ in the computer algebra package MAGMA (see [3], [4], [5] and [6]) that are available in [2] will enable us to determine the fusion of elements within the 2-cores of the maximal parabolic subgroups P_{1234}, P_{135} and P_{235}. For each of these maximal parabolics, an associated representation within MAGMA is also available at [2].

In the case of P_{125}, words for the generators of P_{125} are not available in the online atlas. However, a permutation presentation of this 2-parabolic subgroup is available in the previous version of the online atlas [1].[1]

Before considering each of these maximal parabolics, we record the dimensions of the eigenspaces of elements of order 2, 4 and 8 in $\mathbb{B}$ for the MAGMA representations given in [2]. These are given in Tables 1, 2 and 3 for elements of order 2, 4 and 8 respectively.[2]

$\mathbb{B}$-Conjugacy Class	Dimension of Fixed-Point Space over	
	$GF(2)$	$GF(3)/GF(5)$
$2A$	2510	1939
$2B$	2322	2323
$2C$	2212	2159
$2D$	2202	2195

Table 1: The dimensions of fixed-point spaces for elements of order 2 in $\mathbb{B}$

[1]Having corresponded with Rob Wilson, we have been unable to determine why the permutation representation of P_{125} available in [1] was not transferred to [2].

[2]We note that the codimensions of the eigenspaces over $GF(2)$ have previously been calculated by Wilson in [21].

$\mathbb{B}$-Conjugacy Class	Dimension of Fixed-Point Space over	
	$GF(2)$	$GF(3)/GF(5)$
$4A$	1256	1123
$4B$	1256	1187
$4C$	1178	1171
$4D$	1178	1151
$4E$	1114	1115
$4F$	1168	1155
$4G$	1166	1167
$4H$	1104	1099
$4I$	1106	1079
$4J$	1104	1095

Table 2: The dimensions of fixed-point spaces for elements of order 4 in $\mathbb{B}$

$\mathbb{B}$-Conjugacy Class	Dimension of Fixed-Point Space over	
	$GF(2)$	$GF(3)/GF(5)$
$8A$	596	575
$8B$	632	597
$8C$	632	589
$8D$	592	591
$8E$	632	593
$8F$	590	583
$8G$	560	555
$8H$	590	587
$8I$	584	577
$8J$	558	559
$8K$	552	553
$8L$	584	583
$8M$	552	549
$8N$	552	547

Table 3: The dimensions of fixed-point spaces for elements of order 8 in $\mathbb{B}$

3 Fusion within $P_{1234} \sim 2^{9+16}Sp_8(2)$

There are eight non-trivial P_{1234}-conjugacy classes contained within its 2-core. Representatives of the three largest classes can easily be found using the representation of P_{1234} for MAGMA available in [2]. These classes correspond to the elements of order 4 within $O_2(P_{1234})$. With these classes determined, the five non-trivial classes of involutions may be determined using the fusion data stored in the GAP databases. There are two classes of involutions from $\mathbb{B}$-classes $2B$ and $2D$. The former pair of classes have sizes 255 and 73440, whilst the latter pair of classes have sizes 136 and 1101600. A full summary of the fusion of elements within $O_2(P_{1234})$ is given in Table 4.

$\mathbb{B}$-class	$2A$	$2B$	$2D$	$4B$	$4C$	$4E$
Number of elements in $O_2(P_{1234})$	120	73695	1101736	7833600	7833600	16711680

Table 4: The fusion of elements within the 2-core $O_2(P_{1234})$ of $\mathbb{B}$

