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Strong Linearizations of Rational Matrices

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Abstract

This paper defines for the first time strong linearizations of arbitrary rational matrices, studies in depth properties and different characterizations of such linear matrix pencils, and develops infinitely many examples of strong linearizations that can be explicitly and easily constructed from a minimal state-space realization of the strictly proper part of the considered rational matrix and the coefficients of the polynomial part. As a consequence, the results in this paper establish a rigorous foundation for the numerical computation of the complete structure of zeros and poles, both finite and at infinity, of any rational matrix by applying any well known backward stable algorithm for generalized eigenvalue problems to any of the strong linearizations explicitly constructed in this work. Since the results of this paper require to use several concepts that are not standard in matrix computations, a considerable effort has been done to make the paper as self-contained as possible.

Key words. Linearization, minimal polynomial system matrix, nonlinear eigenvalue problem, rational matrix, strong block minimal bases linearization, strong linearization

AMS subject classification. 65F15, 15A18, 15A22, 15A54, 93B18, 93B20, 93B60

1 Introduction

Given a nonsingular rational matrix $G(\lambda)$ (i.e., a matrix whose entries are rational functions) the rational eigenvalue problem (REP) is to find scalars

λ and nonzero vectors x satisfying

$$G(\lambda)x = 0.$$

The scalars λ and the vectors x are called, respectively, eigenvalues and eigenvectors of the rational matrix $G(\lambda)$. The REP arises in many applications and several approaches can be used to tackle it (see [29] and references therein). Actually, in [29] a new method for solving numerically the REP is given based on the fact that any rational matrix $G(\lambda)$ can be uniquely written as the sum of a polynomial matrix and a strictly proper one. The method consists in applying any well established algorithm for computing the eigenvalues of a linear pencil [16] to a pencil constructed out of a linearization of the polynomial part of $G(\lambda)$ and a realization of its strictly proper part, which preserves the finite zeros of $G(\lambda)$. This method has been formalized and generalized in [1] where a precise definition of linearization of a square rational matrix is given. The definition of linearization in [1] relies on the fact that every rational matrix $G(\lambda)$ admits a right coprime matrix fraction description $G(\lambda) = N(\lambda)D(\lambda)^{-1}$, where $N(\lambda)$ and $D(\lambda)$ are polynomial matrices. Such a decomposition of $G(\lambda)$ has the important property that the finite zeros of $G(\lambda)$ are the eigenvalues of $N(\lambda)$ and the finite poles of $G(\lambda)$ are the eigenvalues of $D(\lambda)$ [19]. A linearization of $G(\lambda)$ is then defined to be a linear pencil

$$L(\lambda) = \begin{bmatrix} \lambda E + A & B \\ C & \lambda Y + X \end{bmatrix} \tag{1}$$

such that $L(\lambda)$ is a linearization of $N(\lambda)$ and $\lambda E + A$ is a linearization of $D(\lambda)$, in the standard sense of linearizations of polynomial matrices (see [20, 9] and the references therein). If such a linearization exists, one has access to the finite zeros and poles of $G(\lambda)$ by solving two linear eigenvalue problems: $L(\lambda)x = 0$ and $(\lambda E + A)y = 0$. The notion of polynomial system matrix introduced by Rosenbrock is then used in [1] to show that Fiedler-type linearizations of square rational matrices always exist.

The linearizations defined in [1] reflect the finite structure of rational matrices but no evidence is given that they preserve also the infinite structure. The main goal of the present paper is to provide a new definition of linearization of rational matrices that preserve the finite as well as the infinite poles and zeros of the original matrix. We emphasize that this goal will be achieved in the general context of arbitrary rational matrices, i.e., square or rectangular, regular or singular, in contrast to the references [1, 29] which only consider square matrices. In addition, infinitely many of such linearizations will be explicitly constructed.

This new definition of linearization takes advantage of the following property of the minimal polynomial system matrices of a rational matrix $G(\lambda)$

(see Theorem 2.1): under very mild conditions if

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix}$$

is a minimal polynomial system matrix giving rise to $G(\lambda)$ (i.e., $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$) then the finite poles and zeros of $G(\lambda)$, counting with multiplicities, are the finite zeros of $A(\lambda)$ and $P(\lambda)$, respectively, counting with multiplicities. With this property in mind, a linear pencil $L(\lambda)$ is said to be a linearization of $G(\lambda)$ if it is a minimal polynomial system matrix of a rational matrix $\widehat{G}(\lambda)$ such that, for some nonnegative integers s_1, s_2, $\mathrm{Diag}(\widehat{G}(\lambda), I_{s_1})$ and $\mathrm{Diag}(G(\lambda), I_{s_2})$ are equivalent via unimodular polynomial matrices. This definition looks very much like the definition of linearization of matrix polynomials that can be found, for example, in [22, 20, 9] and can be generalized to preserve also the poles and zeros of $G(\lambda)$ at infinity (Definition 6.2). We will see that when $G(\lambda)$ is polynomial it reduces to the definition of strong linearization of polynomial matrices.

Notions like polynomial system matrices, least order, realizations, Smith–McMillan forms (finite and at infinity), strict system equivalence, transfer function matrices (which are well-established in the theory of linear systems) will play an important role. They will be reviewed in Section 2. Polynomial system matrices of least order, or minimal, are important in our developments. Since the eigenvalues of the REP $G(\lambda)x = 0$ are the finite zeros of $G(\lambda)$ that are not finite poles, we want linearizations that preserve the poles of $G(\lambda)$ (with their partial multiplicities) but that do not incorporate spurious ones. It will be shown in Section 3 that this leads us to polynomial system matrices of least order. The concept of least order will be discussed in detail in Section 2.

The new definition of linearization of a rational matrix will be given in Section 3 (see Definition 3.3). It will be shown that, when applied to polynomial matrices, it reduces to the usual one [22, 20, 9]. Also, a spectral characterization of the linearizations, in the spirit of [9, Thm. 4.1] will be provided (Theorem 3.9). The relationship between our definition and that of [1] will be analyzed in Section 5 showing that when $L(\lambda)$ in (1) is required to be of least order then the definition of [1] is a particular case of Definition 3.3.

