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2018

MIMS EPrint: **2018.19**

ISSN 1749-9097

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July 30, 2018

Abstract

In this paper the notion of an irreducible cuspidal character for finite groups of Lie type is generalized to any finite group. All the irreducible cuspidal characters for the finite sporadic simple groups are then determined.

1 Introduction

For a finite group G, its complex irreducible character table encodes a diverse range of details relating to the structure and properties of G. For example the structure constants may be extracted from the character table (see [7, Theorem 4.2.12]). While there are results connecting character values with the orders of various subgroups of G (see [7, Theorems 4.2.8 and 4.2.11]). Since the birth and rapid development of character theory by Frobenius, Schur and Burnside ([5]), there has been a fruitful interplay between characters and finite groups. A particular jewel in the crown being Frobenius's theorem ([7, Theorem 5.1]), for which no proof without characters is known. This has resulted in extensive efforts to calculate character tables of "interesting" finite groups. Tables for all the sporadic simple groups are to be found in the ubiquitous $\mathbb{ATLAS}$ [4]. While the case when G is a Lie type group is the subject of the mammoth text by Carter [3], and continues to be a very active area of research.

When G is a group of Lie type, a particular type of irreducible character, called a cuspidal character, plays an important role. This is because of the fact that every irreducible character of G is a constituent of the induced character $\phi_{P_J}^G$ for some cuspidal character ϕ of some P_J, where P_J is a proper parabolic subgroup of G (see [3, Chapter 9] for more details). The aim of the present paper is to generalize the notion of a cuspidal character to an arbitrary finite group, and then to determine all irreducible characters for the sporadic simple groups. Since the complex irreducible character tables are known for all the sporadic simple groups, this begs the question as to whether this is a worthwhile enterprise. Unlike the situation of groups of Lie type where the cuspidal characters are being used to determine further irreducible characters, our motivation here is to better understand the sporadic simple groups in a wider context. We shall return to this issue shortly.

The generalization of cuspidal characters is set against the following backdrop.

Definition 1.1. *Suppose that G is a finite group, X a subgroup of G and I an index set with $|I| = n$. Then an X-*parabolic system of rank n *is a set of pairs of subgroups of G, (P_J, Q_J), indexed by subsets J of I such that*

(i) for each $J \subseteq I$, $X \leq P_J$, $Q_J \trianglelefteq P_J$;

(ii) for $K \subseteq J \subseteq I$, $Q_J \leq Q_K$;

(iii) $P_I = G$ and $Q_I = 1$; and

(iv) $X = P_\emptyset$.

We shall write $\mathfrak{X} = \{(P_J, Q_J)|J \subseteq I\}$ and note that by part (ii) of Definition 1.1 all Q_J are subgroups of $Q_\emptyset$ and that $Q_\emptyset \trianglelefteq P_\emptyset = X$. We allow the possibility that $(P_J, Q_J) = (P_K, Q_K)$ with $J \neq K$, but this will not arise in most of the cases that follow. If our index set is $I = \{1, 2, \ldots, n\}$ for some $n \geq 1$, then given a subset $\{i_1, i_2, \ldots, i_r\} \subseteq I$ with $i_j < i_{j+1}$ for all j, we will often denote the subgroups $P_{\{i_1,i_2,\ldots,i_r\}}$ and $Q_{\{i_1,i_2,\ldots,i_r\}}$ by $P_{i_1 i_2 \cdots i_r}$ and $Q_{i_1 i_2 \cdots i_r}$ respectively. Given an X-parabolic system $\mathfrak{X} = \{(P_J, Q_J)|J \subseteq I\}$ of a finite group G and $J \subseteq I$, we set $\overline{P}_J := P_J/Q_J$. Furthermore, for any subgroup $Q_J \leq Y \leq P_J$, we use the standard bar notation $\overline{Y} := Y/Q_J$. We may use $\mathfrak{X}$ to form an $\overline{X}$-parabolic system, $\overline{\mathfrak{X}}_J$, of rank $|J|$ for $\overline{P}_J$ given by

$$\overline{\mathfrak{X}}_J = \left\{(\overline{P}_J \cap \overline{P}_K, \overline{Q}_K)|(P_K, Q_K) \in \mathfrak{X}, K \subseteq J\right\}.$$

We now describe a particular type of X-parabolic system of interest here. Suppose that G is a finite group, p a prime and $S \in Syl_p(G)$. Set $B = N_G(S)$. A subgroup P of G is called *p-minimal (with respect to B)* if B is a proper subgroup of P and B is contained in a unique maximal subgroup of P. We recall that for H a finite group, $O_p(H)$ is the largest normal p-subgroup of H, and we shall refer to $O_p(H)$ as the p-core of H. Defining

$$\mathscr{M}(G, B) = \{P|P \text{ is a } p\text{-minimal subgroup of } G \text{ (with respect to } B)\},$$

then a set

$$\mathscr{M}_0 = \{P_i|P_i \in \mathscr{M}(G, B), i \in I\}$$

is called a *minimal parabolic system of characteristic p for G* or a *p-minimal parabolic system of G*, if $G = \langle P_i|i \in I\rangle$ and $G \neq \langle P_j|j \in I \setminus \{i\}\rangle$ for any $i \in I$. The *rank* of $\mathscr{M}_0$ is $|I|$. We call $\mathscr{M}_0$ a *geometric* p-minimal parabolic system if for all $J, K \subseteq I$ we have $P_{J\cap K} = P_J \cap P_K$. Otherwise $\mathscr{M}_0$ is called *non-geometric*. Provided $B \neq G$, we always have $G = \langle \mathscr{M}(G, B)\rangle$ and so there is always at least one minimal parabolic system of characteristic p for G.

Now suppose that $O_p(G) = 1$. For a minimal parabolic system $\mathscr{M}_0 = \{P_i|P_i \in \mathscr{M}(G, B), i \in I\}$ of G we define a B-parabolic system $\mathfrak{X} = \{(P_J, Q_J)|J \subseteq I\}$ by

$$P_J = \begin{cases} \langle P_j|j \in J\rangle & \text{if } \emptyset \neq J \subseteq I; \text{ and} \\ B & \text{if } J = \emptyset. \end{cases}$$

and $Q_J = O_p(P_J)$ for all $J \subseteq I$. Should G be a simple group of Lie type of characteristic p, then B would be the Borel subgroup of G and $\{P_J | J \subseteq I\}$ the parabolic subgroups of G (containing B). Further, Q_J would be the unipotent radical of P_J for $J \subseteq I$. Turning to the sporadic simple groups, the p-minimal parabolic systems were catalogued by Ronan and Stroth [14] for groups whose Sylow p-subgroups are non-cyclic. (We note that in [14] they require their p-minimal subgroups to have a non-trivial p-core, which we do not need to assume here.)

For G a finite group, $\mathrm{Irr}(G)$ will denote the set of complex irreducible characters of G. We now give the promised generalization of cuspidal characters.

Definition 1.2. *Let $\mathfrak{X}$ be an X-parabolic system of G where $X \leq G$, and let $\chi \in \mathrm{Irr}(G)$. Then χ is called* $\mathfrak{X}$-cuspidal *if for all $(P_J, Q_J) \in \mathfrak{X}$ with $Q_J \neq 1$ we have*

$$\sum_{g \in Q_J} \chi(g) = 0. \tag{1}$$

The condition (1) will be known as the *cuspidal condition on* Q_J and is equivalent to $(\chi_{Q_J}, 1_{Q_J}) = 0$. In an abuse of terminology, we will also sometimes refer to the cuspidal relation holding for P_J when (1) occurs. Clearly, when the index set $I = \emptyset$, we have $G = P_\emptyset = X$ and $Q_\emptyset = 1$, and hence every irreducible character is vacuously $\mathfrak{X}$-cuspidal. When $\mathfrak{X}$ is a B-parabolic system associated to a p-minimal parabolic system of G, then any $\mathfrak{X}$-cuspidal character will also be called a *p-cuspidal character* of G. Consulting [3, Proposition 9.1.1] we see that this generalizes the situation when G is a simple Lie type group of characteristic p.

Our main theorem is as follows – in its statement the names for the sporadic simple groups and their irreducible characters are as they appear in [4].

Theorem 1.3. *Suppose that G is a finite sporadic simple group with $\mathfrak{X}$ an X-parabolic system of G given by one of the p-minimal parabolic systems of G. Let $\chi \in \mathrm{Irr}(G)$. Then χ is $\mathfrak{X}$-cuspidal for the following pairs $(G, \{\chi_j\})$.*

(i) $p = 2$. $(M_{11}, \{\chi_3, \chi_4\})$, $(M_{22}, \{\chi_3, \chi_4\})$, $(M_{23}, \{\chi_3, \chi_4\})$, $(M_{24}, \{\chi_3, \chi_4\})$, $(Co_2, \{\chi_3, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{16}, \chi_{31}, \chi_{32}\})$, $(Co_1, \{\chi_2, \chi_8, \chi_{11}\})$, $(Ru, \{\chi_2, \chi_3\})$, (Th, χ_2), $(J_4, \{\chi_2, \chi_3\})$.

(ii) $p = 3$. $(M_{11}, \{\chi_6, \chi_7\})$, $(M_{12}, \{\chi_4, \chi_5\})$, $(Fi_{22}, \{\chi_2\})$, $(Th, \{\chi_2\})$.

(iii) $p \geq 5$. $(J_2, \{\chi_6\})$, $(HN, \{\chi_4\})$, $(Th, \{\chi_2\})$, $(Ly, \{\chi_2, \chi_3\})$ $(p = 5)$; $(He, \{\chi_2, \chi_3\})$ $(p = 7)$; $(M_{11}, \{\chi_2, \chi_3, \chi_4\})$ $(p = 11)$; $(M_{23}, \{\chi_2\})$ $(p = 23)$.

We direct the reader to Tables 1, 2 and 3 for which $\mathfrak{X}$-parabolic systems arise in Theorem 1.3 as well as the degrees of the various irreducible $\mathfrak{X}$-cuspidal characters. Note that in these tables we have used the same notation for the p-minimal subgroups as in [14] when the Sylow p-subgroup of G is non-cyclic.

We now return to the question of motivation for this study. As we note shortly, the existence of a number of the $\mathfrak{X}$-cuspidal irreducible characters for the sporadic simple groups coincides with interesting and exceptional behaviour.

For example, when $p = 3$ and $G \cong Th$, we have that χ_2 is an irreducible 3-cuspidal character of degree 248. This representation was instrumental in the original construction of Th in which Th was shown to be a subgroup of $E_8(3)$, but not a subgroup of $E_8(q)$ for any other prime $q \neq 3$ (see [16], [17]). In [11], Margolin looked at a geometry for the Mathieu group M_{24}. Margolin's interest stemmed from the two 1333-dimensional irreducible $GF(2)J_4$-representations. Since $2^{11} : M_{24}$ is a maximal subgroup of J_4, Margolin considered the restriction of these representations to $2^{11} : M_{24}$, namely as a faithful 1288-dimensional representation and a 45-dimensional representation having kernel 2^{11}. Hence Margolin sought to find a simple explanation for this 45-dimensional representation and this resulted in the construction of a geometry. We note that both 1333-dimensional J_4-characters are 2-cuspidal, as are both of the resulting irreducible M_{24}-characters of degree 45, along with their irreducible restrictions to M_{22} and M_{23}. So the work presented here, may further highlight certain characters and/or $\mathfrak{X}$-parabolic systems (and associated geometries) where one might prospect for interesting nuggets.

The sporadic simple groups are something of an unruly bunch, so we can not expect the light cuspidal characters shines on them to reveal all their secrets. Indeed, looking at Theorem 1.3(i) we see that though M^cL (and to a lesser extent Fi_{23}) has more minimal parabolic systems than you can shake a stick at, it (and Fi_{23}) fail to have any cuspidal characters. Moreover, the minimal parabolic systems for M_{24} and He (with $p = 2$) have the same diagram (see [14]) yet M_{24} has 2-cuspidal characters, but He does not. For the tuples (G, p) given by $(M_{11}, 11)$ and $(M_{23}, 23)$, it is unlikely that the p-cuspidal characters of G will give rise to any interesting geometries. Indeed, in both cases a Sylow p-subgroup of G is cyclic of order p, and the resulting p-cuspidal characters have degree $p-1$. Similarly, when (G, p) is $(M_{11}, 3)$ or $(Th, 5)$, there is a unique class of elements of order p, and a Sylow p-subgroup has exponent p. Hence it is improbable that the resulting p-cuspidal characters lead to interesting geometries. As an aside, we mention that representation theory and minimal parabolic systems have interacted in the modular case (see [13]). Finally we note that Co_1 and Co_2 are the proud owners of many irreducible cuspidal characters for $p = 2$, and these are definitely worthy of further scrutiny.

This paper is organized as follows. Section 2 consists of some general results on cuspidal characters. Theorem 2.3 and Proposition 2.5 are the exact analogues of [3, 9.1.3] and [3, 9.1.2] respectively. But other results do not generalize from the Lie type group case (see Example 2.4). The last two results of this section are useful for our later calculations. Sections 3, 4, 5 and 6 determine the p-cuspidal characters for the sporadic groups in the cases where, respectively, $p = 2$, $p = 3$, $p = 5$ and $p > 5$.