4 Fusion within $P_{125} \sim 2^{3+32}(L_3(2) \times \mathrm{Sym}(5))$

A permutation representation of $P_{125} \sim 2^{3+32}(L_3(2) \times \mathrm{Sym}(5))$ on 21504 points for MAGMA is given in [1]. This allows us to determine the P_{125}-classes of elements that are present in $O_2(P_{125})$. By forming a Sylow 2-subgroup, S_{125} of P_{125}, we may coerce representatives of each of these classes into S_{125}. Using the command `LMGSylow` in MAGMA, we can form Sylow 2-subgroups of P_{125} sitting within the matrix representations over $GF(2)$ and $GF(3)$ that are available in [2]. Forming isomorphisms from S_{125} onto each of the Sylow subgroups allows us to then look at the dimension of eigenspaces of the images of the representatives of classes contained in the permutation representation of $O_2(P_{125})$. Appealing to Tables 1, 2 and 3 and utilising the class information that is easily obtained from the permutation representation of P_{125} allows us to determine which $\mathbb{B}$-class each of the P_{125}-classes contained in $O_2(P_{125})$ belongs to. This information is given in Table 5. A summary of the fusion of elements in the maximal 2-parabolic P_{125} is given in Table 6

Order	$\mathbb{B}$-class	Number of P_{125}-classes in $O_2(P_{125})$	Sizes of P_{125}-classes
	$2A$	3	8, 560, 8960
	$2B$	6	7, 120, 1120, 4480, 53760, 107520
	$2C$	2	1146880, 5160960
2	$2D$	19	120, 560, 4480, 6720, 8960, 26880, 53760, 107520, 161280, 322560 (2 classes), 430080, 573440, 860160 (3 classes), 1376256, 2580480, 10321920
	$4A$	9	53760, 107520 (2 classes), 430080 (2 classes), 645120, 860160, 2580480, 10321920
	$4B$	19	35840, 53760, 107520, 322560, 344064, 430080, 645120, 860160 (3 classes), 1032192, 1290240 (2 classes), 1720320, 2580480, 5160960, 6881280 (2 classes), 10321920
	$4C$	16	215040, 860160, 1720320 (2 classes), 5160960 (2 classes), 6881280 (2 classes), 10321920 (2 classes), 13762560, 41287680 (4 classes), 110100480
	$4D$	11	1146880, 3440640, 6881280, 10321920, 13762560, 20643840, 41287680, 82575360 (2 classes), 110100480, 165150720
	$4E$	5	6881280, 13762560 (2 classes), 41287680, 55050240
4	$4F$	30	430080 (2 classes), 573440, 1290240, 1720320 (2 classes), 5160960 (5 classes), 6881280 (2 classes), 10321920 (6 classes), 13762560, 20643840 (2 classes), 41287680 (4 classes), 82575360, 165150720, 330301440 (2 classes)
	$4G$	17	3440640, 5160960, 13762560, 20643840 (3 classes), 27525120, 41287680, 82575360 (3 classes), 165150720 (3 classes), 330301440 (3 classes)
	$4H$	19	13762560 (2 classes), 20643840, 27525120, 41287680, 55050240, 82575360 (4 classes), 110100480, 165150720 (3 classes), 330301440 (3 classes), 660602880, 1321205760
	$4J$	9	55050240, 82575360, 110100480 (2 classes), 165150720, 330301440 (2 classes), 660602880, 1321205760
	$8A$	1	110100480
	$8B$	1	2642411520
	$8C$	1	2642411520
8	$8D$	2	330301440, 1321205760
	$8E$	1	5284823040
	$8F$	2	440401920, 2642411520
	$8H$	2	2642411520, 1321205760
	$8I$	1	3523215360

Table 5: The identification of P_{125}-classes contained in $O_2(P_{125})$ and their corresponding $\mathbb{B}$-classes

$\mathbb{B}$-class	$2A$	$2B$	$2C$	$2D$
Number of elements in $O_2(P_{125})$	9528	167007	6307840	18878056

$\mathbb{B}$-class	$4A$	$4B$	$4C$	$4D$	$4E$
Number of elements in $O_2(P_{125})$	15536640	41678336	338257920	537886720	130744320