Furthermore, in view of the proposed definition of linearization, it is important to determine when two polynomial system matrices give rise to equivalent rational matrices. Based on the strict system equivalence introduced in [27], a new equivalence relation is defined in Section 4 whose equivalence classes are formed by the polynomial system matrices having equivalent transfer function matrices. For a given rational matrix $G(\lambda)$, this equivalence relation gives us the precise amount of freedom to obtain linearizations out of any polynomial system matrix whose transfer function

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matrix is $G(\lambda)$. A general procedure is then proposed to construct linearizations for any rational matrix. Practical implementations of this procedure were used in [29] and [1] to show the existence of Frobenius companion-type and Fiedler-type linearizations of square rational matrices. This procedure will be fundamental to construct in Section 8 infinitely many linearizations of arbitrary rational matrices. More precisely, we will show how any strong block minimal bases linearization of a matrix polynomial [11, Def. 3.1] can be used very easily to construct a linearization of any rational matrix. Since strong block minimal bases linearizations of matrix polynomials form a very wide class containing infinitely many linearizations which include, among many others, Frobenius companion forms, all Fiedler linearizations [12, 5, 8] modulo permutations, and all block Kronecker linearizations [11, Def. 5.1], we construct in this way the widest class known so far of linearizations of arbitrary rational matrices.

Strong linearizations of rational matrices are introduced in Section 6 (see Definition 6.2). As for the case of linearizations which are not necessarily strong, we show that this new notion is a natural extension of strong linearizations for matrix polynomials and present a spectral characterization of strong linearizations, which establishes that such linearizations preserve not only the finite but also the infinite structure of rational matrices. Also, a new equivalence relation is introduced in Section 7 that classifies system matrices that give rise to equivalent at infinity rational matrices. This equivalence relation together with the one defined in Section 4 allows us to show in Section 8 that all the linearizations of rational matrices constructed from strong block minimal bases linearizations of matrix polynomials are always strong linearizations in the rational case. In addition, two examples are taken from [29] to practically implement the construction of strong linearizations of real symmetric rational matrices that preserve the symmetry of the original problem. Finally, we discuss in Section 9 the main conclusions of this work and some possible lines of future research motivated by the results in this paper.

All along this paper polynomial matrix and matrix polynomial will be used as synonymous terms.

2   Preliminaries

In this section we review the basic notions of linear system theory that we will use in the subsequent sections. Our basic references are [27, 19, 32].

Although for practical purposes the rational matrices of interest are those whose elements have real or complex coefficients, the results in this paper are of algebraic nature and apply for matrices with coefficients in arbitrary fields. Thus $\mathbb{F}$ will denote any arbitrary field, $\mathbb{F}[\lambda]$ the ring of polynomials with coefficients in $\mathbb{F}$ and $\mathbb{F}(\lambda)$ the field of *rational functions*, i.e., quotients

of coprime polynomials of $\mathbb{F}[\lambda]$. A rational function $r(\lambda) = \frac{n(\lambda)}{d(\lambda)}$ is said to be *proper* if $\deg(n(\lambda)) \leq \deg(d(\lambda))$, where $\deg(\cdot)$ stands for degree. If $\deg(n(\lambda)) < \deg(d(\lambda))$ then $r(\lambda)$ is called *strictly proper*. Let $\mathbb{F}(\lambda)^{p \times m}$ be the set of $p \times m$ matrices with elements in $\mathbb{F}(\lambda)$. Any rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ can be written as

$$G(\lambda) = D(\lambda) + C(\lambda) A(\lambda)^{-1} B(\lambda), \tag{2}$$

for some nonsingular matrix polynomial $A(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ and matrix polynomials $B(\lambda) \in \mathbb{F}[\lambda]^{n \times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p \times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$ with $n \geq \deg(\det A(\lambda))$ (see [27]). The matrix polynomial

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \tag{3}$$

is called a *polynomial system matrix* of (or giving rise to) $G(\lambda)$. In other words, a polynomial matrix $P(\lambda)$ with the form of (3) and such that $A(\lambda)$ is nonsingular and $\deg(\det A(\lambda)) \leq n$ will be said to be a polynomial system matrix of the rational matrix $G(\lambda)$ if $G(\lambda)$ is the Schur complement of $A(\lambda)$ in $P(\lambda)$. Then $G(\lambda)$ is called the *transfer function matrix* of $P(\lambda)$ and $\deg(\det A(\lambda))$ is its *order*.

The reason why condition $n \geq \deg(\det A(\lambda))$ is required will be clarified later on (see Remark 4.5). Nevertheless notice that if $n < \deg(\det A(\lambda)) = d$ in (2) then by putting

$$\widetilde{A}(\lambda) = \begin{bmatrix} I_{d-n} & 0 \\ 0 & A(\lambda) \end{bmatrix}, \quad \widetilde{B}(\lambda) = \begin{bmatrix} 0_{(d-n) \times m} \\ B(\lambda) \end{bmatrix}, \quad \widetilde{C}(\lambda) = \begin{bmatrix} 0_{p \times (d-n)} & C(\lambda) \end{bmatrix},$$

we get $G(\lambda) = D(\lambda) + \widetilde{C}(\lambda) \widetilde{A}(\lambda)^{-1} \widetilde{B}(\lambda)$, with $\widetilde{A}(\lambda) \in \mathbb{F}[\lambda]^{d \times d}$. Therefore we will assume that the order of $P(\lambda)$ is always not bigger than the size of $A(\lambda)$. The matrix $A(\lambda)$ will be called the *state matrix* of the system.

When $A(\lambda)$ is a monic linear matrix polynomial, say $A(\lambda) = \lambda I_n - A$, $B(\lambda) = B$ and $C(\lambda) = C$ are constant matrices, $P(\lambda)$ is said to be a polynomial system matrix of $G(\lambda)$ in *state-space form*.