Family	Group	2-Minimal Parabolic System	Rank	Geometric/ Non-Geometric	2-Cuspidal Characters (character degrees)
Mathieu Groups	M_{11}	$\{P_1 \sim 2_-^{1+2}.\,\mathrm{Sym}(3), P_2 \sim 3^2.SD_{16}\}$	2	Geometric	None
		$\{P_1 \sim 2_-^{1+2}.\,\mathrm{Sym}(3), P_3 \sim \mathrm{Alt}(6).2\}$	2	Geometric	None
		$\{P_2 \sim 3^2.SD_{16}, P_3 \sim \mathrm{Alt}(6).2\}$	2	Geometric	χ_3 (10), χ_4 (10)
	M_{12}	$\{P_1 \sim 4^2.2.\,\mathrm{Sym}(3), P_2 \sim 2_+^{1+4}.\,\mathrm{Sym}(3)\}$	2	Geometric	None
	M_{22}	$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_2 \sim 2^4.\,\mathrm{Sym}(5)\}$	2	Geometric	χ_3 (45), χ_4 (45)
	M_{23}	$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_3 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_7 \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Geometric	χ_3 (45), χ_4 (45)
		$\{P_3 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_4 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_7 \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Geometric	χ_3 (45), χ_4 (45)
		$\{P_2 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_3 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_7 \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Geometric	χ_3 (45), χ_4 (45)
		$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_6 \sim 2^4.\,\mathrm{Sym}(5), P_7 \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	χ_3 (45), χ_4 (45)
		$\{P_2 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_6 \sim 2^4.\,\mathrm{Sym}(5), P_7 \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	χ_3 (45), χ_4 (45)
		$\{P_3 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_6 \sim 2^4.\,\mathrm{Sym}(5), P_7 \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	χ_3 (45), χ_4 (45)
		$\{P_4 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_6 \sim 2^4.\,\mathrm{Sym}(5), P_7 \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	χ_3 (45), χ_4 (45)
	M_{24}	$\{P_1 \sim 2^{6+3}.\,\mathrm{Sym}(3), P_2 \sim 2^{6+3}.\,\mathrm{Sym}(3), P_3 \sim 2^{6+3}.\,\mathrm{Sym}(3)\}$	3	Geometric	χ_3 (45), χ_4 (45)
Leech Lattice and Conway Groups	HS	$\{P_1 \sim 4.2^4.\,\mathrm{Sym}(5), P_2 \sim 4^3.2^2.\,\mathrm{Sym}(3)\}$	2	Geometric	None
	J_2	$\{P_1 \sim 2^{2+4}.3.\,\mathrm{Sym}(3), P_2 \sim 2_+^{1+4}.L_2(4)\}$	2	Geometric	None
	Co_1	$\{P_1 \sim [2^{20}].\,\mathrm{Sym}(3), P_2 \sim [2^{20}].\,\mathrm{Sym}(3), P_3 \sim [2^{20}].\,\mathrm{Sym}(3), P_4 \sim [2^{20}].\,\mathrm{Sym}(3)\}$	4	Geometric	χ_2 (276), χ_8 (37674), χ_{11} (94875)
	Co_2	$\{P_1 \sim [2^{15}].\,\mathrm{Sym}(5), P_2 \sim [2^{17}].\,\mathrm{Sym}(3), P_3 \sim [2^{17}].\,\mathrm{Sym}(3)\}$	3	Geometric	χ_3 (253), χ_{10} (9625), χ_{11} (9625), χ_{12} (10395), χ_{13} (10395), χ_{16} (31625), χ_{31} (239085), χ_{32} (239085)
	Co_3	$\{P_1 \sim 2^{4+4+1}.\,\mathrm{Sym}(3), P_2 \sim 2^{4+4+1}.\,\mathrm{Sym}(3), P_3 \sim 2^{4+4+1}.\,\mathrm{Sym}(3)\}$	3	Geometric	None
	M^cL	$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5 \sim 2^4.\,\mathrm{Sym}(5), P_5^\sigma \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5 \sim 2^4.\,\mathrm{Sym}(5), P_5^\sigma \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_2 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5 \sim 2^4.\,\mathrm{Sym}(5), P_5^\sigma \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_2^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5 \sim 2^4.\,\mathrm{Sym}(5), P_5^\sigma \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_3 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5 \sim 2^4.\,\mathrm{Sym}(5), P_5^\sigma \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_2^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5^\sigma \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_2 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5 \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_3 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5^\sigma \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_3 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5 \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_2 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5^\sigma \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Geometric	None
		$\{P_1^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_2^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5 \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Geometric	None
		$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_1^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5 \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Geometric	None
		$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_1^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_5^\sigma \sim 2^4.\,\mathrm{Sym}(5)\}$	3	Geometric	None
		$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_1^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_2 \sim 2^{4+2}.\,\mathrm{Sym}(3)\}$	3	Geometric	None
		$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_1^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_2^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3)\}$	3	Geometric	None
		$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_1^\sigma \sim 2^{4+2}.\,\mathrm{Sym}(3), P_3 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_4 \sim 2^{4+2}.\,\mathrm{Sym}(3)\}$	4	Geometric	None
	Suz	$\{P_1 \sim 2^{4+6+1}.L_2(4), P_2 \sim 2^{4+6+2}.(3 \times \mathrm{Sym}(3)), P_3 \sim 2^{6+4+2}.(3 \times \mathrm{Sym}(3))\}$	3	Geometric	None

Table 1: The 2-minimal parabolic systems and their associated 2-cuspidal characters for the Mathieu, Leech Lattice and Conway sporadic simple groups.

Family	Group	2-Minimal Parabolic System	Rank	Geometric/ Non-Geometric	2-Cuspidal Characters (character degrees)
Monster Group and Subquotients	He	$\{P_1 \sim 2^{6+3}.\operatorname{Sym}(3), P_2 \sim 2^{6+3}.\operatorname{Sym}(3), P_4 \sim 2^{6+3}.\operatorname{Sym}(3)\}$	3	Geometric	None
		$\{P_1 \sim 2^{6+3}.\operatorname{Sym}(3), P_3 \sim 2^{6+3}.\operatorname{Sym}(3), P_4 \sim 2^{6+3}.\operatorname{Sym}(3)\}$	3	Geometric	None
	HN	$\{P_1 \sim 2_+^{1+8}.\operatorname{Alt}(5) \wr \mathbb{Z}_2, P_2 \sim 2^{2+3+6+2}.3.\operatorname{Sym}(3)\}$	2	Geometric	None
	Th	$\{P_1 \sim 2^{1+8}.\operatorname{Alt}(9), P_2 \sim 2^{5+6+2+1}.\operatorname{Sym}(3)\}$	2	Geometric	χ_2 (248)
	Fi_{22}	$\{P_1 \sim [2^{16}].\operatorname{Sym}(3), P_2 \sim [2^{16}].\operatorname{Sym}(3), P_3 \sim [2^{14}].\operatorname{Sym}(5)\}$	3	Geometric	None
	Fi_{23}	$\{P_1 \sim [2^{17}].\operatorname{Sym}(3), P_3 \sim [2^{17}].\operatorname{Sym}(3), P_5 \sim [2^{17}].\operatorname{Sym}(3), P_8 \sim [2^{15}].\operatorname{Sym}(5)\}$	4	Geometric	None
		$\{P_1 \sim [2^{17}].\operatorname{Sym}(3), P_4 \sim [2^{17}].\operatorname{Sym}(3), P_5 \sim [2^{17}].\operatorname{Sym}(3), P_8 \sim [2^{15}].\operatorname{Sym}(5)\}$	4	Geometric	None
		$\{P_1 \sim [2^{17}].\operatorname{Sym}(3), P_2 \sim [2^{17}].\operatorname{Sym}(3), P_5 \sim [2^{17}].\operatorname{Sym}(3), P_8 \sim [2^{15}].\operatorname{Sym}(5)\}$	4	Non-Geometric	None
		$\{P_1 \sim [2^{17}].\operatorname{Sym}(3), P_7 \sim [2^{15}].\operatorname{Sym}(5), P_8 \sim [2^{15}].\operatorname{Sym}(5)\}$	3	Non-Geometric	None
	Fi'_{24}	$\{P_1 \sim [2^{20}].L_2(2), P_2 \sim [2^{20}].L_2(2), P_3 \sim [2^{20}].L_2(2), P_4 \sim [2^{20}].L_2(2)\}$	4	Geometric	None
	$\mathbb{B}$	$\{P_1 \sim [2^{40}].\operatorname{Sym}(3), P_2 \sim [2^{40}].\operatorname{Sym}(3), P_3 \sim [2^{40}].\operatorname{Sym}(3), P_5 \sim [2^{38}].\operatorname{Sym}(5)\}$	4	Geometric	None
	$\mathbb{M}$	$\left\{\begin{array}{c} P_1 \sim [2^{45}].L_2(2), P_2 \sim [2^{45}].L_2(2), P_3 \sim [2^{45}].L_2(2) \\ P_4 \sim [2^{45}].L_2(2), P_5 \sim [2^{45}].L_2(2) \end{array}\right\}$	5	Geometric	None
Pariahs	J_1	$\{J_1\}$	1	Geometric	None
	$O'N$	$\{P_1 \sim 4^3.2^2.\operatorname{Sym}(3), P_2 \sim 4.L_3(4).2\}$	2	Geometric	None
	J_3	$\{P_1 \sim 2^{2+4}.(3 \times \operatorname{Sym}(3)), P_2 \sim 2_-^{1+4}.L_2(4)\}$	2	Geometric	None
	Ru	$\{P_1 \sim 2^{5+6}.\operatorname{Sym}(5), P_2 \sim 2^{5+6+2}.\operatorname{Sym}(3)\}$	2	Geometric	χ_2 (378), χ_3 (378)
	J_4	$\{P_1 \sim [2^{20}].\operatorname{Sym}(3), P_2 \sim [2^{20}].\operatorname{Sym}(3), P_3 \sim [2^{18}].\operatorname{Sym}(5)\}$	3	Geometric	χ_2 (1333), χ_3 (1333)
	Ly	$\{P_1 \sim [2^7].\operatorname{Sym}(3), P_2 \sim [2^5].\operatorname{Sym}(5)\}$	2	Geometric	None
		$\{P_1 \sim [2^7].\operatorname{Sym}(3), P_3 \sim 2.\operatorname{Sym}(9)\}$	2	Geometric	None

Table 2: The 2-minimal parabolic systems and their associated 2-cuspidal characters for the Monster group, its subquotients and the pariah sporadic simple groups.

Family	Group	Prime, p	p-Minimal Parabolic System	Rank	Geometric/ Non-Geometric	p-Cuspidal Characters (character degrees)
Mathieu Groups	M_{11}	3	$\{M_{11}\}$	1	Geometric	χ_6 (16), χ_7 (16)
		11	$\{M_{11}\}$	1	Geometric	χ_2 (10), χ_3 (10), χ_4 (10)
	M_{12}	3	$\{P_1 \sim 3^2.GL_2(3), P_2 \sim 3^2.GL_2(3)\}$	2	Geometric	χ_4 (16), χ_5 (16)
	M_{23}	23	$\{M_{23}\}$	1	Geometric	χ_2 (22)
Leech Lattice and Conway Groups	J_2	5	$\{J_2\}$	1	Geometric	χ_6 (36)
Monster Group and Subquotients	He	7	$\{He\}$	1	Geometric	χ_2 (51), χ_3 (51)
	HN	5	$\{P_1 \sim 5^{1+4}.(2^{1+4}.5.4), P_2 \sim 5^{2+1+2}.4.\,\mathrm{Alt}(5)\}$	2	Geometric	χ_4 (760)
	Th	3	$\{P_1 \sim 3^{(1+2)+4+2}.GL_2(3), P_2 \sim 3^{(2+3)+4}.GL_2(3)\}$	2	Geometric	χ_2 (248)
		5	$\{Th\}$	1	Geometric	χ_2 (248)
	Fi_{22}	3	$\{P_1 \sim [3^8].2.PGL_2(3), P_2 \sim [3^8].2.PGL_2(3), P_3 \sim [3^8].2.PGL_2(3)\}$	3	Geometric	χ_2 (78)
Pariahs	Ly	5	$\{P_1 \sim 5_+^{1+4}.4.PGL_2(5), P_2 \sim 5^{3+2}.4.PGL_2(5), P_3 \sim 5_+^{1+4}.4.PGL_2(5)\}$	3	Geometric	χ_2 (2480) χ_3 (2480)

Table 3: The p-cuspidal characters and associated p-minimal parabolic systems of the sporadic simple groups in the case that $p > 2$.

2 Elementary Properties of $\mathfrak{X}$-Cuspidal Characters

We recall the notion of the intertwining number of two modules.

Definition 2.1. *[10] Let F be a field of characteristic 0, with algebraic closure $\overline{F}$, let G be a finite group and let V and W be $\overline{F}G$-modules. The* intertwining number*, denoted $i(V,W)$, is defined by*

$$i(V,W) := \dim_F \operatorname{Hom}_{\overline{F}G}(V,W)$$

The intertwining number of modules will be of importance due to its connection with the inner product of the associated characters.

Theorem 2.2. *[10, Chapter 3, Theorem 1.1] Let F be an arbitrary field of characteristic 0 and let λ and μ be arbitrary characters of G afforded by $\overline{F}G$-modules V and W respectively. Then*

$$(\lambda, \mu) = i(V,W).$$

We may use Theorem 2.2 to prove an analogue of Proposition 9.1.3 of [3], the proof of which is almost identical to that used in Carter's Proposition.

Theorem 2.3. *Let $\mathfrak{X}$ be an X-parabolic system of G and $\chi \in \operatorname{Irr}(G)$. Then there exists $(P_J, Q_J) \in \mathfrak{X}$ and an $\overline{\mathfrak{X}}_J$-cuspidal character ψ of $\overline{P_J} = P_J/Q_J$ such that $(\chi, \psi^G) \neq 0$.*

Proof. Let $\mathcal{S} = \{J \subseteq I | (\chi_{Q_J}, 1_{Q_J}) \neq 0\}$. Note that $\mathcal{S} \neq \emptyset$ as $Q_I = 1$. Let J be a minimal element of $\mathcal{S}$ and let V be an irreducible $\mathbb{C}G$-module that affords χ. Define

$$V' = \{v \in V | v \cdot u = v \text{ for all } u \in Q_J\}.$$

By Theorem 2.2, as $(\chi_{Q_J}, 1_{Q_J}) \neq 0$, there exists a non-zero $\mathbb{C}Q_J$-homomorphism from the trivial $\mathbb{C}Q_J$-module to V, and hence V' is non-empty.

Clearly V' is a linear subspace of V, and given $g \in P_J$ and $u \in Q_J$ we have that

$$(vg)u = vgug^{-1}g = vg,$$

as $Q_J \trianglelefteq P_J$. Hence V' is a $\mathbb{C}P_J$-module.

Consider V' as a $\mathbb{C}\overline{P}_J$-module, having associated character $\phi = \sum_i \phi_i$ (with the ϕ_i irreducible $\mathbb{C}\overline{P}_J$-characters). So V' affords $\phi_{P_J} = \sum_i (\phi_i)_{P_J}$ and V affords the character χ_{P_J}. Now V' is a $\mathbb{C}P_J$-submodule of V, hence each $(\phi_i)_{P_J}$ is a component of χ_{P_J}. Consequently

$$((\phi_i)^G, \chi) = ((\phi_i)_{P_J}, \chi_{P_J}) \neq 0,$$

and χ is a component of $(\phi_i)^G$. Thus it remains to prove that ϕ_i is a cuspidal character.

If ϕ_i is not cuspidal, then $((\phi_i)_{Q_K}, 1_{Q_K}) \neq 0$ for some $K \subsetneq J$. It follows that $\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}Q_K}(1, V) \neq 0$ and hence $(\chi_{Q_K}, 1_{Q_K}) \neq 0$. Hence $K \in \mathcal{S}$, contradicting the minimality of J. Thus the result holds true. $\square$

We illustrate this behaviour with an example.