$\mathbb{B}$-class	$4F$	$4G$	$4H$	$4I$	$4J$
Number of elements in $O_2(P_{125})$	1236193280	1887191040	4080599040	0	3165388800

$\mathbb{B}$-class	$8A$	$8B$	$8C$	$8D$	$8E$
Number of elements in $O_2(P_{125})$	110100480	2642411520	2642411520	1651507200	5284823040

$\mathbb{B}$-class	$8F$	$8G$	$8H$	$8I$	$8J$
Number of elements in $O_2(P_{125})$	3082813440	0	3963617280	3523215360	0

$\mathbb{B}$-class	$8K$	$8L$	$8M$	$8N$
Number of elements in $O_2(P_{125})$	0	0	0	0

Table 6: The fusion of elements within the 2-core $O_2(P_{125})$ of $\mathbb{B}$

5 Fusion within $P_{135} \sim 2^{2+10+20}(\mathrm{Sym}(3) \times M_{22}2)$

To determine the fusion within P_{135} we use a similar approach to that used for the maximal 2-parabolic P_{125}. Equipped with the permutation representation of P_{135} over 6144 points available in [1] we may determine the P_{135}-conjugacy classes present in $O_2(P_{135})$ in the permutation setting. Forming an isomorphism from a Sylow 2-subgroup of this permutation representation into a Sylow 2-subgroup of the matrix representations of P_{135} over $GF(2)$ and $GF(3)$ sitting within $\mathbb{B}$ (from [2]) allows us to associate these P_{135}-classes with their corresponding $\mathbb{B}$-classes.

We summarise the association between the P_{135}-classes and their corresponding $\mathbb{B}$-classes in Table 7, whilst the fusion of elements of P_{135} is summarised in Table 8.

Order	$\mathbb{B}$-class	Number of P_{135}-classes in $O_2(P_{135})$	Sizes of P_{135}-classes
2	$2A$	2	88, 7392
	$2B$	4	3, 924, 14784, 126720
	$2C$	1	3784704
	$2D$	7	3080,7392,126720,443520,1774080,3784704,7096320
	$4A$	4	236544, 253440, 946176, 7096320
	$4B$	6	236544, 946176, 1774080, 1892352, 7096320, 14192640
	$4C$	3	7096320, 28385280, 113541120
	$4D$	2	7569408, 227082240
	$4E$	1	75694080
	$4F$	5	14192640 (2 classes), 42577920, 113541120, 340623360
	$4G$	3	3784704, 227082240, 681246720
	$4H$	2	75694080, 1362493440
	$4J$	1	908328960

Table 7: The identification of P_{135}-classes contained in $O_2(P_{135})$ and their corresponding $\mathbb{B}$-classes

$\mathbb{B}$-class	$2A$	$2B$	$2C$	$2D$
Number of elements in $O_2(P_{135})$	7480	142431	3784704	13235816

$\mathbb{B}$-class	$4A$	$4B$	$4C$	$4D$	$4E$
Number of elements in $O_2(P_{135})$	8532480	26138112	149022720	234651648	75694080

$\mathbb{B}$-class	$4F$	$4G$	$4H$	$4I$	$4J$
Number of elements in $O_2(P_{135})$	525127680	912113664	1438187520	0	908328960

Table 8: The fusion of elements within the 2-core $O_2(P_{135})$ of $\mathbb{B}$

6 Fusion within $P_{235} \sim 2_+^{1+22}.Co_2$

The extra-special nature of the 2-core $O_2(P_{235})$ combined with its inferior size mean that the calculation of fusion of $O_2(P_{235})$-classes within P_{235} can be obtained as an elementary school exercise using the information available from GAP. This is given in Table 9.

$\mathbb{B}$-class	$2A$	$2B$	$2D$	$4A$	$4B$
Number of elements in $O_2(P_{235})$	4600	93151	4098600	953856	3238400

Table 9: The fusion of elements within the 2-core $O_2(P_{235})$ of $\mathbb{B}$

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