As already seen, the integer n or the polynomial matrices of (2) are not uniquely determined by $G(\lambda)$. It turns out that different polynomial system matrices may exist with different orders giving rise to the same transfer function matrix. For example, for any nonsingular polynomial matrix $\widehat{A}(\lambda)$, the rational matrix (2) can be written as follows:

$$G(\lambda) = D(\lambda) + \begin{bmatrix} C(\lambda) & 0 \end{bmatrix} \begin{bmatrix} A(\lambda) & 0 \\ 0 & \widehat{A}(\lambda) \end{bmatrix}^{-1} \begin{bmatrix} B(\lambda) \\ 0 \end{bmatrix}.$$

A polynomial system matrix of $G(\lambda)$ is said to have *least order*, or to be *minimal*, if its order is the smallest integer for which matrix polynomials

$A(\lambda)$ (nonsingular, with size $n \times n$, $n \geq \deg(\det A(\lambda))$), $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ satisfying (2) exist. The least order is uniquely determined by $G(\lambda)$ and is denoted by $\nu(G(\lambda))$. It is called the *least order* of $G(\lambda)$ ([27, Ch. 3, Sec. 5.1] or [32, Sec. 1.10]). Let us recall three equivalent conditions that characterize when the polynomial system matrix in (3) has least order ([27, Ch. 3]):

(i) $P(\lambda)$ has no decoupling zeros.

(ii) $A(\lambda)$ and $B(\lambda)$ are left coprime and $A(\lambda)$ and $C(\lambda)$ are right coprime.

(iii) (A, B) is controllable and (A, C) is observable assuming that $P(\lambda)$ is in state-space form.

The meaning of these three conditions is well-known in the theory of linear control systems but we will not go into the details. Only property (ii) will be analyzed: Two polynomial matrices $A(\lambda) \in \mathbb{F}[\lambda]^{p \times n}$, $B(\lambda) \in \mathbb{F}[\lambda]^{q \times n}$ are called *right coprime* if their only right common divisors are *unimodular* matrices (polynomial matrices with nonzero constant determinant). That is to say, if there exist $\bar{A}(\lambda) \in \mathbb{F}[\lambda]^{p \times n}, \bar{B}(\lambda) \in \mathbb{F}[\lambda]^{q \times n}, X(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ such that

$$\begin{array}{c} A(\lambda) = \bar{A}(\lambda)X(\lambda) \\ B(\lambda) = \bar{B}(\lambda)X(\lambda) \end{array} \Rightarrow X(\lambda) \text{ unimodular.}$$

A useful characterization is that $A(\lambda) \in \mathbb{F}[\lambda]^{p \times n}$ and $B(\lambda) \in \mathbb{F}[\lambda]^{q \times n}$ are right coprime if and only if the Smith form of $\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix}$ is $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$ (see [27, Ch. 2, Sec. 6]). On the other hand, $A(\lambda) \in \mathbb{F}[\lambda]^{n \times p}$, $B(\lambda) \in \mathbb{F}[\lambda]^{n \times q}$ are *left coprime* if their transposes $A(\lambda)^T$ and $B(\lambda)^T$ are right coprime.

Any rational function $r(\lambda) \in \mathbb{F}(\lambda)$ can be uniquely written as

$$r(\lambda) = p(\lambda) + r_{sp}(\lambda)$$

with $p(\lambda)$ a polynomial and $r_{sp}(\lambda)$ a strictly proper rational function. Using this decomposition for all entries of $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ we find that

$$G(\lambda) = D(\lambda) + G_{sp}(\lambda) \tag{4}$$

where $D(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$ is a polynomial matrix and $G_{sp}(\lambda)$ is a *strictly proper rational matrix*, i.e., the entries of $G_{sp}(\lambda)$ are strictly proper rational functions. Now, it is a well-known fact that any strictly proper rational matrix admits *realizations* (see, for example, [27, Ch. 3, Sec. 5.2] or [19, Sec. 6.4]). This means that for some positive integer n there exist matrices $A(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$, $B(\lambda) \in \mathbb{F}[\lambda]^{n \times m}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{p \times n}$ such that $G_{sp}(\lambda) = C(\lambda)A(\lambda)^{-1}B(\lambda)$ and

$$\begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix}$$

is a polynomial system matrix of $G(\lambda)$. Notice that, when $G(\lambda)$ is decomposed as in (4), the polynomial part $D(\lambda)$ is uniquely determined by $G(\lambda)$ while the integer n and the matrices $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are not. Furthermore, it is also well-known that any strictly proper rational matrix admits state-space realizations. Therefore any rational matrix $G(\lambda)$ can be written as $G(\lambda) = D(\lambda) + C(\lambda I_n - A)^{-1}B$ where $D(\lambda)$ is polynomial and $C(\lambda I_n - A)^{-1}B$ is strictly proper. Moreover, the realization may always be taken of least order (i.e., such that the corresponding polynomial system matrix in state-space form is of least order). Such realizations are called *minimal*.

Minimal polynomial system matrices convey precise information about the finite poles and zeros of their transfer function matrices. Recall (see, for example, [27, Ch. 3, Sec. 4] or [19, Sec. 6.5.2]) that any rational matrix is *(finite) equivalent*[1] to its *(finite) Smith–McMillan form*. That is to say, if $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ then there are unimodular matrices $U(\lambda) \in \mathbb{F}[\lambda]^{p \times p}$ and $V(\lambda) \in \mathbb{F}[\lambda]^{m \times m}$ such that

$$M(\lambda) = U(\lambda)G(\lambda)V(\lambda) = \begin{bmatrix} \mathrm{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)} \right) & 0 \\ 0 & 0 \end{bmatrix} \quad (5)$$

where $r = \mathrm{rank}\, G(\lambda)$, $\epsilon_1(\lambda), \ldots, \epsilon_r(\lambda), \psi_1(\lambda), \ldots, \psi_r(\lambda)$ are nonzero monic polynomials, $\epsilon_i(\lambda), \psi_i(\lambda)$ are coprime for all $i = 1, \ldots, r$, and $\epsilon_1(\lambda) \mid \cdots \mid \epsilon_r(\lambda)$ while $\psi_r(\lambda) \mid \cdots \mid \psi_1(\lambda)$, where $\mid$ stands for divisibility. The irreducible fractions $\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}$ are called the *(finite) invariant rational functions* of $G(\lambda)$. In addition, $\psi_1(\lambda)$ is the monic least common denominator of the entries in $G(\lambda)$ and so, $G(\lambda)$ is polynomial if and only if $\psi_1(\lambda) = 1$. In this case (i.e., if $G(\lambda)$ is a polynomial matrix), $M(\lambda)$ is called the *(finite) Smith normal form* of $G(\lambda)$ and the monic polynomials $\epsilon_1(\lambda) \mid \cdots \mid \epsilon_r(\lambda)$ are called the *invariant polynomials* of $G(\lambda)$.