Example 2.4. $G = \mathrm{Alt}(7)$ *has a 2-minimal parabolic system of rank 2 given by* $\{P_1, P_2\}$ $(\subseteq \mathscr{M}(G,B))$ *with* $B \cong \mathrm{Dih}(8)$, $P_1 \cong \mathrm{Dih}(8) : C_3$ *and* $P_2 \cong \mathrm{Sym}(4)$. *Take* $B = \langle (1,2)(3,4), (1,3)(5,6)\rangle$, $P_1 = \langle B, (5,6,7)\rangle$ *and* $P_2 = \langle B, (1,2,5)(3,4,6)\rangle$. *Let* $\mathfrak{X}$ *be the* B*-parabolic system given by* $\{(P_J, Q_J) | J \subseteq \{1,2\}\}$ *where* $P_\emptyset = B$. *So* $Q_\emptyset = B$, $Q_1 = \langle (1,2)(3,4), (1,3)(2,4)\rangle$, $Q_2 = \langle (1,3)(5,6), (2,4)(5,6)\rangle$ *and* $Q_{\{1,2\}} = 1$. *It follows using [4] that for* $i = 1, 2$ *we have*

$$\sum_{g \in Q_i} \chi(g) = \chi(1) + 3\chi(2A) \neq 0$$

for any $\chi \in \mathrm{Irr}(G)$. *Thus* G *has no* $\mathfrak{X}$*-cuspidal characters.*

For $\overline{P_i} = P_i/Q_i \cong \mathrm{Sym}(3)$ *we have that* $\overline{Q_\emptyset} \cong C_2$. *It follows that there is one* $\overline{\mathfrak{X}_i}$*-cuspidal character, namely the sign character. We denote this character by* ϕ_i *and also think of it as a* P_i*-character. Using the notation from [4], calculations show that the constituent characters of* ϕ_1^G *are* $\chi_3, \chi_4, \chi_7, \chi_9$. *Meanwhile the constituent characters of* ϕ_2^G *are* $\chi_3, \chi_4, \chi_5, \chi_7, \chi_8, \chi_9$.

For $\overline{P_\emptyset} = 1$ *we note that trivial character will be* $\overline{\mathfrak{X}_\emptyset}$*-cuspidal, and it lifts to the trivial character* 1_B. *We have that the constituent characters of* 1_B^G *are* $\chi_1, \chi_2, \chi_5, \chi_6, \chi_7, \chi_8, \chi_9$. *Thus we observe - not very surprisingly - that Proposition 9.1.5 of [3] does not extend to our more general situation, since* $(\phi_1^G, \phi_2^G) = 5$, *whilst* $\phi_1^G \neq \phi_2^G$. •

Although by definition, to determine whether or not a character is cuspidal we must check the cuspidal condition for every subgroup in the X-parabolic system, we shall shortly see that this is not actually necessary. First we give the analogue of [3, Proposition 9.1.2].

Proposition 2.5. *Let* G *be a group,* $X \leq G$ *and* $\chi \in \mathrm{Irr}(G)$. *If* $\mathfrak{X}$ *is an* X*-parabolic system of* G *of rank* n *having underlying indexing set* I*, then the following are equivalent:*

(i) χ *is a* $\mathfrak{X}$*-cuspidal character of* G.

(ii) $(\chi_{Q_J}, 1_{Q_J}) = 0$ *for all* $J \subseteq I$ *such that* $Q_J \neq 1$.

(iii) $(\chi, 1_{Q_J}^G) = 0$ *for all* $J \subseteq I$ *such that* $Q_J \neq 1$.

(iv) $\sum_{x \in Q_J} \chi(xg) = 0$ *for all* $J \subseteq I$ *such that* $Q_J \neq 1$ *and all* $g \in G$.

(v) $\sum_{x \in Q_J} \chi(gx) = 0$ *for all* $J \subseteq I$ *such that* $Q_J \neq 1$ *and all* $g \in G$.

Proof. $(i) \Rightarrow (ii)$. Assume that χ is a $\mathfrak{X}$-cuspidal character of G. Thus for each pair of subgroups (P_J, Q_J) for $J \subseteq I$ we have that either $Q_J = 1$ or

$$\sum_{x \in Q_J} \chi(x) = 0.$$

In particular

$$\sum_{x \in Q_J} \chi(x) 1_{Q_J}(x) = 0$$

and hence $(\chi_{Q_J}, 1_{Q_J}) = 0$ for all $J \subseteq I$ such that $Q_J \neq 1$.

(ii) $\Rightarrow$ *(iv)* Let Q_J be such that $Q_J \neq 1$ (if no such Q_J exists, the result is vacuously true). Let ρ be an irreducible representation corresponding to χ, let ρ' be an irreducible constituent of $\rho|_{Q_J}$ and let d denote the degree of ρ'. The module corresponding to ρ' has basis $\{e_1, \ldots, e_d\}$ and hence we may define coefficient functions ρ'_{ij} for $i, j = 1, \ldots, d$ by

$$e_i g = \sum_{j=1}^{d} \rho'_{ij}(g) e_j.$$

By the orthogonality relations for the coefficient functions (as given in [3, Section 6.1]), it follows that

$$(\rho'_{ij}, (1_{Q_J})_{11}) = 0$$

for all $i, j = 1, \ldots, d$, as 1_{Q_J} is not an irreducible constituent of χ_{Q_J}. Thus

$$\sum_{x \in Q_J} \rho'_{ij}(x) = 0$$

for all i, j and hence

$$\sum_{x \in Q_J} \rho'(x) = 0$$

for all i, j. Since this holds for all irreducible components ρ' of χ_{Q_J} we deduce that

$$\sum_{x \in Q_J} \rho(x) = 0. \tag{2}$$

Now let $g \in G$ be given. Multiplying (2) on the right by $\rho(g)$ gives

$$\sum_{x \in Q_J} \rho(xg) = \left(\sum_{x \in Q_J} \rho(x) \right) \rho(g) = 0.$$

Consequently, taking traces we obtain

$$\sum_{x \in Q_J} \chi(xg) = 0.$$

(iv) $\Rightarrow$ *(i)* Taking $g = 1$ we see that χ is a $\mathfrak{X}$-cuspidal character of G.

(ii) $\Rightarrow$ *(v)* $\Rightarrow$ *(i)* This follows analogously by multiplying on the left by $\rho(g)$ in (2).

(ii) $\Leftrightarrow$ *(iii)* This follows by Frobenius reciprocity. $\square$

Proposition 2.5 infers that we only have to check that the cuspidal condition holds for certain "maximal" subgroups of a parabolic system to ascertain whether a character is cuspidal.

Corollary 2.6. *Let G be a group, $X \leq G$, $I = \{1, \ldots, n\}$ and let $\mathfrak{X} = \{(P_J, Q_J) | J \subseteq I\}$ be an X-parabolic system of G. Define*

$$\mathfrak{Y} := \{J \subseteq I | Q_J \neq 1 \text{ and if } J \subsetneq K \subseteq I, \text{ then } Q_K = 1\}.$$

Then $\chi \in \mathrm{Irr}(G)$ is cuspidal precisely when

$$\sum_{x \in Q_J} \chi(x) = 0$$

for all Q_J such that $J \in \mathfrak{Y}$.

Proof. The condition is clearly necessary. To see that it is sufficient, let $J' \subseteq I$ be such that $Q_{J'} \neq 1$. We shall show that

$$\sum_{x \in Q_{J'}} \chi(x) = 0.$$

Since $Q_{J'} \neq 1$, we see that $J' \subsetneq I$ and there exists some $J \in \mathfrak{Y}$ such that $J' \subseteq J$. Consequently $Q_J \leq Q_{J'}$. By assumption

$$\sum_{x \in Q_J} \chi(x) = 0,$$

and so $(\chi|_{Q_J}, 1_{Q_J}) = 0$. The proof of Proposition 2.5 asserts that

$$\sum_{x \in Q_J} \chi(xg) = 0$$

for all $g \in G$.

Let T denote a right transversal of Q_J in $Q_{J'}$. Then

$$\sum_{x \in Q_{J'}} \chi(q) = \sum_{t \in T} \left(\sum_{x \in Q_J} \chi(xt) \right) = 0$$

as required. □

The final result that we will use in classifying the p-cuspidal characters of the sporadic simple groups concerns irreducible characters of odd degree.

Lemma 2.7. *Let G be a finite group and p an odd prime such that $|G| = p^a m$ for some $a \geq 1$ with $(p, m) = 1$. Assume that G has a p-minimal parabolic system containing a parabolic subgroup with non-trivial p-core. If for each G-conjugacy class, C, of elements of order p^b for $b \leq a$ and all $g \in C$ we have that*

$$\langle g \rangle \cap \left\{ g \in G \mid |g| = p^b \right\} \subseteq C, \tag{3}$$

then every p-cuspidal character of G has even degree.

Proof. Assume that condition (3) holds for all non-trivial powers of p. Then a non-trivial p-core, Q, of a parabolic subgroup will intersect every conjugacy class of p-elements in a set of even order. Thus if the degree of $\chi \in \mathrm{Irr}(G)$ is odd, then the same is true of

$$\sum_{g \in Q} \chi(g),$$

and hence χ is not a p-cuspidal character of G. □

3 2-Cuspidal Characters

We now work systematically through the sporadic simple groups, determining for each group G and each 2-minimal parabolic system of G, which characters $\chi \in \mathrm{Irr}(G)$ are 2-cuspidal. A summary of our results is given in Tables 1 and 2. Throughout, the notation $\chi_i \in \mathrm{Irr}(G)$ is the same as that used in [4]. We shall also use the standard notation from [4] for the conjugacy classes of G.

3.1 The Mathieu Groups

M_{11}

There are three 2-minimal parabolic subgroups of M_{11}, namely

$$P_1 \sim 2_-^{1+2}.\,\mathrm{Sym}(3), \qquad P_2 \sim 3^2.SD_{16}, \qquad \text{and} \qquad P_3 \sim \mathrm{Alt}(6).2,$$

(where SD_{16} is the semidihedral group of order 16) and these give rise to three 2-minimal parabolic systems, each of rank 2. Since $O_2(P_2) = O_2(P_3) = 1$, we must consider a Sylow 2-subgroup of M_{11}. Such a subgroup will intersect the M_{11}-conjugacy classes $1A$, $2A$, $4A$, $8A$ and $8B$ in 1, 5, 6, 2 and 2 elements respectively. It follows that the cuspidal relation for a Sylow 2-subgroup holds for $\chi_3, \chi_4 \in \mathrm{Irr}(M_{11})$ (both of degree 10). Consequently, χ_3 and χ_4 are 2-cuspidal characters of the minimal parabolic system $\{P_2, P_3\}$. Finally, as $O_2(P_1)$ contains 1, 1 and 6 elements from the classes $1A$, $2A$ and $4A$ respectively and

$$\chi_i(1A) + \chi_i(2A) + 6 \cdot \chi_i(4A) = 8$$

for $i = 3, 4$, we see that the minimal parabolic systems containing P_1 admit no 2-cuspidal characters.

M_{12}

There are no 2-cuspidal characters for the unique 2-minimal parabolic system of M_{12} given by

$$\left\{P_1 \sim 4^2.2.\,\mathrm{Sym}(3), P_2 \sim 2_+^{1+4}.\,\mathrm{Sym}(3)\right\}.$$

To see this, we observe that $O_2(P_1)$ intersects the M_{12}-conjugacy classes $1A$, $2A$, $2B$, $4A$ and $4B$ in 1, 4, 15, 6 and 6 elements respectively, whilst $O_2(P_2)$ intersects these classes in 1, 12, 7, 6 and 6 elements respectively. Consequently the only character satisfying the cuspidal relation for $O_2(P_1)$ is χ_{13} (of degree 120). However

$$\chi_{13}(1A) + 12 \cdot \chi_{13}(2A) + 7 \cdot \chi_{13}(2B) + 6 \cdot \chi_{13}(4A) + 6 \cdot \chi_{13}(4B) = 64.$$

M_{22}

There is a unique 2-minimal parabolic system for M_{22}, namely

$$\left\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_2 \sim 2^4.\,\mathrm{Sym}(5)\right\}.$$

Parabolic Subgroup	2-core	Order of intersection with M_{24}-class				
		$1A$	$2A$	$2B$	$4A$	$4B$
P_{12}	2^6	1	45	18	0	0
P_{13}	2^{6+2}	1	57	54	72	72
P_{23}	2^{4+3}	1	29	42	56	0

Table 4: The 2-cores of maximal parabolic subgroups of M_{24}.

The 2-cores $O_2(P_1)$ and $O_2(P_2)$ intersect the M_{22}-conjugacy classes $1A$, $2A$, $4A$ and $4B$ in 1, 27, 12, 24 and 1, 15, 0 and 0 elements respectively. The only elements $\chi \in \mathrm{Irr}(M_{22})$ satisfying

$$\chi(1A) + 27 \cdot \chi(2A) + 12 \cdot \chi(4A) + 24 \cdot \chi(4B) = \chi(1A) + 15 \cdot \chi(2A) = 0$$

are the two characters of degree 45, χ_3 and χ_4.

M_{23}

The group M_{23} has seven conjugacy classes of 2-minimal parabolic subgroups, six of which feature in 2-minimal parabolic systems of M_{23}. Using the notation of [14], these subgroups are

$$\begin{array}{lll} P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), & P_2 \sim 2^{4+2}.\,\mathrm{Sym}(3), & P_3 \sim 2^{4+2}.\,\mathrm{Sym}(3) \\ P_4 \sim 2^{4+2}.\,\mathrm{Sym}(3), & P_6 \sim 2^{4}.\,\mathrm{Sym}(5), & P_7 \sim 2^{4}.\,\mathrm{Sym}(5). \end{array}$$

The 2-minimal parabolic systems are given by $\{P_1, P_3, P_7\}$, $\{P_3, P_4, P_7\}$, $\{P_2, P_3, P_7\}$, $\{P_1, P_6, P_7\}$, $\{P_2, P_6, P_7\}$, $\{P_3, P_6, P_7\}$ and $\{P_4, P_6, P_7\}$.

Considering the maximal 2-parabolic subgroups of these systems, we see that the maximal 2-parabolic subgroups involving P_6 and P_7 have trivial 2-cores. Thus we need to check sub-maximal parabolics in order to apply Corollary 2.6. The 2-cores $O_2(P_i)$ for $i = 1, 2, 3, 4$ intersect the M_{23}-classes $1A$, $2A$ and $4A$ in 1, 27 and 36 elements respectively. The remaining sub-maximal parabolics, namely P_6, P_7, P_{13} and P_{34}, have rank 4 elementary abelian 2-groups for their 2-cores. Thus the non-trivial elements of their 2-cores lie in the M_{23}-class $2A$. For each sub-maximal parabolic, the only irreducible M_{23}-characters satisfying the cuspidal relation are χ_3 and χ_4. Hence for each of the 2-minimal parabolic systems, the characters χ_3 and χ_4 of degree 45 are 2-cuspidal characters.

M_{24}

The Mathieu group M_{24} has a unique 2-minimal parabolic system given by

$$\left\{P_1 \sim 2^{6+3}.\,\mathrm{Sym}(3), P_2 \sim 2^{6+3}.\,\mathrm{Sym}(3), P_3 \sim 2^{6+3}.\,\mathrm{Sym}(3)\right\}.$$

The maximal parabolic subgroups P_{12}, P_{13} and P_{23} all have non-trivial 2-cores, and their intersections with the M_{24}-conjugacy classes are summarised in Table 4. It follows that the characters χ_3, χ_4, χ_{12}, χ_{13}, χ_{15} and χ_{16} satisfy the cuspidal relation for P_{12}, as do χ_3, χ_4, χ_5, χ_6, χ_{12}, χ_{13}, χ_{15} and χ_{16} for P_{13} and χ_3, χ_4, χ_5, χ_6 and χ_8 for P_{23}. We conclude that χ_3 and χ_4 - both of degree 45 - are the only 2-cuspidal characters of M_{24}.

Conjugacy Class	$1A$	$2A$	$2B$	$2C$	$4A$	$4B$	$4C$	$4D$	$4E$	$4F$
P_{123}	1	1095	1344	4984	336	22512	18816	38976	0	43008
P_{124}	1	1095	576	6264	720	13680	17280	25920	0	0
P_{134}	1	759	0	1288	0	0	0	0	0	0
P_{234}	1	551	0	2520	240	15120	896	13440	0	0

Table 5: The intersections of the 2-cores of the maximal 2-parabolic subgroups of Co_1 with the Co_1-conjugacy classes.

3.2 The Leech Lattice and Conway Groups

HS

The Higman-Sims group has a unique 2-minimal parabolic system of the form

$$\left\{P_1 \sim 4.2^4.\operatorname{Sym}(5), P_2 \sim 4^3.2^2.\operatorname{Sym}(3)\right\}.$$

Considering $O_2(P_1)$, we see that it intersects the HS-conjugacy classes $1A$, $2A$, $2B$, $4A$, $4B$ and $4C$ in 1, 31, 0, 2, 30 and 0 elements respectively. Consequently, there are no 2-cuspidal characters of HS, as the cuspidal relation does not hold for P_1.

J_2

There is a unique 2-minimal parabolic system of J_2 given by

$$\left\{P_1 \sim 2^{2+4}.3.\operatorname{Sym}(3), P_2 \sim 2_+^{1+4}.L_2(4)\right\}.$$

The intersections of $O_2(P_1)$ with the J_2-classes $1A$, $2A$, $2B$ and $4A$ have orders 1, 3, 24 and 36 respectively. Consequently, the cuspidal relation on P_1 holds for the irreducible characters χ_4, χ_5, χ_{14} and χ_{15}. Meanwhile, the 2-core $O_2(P_2)$ intersects the given J_2-classes in 1, 11, 0 and 20 elements respectively, meaning that the cuspidal relation holds on P_2 for the characters χ_8, χ_9 and χ_{18}. We conclude that J_2 admits no 2-cuspidal characters.

Co_1

The largest Conway group, Co_1, admits a unique 2-minimal parabolic system, having rank 4. Its minimal parabolic subgroups are given by $P_i \sim [2^{20}].\operatorname{Sym}(3)$ for $i = 1, \ldots, 4$ and the corresponding 2-maximal parabolic subgroups have the form $P_{123} \sim 2^{2+12+3}.(\operatorname{Sym}(3) \times L_3(2))$, $P_{124} \sim 2^{4+12}.(\operatorname{Sym}(3) \times 3.Sp_4(2))$, $P_{134} \sim 2^{11}.M_{24}$ and $P_{234} \sim 2^{1+8+6}.L_4(2)$. The orders of the intersections of the 2-cores of the maximal parabolic subgroups with the Co_1-conjugacy classes are given in Table 5. A summary of the elements of $\operatorname{Irr}(Co_1)$ which satisfy the cuspidal relation for each of the maximal parabolics is given in Table 6. We conclude that Co_1 admits three 2-cuspidal characters, namely χ_2, χ_8 and χ_{11}.

Co_2

The group Co_2 has a 2-minimal parabolic system of the form $\{P_1, P_2, P_3\}$, where $P_1 \sim [2^{15}].\operatorname{Sym}(5)$ and $P_i \sim [2^{17}].\operatorname{Sym}(3)$ for $i = 2, 3$. This system has maximal

Parabolic Subgroup	Characters satisfying the cuspidal relation (character degrees)
P_{123}	χ_2 (276), χ_4 (1771), χ_5 (8855), χ_8 (37674), χ_{11} (94875), χ_{13} (345345), χ_{15} (483000), χ_{21} (1434510), χ_{23} (1771000), χ_{27} (2464749), χ_{28} (2464749)
P_{124}	χ_2 (276), χ_8 (37674), χ_{11} (94875), χ_{21} (1434510), χ_{27} (2464749), χ_{28} (2464749)
P_{134}	χ_2 (276), χ_4 (1771), χ_5 (8855), χ_8 (37674), χ_{11} (94875), χ_{13} (345345), χ_{15} (483000), χ_{21} (1434510), χ_{23} (1771000), χ_{27} (2464749), χ_{28} (2464749)
P_{234}	χ_2 (276), χ_4 (1771), χ_8 (37674), χ_9 (44275), χ_{11} (94875), χ_{13} (345345)

Table 6: The elements of $\mathrm{Irr}(Co_1)$ satisfying the cuspidal relation for each maximal 2-parabolic subgroup of Co_1.

2-core	Order of intersection with Co_2-conjugacy class											
	$1A$	$2A$	$2B$	$2C$	$4A$	$4B$	$4C$	$4D$	$4E$	$4F$	$4G$	$8A$
$O_2(P_{12})$	1	125	490	2328	240	1440	2400	1680	1920	5760	0	0
$O_2(P_{13})$	1	77	330	616	0	0	0	0	0	0	0	0
$O_2(P_{23})$	1	141	634	1848	240	3808	2464	336	8064	8064	0	7168

Table 7: The 2-cores of the maximal parabolic subgroups of Co_2.

parabolic subgroups $P_{12} \sim 2^{4+10}.(\mathrm{Sym}(3) \times \mathrm{Sym}(5))$, $P_{13} \sim 2^{10}.M_{22}.2$ and $P_{23} \sim 2^{1+8+6}.L_3(2)$. The orders of the intersections of the 2-cores of these maximal parabolics with the relevant Co_2-conjugacy classes are given in Table 7. It follows that the cuspidal relation holds on P_{12} for the characters χ_2, χ_3, χ_{10}, χ_{11}, χ_{12}, χ_{13}, χ_{16}, χ_{21}, χ_{22}, χ_{23}, χ_{31}, χ_{32} and χ_{37}, it holds on P_{13} for χ_3, χ_5, χ_9, χ_{10}, χ_{11}, χ_{12}, χ_{13}, χ_{16}, χ_{25}, χ_{31} and χ_{32}, and the cuspidal relation holds on P_{23} for the irreducible characters χ_2, χ_3, χ_5, χ_7, χ_8, χ_9, χ_{10}, χ_{11}, χ_{12}, χ_{13}, χ_{16}, χ_{20}, χ_{21}, χ_{22}, χ_{23}, χ_{25}, χ_{31}, χ_{32}, χ_{36}, χ_{37} and χ_{40}. Consequently, Co_2 admits eight 2-cuspidal characters namely χ_3 (of degree 253), χ_{10} (9625), χ_{11} (9625), χ_{12} (10395), χ_{13} (10395), χ_{16} (31625), χ_{31} (239085) and χ_{32} (239085).

Co_3

There is a unique 2-minimal parabolic system of Co_3 given by

$$\left\{P_1 \sim 2^{4+4+1}.\,\mathrm{Sym}(3), P_2 \sim 2^{4+4+1}.\,\mathrm{Sym}(3), P_3 \sim 2^{4+4+1}.\,\mathrm{Sym}(3)\right\},$$

which has maximal parabolic subgroups $P_{12} \sim 2^{2+6}.3.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$, $P_{13} \sim 4^3.2.L_3(2)$ and $P_{23} \sim 4.2^4.Sp_4(2)$. We have that $O_2(P_{23})$ intersects the Co_3-conjugacy classes $1A$, $2A$, $2B$, $4A$ and $4B$ in 1, 31, 0, 2 and 30 elements respectively, and hence the cuspidal condition does not hold for P_{23}. Thus Co_3 admits no 2-cuspidal characters.

M^cL

The McLaughlin group has a multitude of 2-minimal parabolic systems comprising of the 2-minimal parabolic subgroups P_i and P_i^σ for $i = 1, \ldots, 5$, where σ is a non-trivial outer automorphism of M^cL of order 2. Here $P_3^\sigma = P_3$, $P_4^\sigma = P_4$,

$P_i \sim 2^{4+2}.\,\mathrm{Sym}(3)$ for $i = 1, 2, 3, 4$ and $P_5 \sim 2^4.\,\mathrm{Sym}(5)$. These subgroups give rise to the minimal parabolic systems

$$\begin{array}{llll}
\{P_1, P_5, P_5^\sigma\}, & \{P_1^\sigma, P_5, P_5^\sigma\}, & \{P_2, P_5, P_5^\sigma\}, & \{P_2^\sigma, P_5, P_5^\sigma\}, \\
\{P_3, P_5, P_5^\sigma\}, & \{P_1, P_2^\sigma, P_5^\sigma\}, & \{P_1^\sigma, P_2, P_5\}, & \{P_1, P_3, P_5^\sigma\}, \\
\{P_1^\sigma, P_3, P_5\}, & \{P_1, P_2, P_5^\sigma\}, & \{P_1^\sigma, P_2^\sigma, P_5\}, & \{P_1, P_1^\sigma, P_5\}, \\
\{P_1, P_1^\sigma, P_5^\sigma\}, & \{P_1, P_1^\sigma, P_2\}, & \{P_1, P_1^\sigma, P_2^\sigma\}, & \{P_1, P_1^\sigma, P_3, P_4\}.
\end{array}$$

Since the 2-cores of P_5 and P_5^σ are elementary abelian of rank 4, we see that any minimal parabolic system containing either of these minimal parabolics will not admit a 2-cuspidal character. Conversely, any 2-minimal parabolic system not containing these subgroups will contain the parabolic subgroup $P_{11^\sigma} := \langle P_1, P_1^\sigma \rangle$. Since $O_2(P_{11^\sigma})$ intersects the M^cL-conjugacy classes $1A$, $2A$ and $4A$ in 1, 19 and 12 elements respectively, we see that the cuspidal relation does not hold for P_{11^σ} and hence none of the 2-minimal parabolic systems of M^cL admit a 2-cuspidal character.

Suz

The group Suz has a unique 2-minimal parabolic system, which has rank 3. Its minimal parabolic subgroups satisfy $P_1 \sim 2^{4+6+1}.L_2(4)$, $P_2 \sim 2^{4+6+2}.(3 \times L_2(2))$ and $P_3 \sim 2^{6+4+2}.(3 \times L_2(2))$. The maximal parabolic subgroups are given by $P_{12} \sim 2^{1+6}.U_4(2)$, $P_{13} \sim 2^{2+8}.(\mathrm{Sym}(3) \times L_2(4))$ and $P_{23} \sim 2^{4+6}.3.Sp_4(2)'$. The 2-core $O_2(P_{12})$ intersects the Suz-conjugacy classes $1A$, $2A$ and $4A$ in 1, 55 and 72 elements respectively. We deduce that there are no 2-cuspidal characters of Suz.

3.3 The Monster Group and its Subquotients

He

There are four 2-minimal parabolic subgroups of He given by $P_1 \cong P_4 \sim 2^{6+3}.\,\mathrm{Sym}(3)$ and $P_2 \cong P_3 \sim 2^{6+3}.\,\mathrm{Sym}(3)$. These give rise to the 2-minimal parabolic systems $\{P_1, P_2, P_4\}$ and $\{P_1, P_3, P_4\}$. Considering the maximal parabolic subgroups P_{14} and $P_{13} \cong P_{24}$ we see that $O_2(P_{14})$ intersects the He-conjugacy classes $1A$, $2A$, $2B$, $4A$, $4B$ and $4C$ in 1, 18, 45, 0, 0 and 0 elements respectively, whilst $O_2(P_{13})$ intersects the respective conjugacy classes in 1, 42, 29, 0, 56 and 0 elements. It follows that the cuspidal relation for P_{14} holds for the characters $\chi_7, \chi_8 \in \mathrm{Irr}(He)$, whilst for $P_{13} \cong P_{24}$ the cuspidal relation holds for $\chi_4, \chi_5 \in \mathrm{Irr}(He)$. Since each 2-minimal parabolic system contains P_{14} and either P_{13} or P_{24}, we conclude that there are no 2-cuspidal characters of He.

HN

There is a unique 2-minimal parabolic system of HN given by

$$\left\{P_1 \sim 2_+^{1+8}.\,\mathrm{Alt}(5) \wr \mathbb{Z}_2, P_2 \sim 2^{2+3+6+2}.3.\,\mathrm{Sym}(3)\right\}.$$

We consider the minimal parabolic subgroup P_1, whose character table is given in [8, TABLE IV], and we adopt the notation given in [8] for the P_1-conjugacy classes. We have that each P_1-class is either contained in, or is disjoint from $O_2(P_1)$. It follows that

$$O_2(P_1) = 1_1 \cup 2_1 \cup 2_2 \cup 2_3 \cup 4_1.$$

Considering the centralizer orders of 1_1, 2_1, 2_2, 2_3 and 4_1 in P_1 and the orders of the centralizers of 2-elements in HN, we see that the P_1-classes 2_1 and 2_2 are contained in the HN-class $2B$, the P_1-class 2_3 lies in either HN-class $2A$ or $2B$, and the P_1-class 4_1 is contained in the HN-class $4A$. It follows that $|O_2(P_1) \cap 2A| = 0$ or 120, $|O_2(P_1) \cap 2B| = 151$ or 271 and $|O_2(P_1) \cap 4A| = 240$. It follows that the cuspidal relation on P_1 does not hold for any $\chi \in \mathrm{Irr}(HN)$, and hence there are no 2-cuspidal characters of HN.