The *(finite) poles* of $G(\lambda)$ are the roots in $\overline{\mathbb{F}}$ (the algebraic closure of $\mathbb{F}$) of $\psi_1(\lambda)$ and its *(finite) zeros* are the roots in $\overline{\mathbb{F}}$ of $\epsilon_r(\lambda)$. If $\lambda_0 \in \overline{\mathbb{F}}$ is a zero of $G(\lambda)$ then, for $i = 1, \ldots, r$, we can write $\epsilon_i(\lambda) = (\lambda - \lambda_0)^{m_i}\widehat{\epsilon_i}(\lambda)$ with $\widehat{\epsilon_i}(\lambda_0) \neq 0$ and $m_i \geq 0$. The nonzero elements in $(m_1, \ldots, m_r)$ are called the *partial multiplicities of λ_0 as a zero* of $G(\lambda)$. The *partial multiplicities of the poles* of $G(\lambda)$ are defined similarly. Notice that although $\epsilon_i(\lambda)$ and $\psi_i(\lambda)$ are coprime polynomials for all $i = 1, \ldots, r$, $G(\lambda)$ may have zeros and poles at the same points.

When $G(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$ is a polynomial matrix then $\psi_i(\lambda) = 1$ for $i = 1, \ldots, r$ and the polynomials $(\lambda - \lambda_0)^{m_i}$ with $m_i \neq 0$ are the *finite elementary divisors* of $G(\lambda)$ with respect to, or associated to, λ_0.

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[1]In this manuscript, two rational matrices $G_1(\lambda)$ and $G_2(\lambda)$ are said to be equivalent if there exist two unimodular polynomial matrices $U(\lambda)$ and $V(\lambda)$ such that $G_1(\lambda) = U(\lambda)G_2(\lambda)V(\lambda)$. Other types of equivalence relations are often used in this paper, but in those cases the corresponding type of equivalence will be always explicitly mentioned.

As announced, the finite poles and zeros of any rational matrix can be found through any of its polynomial system matrices of least order. This follows from the following result by Rosenbrock ([27, Ch. 3, Thm. 4.1]). In what follows we will use a notation like

$$\begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

to mean that $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$.

Theorem 2.1 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix of rank r and let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)} \tag{6}$$

be a polynomial system matrix of least order whose transfer function matrix is $G(\lambda)$ such that $n \geq r = \operatorname{rank} G(\lambda)$. Let the Smith–McMillan form of $G(\lambda)$ be the matrix $M(\lambda)$ in (5). Then, the invariant polynomials of $A(\lambda)$ are $1|\cdots|1|\psi_r(\lambda)|\cdots|\psi_1(\lambda)$ with at least $n - r$ invariant polynomials equal to 1, and the invariant polynomials of $P(\lambda)$ are $1|\cdots|1|\epsilon_1(\lambda)|\cdots|\epsilon_r(\lambda)$ with at least n invariant polynomials equal to 1.

Remark 2.2 We have stated Theorem 2.1 exactly with the hypotheses assumed originally by Rosenbrock. However, we emphasize that Theorem 2.1 also holds if the assumption $n \geq r = \operatorname{rank} G(\lambda)$ is replaced by the weaker one $n \geq t$, where t is the number of denominators $\psi_1(\lambda), \ldots, \psi_r(\lambda)$ different from 1 in the Smith–McMillan form of $G(\lambda)$. In order to check this, let $P(\lambda)$ in (6) be a polynomial system matrix of least order of $G(\lambda)$ with $n < r = \operatorname{rank} G(\lambda)$. Then also

$$\widetilde{P}(\lambda) = \left[\begin{array}{cc|c} I_{r-n} & 0 & 0 \\ 0 & A(\lambda) & B(\lambda) \\ \hline 0 & -C(\lambda) & D(\lambda) \end{array} \right]$$

is a polynomial system matrix of least order of $G(\lambda)$. Therefore, Theorem 2.1 can be applied to $\widetilde{P}(\lambda)$ by replacing n by r, $A(\lambda)$ by $\begin{bmatrix} I_{r-n} & 0 \\ 0 & A(\lambda) \end{bmatrix}$ and $P(\lambda)$ by $\widetilde{P}(\lambda)$. This leads to the following result for $P(\lambda)$ in (6) with $n < r = \operatorname{rank} G(\lambda)$: "the invariant polynomials of $A(\lambda)$ are $1|\cdots|1|\psi_t(\lambda)|\cdots|\psi_1(\lambda)$ with $n - t$ invariant polynomials equal to 1, and the invariant polynomials of $P(\lambda)$ are $1|\cdots|1|\epsilon_1(\lambda)|\cdots|\epsilon_r(\lambda)$ with at least n invariant polynomials equal to 1. This remark is relevant because very often the REPs arising in applications (see [29]) satisfy $t \ll \operatorname{rank} G(\lambda)$, as a consequence of having strictly proper parts $G_{sp}(\lambda)$ in (4) with rank much smaller than $\operatorname{rank} G(\lambda)$.

A consequence of Theorem 2.1 is that the order of any polynomial system matrix of least order giving rise to $G(\lambda)$ is the degree of the polynomial

$\psi(\lambda) = \psi_1(\lambda) \cdots \psi_r(\lambda)$, i.e., the product of the denominators in the Smith–McMillan form of $G(\lambda)$ (see [27, Ch. 3, Sec. 5.1]). Hence $\nu(G(\lambda)) = \deg(\psi(\lambda))$.

Also, if $P(\lambda)$ is a polynomial system matrix of least order of $G(\lambda)$ then the finite poles of $G(\lambda)$ are the finite zeros of $A(\lambda)$ and the finite zeros of $G(\lambda)$ are the finite zeros of $P(\lambda)$ (counting in all cases the corresponding partial multiplicities). In particular, if $P(\lambda)$ is a minimal polynomial system matrix in state-space form and $D(\lambda)$ is a linear polynomial, then $P(\lambda)$ is a linear pencil, its finite zeros are the finite zeros of $G(\lambda)$ and the finite zeros of $A(\lambda) = \lambda I - A$ are the finite poles of $G(\lambda)$.

$G(\lambda)$ may also have *poles and zeros at infinity*, which are the poles and zeros at $\lambda = 0$ of $G(1/\lambda)$ (see [19]). Let $\mathbb{F}_{pr}(\lambda)$ denote the ring of proper rational functions. Its units are called *biproper rational functions*, that is, rational functions having the same degree of numerator and denominator. $\mathbb{F}_{pr}(\lambda)^{p \times m}$ denotes the set of $p \times m$ *proper matrices*, i.e., matrices with entries in $\mathbb{F}_{pr}(\lambda)$. A *biproper matrix* is a square proper matrix whose determinant is a biproper rational function. Biproper matrices are also called *bicausal*. Two rational matrices $G_1(\lambda), G_2(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ are *equivalent at infinity* if there exist biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p \times p}$, $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m \times m}$ such that $G_2(\lambda) = B_1(\lambda)G_1(\lambda)B_2(\lambda)$. Every rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ is equivalent at infinity to its *Smith–McMillan form at infinity*:

$$\begin{bmatrix} \text{Diag}\left(\frac{1}{\lambda^{q_1}}, \ldots, \frac{1}{\lambda^{q_r}}\right) & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{F}(\lambda)^{p \times m}$$

where $r = \text{rank}\, G(\lambda)$ and $q_1 \leq \cdots \leq q_r$ are integers (see [4] or [32]). The rational functions $\frac{1}{\lambda^{q_1}}, \ldots, \frac{1}{\lambda^{q_r}}$ are called the *invariant rational functions at infinity* of $G(\lambda)$. The integers $q_1, \ldots, q_r$ are called the *invariant orders at infinity* of $G(\lambda)$. The invariant orders at infinity form a complete system of invariants for the equivalence at infinity in $\mathbb{F}(\lambda)^{p \times m}$ and they determine the zeros and poles at infinity of $G(\lambda)$ (see [4, Prop. 6.11]). Namely, if $q_1 \leq \cdots \leq q_k < 0 = q_{k+1} = \cdots = q_{u-1} < q_u \leq \cdots \leq q_r$ are the invariant orders at infinity of $G(\lambda)$ then $G(\lambda)$ has $r - u + 1$ zeros at infinity each one of order $q_u, \ldots, q_r$ and k poles at infinity each one of order $-q_1, \ldots, -q_k$.

Notice that $G(\lambda)$ is proper if and only if $q_1, \ldots, q_r$ are nonnegative integers, that is, proper matrices do not have poles at infinity (they are analytic at ∞ when $\mathbb{F} = \mathbb{C}$). However, non-constant polynomial matrices always have poles at infinity (they are never analytic at ∞ when $\mathbb{F} = \mathbb{C}$) and they may have zeros at infinity as well. Moreover for any non strictly proper rational matrix $-q_1$ is the degree of the polynomial part of the matrix in the expression (4) ([4, 32]). The *degree* of a polynomial matrix is the degree of the entries of highest degree.

In addition to finite elementary divisors, matrix polynomials may have elementary divisors at infinity as well [15, p. 185]. The *elementary divisors at infinity* or *infinite elementary divisors* of a matrix polynomial $Q(\lambda)$ are

defined as follows: Consider the *reversal* of $Q(\lambda)$, i.e., the matrix polynomial $\operatorname{rev} Q(\lambda) := \lambda^d Q\left(\frac{1}{\lambda}\right)$ where $d = \deg(Q(\lambda))$. This matrix polynomial may or may not have 0 as an eigenvalue. If it has 0 as an eigenvalue then $Q(\lambda)$ is said to have ∞ as an eigenvalue or to have *eigenvalues at infinity*. The infinite elementary divisors of $Q(\lambda)$ are the elementary divisors associated to the eigenvalue 0 of the reversal of $Q(\lambda)$. Let $q_1, \ldots, q_r$ be the invariant orders at infinity of the polynomial matrix $Q(\lambda)$ of degree d and rank r and let $\lambda^{e_1}, \ldots, \lambda^{e_r}$ be its infinite elementary divisors (including possible exponents equal to zero). Then (see [4])

$$e_i = q_i - q_1 = d + q_i, \quad i = 1, \ldots, r. \tag{7}$$

Similarly to the finite case, the zeros and poles at infinity of a rational matrix $G(\lambda)$ can be determined by its polynomial system matrices (3). However, while for the finite case the minimality of the polynomial system matrix is required (Theorem 2.1), for the infinite case the matrices $A(\lambda)^{-1}B(\lambda)$ and $C(\lambda)A(\lambda)^{-1}$ must both be proper. In fact, we can prove the following lemma.

Lemma 2.3 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ and let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p) \times (n+m)}$$

be a polynomial system matrix of $G(\lambda)$ such that both $A(\lambda)^{-1}B(\lambda)$ and $C(\lambda)A(\lambda)^{-1}$ are proper rational matrices. Then $P(\lambda)$ is equivalent at infinity to

$$\begin{bmatrix} A(\lambda) & 0 \\ 0 & G(\lambda) \end{bmatrix}.$$

Proof.- The desired result is obtained by pre and post multiplying $P(\lambda)$ by the biproper matrices $\begin{bmatrix} I_n & 0 \\ C(\lambda)A(\lambda)^{-1} & I_p \end{bmatrix}$ and $\begin{bmatrix} I_n & -A(\lambda)^{-1}B(\lambda) \\ 0 & I_m \end{bmatrix}$, respectively. ∎

Corollary 2.4 *Under the conditions of the previous lemma, if $q_1^A, \ldots, q_n^A$ are the invariant orders at infinity of $A(\lambda)$ and $q_1^G, \ldots, q_r^G$ are the invariant orders at infinity of $G(\lambda)$ then the invariant orders at infinity of $P(\lambda)$ are $q_1^P, \ldots, q_{n+r}^P$ where*

$$(q_{n+r}^P, \cdots, q_1^P) = (q_n^A, \cdots, q_1^A) \cup (q_r^G, \cdots, q_1^G).$$

In words, the invariant orders at infinity of $P(\lambda)$ are the ordered reunion of the invariant orders at infinity of $A(\lambda)$ and of $G(\lambda)$. This means that the invariant orders at infinity of $G(\lambda)$ are determined by those of $P(\lambda)$ and $A(\lambda)$. Therefore, the infinite poles of $G(\lambda)$ are determined by the infinite

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poles of $P(\lambda)$ and $A(\lambda)$ while the infinite zeros of $G(\lambda)$ are determined by the infinite zeros of $P(\lambda)$ and $A(\lambda)$.