Th

The 2-minimal parabolic system

$$\{P_1 \sim 2^{1+8}.\,\mathrm{Alt}(9), P_2 \sim 2^{5+6+2+1}.\,\mathrm{Sym}(3)\}$$

of Th is unique. Considering fusion within the maximal subgroup $2^5.L_5(2) > P_2$ we find that $O_2(P_2)$ intersects the Th-conjugacy classes $1A$, $2A$, $4A$, $4B$, $8A$ and $8B$ in 1, 687, 656, 7104, 4864 and 3072 elements respectively. It follows that the cuspidal relation holds on P_2 for $\chi_2, \chi_6 \in \mathrm{Irr}(Th)$. Considering the normal subgroups of a Sylow 2-subgroup of $2^5.L_5(2)$ having order 2^9 and exponent 4, we see that for each such subgroup the only element of $\mathrm{Irr}(Th)$ for which the cuspidal relation holds is χ_2 of degree 248. Thus χ_2 is the unique 2-cuspidal character of Th.

Fi_{22}

There is a unique 2-minimal parabolic system of Fi_{22} given by $\{P_1, P_2, P_3\}$ where $P_i \sim [2^{16}].\,\mathrm{Sym}(3)$ for $i = 1, 2$ and $P_3 \sim [2^{14}].\,\mathrm{Sym}(5)$. This system has maximal parabolic subgroups $P_{12} \sim 2^{9+4+2}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$, $P_{13} \sim 2^{2+8}.U_4(2).2$ and $P_{23} \sim 2^{10}.M_{22}$. The 2-core $O_2(P_{13})$ intersects the Fi_{22}-conjugacy classes $1A$, $2A$, $2B$, $2C$, $4A$, $4B$, $4C$, $4D$ and $4E$ in 1, 2, 271, 270, 480, 0, 0, 0 and 0 elements respectively. We see that the cuspidal relation does not hold on P_{13} and hence Fi_{22} admits no 2-cuspidal characters.

Fi_{23}

The group Fi_{23} has eight 2-minimal parabolic subgroups, seven of which feature in 2-minimal parabolic systems. Using the notation of [14] these have the form $P_i \sim [2^{17}].\,\mathrm{Sym}(3)$ for $i = 1, \ldots, 5$ and $P_i \sim [2^{15}].\,\mathrm{Sym}(5)$ for $i = 7, 8$. These give rise to the geometric 2-minimal parabolic systems $\{P_1, P_3, P_5, P_8\}$ and $\{P_1, P_4, P_5, P_8\}$ and the non-geometric systems $\{P_1, P_2, P_5, P_8\}$ and $\{P_1, P_7, P_8\}$. The maximal parabolic subgroups of these systems are

$$\begin{array}{cc}
P_{125} \sim 2^{10+4}.(\mathrm{Sym}(3) \times \mathrm{Alt}(7)), & P_{128} = P_{138} = P_{148} \sim 2^2 \times 2^{1+8}.(3 \times U_4(2)).2, \\
P_{135} \sim [2^{14}].(\mathrm{Sym}(3) \times L_3(2)), & P_{145} \sim [2^{14}].(\mathrm{Sym}(3) \times L_3(2)), \\
P_{158} \sim 2.Fi_{22}, & P_{17} \sim [2^{14}].(\mathrm{Sym}(3) \times \mathrm{Sym}(5)), \\
P_{18} \sim [2^{11}].U_4(2).2, & P_{258} = P_{358} = P_{458} \sim 2^{11}.M_{23}, \\
P_{78} \sim 2^{11}.M_{21}.2. &
\end{array}$$

It is easy to check that the cuspidal relation does not hold for P_{158}, and hence the three 2-minimal parabolic systems of rank 4 do not admit any 2-cuspidal characters.

Finally, we consider the maximal parabolic subgroup $P_{18} < P_{158}$. We see that the 2-core $O_2(P_{18})$ intersects the Fi_{23}-conjugacy classes $1A$, $2A$, $2B$, $2C$, $4A$, $4B$, $4C$ and $4D$ in 1, 3, 273, 811, 0, 960, 0 and 0 elements respectively. Consequently, the cuspidal relation does not hold for P_{18}, and hence there are no 2-cuspidal characters of Fi_{23}.

Fi'_{24}

There is a unique 2-minimal parabolic system of Fi'_{24}, which has rank 4. The maximal parabolic subgroups are $P_a \sim 2^{1+12}.(3.U_4(3).2)$, $P_b \sim 2^{3+12+2}.(\mathrm{Sym}(3)\times Sp_4(2)')$, $P_c \sim 2^{8+6+3}.(L_3(2) \times \mathrm{Sym}(3))$ and $P_d \sim 2^{11}.M_{24}$. Since $O_2(P_d)$ is elementary abelian, we consider the minimum value that each $\chi \in \mathrm{Irr}(Fi'_{24})$ takes on elements of order 2. We immediately deduce that the only possible 2-cuspidal character of Fi'_{24} is χ_2 of degree 8671. For χ_2 to be 2-cuspidal, we would require an integer solution to

$$\chi_2(1A) + j \cdot \chi_2(2A) + (2^{11} - j - 1) \cdot \chi_2(2B) = 0.$$

Since no such solution exists, we conclude that there are no 2-cuspidal characters of Fi'_{24}.

$\mathbb{B}$

The baby monster has five conjugacy classes of 2-minimal parabolic subgroups having representatives $P_i \sim [2^{40}].\mathrm{Sym}(3)$ for $i = 1, \ldots, 4$ and $P_5 \sim [2^{38}].\mathrm{Sym}(5)$. These give rise to a unique 2-minimal parabolic system $\{P_1, P_2, P_3, P_5\}$. The maximal parabolic subgroups of this system are given by $P_{123} \sim 2^{9+16+6+4}.L_4(2)$, $P_{125} \sim 2^{3+32}.(L_3(2) \times \mathrm{Sym}(5))$, $P_{135} \sim 2^{2+10+20}.(\mathrm{Sym}(3) \times M_{22}2)$ and $P_{235} \sim 2_+^{1+22}.Co_2$. All of these maximal parabolic subgroups are 2-radical. Indeed, from [20] we observe that all 2-parabolic subgroups generated by $P_1, \ldots, P_5$ are 2-radical with the exception of P_3, P_4 and P_{34}. The reader can find further information regarding the structure of the 2-radical parabolic subgroups in [20].

The fusion of elements within the 2-cores of P_{125}, P_{135} and P_{235} is given in [15]. We see that there are no characters of $\mathbb{B}$ satisfying the cuspidal condition for the 2-core $O_2(P_{135})$ and hence the baby monster admits no 2-cuspidal characters.

$\mathbb{M}$

The monster group has a unique 2-minimal parabolic system, $\{P_1, P_2, P_3, P_4, P_5\}$, where $P_i \sim [2^{45}].L_2(2)$ for $i = 1, \ldots, 5$. The maximal parabolic subgroups are given by $P_{1234} \sim 2^{5+5+16+10}.L_5(2)$, $P_{1235} \sim 2^{4+1+2+8+8+12+4}.(L_4(2) \times \mathrm{Sym}(3))$, $P_{1245} \sim 2^{3+36}.(L_3(2) \times 3.Sp_4(2))$, $P_{1345} \sim 2^{2+11+22}.(\mathrm{Sym}(3) \times M_{24})$ and $P_{2345} \sim 2^{1+24}.Co_1$.

We observe that there is no $\chi \in \mathrm{Irr}(\mathbb{M})$ that satisfies the cuspidal relation for $O_2(P_{2345})$. Indeed, let z be an involution of $\mathbb{M}$ in class $2B$ and let Λ be the Leech lattice as defined in [1]. Moreover, let Λ_i be the set of all vectors in Λ of *type i* defined as

$$\Lambda_i := \{v \in \Lambda | (v,v)/16 = i\}.$$

Then calculations in [1] show that

$$\begin{aligned} |\Lambda_2| &= 196,560 &&= 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13, \\ |\Lambda_3| &= 2^{12}(2^{12}-1) &&= 2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13, \text{ and} \\ |\Lambda_4| &= 398,034,000 &&= 2^4 \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13. \end{aligned}$$

Let $G := \cdot 0$ - the automorphism group of Λ, and let $\widetilde{G} := G/\langle \varepsilon_X \rangle$ (where ε_X is the scalar map defined on Λ by -1). Thus $\widetilde{G}$ is equal to Co_1. Since ε_X acts trivially on $\widetilde{\Lambda} := \Lambda/2\Lambda$, and G acts transitively on the set Λ_2 ([1, Lemma 22.12(1)]) and on each of the sets Λ_3, Λ_4 ([1, Lemma 22.14(1)]), it follows that $\widetilde{G}$ acts transitively on the sets $\widetilde{\Lambda_2}$, $\widetilde{\Lambda_3}$ and $\widetilde{\Lambda_4}$ (where $\widetilde{\Lambda_i}$ is the image of Λ_i in $\widetilde{\Lambda}$).

Without loss, we may assume that z is the central involution of the extra-special group $O_2(P_{2345})$ and we define $\overline{P_{2345}} := P_{2345}/\langle z \rangle$. By considering the action of G on $\widetilde{\Lambda}$ we have that

$$\begin{aligned} \left|\mathrm{Stab}_G(\widetilde{\lambda_2})\right| &= 2^{17} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\ \left|\mathrm{Stab}_G(\widetilde{\lambda_3})\right| &= 2^9 \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \text{ and} \\ \left|\mathrm{Stab}_G(\widetilde{\lambda_4})\right| &= 2^{17} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \end{aligned}$$

(where $\lambda_i \in \Lambda_i$). It follows that

$$\begin{aligned} \left|\mathrm{Stab}_{\overline{P_{2345}}}(\widetilde{\lambda_2})\right| &= 2^{41} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\ \left|\mathrm{Stab}_{\overline{P_{2345}}}(\widetilde{\lambda_3})\right| &= 2^{33} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \text{ and} \\ \left|\mathrm{Stab}_{\overline{P_{2345}}}(\widetilde{\lambda_4})\right| &= 2^{41} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23. \end{aligned} \tag{4}$$

The question remains, how does the element λ_i lift to the extra-special group $2^{1+24} = O_2(P_{2345})$? Once this is established, we may then use the centralizer orders

$$\begin{aligned} |C_{\mathbb{M}}(2A)| &= 2^{42} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47, \\ |C_{\mathbb{M}}(2B)| &= 2^{46} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23, \\ |C_{\mathbb{M}}(4A)| &= 2^{34} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \\ |C_{\mathbb{M}}(4B)| &= 2^{27} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17, \\ |C_{\mathbb{M}}(4C)| &= 2^{34} \cdot 3^4 \cdot 5^7, \text{ and} \\ |C_{\mathbb{M}}(4D)| &= 2^{27} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13 \end{aligned} \tag{5}$$

to determine the fusion within $O_2(P_{2345})$. Indeed, we have that $C_{\overline{P_{2345}}}(\widetilde{\lambda_2}) \sim 2^{24}.Co_2$, $C_{\overline{P_{2345}}}(\widetilde{\lambda_3}) \sim 2^{24}.Co_3$ and $C_{\overline{P_{2345}}}(\widetilde{\lambda_4}) \sim 2^{24}.(2^{11} : M_{24})$.

There are two possible ways in which a $\widetilde{\lambda_i}$ can lift into 2^{1+24}, namely to an abelian subgroup of order 4 of the form $\langle \lambda_i, z \rangle$ having exponent 2 or 4. The former case occurs when $|\lambda_i| = 2$, whilst the latter case occurs when $|\lambda_i| = 4$

and hence $\lambda_i^2 = z$. Since $(4A_{\mathbb{M}})^3 = 4A_{\mathbb{M}}$, $(4B_{\mathbb{M}})^3 = 4B_{\mathbb{M}}$, $(4C_{\mathbb{M}})^3 = 4C_{\mathbb{M}}$ and $(4D_{\mathbb{M}})^3 = 4D_{\mathbb{M}}$, we may use the centralizer and stabilizer orders from (5) and (4) to see that the only possible elements of order 4 in P_{2345} must lie in the $\mathbb{M}$-conjugacy class $4A$. Since the exponent of 2^{1+24} is 4, we conclude that the elements of the orbit $\widetilde{\lambda_3}^{\widetilde{G}}$ lift to cyclic groups of order 4 containing 1, z and two elements from the $\mathbb{M}$-class $4A$. This means that

$$|O_2(P_{2345}) \cap 4A| = 2 \cdot \left|\widetilde{\Lambda_3}\right| = |\Lambda_3| = 2^{12}(2^{12}-1) = 2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 = 16,773,120.$$

Next we consider the lifts of $\widetilde{\lambda_2}$ and $\widetilde{\lambda_4}$. We see that these must lift to the elementary abelian subgroups $\langle \lambda_2, z\rangle$ and $\langle \lambda_4, z\rangle$ respectively. Since there are only two $\mathbb{M}$-classes of involutions, we have that $(\lambda_2 z)^{g_2} = \lambda_2$ and $(\lambda_4 z)^{g_4} = \lambda_4$ for some $g_2, g_4 \in \mathbb{M}$. To determine which element lifts to class $2A$ and which element lifts to $2B$, we note that by [12, Lemma 4.4] for $x \neq z$ an involution of P_{2345}, either $C_{P_{2345}}(x) \sim 2^{1+23}.Co_2$ (if x is not 2-central) or $C_{P_{2345}}(x) \sim 2^{1+23}.(2^{11} : M_{24})$ if x is 2-central. Here

$$\begin{aligned} \left|2^{1+23}.Co_2\right| &= 2^{42} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23 \text{ and} \\ \left|2^{1+23}.(2^{11} : M_{24})\right| &= 2^{45} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23. \end{aligned} \tag{6}$$

Combining (6) with (4) and the fact that the 2-central elements of P_{2345} lie in $2B$, we have that $\widetilde{\lambda_2}$ lifts to an elementary abelian subgroup generated by z and an involution of the $\mathbb{M}$-class $2A$, whilst $\widetilde{\lambda_4}$ lifts to a subgroup generated by z and an element of $2B$. Since z also lies in $2B$, we conclude that

$$\begin{aligned} |O_2(P_{2345}) \cap 2A| &= 2 \cdot \left|\widetilde{\Lambda_2}\right| &&= |\Lambda_2| &&= 196,560, \\ |O_2(P_{2345}) \cap 2B| &= 2 \cdot \left|\widetilde{\Lambda_4}\right| + 1 &&= |\Lambda_4|/24 + 1 &&= 16,584,751 \quad \text{and} \\ |O_2(P_{2345}) \cap 4A| &= 2 \cdot \left|\widetilde{\Lambda_3}\right| &&= |\Lambda_3| &&= 16,773,120. \end{aligned}$$

As there is no $\chi \in \mathrm{Irr}(\mathbb{M})$ satisfying

$$\chi(1A) + 196560 \cdot \chi(2A) + 16584751 \cdot \chi(2B) + 16773120 \cdot \chi(4A) = 0,$$

it follows that there are no 2-cuspidal characters of $\mathbb{M}$.