Nevertheless there is an important difference between the finite and infinite cases. While the finite zeros of $G(\lambda)$ are those of $P(\lambda)$ and the finite poles of $G(\lambda)$ are the finite zeros of $A(\lambda)$, an analogous distinction cannot be made in the infinite case, i.e., the infinite zeros of $G(\lambda)$ come from those of both $P(\lambda)$ and $A(\lambda)$ and so do the infinite poles.

3 Linearizations of rational matrices

As said in the introduction, given a nonsingular rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{m \times m}$, the *rational eigenvalue problem (REP)* is to find scalars λ and nonzero vectors x satisfying

$$G(\lambda)x = 0.$$

The scalars λ and the vectors x are called, respectively, *eigenvalues* and *(right) eigenvectors* of the rational matrix $G(\lambda)$. Any rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ can be written as

$$G(\lambda) = \frac{N(\lambda)}{\psi_1(\lambda)}$$

where $N(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$ is polynomial and $\psi_1(\lambda)$ is the monic least common denominator of the entries of $G(\lambda)$, which coincides with the denominator of the first finite invariant rational function of $G(\lambda)$. Thus, if $G(\lambda) \in \mathbb{F}(\lambda)^{m \times m}$ is nonsingular, λ_0 is an eigenvalue of $G(\lambda)$ if $G(\lambda_0)x = 0$ for some nonzero x, that is, if there exists $x \neq 0$ such that $N(\lambda_0)x = 0$ and $\psi_1(\lambda_0) \neq 0$. This condition is equivalent to $\psi_1(\lambda_0) \neq 0$ and $\operatorname{rank} N(\lambda_0) < m$ or to $\psi_1(\lambda_0) \neq 0$ and $\operatorname{rank} G(\lambda_0) < m$. Therefore, we can define the set of eigenvalues of $G(\lambda)$ as (see [1])

$$\operatorname{Eig}(G) = \{\lambda_0 \in \overline{\mathbb{F}} : \psi_1(\lambda_0) \neq 0, \operatorname{rank} G(\lambda_0) < m\}.$$

Let $\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\epsilon_m(\lambda)}{\psi_m(\lambda)}$ be the finite invariant rational functions of $G(\lambda)$. The set of finite zeros of $G(\lambda)$ is

$$\operatorname{Zeros}(G) = \{\lambda_0 \in \overline{\mathbb{F}} : \epsilon_m(\lambda_0) = 0\}.$$

Since $\operatorname{rank} G(\lambda_0) < m$ if and only if $\epsilon_m(\lambda_0) = 0$, the set of eigenvalues of $G(\lambda)$ can be also defined as

$$\operatorname{Eig}(G) = \{\lambda_0 \in \overline{\mathbb{F}} : \psi_1(\lambda_0) \neq 0, \epsilon_m(\lambda_0) = 0\}.$$

It may happen $\psi_1(\lambda_0) = \epsilon_m(\lambda_0) = 0$ (as in Example 2.6 of [1]). Hence $\operatorname{Eig}(G) \subseteq \operatorname{Zeros}(G)$. However, if $G(\lambda)$ is polynomial (i.e., $\psi_1(\lambda) = 1$) then $\operatorname{Eig}(G) = \operatorname{Zeros}(G)$.

The approach proposed in [29] to solve the REP $G(\lambda)x = 0$ is to construct a linearization of $G(\lambda)$ out of a polynomial system matrix of $G(\lambda)$ in state-space form. In fact, the following proposition shows that, for any polynomial system matrix $P(\lambda)$ of $G(\lambda)$, the REP $G(\lambda)x = 0$ and the polynomial eigenvalue problem (PEP) $P(\lambda)x = 0$ are very closely related.

Proposition 3.1 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{m \times m}$ be a nonsingular rational matrix and*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+m) \times (n+m)}$$

be any polynomial system matrix with $G(\lambda)$ as transfer function matrix. Assume that (λ_0, x_0) is a solution of the REP $G(\lambda)x = 0$ such that $\det A(\lambda_0) \neq 0$; and define y_0 as the unique solution of $A(\lambda_0)y_0 + B(\lambda_0)x_0 = 0$. Then $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} \right)$ is a solution of the PEP $P(\lambda)z = 0$.

And conversely, if $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} \right)$ is a solution of the PEP $P(\lambda)z = 0$ such that $\det A(\lambda_0) \neq 0$, then (λ_0, x_0) is a solution of the REP $G(\lambda)x = 0$.

Proof.- Let $G(\lambda_0)x_0 = 0$ with $x_0 \neq 0$, $\det A(\lambda_0) \neq 0$, and define $y_0 = -A(\lambda_0)^{-1}B(\lambda_0)x_0$. Then

$$\begin{bmatrix} I_n & 0 \\ 0 & G(\lambda_0) \end{bmatrix} \begin{bmatrix} A(\lambda_0)y_0 + B(\lambda_0)x_0 \\ x_0 \end{bmatrix} = 0.$$

Thus

$$\begin{bmatrix} A(\lambda_0) & B(\lambda_0) \\ 0 & G(\lambda_0) \end{bmatrix} \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} = 0$$

and

$$P(\lambda_0) \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -C(\lambda_0)A(\lambda_0)^{-1} & I_p \end{bmatrix} \begin{bmatrix} A(\lambda_0) & B(\lambda_0) \\ 0 & G(\lambda_0) \end{bmatrix} \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} = 0.$$

The converse is proved similarly. ∎

In this proposition $P(\lambda)$ is not required to have least order. However, if we want to use the PEP to find all eigenvalues of $G(\lambda)$ (i.e., the finite zeros of $G(\lambda)$ which are not poles) then using polynomial system matrices of least order is advisable. In fact, if $P(\lambda)$ is a polynomial system matrix of least order giving rise to $G(\lambda)$ then, by Theorem 2.1, all finite poles of $G(\lambda)$ are roots of $\det A(\lambda)$, and conversely. But if $P(\lambda)$ is not of least order then there may be roots of $\det A(\lambda)$ which are not poles of $G(\lambda)$. Spurious poles are then introduced that may coincide with some zeros of $G(\lambda)$. Such zeros would not be computed. The following example illustrates this situation.