3.4 The Pariahs

J_1

The normalizer of a Sylow 2-subgroup of J_1 is maximal. Thus as the cuspidal relation does not hold for such a Sylow subgroup, J_1 has no 2-cuspidal characters.

$O'N$

The group $O'N$ admits a unique 2-minimal parabolic system of the form

$$\left\{P_1 \sim 4^3.2^2.\mathrm{Sym}(3), P_2 \sim 4.L_3(4).2\right\}.$$

The generators of $O_2(P_2)$ are elements of the $O'N$-conjugacy class $4A$. Thus we see that the cuspidal relation does not hold on P_2 for any $\chi \in \mathrm{Irr}(O'N)$. Hence $O'N$ admits no 2-cuspidal characters.

J_4-conjugacy class, C	$1A$	$2A$	$2B$	$4A$	$4B$	$4C$
$\lvert O_2(P_{12}) \cap C\rvert$	1	3579	4868	22848	100800	72704
$\lvert O_2(P_{13}) \cap C\rvert$	1	3067	5892	21312	54720	46080
$\lvert O_2(P_{23}) \cap C\rvert$	1	1387	2772	4032	0	0

J_4-conjugacy class, C	$8A$	$8B$	$8C$
$\lvert O_2(P_{12}) \cap C\rvert$	0	57344	0
$\lvert O_2(P_{13}) \cap C\rvert$	0	0	0
$\lvert O_2(P_{23}) \cap C\rvert$	0	0	0

Table 8: The intersections of the 2-cores of the maximal parabolic subgroups of J_4 with the J_4-conjugacy classes.

J_3

There is a unique 2-minimal parabolic system of J_3, given by

$$\left\{P_1 \sim 2^{2+4}.(3 \times \mathrm{Sym}(3)), P_2 \sim 2_-^{1+4}.L_2(4)\right\}.$$

Considering the 2-core $O_2(P_2)$, it contains 1, 11 and 20 elements from the J_3-conjugacy classes $1A$, $2A$ and $4A$ respectively. It follows that J_3 does not admit any 2-cuspidal characters.

Ru

There are three 2-minimal parabolic subgroups of Ru given by $P_1 \sim 2^{5+6}.\mathrm{Sym}(5)$ and $P_i \sim 2^{5+6+2}.\mathrm{Sym}(3)$ for $i = 2, 3$. Since $P_3 \leq P_1$, we obtain a unique 2-minimal parabolic system, namely $\{P_1, P_2\}$. Considering the 2-cores of P_1 and P_2, we see that $O_2(P_1)$ intersects the Ru-conjugacy classes $1A$, $2A$, $2B$, $4A$, $4B$, $4C$, $4D$, $8A$, $8B$ and $8C$ in 1, 271, 0, 512, 64, 240, 960, 0, 0 and 0 elements respectively, whilst $O_2(P_2)$ intersects the given Ru-classes in respectively 1, 367, 192, 608, 448, 1296, 1440, 1536, 768 and 1536 elements. It follows that the cuspidal relation holds on P_1 for $\chi_2, \chi_3 \in \mathrm{Irr}(Ru)$ and on P_2 for $\chi_2, \chi_3, \chi_4 \in \mathrm{Irr}(Ru)$. We conclude that the two characters χ_2 and χ_3 of degree 378 are the only 2-cuspidal characters of Ru.

J_4

There is a unique 2-minimal parabolic system of J_4 given by $\{P_1, P_2, P_3\}$ where $P_i \sim [2^{20}].\mathrm{Sym}(3)$ for $i = 1, 2$ and $P_3 \sim [2^{18}].\mathrm{Sym}(5)$. The maximal 2-parabolic subgroups of this system are given by $P_{12} \sim [2^{18}].L_3(2)$, $P_{13} \sim [2^{17}].(\mathrm{Sym}(3) \times \mathrm{Sym}(5))$ and $P_{23} \sim 2^{1+12}.3.M_{22}.2$. By considering centralizer orders, powering up classes and conjugation of representatives of certain P_{12}-, P_{13}- and P_{23}-conjugacy classes by random elements in J_4, we may determine the fusion of $O_2(P_{12})$-, $O_2(P_{13})$- and $O_2(P_{23})$-classes in J_4. We detail the orders of the intersections of the 2-cores of the maximal parabolic subgroups with the J_4-conjugacy classes in Table 8. Consequently, the cuspidal relation holds on P_{12} for the characters χ_2, χ_3, χ_4 and χ_5, it holds on P_{13} for χ_2, χ_3, χ_4, χ_5, χ_6, χ_7, χ_9, χ_{10}, χ_{12} and χ_{13}, whilst the cuspidal relation holds on P_{23} for the irreducible characters χ_2 and χ_3. We conclude that J_4 admits two 2-cuspidal characters, χ_2 and χ_3, both of degree 1333.

Ly

There are six 2-minimal parabolic subgroups of Ly, three of which feature in the two 2-minimal parabolic systems of Ly. These are $P_1 \sim [2^7].\,\mathrm{Sym}(3)$, $P_2 \sim [2^5].\,\mathrm{Sym}(5)$, $P_3 \sim 2.\,\mathrm{Sym}(9)$ and they give rise to the systems $\{P_1, P_2\}$ and $\{P_1, P_3\}$. Since $|O_2(P_1)| = 2^7$, $|O_2(P_2)| = 2^5$ and $|O_2(P_3)| = 2$, it is easy to see that the cuspidal relation does not hold for any of the 2-minimal parabolic subgroups of Ly, and hence there are no 2-cuspidal characters of Ly.

4 3-Cuspidal Characters

We now describe the 3-cuspidal characters for each of the sporadic groups.

4.1 The Mathieu Groups

M_{11}

The normalizer of a Sylow 3-subgroup of M_{11} is the maximal subgroup $M_9 : 2$ of M_{11}. Consequently, we see that M_{11} admits two 3-cuspidal characters, χ_6 and χ_7, both of degree 16.

M_{12}

The group M_{12} has a unique 3-minimal parabolic system

$$\left\{P_1 \sim 3^2.GL_2(3), P_2 \sim 3^2.GL_2(3)\right\}.$$

The non-trivial elements of the 3-cores $O_3(P_1)$ and $O_3(P_2)$ lie in the M_{12}-conjugacy class $3A$. It follows that the 3-cuspidal characters of M_{12} are χ_4 and χ_5, both of which have degree 16.

M_{22}

There is a unique 3-minimal parabolic system of M_{22}, namely

$$\{P_1 \cong M_{10}, P_2 \cong L_3(4)\}.$$

Since $O_3(P_i) = 1$ for $i = 1, 2$, an element $\chi \in \mathrm{Irr}(M_{22})$ will be 3-cuspidal if and only if the cuspidal relation holds for a Sylow 3-subgroup. Since this is never the case, M_{22} has no 3-cuspidal characters.

M_{23}

The 3-cores of the two 3-minimal parabolic subgroups comprising the unique 3-minimal parabolic system

$$\{P_1 \cong M_{11}, P_2 \sim L_3(4) : 2_2\}$$

of M_{23} are both trivial. Since the cuspidal relation does not hold for a Sylow 3-subgroup, we conclude that M_{23} admits no 3-cuspidal characters.

M_{24}

There are three 3-minimal parabolic subgroups of M_{24} given by $P_1 \sim 3.\,\mathrm{Sym}(6)$, $P_2 \sim 2^6.3_+^{1+2}.2_+^{1+2}$ and $P_3 \sim M_{12} : 2$. These give rise to two 3-minimal parabolic systems of M_{24}, namely $\{P_1, P_3\}$ and $\{P_2, P_3\}$. It is easy to observe that there is no $\chi \in \mathrm{Irr}(M_{24})$ satisfying the cuspidal relation for a Sylow 3-subgroup of M_{24}. It follows that neither 3-minimal parabolic system of M_{24} admits a 3-cuspidal character.

4.2 The Leech Lattice and Conway Groups

HS

As a Sylow 3-subgroup of HS has order 9, it is easy to see that the cuspidal relation will not hold for such a subgroup, and hence HS admits no 3-cuspidal characters.

J_2

There is a unique 3-minimal parabolic system of J_2 given by

$$\{P_1 \sim 3.\,\mathrm{Alt}(6).2, P_2 \cong U_3(3)\}\,.$$

The non-trivial elements of $O_3(P_1)$ are contained in the J_2-conjugacy class $3A$. We immediately see that J_2 has no 3-cuspidal characters.

Co_1

There is a solitary 3-minimal parabolic system of Co_1, which has rank 3. Its maximal parabolic subgroups are $P_{12} \sim 3^{1+4}.Sp_4(3).2$, $P_{13} \sim 3^{3+4}.GL_2(3)^2$ and $P_{23} \sim 3^6.2.M_{12}$.

The Co_1-fusion of the 3-core of P_{23} is described in [6]. We see that $O_3(P_{23})$ intersects the Co_1-conjugacy classes $1A$, $3A$, $3B$, $3C$ and $3D$ in 1, 24, 264, 440 and 0 elements respectively. It follows that the cuspidal relation does not hold for P_{23} and hence Co_1 admits no 3-cuspidal characters.

Co_2

The unique 3-minimal parabolic system of Co_2 has the form

$$\left\{P_1 \sim 3_+^{1+4}.2_-^{1+4}.\,\mathrm{Sym}(5), P_2 \sim 3^4.L_2(9).\,\mathrm{Dih}(8)\right\}.$$

Since $O_3(P_2)$ intersects the respective Co_2-conjugacy classes $1A$, $3A$ and $3B$ in 1, 20 and 60 elements, we see that there are no 3-cuspidal characters of Co_2.

Co_3

There are two 3-minimal parabolic subgroups of Co_3 and they form the unique 3-minimal parabolic system

$$\left\{P_1 \sim 3_+^{1+4}.4.\,\mathrm{Sym}(6), P_2 \sim 3^5.(M_{11} \times 2)\right\}.$$

The 3-cores $O_3(P_1)$ and $O_3(P_2)$ intersect the Co_3-classes $1A$, $3A$, $3B$ and $3C$ in 1, 2, 240 and 0 and 1, 110, 132 and 0 elements respectively. It follows that the cuspidal relation holds on P_1 for $\chi_6, \chi_7 \in \mathrm{Irr}(Co_3)$ and on P_2 for $\chi_{10}, \chi_{11} \in \mathrm{Irr}(Co_3)$. Consequently, there are no 3-cuspidal characters of Co_3.

M^cL

The McLaughlin group has a unique 3-minimal parabolic system given by

$$\left\{P_1 \sim 3^4.M_{10}, P_2 \sim 3_+^{1+4}.2.\,\mathrm{Sym}(5)\right\}.$$

The 3-core $O_3(P_1)$ intersects the M^cL-conjugacy classes $1A$, $3A$ and $3B$ in 1, 20 and 60 elements respectively. It follows that no $\chi \in \mathrm{Irr}(M^cL)$ satisfies the cuspidal relation for P_1 and hence M^cL has no 3-cuspidal characters.

Suz

The unique 3-minimal parabolic system

$$\left\{P_1 \sim 3^5.M_{11}, P_2 \sim 3^{2+4}.2.(\mathrm{Alt}(4) \times 2^2).2\right\}$$

of Suz does not admit any 3-cuspidal characters. To see this, we note that the 3-core $O_3(P_1)$ intersects the Suz-classes $1A$, $3A$, $3B$ and $3C$ in 1, 22, 220 and 0 elements respectively, meaning that the cuspidal relation does not hold on P_1 for any $\chi \in \mathrm{Irr}(Suz)$.

4.3 The Monster Group and its Subquotients

He

Since a Sylow 3-subgroup of He has order 27 and exponent 3, we easily observe that there are no 3-cuspidal characters of He.

HN

The unique 3-minimal parabolic system of HN is

$$\left\{P_1 \sim 3_+^{1+4}.2.\,\mathrm{Sym}(5), P_2 \sim 3^4.2.(\mathrm{Alt}(4) \times \mathrm{Alt}(4)).4\right\}.$$

The character table of the subgroup $M = 3^{1+4}.SL_2(5) \leq HN$ is given as [8, TABLE II]. Since $M \leq P_1$, it follows that $O_3(M) = O_3(P_1)$. Moreover, every M-conjugacy class is either contained in, or disjoint from $O_3(M)$. Using the notation from [8], we have that

$$O_3(M) = 1 \cup 3_1 \cup 3_1^2 \cup 3_2 \cup 3_3.$$

Considering centralizer orders in M and HN, we see that 3_1 and 3_1^2 are contained in the HN-class $3B$, whilst 3_2 and 3_3 lie in either $3A$ or $3B$. It follows that $|O_3(P_1) \cap 3A| = |O_3(M) \cap 3A| = 0, 120$ or 240 and $|O_3(P_1) \cap 3B| = |O_3(M) \cap 3B| = 2, 122$ or 242. For each of these possibilities, we see that the cuspidal relation would not hold for P_1, and hence there are no 3-cuspidal characters of HN.

Th

The Thompson group has a single 3-minimal parabolic system. It is of rank 2 and has the form

$$\left\{P_1 \sim 3^{(1+2)+4+2}.GL_2(3), P_2 \sim 3^{(2+3)+4}.GL_2(3)\right\}.$$

Conjugacy Class, C	$1A$	$3A$	$3B$	$3C$	$3D$
$\lvert O_3(P_{12}) \cap C \rvert$	1	0	260	234	234
$\lvert O_3(P_{13}) \cap C \rvert$	1	0	260	234	234
$\lvert O_3(P_{23}) \cap C \rvert$	1	72	386	576	1152

Table 9: The orders of the intersections of the Fi_{22}-conjugacy classes with the 3-cores of maximal 3-parabolic subgroups.