Example 3.2 Consider the 1×1 rational matrix $G(\lambda) = \frac{\lambda^2 - 1}{\lambda + 2}$, which has one finite pole $\lambda_0 = -2$ and two finite zeros: $+1$ and -1. The least order of $G(\lambda)$ is 1. Consider the following polynomial system matrices of $G(\lambda)$

$$P(\lambda) = \begin{bmatrix} \lambda + 2 & 1 \\ -3 & \lambda - 2 \end{bmatrix},$$

with $A(\lambda) = \lambda + 2$, $B(\lambda) = 1$, $C(\lambda) = 3$ and $D(\lambda) = \lambda - 2$, and

$$\widehat{P}(\lambda) = \begin{bmatrix} \lambda + 1 & 0 & 0 \\ 0 & \lambda + 2 & 1 \\ 0 & -3 & \lambda - 2 \end{bmatrix},$$

with $\widehat{A}(\lambda) = \begin{bmatrix} \lambda + 1 & 0 \\ 0 & \lambda + 2 \end{bmatrix}$, $\widehat{B}(\lambda) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\widehat{C}(\lambda) = \begin{bmatrix} 0 & 3 \end{bmatrix}$ and $\widehat{D}(\lambda) = \lambda - 2$. The order of $P(\lambda)$ is 1, which is minimal, and the order of $\widehat{P}(\lambda)$ is 2. The root of $\det A(\lambda)$ is the pole of $G(\lambda)$ while -1 is a root of $\det \widehat{A}(\lambda)$. Since $\det \widehat{A}(-1) = 0$, Proposition 3.1 cannot be used with $\widehat{P}(\lambda)$ to compute the eigenvalue -1 of $G(\lambda)$. It must be remarked that while $P(\lambda)$ is a linearization of $G(\lambda)$ in the sense of [1], $\widehat{P}(\lambda)$ is not.

Example 3.2 illustrates that if Proposition 3.1 is applied with non-minimal polynomial system matrices, some eigenvalues of $G(\lambda)$, which are not poles, may not be computed. As a conclusion, in order to obtain all elements of $\mathrm{Eig}(G)$ through the polynomial eigenvalue problem $P(\lambda)z = 0$, the eigenvalues of the submatrix $A(\lambda)$ of $P(\lambda)$ must be exactly the poles of $G(\lambda)$. Polynomial system matrices of least order guarantee that this condition is always satisfied.

As mentioned, Proposition 3.1 shows the equivalence between the REP for $G(\lambda)$ and the PEP for any of its polynomial system matrices for those scalars that are not eigenvalues of $A(\lambda)$. Note that this property does not depend on whether $P(\lambda)$ is a linear polynomial or not. It will be shown next how to obtain an equivalent linear eigenvalue problem (LEP). For this purpose, we will need a precise definition of linearization for rational matrices. It will be a natural extension of the usual definition for polynomial matrices. Recall (see [20, 9] for example) that for a given matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$, a *linearization* of $P(\lambda)$ is any linear matrix polynomial (linear pencil) $L(\lambda) = L_1 \lambda + L_0 \in \mathbb{F}[\lambda]^{q \times r}$ for which there are integers $s_1, s_2 \geq 0$ and unimodular matrices $U(\lambda) \in \mathbb{F}[\lambda]^{(p+s_1) \times (p+s_1)}$ and $V(\lambda) \in \mathbb{F}[\lambda]^{(m+s_1) \times (m+s_1)}$ such that $s_1 - s_2 = q - p = r - m$ and

$$U(\lambda) \, \mathrm{Diag}(P(\lambda), I_{s_1}) V(\lambda) = \mathrm{Diag}(L(\lambda), I_{s_2}).$$

Thus linearizations of matrix polynomials preserve the finite elementary divisors of $P(\lambda)$. Linearizations of matrix polynomials that also preserve

the infinite elementary divisors are called *strong linearizations*. In this and next section we will focus on linearizations that preserve the finite zeros and poles (with their partial multiplicities) of rational matrices. Strong linearizations that preserve both the finite and infinite zeros and poles will be introduced and studied in Section 6.

Our definition of linearization of a rational matrix follows a similar pattern to that of a matrix polynomial. As announced in the introduction, we allow rational matrices of arbitrary size, in contrast to the references [29, 1] which consider only square rational matrices.

Definition 3.3 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$. A linearization of $G(\lambda)$ is a linear pencil of the form*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)}, \tag{8}$$

with $n \geq 0$, such that the following conditions hold:

(a) if $n > 0$ then $\det(A_1\lambda + A_0) \neq 0$, and

(b) if $\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$ then:

(i) $L(\lambda)$ is a minimal polynomial system matrix of $\widehat{G}(\lambda)$, and

(ii) there are integers $s_1, s_2 \geq 0$ and unimodular matrices $U(\lambda) \in \mathbb{F}[\lambda]^{(p+s_1)\times(p+s_1)}$ and $V(\lambda) \in \mathbb{F}[\lambda]^{(m+s_1)\times(m+s_1)}$ such that $s_1 - s_2 = q - p = r - m$ and

$$U(\lambda)\operatorname{Diag}(G(\lambda), I_{s_1})V(\lambda) = \operatorname{Diag}(\widehat{G}(\lambda), I_{s_2}).$$

Notice that $L(\lambda)$ is always a polynomial system matrix because $n \geq \deg(\det(A_1\lambda + A_0))$. Now, by the minimality of $L(\lambda)$, the degree of $\det(A_1\lambda + A_0)$ is the least order of $\widehat{G}(\lambda)$, that is, $\deg(\det(A_1\lambda + A_0)) = \nu(\widehat{G}(\lambda))$, which is the sum of the degrees of the denominators in the (finite) Smith–McMillan form of $\widehat{G}(\lambda)$. On the other hand, condition *(ii)* means that the matrices $\operatorname{Diag}(\widehat{G}(\lambda), I_{s_2})$ and $\operatorname{Diag}(G(\lambda), I_{s_1})$ have the same Smith–McMillan form. A natural question is whether $\nu(\widehat{G}(\lambda)) = \nu(G(\lambda))$ and, in general, how the invariant rational functions of $\widehat{G}(\lambda)$ and $G(\lambda)$ are related. We answer this question in the following lemma.