The 3-cores $O_3(P_1)$ and $O_3(P_2)$ intersect the respective Th-conjugacy classes $1A$, $3A$, $3B$, $3C$, $9A$, $9B$ and $9C$ in 1, 270, 2186, 4104, 2106, 5184 and 5832 elements and 1, 756, 2672, 4590, 648, 5184 and 5832 elements respectively. It follows that for both P_1 and P_2 there is a unique element of $\mathrm{Irr}(Th)$ satisfying the cuspidal relation, namely χ_2 of degree 248. We conclude that χ_2 is the unique 3-cuspidal character of Th.

Fi_{22}

There is a unique 3-minimal parabolic system of Fi_{22}, which has rank 3 and minimal parabolic subgroups of the form $P_i \sim [3^8].2.PGL_2(3)$ for $i = 1, 2, 3$. The maximal parabolic subgroups of this system are

$$P_{12} \cong P_{13} \sim 3^{4+2}.L_3(3) \quad \text{and} \quad P_{23} \sim 3_+^{1+6}.2^2.SL_2(3).\operatorname{Sym}(4).$$

The maximal parabolics P_{12} and P_{13} are submaximal subgroups of Fi_{22}, being contained in the maximal subgroups isomorphic to $O_7(3)$, and hence their 3-cores can be easily computed. Meanwhile, by [2, (39.6)], the 3-core $O_3(P_{23})$ is isomorphic to the Fitting subgroup of the normalizer in Fi_{22} of an element of the Fi_{22}-class $3B$. The orders of the intersections of the 3-cores of these maximal parabolics with the Fi_{22}-conjugacy classes is summarized in Table 9.

We see that the cuspidal relation holds on P_{12} and P_{13} for the characters $\chi_2, \chi_5 \in \mathrm{Irr}(Fi_{22})$, and it holds on P_{23} for χ_2. Consequently, χ_2 (of degree 78) is the unique 3-cuspidal character of Fi_{22}.

Fi_{23}

The group Fi_{23} has a unique 3-minimal parabolic system given by

$$\left\{P_1 \sim [3^{12}].2^2.PGL_2(3), P_2 \sim [3^{12}].2^2.PGL_2(3), P_3 \sim [3^9].2.L_2(3)^3.2.\operatorname{Sym}(3)\right\}.$$

The corresponding 3-maximal parabolic subgroups are

$$P_{12} \sim 3^{3+7}.GL_3(3), \quad P_{13} \sim 3_+^{1+8} : 2_-^{1+6} : 3_+^{1+2} : 2.\operatorname{Sym}(4) \quad \text{and} \quad P_{23} \sim D_4(3).\operatorname{Sym}(3).$$

Using the information on Fi_{23}-fusion within $O_3(P_{13})$ given in [18, Table 2], and the fact that the non-trivial elements of $Z(3_+^{1+8})$ lie in the Fi_{23}-class $3B$, we see that $O_3(P_{13})$ intersects the Fi_{23}-classes $3A$, $3B$, $3C$ and $3D$ in 864, 1538, 3456 and 13824 elements respectively. (We note that the above fusion can also be calculated within P_{23}. Indeed, there are nine P_{23}-classes of elements of order 3, say $3a, \ldots, 3i$, having centralizer orders in P_{23} of 408146688, 37791360, 37791360, 12737088, 2834352, 944784, 314928, 78732 and 17496 respectively.

Element, m	$\mathrm{Stab}_{U_5(2)}(m)$	$\lvert\mathrm{Stab}_{U_5(2)}(m)\rvert$	$\lvert m^{U_5(2)}\rvert$
m_1	$3^4.\mathrm{Alt}(5)$	4860	2816
m_2	$3.\mathrm{Alt}(6)$	1080	12672
m_3	$2^{1+6}.3^{1+2}.3$	10368	1320
m_4	$3^3.\mathrm{Alt}(4)$	324	42240

Table 10: The non-trivial orbits of the unique irreducible 10-dimensional $GF(3)U_5(2)$-module.

It follows that the P_{23}-classes satisfy the following inclusions; $3c, 3d \subset 3A$, $3a, 3e \subset 3B$, $3b, 3g \subset 3C$ and $3f, 3h, 3i \subset 3D$.) Consequently, the cuspidal relation does not hold on P_{13} for any $\chi \in \mathrm{Irr}(Fi_{23})$, and Fi_{23} admits no 3-cuspidal characters.

Fi_{24}'

As with Fi_{23} we see that there is a unique 3-minimal parabolic system of Fi_{24}', namely

$$\left\{P_1 \sim [3^{15}].2^2.PGL_2(3), P_2 \sim [3^{15}].2^2.PGL_2(3), P_3 \sim [3^{15}].2.\mathrm{Sym}(5)\right\},$$

having maximal parabolics $P_{12} \sim 3^{3+7+3}.L_3(3).2$, $P_{13} \sim 3^{2+4+8}.(SL_2(3) \times \mathrm{Alt}(5)).2$ and $P_{23} \sim 3_+^{1+10}.U_5(2).2$.

Consider the extra-special 3-core, $O_3(P_{23}) \cong 3_+^{1+10}$. Since there is a unique irreducible 10-dimensional $GF(3)U_5(2)$-module, M, we can explicitly determine the sizes of the orbits of elements of this module. These are summarized in Table 10. We see that there are at most four classes of non-central elements of 3_+^{1+10}.

Let $z \in Z(3_+^{1+10}) \setminus \{1\}$ and let $x \in 3_+^{1+10} \setminus Z(3_+^{1+10})$. Thus x represents a non-zero vector in M. Then x, x^2, zx, zx^2, z^2x and z^2x^2 are all Fi_{24}'-conjugate. Indeed, from the $\mathbb{ATLAS}$ we have that for any 3-element $w \in Fi_{24}'$, the elements w and w^2 are Fi_{24}'-conjugate. Suppose that $g \in Fi_{24}'$ is such that $x^g = x^2$. Then $(zx)^g = zx^2$ as z is central. Thus zx and zx^2 lie in a common Fi_{24}'-class, and are joined by $(zx)^2 = z^2x^2$ and $(zx^2)^2 = z^2x$. Finally, as zx and z^2x are Fi_{24}'-conjugate, there exists $h \in Fi_{24}'$ satisfying $z^2x = (zx)^h = zx^h$ and hence $zx = x^h$. We conclude that the orbits of m_1, m_2, m_3 and m_4 give rise to orbits of $3_+^{1+10} \setminus Z(3_+^{1+10})$ of respective sizes 8448, 38016, 3960 and 126720.

Label the orbit of 3_+^{1+10} arising from m_i by M_i for $i = 1, \ldots, 4$. Considering the orders of stabilizers given in Table 10 together with the centralizer orders of elements of order 3 in Fi_{24}' given in the $\mathbb{ATLAS}$, we deduce that $M_1, M_2, M_3 \subset 3A \cup 3B \cup 3C$, whilst elements in M_4 could form a subset of $3A$, $3B$, $3C$, $3D$ or $3E$. Since $N(3B) \cong P_{23}$, it follows that the non-trivial central elements of 3_+^{1+10} lie in the Fi_{24}'-class $3B$.

The menagerie of information obtained above results in 135 different possibilities for the fusion of 3-elements within $O_3(P_{23}) \cong 3_+^{1+10}$. Feeding each possibility into MAGMA and allowing it to roam over all 108 complex characters of Fi_{24}', we see that the cuspidal relation never holds for P_{23}, and hence Fi_{24}' admits no 3-cuspidal characters.

Aside 4.1. *We note that the Fi_{24}'-fusion within the* 3*-core* $O_3(P_{23})$ *has previously been studied by Wilson. Indeed, in [18, Section 2.2] Wilson calculates that* $O_3(P_{23})$ *contains* 3960*,* 8450*,* 38016 *and* 126720 *elements from the respective* Fi_{24}'*-classes* $3A$*,* $3B$*,* $3C$ *and* $3D$*. However, these calculations are based heavily on an unpublished preprint, and we have been unable to verify them.*

$\mathbb{B}$

There is a unique 3-minimal parabolic system of $\mathbb{B}$, which has rank 3. The maximal parabolic subgroups of this system are $P_{12} \sim 3^{3+7}.GL_3(3)$, $P_{13} \sim 3^{2+3+6}.GL_2(3)^2$ and $P_{23} \sim 3^{1+8}.2^{1+6}.PSp_4(3).2$. Considering the minimum value that each $\chi \in \mathrm{Irr}(\mathbb{B})$ takes on 3-elements, we see that the only possible 3-cuspidal character of $\mathbb{B}$ is χ_2 of degree 4371. However, as $\mathbb{B}$ satisfies the condition of Lemma 2.7, we see that χ_2 cannot be 3-cuspidal, and hence $\mathbb{B}$ admits no 3-cuspidal characters.

$\mathbb{M}$

The monster group, $\mathbb{M}$, has a unique 3-minimal parabolic system, which has rank 3. Its maximal parabolic subgroups are given by

$$P_{12} \sim 3^{3+8+6}.2^4.L_3(3),\ P_{13} \sim 3^{2+5+10}.(GL_2(3) \times M_{11}) \text{ and } P_{23} \sim 3_+^{1+12}.2.\,\mathrm{Suz}\,2.$$

By considering the 3-core $O_3(P_{23})$, we see that it has exponent 3. Moreover, appealing to [9, Lemma 1.5] we see that

$$|O_3(P_{23}) \cap 3A| = 196,560 \qquad \text{and} \qquad |O_3(P_{23}) \cap 3B| = 1,397,762.$$

It immediately follows that $\mathbb{M}$ admits no 3-cuspidal characters.

4.4 The Pariahs

J_1

The cuspidal relation does not hold for a Sylow 3-subgroup of J_1. Hence there are no 3-cuspidal characters for the unique 3-minimal parabolic system, $\{J_1\}$, of J_1.

$O'N$

A Sylow 3-subgroup of $O'N$ has order 81. Moreover, since there is a unique $O'N$-conjugacy class of non-trivial 3-elements, we see that the cuspidal relation does not hold for a Sylow 3-subgroup of $O'N$. Consequently, $O'N$ admits no 3-cuspidal characters.

J_3

The normalizer of a Sylow 3-subgroup of J_3 is maximal. Moreover, such a Sylow subgroup intersects the J_3-conjugacy classes $1A$, $3A$, $3B$, $9A$, $9B$ and $9C$ in 1, 18, 8, 72, 72 and 72 elements respectively. Checking the cuspidal relation for each $\chi \in \mathrm{Irr}(J_3)$ for a Sylow 3-subgroup, we see that there are no 3-cuspidal characters of J_3.

Ru

Since a Sylow 3-subgroup of Ru has order 27 and there is a unique Ru-conjugacy class of non-trivial 3-elements, it is easy to see that Ru admits no 3-cuspidal characters.

J_4

The non-trivial elements of a Sylow 3-subgroup of J_4 lie in the J_4-class $3A$. Since such a subgroup has order 27, it is easy to see that the cuspidal relation does not hold for a Sylow 3-subgroup of J_4. Hence, J_4 has no 3-cuspidal characters.

Ly

There is a unique 3-minimal parabolic system of Ly given by

$$\left\{P_1 \sim 3^{2+4}.8.\,\mathrm{Sym}(5), P_2 \sim 3^5.(M_{11} \times 2)\right\}.$$

Considering the minimum value that each $\chi \in \mathrm{Irr}(Ly)$ takes on elements of order 3, we see that the only possible candidates for 3-cuspidal characters are χ_7 and χ_8 of degree 120064. However, these characters take strictly negative values on all 3-elements, and hence the cuspidal relation cannot hold for them for both $O_3(P_2)$ and a Sylow 3-subgroup of Ly. We conclude that there are no 3-cuspidal characters of Ly.

5 5-Cuspidal Characters

The groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, Suz, He, Fi_{22}, Fi_{23}, Fi_{24}', J_1, $O'N$, J_3, Ru and J_4 have Sylow 5-subgroups of exponent 5 and of order at most 5^3. Thus, considering the minimal values that an irreducible character takes on elements of order 5 together with the character degree, we see that none of these groups admit a 5-cuspidal character. We now consider the remaining eleven sporadic groups in turn.

HS

A Sylow 5-subgroup of HS intersects the HS-conjugacy classes $1A$, $5A$, $5B$ and $5C$ in 1, 4, 40 and 80 elements respectively. It follows immediately that HS has no 5-cuspidal characters.

J_2

The normalizer in J_2 of a Sylow 5-subgroup, S, is maximal and S intersects each of the J_2-conjugacy classes $5A$, $5B$, $5C$ and $5D$ in 6 elements. From this we deduce that J_2 has a unique 5-cuspidal character given by χ_6 of degree 36.

Co_1

There is a unique 5-minimal parabolic system of Co_1 given by

$$\left\{P_1 \sim 5^3.(4 \times \mathrm{Alt}(5)).2, P_2 \sim 5^{1+2}.GL_2(5)\right\}.$$

Considering the minimum value that each $\chi \in \mathrm{Irr}(Co_1)$ takes on elements of order 5, and the order of the 5-cores of the minimal parabolic subgroups, we see that there are no 5-cuspidal characters of Co_1.

Co_2, Co_3, M^cL

If $G \in \{Co_2, Co_3, M^cL\}$, then a Sylow 5-subgroup of G intersects the G-conjugacy classes $1A$, $5A$ and $5B$ in 1, 4 and 120 elements respectively. It follows that the cuspidal relation does not hold for a Sylow 5-subgroup for any $\chi \in \mathrm{Irr}(G)$ and hence G has no 5-cuspidal characters.

HN

There is a unique 5-minimal parabolic system of HN given by

$$\left\{P_1 \sim 5^{1+4}.(2^{1+4}.5.4), P_2 \sim 5^{2+1+2}.4.\,\mathrm{Alt}(5)\right\}.$$

The character table of P_1 is given as [8, Table III], whilst a partial character table of P_2 - featuring the conjugacy classes of elements of order 2 and classes contained in $O_5(P_2)$ - is given in Table 11. By considering the restriction of $\chi_2 \in \mathrm{Irr}(HN)$ to P_1 and P_2, we may calculate the HN-fusion within $O_5(P_1)$ and $O_5(P_2)$. We see that $O_5(P_1)$ intersects the HN-classes $1A$, $5A$, $5B$, $5C$, $5D$ and $5E$ in 1, 400, 324, 800, 800 and 800 elements respectively, whilst $O_5(P_2)$ intersects the given classes in 1, 0, 624, 650, 650, and 1200 elements respectively. Consequently, the cuspidal relation holds on P_1 for $\chi_4 \in \mathrm{Irr}(HN)$ and on P_2 for $\chi_4, \chi_5 \in \mathrm{Irr}(HN)$. We conclude that χ_4 (of degree 760) is the unique 5-cuspidal character of HN.

Th

The normalizer of a Sylow 5-subgroup of Th is a maximal subgroup. Thus a character $\chi \in \mathrm{Irr}(Th)$ will be 5-cuspidal precisely when the cuspidal condition holds for the Sylow subgroup. Since Th has a unique conjugacy class of elements of order 5, we observe that there is a unique 5-cuspidal character of Th, namely χ_2 of degree 248.

$\mathbb{B}$

There is a unique 5-minimal parabolic system of $\mathbb{B}$ given by

$$\left\{P_1 \sim 5^{1+4}.2^{1+4}.\,\mathrm{Sym}(5).2, P_2 \sim 5^{2+1+2}.GL_2(5)\right\}.$$

Considering the minimum value that each $\chi \in \mathrm{Irr}(\mathbb{B})$ takes on $\mathbb{B}$-classes of 5-elements, we see that the only possible 5-cuspidal character of $\mathbb{B}$ is χ_2. However, $\deg(\chi_2) = 4371$, and $\mathbb{B}$ satisfies the conditions of Lemma 2.7. Thus there are no 5-cuspidal characters of $\mathbb{B}$.

$\mathbb{M}$

The monster has a unique 5-minimal parabolic system, namely

$$\left\{P_1 \sim 5^{1+6}.2.(J_2 \times 2).2, P_2 \sim 5^{2+2+4}.\,\mathrm{Sym}(3).GL_2(5)\right\}.$$

Class	1_1	2_1	2_2	5_1	5_2	5_3	5_7	5_8	5_9	5_{10}
Size	1	625	3750	24	50	50	600	600	600	600
Order	1	2	2	5	5	5	5	5	5	5
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	1	1	1	1	1	1
χ_3	2	-2	0	2	2	2	2	2	2	2
χ_4	2	-2	0	2	2	2	2	2	2	2
χ_5	2	-2	0	2	2	2	2	2	2	2
χ_6	2	-2	0	2	2	2	2	2	2	2
χ_7	3	3	1	3	3	3	3	3	3	3
χ_8	3	3	1	3	3	3	3	3	3	3
χ_9	3	3	-1	3	3	3	3	3	3	3
χ_{10}	3	3	-1	3	3	3	3	3	3	3
χ_{11}	4	4	0	4	4	4	4	4	4	4
χ_{12}	4	4	0	4	4	4	4	4	4	4
χ_{13}	4	-4	0	4	4	4	4	4	4	4
χ_{14}	4	-4	0	4	4	4	4	4	4	4
χ_{15}	5	5	1	5	5	5	5	5	5	5
χ_{16}	5	5	-1	5	5	5	5	5	5	5
χ_{17}	6	-6	0	6	6	6	6	6	6	6
χ_{18}	6	-6	0	6	6	6	6	6	6	6
χ_{19}	10	2	0	10	$5 \cdot w^3 + 5 \cdot w^2$	$-5 \cdot w^3 - 5 \cdot w^2 - 5$	0	0	0	0
χ_{20}	10	2	0	10	$-5 \cdot w^3 - 5 \cdot w^2 - 5$	$5 \cdot w^3 + 5 \cdot w^2$	0	0	0	0
χ_{21}	20	-4	0	20	$10 \cdot w^3 + 10 \cdot w^2$	$-10 \cdot w^3 - 10 \cdot w^2 - 10$	0	0	0	0
χ_{22}	20	-4	0	20	$-10 \cdot w^3 - 10 \cdot w^2 - 10$	$10 \cdot w^3 + 10 \cdot w^2$	0	0	0	0
χ_{23}	20	-4	0	20	$-10 \cdot w^3 - 10 \cdot w^2 - 10$	$10 \cdot w^3 + 10 \cdot w^2$	0	0	0	0
χ_{24}	20	-4	0	20	$10 \cdot w^3 + 10 \cdot w^2$	$-10 \cdot w^3 - 10 \cdot w^2 - 10$	0	0	0	0
χ_{25}	24	0	4	24	24	24	-1	-1	-1	-1
χ_{26}	24	0	-4	24	24	24	-1	-1	-1	-1
χ_{27}	30	6	0	30	$-15 \cdot w^3 - 15 \cdot w^2 - 15$	$15 \cdot w^3 + 15 \cdot w^2$	0	0	0	0
χ_{28}	30	6	0	30	$-15 \cdot w^3 - 15 \cdot w^2 - 15$	$15 \cdot w^3 + 15 \cdot w^2$	0	0	0	0
χ_{29}	30	6	0	30	$15 \cdot w^3 + 15 \cdot w^2$	$-15 \cdot w^3 - 15 \cdot w^2 - 15$	0	0	0	0
χ_{30}	30	6	0	30	$15 \cdot w^3 + 15 \cdot w^2$	$-15 \cdot w^3 - 15 \cdot w^2 - 15$	0	0	0	0
χ_{31}	40	-8	0	40	$-20 \cdot w^3 - 20 \cdot w^2 - 20$	$20 \cdot w^3 + 20 \cdot w^2$	0	0	0	0
χ_{32}	40	8	0	40	$20 \cdot w^3 + 20 \cdot w^2$	$-20 \cdot w^3 - 20 \cdot w^2 - 20$	0	0	0	0
χ_{33}	40	-8	0	40	$20 \cdot w^3 + 20 \cdot w^2$	$-20 \cdot w^3 - 20 \cdot w^2 - 20$	0	0	0	0
χ_{34}	40	8	0	40	$-20 \cdot w^3 - 20 \cdot w^2 - 20$	$20 \cdot w^3 + 20 \cdot w^2$	0	0	0	0
χ_{35}	48	0	0	48	48	48	-2	-2	-2	-2
χ_{36}	48	0	0	48	48	48	-2	-2	-2	-2
χ_{37}	50	10	0	50	$25 \cdot w^3 + 25 \cdot w^2$	$-25 \cdot w^3 - 25 \cdot w^2 - 25$	0	0	0	0
χ_{38}	50	10	0	50	$-25 \cdot w^3 - 25 \cdot w^2 - 25$	$25 \cdot w^3 + 25 \cdot w^2$	0	0	0	0
χ_{39}	60	-12	0	60	$30 \cdot w^3 + 30 \cdot w^2$	$-30 \cdot w^3 - 30 \cdot w^2 - 30$	0	0	0	0
χ_{40}	60	-12	0	60	$-30 \cdot w^3 - 30 \cdot w^2 - 30$	$30 \cdot w^3 + 30 \cdot w^2$	0	0	0	0
χ_{41}	120	0	-4	-5	0	0	-5	0	5	-5
χ_{42}	120	0	4	-5	0	0	-5	0	5	-5
χ_{43}	120	0	-4	-5	0	0	0	5	$-5 \cdot w^3 - 5 \cdot w^2 - 5$	0
χ_{44}	120	0	4	-5	0	0	$5 \cdot w^3 + 5 \cdot w^2 + 5$	-5	0	$-5 \cdot w^3 - 5 \cdot w^2$
χ_{45}	120	0	4	-5	0	0	0	5	$5 \cdot w^3 + 5 \cdot w^2$	0
χ_{46}	120	0	-4	-5	0	0	$-5 \cdot w^3 - 5 \cdot w^2$	-5	0	$5 \cdot w^3 + 5 \cdot w^2 + 5$
χ_{47}	120	0	-4	-5	0	0	$5 \cdot w^3 + 5 \cdot w^2 + 5$	-5	0	$-5 \cdot w^3 - 5 \cdot w^2$
χ_{48}	120	0	4	-5	0	0	0	5	$-5 \cdot w^3 - 5 \cdot w^2 - 5$	0
χ_{49}	120	0	-4	-5	0	0	0	5	$5 \cdot w^3 + 5 \cdot w^2$	0
χ_{50}	120	0	4	-5	0	0	$-5 \cdot w^3 - 5 \cdot w^2$	-5	0	$5 \cdot w^3 + 5 \cdot w^2 + 5$

Table 11: A partial character table of the 5-minimal parabolic subgroup $P_2 \sim 5^{2+1+2}.4.\mathrm{Alt}(5)$ of HN (where $w = \exp(2\pi i/5)$).

Using a similar approach to that used for the baby monster, with the 5-core $O_5(P_1)$, we deduce that there are no 5-cuspidal characters of $\mathbb{M}$.

Ly

From [14] we see that there is a unique 5-minimal parabolic system of Ly, having rank 3. Its minimal parabolic subgroups are $P_1 \sim 5_+^{1+4}.4.PGL_2(5)$, $P_2 \sim 5^{3+2}.4.PGL_2(5)$ and $P_3 \sim 5_+^{1+4}.4.PGL_2(5)$. The maximal 5-parabolic subgroups are given by

$$P_{12} \sim 5^3.SL_3(5), \quad P_{13} \sim 5_+^{1+4}.2.\operatorname{Alt}(6).4 \quad \text{and} \quad P_{23} \cong G_2(5).$$

By considering the elementary abelian subgroup $O_5(P_{12}) \cong 5^3$, we see that the only possible 5-cuspidal characters of Ly are χ_2 and χ_3 (both of degree 2480), and that for these characters to be 5-cuspidal, we must have that the non-trivial elements of $O_5(P_{12})$ are contained in the Ly-conjugacy class $5A$. Defining S to be our given Sylow 5-subgroup of Ly, we see that S has a unique normal elementary abelian 5-subgroup of order 5^3. Considering this within the maximal subgroup $G_2(5) \leq Ly$, we see that the non-trivial elements of $O_5(P_{12})$ are indeed contained in the Ly-class $5A$. It remains to check the cuspidal relation for χ_2 and χ_3 for the extra-special 5-core $O_5(P_{13})$ of order 5^5.

By constructing $O_5(P_{13})$ within both P_{13} and P_{23} we may deduce that it intersects the Ly-conjugacy classes $1A$, $5A$ and $5B$ in 1, 724 and 2400 elements respectively. It follows that the cuspidal relation holds on $O_5(P_{13})$ for both χ_2 and χ_3, and hence we see that χ_2 and χ_3 are 5-cuspidal characters of Ly.

6 p-Cuspidal Characters ($p > 5$)

In the case that $p > 5$, most sporadic groups with order divisible by p have a cyclic Sylow p-subgroup of order p. The exceptions are $(Co_1, p = 7)$, $(He, p = 7)$, $(Th, p = 7)$, $(Fi'_{24}, p = 7)$, $(\mathbb{B}, p = 7)$, $(\mathbb{M}, p = 7, 11, 13)$, $(O'N, p = 7)$ and $(J_4, p = 11)$.

The normalizer of a Sylow 7-subgroup, S, of He is maximal in He, and hence there is a unique 7-minimal parabolic system given by $\{He\}$. Consequently, an element $\chi \in \operatorname{Irr}(He)$ will be 7-cuspidal precisely when the cuspidal condition holds for S. We have that S contains 1, 42, 42, 132, 63 and 63 elements from the He-classes $1A$, $7A$, $7B$, $7C$, $7D$ and $7E$ respectively. It follows that the 7-cuspidal characters of He are χ_2 and χ_3 of degree 51.

In all other cases, since the sporadic group in question contains no elements of order p^a for $a > 1$, we may consider the minimum value that each irreducible character takes on elements of order p, to conclude that there are no p-cuspidal characters.

We conclude by considering the p-cuspidal characters arising from sporadic groups having a cyclic Sylow p-subgroup of order p. Since such a subgroup will necessarily be the p-core of its normalizer, it is easy to see that the cuspidal relation must hold for the Sylow subgroup. Moreover, a character will be cuspidal precisely when this is true. This gives an additional four cuspidal characters for the sporadic groups; the 10-dimensional characters $\chi_2, \chi_3, \chi_4 \in \operatorname{Irr}(M_{11})$ are 11-cuspidal and the 22-dimensional character $\chi_2 \in \operatorname{Irr}(M_{23})$ is 23-cuspidal.

References

[1] M. Aschbacher. *Sporadic Groups.* Cambridge University Press, Cambridge Tracts in Mathematics 104, 1994.

[2] M. Aschbacher. 3-*Transposition Groups.* Cambridge University Press, Cambridge Tracts in Mathematics 124, 1997.

[3] R.W. Carter. *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters.* John Wiley and Sons. 1993.

[4] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson. $\mathbb{ATLAS}$ *of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Clarendon Press, 2009.

[5] C. W. Curtis. *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer.* History of Mathematics, 15. American Mathematical Society, Providence, RI; London Mathematical Society, London, 1999. xvi+287 pp.

[6] R.T. Curtis. *On Subgroups of* $\cdot 0$ *II. Local Structure.* J. Algebra, 63(1980), 413–434.

[7] D. Gorenstein. *Finite Groups.* Harper and Row, Publishers, New York-London, 1968. xv+527 pp.

[8] K. Harada. *On the Simple Group* F *of Order* $2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11\cdot 19$. Proceedings of the Conference on Finite Groups, Academic Press, 1976.

[9] A.A. Ivanov and U. Meierfrankenfeld. *Simple Connectedness of the* 3-*Local Geometry of the Monster.* J. Algebra, 194(1997), 383–407.

[10] G. Karpilovsky. *Group Representations.* Elsevier Science, North-Holland mathematics studies 3, 1994.

[11] R.S. Margolin. *A Geometry for* M_{24}. J. Algebra, 156(1993), 370–384.

[12] U. Meierfrankenfeld and S. Shpectorov. *Maximal* 2-*local subgroups of the Monster and Baby Monster.* 2002. `http://users.math.msu.edu/users/meier/Preprints/2monster/maxmon.pdf`.

[13] M.A. Ronan and S.D. Smith. *Sheaves on buildings and modular representations of Chevalley groups.* J. Algebra 96(1985), 319–346.

[14] M.A. Ronan and G. Stroth. *Minimal Parabolic Geometries for the Sporadic Groups.* European J. Combin., 5(1984), 59–91.

[15] P. Rowley and D. Ward. *Fusion in* 2-*cores of Maximal Parabolic Subgroups of the Baby Monster.* MIMS EPrints.

[16] P.E. Smith. *A simple subgroup of* $\mathbb{M}$*?* *and* $E_8(3)$. Bull. London Math. Soc. 8(1976), 161–165.

[17] J.G. Thompson. *A conjugacy theorem for* E_8. J. Algebra, 38(1976), 525–530.

[18] R.A. Wilson. *The local subgroups of the Fischer groups*. J. London Math. Soc., 36(1987), 77–94.

[19] R.A. Wilson. *The Finite Simple Groups*. Springer-Verlag, Graduate Texts in Mathematics 251, 2009.

[20] S. Yoshiara. *Radical* 2*-subgroups of the Monster and the Baby Monster*. J. Algebra, 287(2005), 123–139.