Notice that in the definition of linearization we can always take $s_1 = 0$ or $s_2 = 0$ according as $p \geq q$ and $m \geq r$ or $q \geq p$ and $r \geq m$. In what follows we will assume, for notational simplicity and because it is the most usual case, $s := s_1 \geq 0$ and $s_2 = 0$.

Lemma 3.4 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ be a rational matrix with Smith–McMillan form*

$$\begin{bmatrix} \operatorname{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}\right) & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{F}(\lambda)^{p \times m}.$$

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Then the Smith–McMillan form of $\mathrm{Diag}(G(\lambda), I_s)$ *is*

$$\left[\begin{array}{cc} \mathrm{Diag}\left(\frac{\widetilde{\epsilon}_1(\lambda)}{\widetilde{\psi}_1(\lambda)}, \ldots, \frac{\widetilde{\epsilon}_{r+s}(\lambda)}{\widetilde{\psi}_{r+s}(\lambda)}\right) & 0 \\ 0 & 0 \end{array} \right] \in \mathbb{F}(\lambda)^{(p+s)\times(m+s)}, \qquad (9)$$

where

$$\widetilde{\epsilon}_1(\lambda) = \cdots = \widetilde{\epsilon}_s(\lambda) = 1, \qquad \widetilde{\epsilon}_{s+i}(\lambda) = \epsilon_i(\lambda), \ i = 1, \ldots, r,$$
$$\widetilde{\psi}_i(\lambda) = \psi_i(\lambda), \ i = 1, \ldots, r, \quad \widetilde{\psi}_{r+1}(\lambda) = \cdots = \widetilde{\psi}_{r+s}(\lambda) = 1.$$

Proof.- Observe that if $s = 0$ then there is nothing to prove. Moreover, if $s > 0$, we only need to prove the result for $s = 1$ because the result for $s > 1$ follows from the result for $s = 1$ applied to $\mathrm{Diag}(G(\lambda), I_{s-1})$ instead of $G(\lambda)$. Note also that from the divisibility relations of the Smith–McMillan form of $G(\lambda)$ it follows that $\epsilon_i(\lambda)$ and $\psi_j(\lambda)$ are coprime if $i < j$ and, so, (9) indeed defines a Smith–McMillan form, i.e., the fractions in (9) are irreducible. Obvious unimodular transformations allow to see that $\mathrm{Diag}(G(\lambda), 1)$ is equivalent to

$$\left[\begin{array}{cc} \mathrm{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}, 1\right) & 0 \\ 0 & 0 \end{array} \right] = \frac{Q(\lambda)}{\psi_1(\lambda)}$$

with

$$Q(\lambda) = \left[\begin{array}{cc} \mathrm{Diag}\left(\epsilon_1(\lambda), \epsilon_2(\lambda)\frac{\psi_1(\lambda)}{\psi_2(\lambda)}, \ldots, \epsilon_r(\lambda)\frac{\psi_1(\lambda)}{\psi_r(\lambda)}, \psi_1(\lambda)\right) & 0 \\ 0 & 0 \end{array} \right].$$

Thus if $\alpha_i(\lambda) = \frac{\psi_1(\lambda)}{\psi_i(\lambda)}$ for $i = 1, \ldots, r$ then

$$Q(\lambda) = \left[\begin{array}{cc} \mathrm{Diag}\left(\epsilon_1(\lambda), \epsilon_2(\lambda)\alpha_2(\lambda), \ldots, \epsilon_r(\lambda)\alpha_r(\lambda), \psi_1(\lambda)\right) & 0 \\ 0 & 0 \end{array} \right].$$

Note that

$$\epsilon_1(\lambda) \mid \epsilon_2(\lambda)\alpha_2(\lambda) \mid \cdots \mid \epsilon_r(\lambda)\alpha_r(\lambda)$$

and

$$\gcd\left(\epsilon_j(\lambda)\alpha_j(\lambda), \psi_1(\lambda)\right) = \gcd(\epsilon_j(\lambda)\alpha_j(\lambda), \psi_j(\lambda)\alpha_j(\lambda)) = \alpha_j(\lambda), \ j = 2, \ldots, r,$$

where gcd stands for greatest common divisor. This implies that if for $j = 2, \ldots, r$ $D_j(\lambda)$ is the determinantal divisor of order j (i.e., the greatest common divisor of all $j \times j$ minors) of $Q(\lambda)$, then

$$\begin{aligned} D_j(\lambda) &= \gcd(\epsilon_1(\lambda)\epsilon_2(\lambda)\alpha_2(\lambda)\cdots\epsilon_{j-1}(\lambda)\alpha_{j-1}(\lambda)\psi_1(\lambda), \\ &\quad \epsilon_1(\lambda)\epsilon_2(\lambda)\alpha_2(\lambda)\cdots\epsilon_j(\lambda)\alpha_j(\lambda)). \end{aligned}$$

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Thus

$$D_j(\lambda) = \epsilon_1(\lambda)\epsilon_2(\lambda)\alpha_2(\lambda)\cdots\epsilon_{j-1}(\lambda)\alpha_{j-1}(\lambda)\gcd(\psi_1(\lambda), \epsilon_j(\lambda)\alpha_j(\lambda))$$

$$= \epsilon_1(\lambda)\epsilon_2(\lambda)\alpha_2(\lambda)\cdots\epsilon_{j-1}(\lambda)\alpha_{j-1}(\lambda)\alpha_j(\lambda),$$

as already seen above. In addition $D_1(\lambda) = \gcd(\epsilon_1(\lambda), \psi_1(\lambda)) = 1$. Hence the invariant polynomials of $Q(\lambda)$ are

$$1, \; \epsilon_1(\lambda)\left(\frac{\psi_1(\lambda)}{\psi_2(\lambda)}\right), \ldots, \epsilon_{r-1}(\lambda)\left(\frac{\psi_1(\lambda)}{\psi_r(\lambda)}\right), \; \epsilon_r(\lambda)\,\psi_1(\lambda)\,.$$

The result follows by dividing these polynomials by $\psi_1(\lambda)$. ∎

Example 3.5 Take

$$G(\lambda) = \begin{bmatrix} \frac{\lambda}{(\lambda-1)(\lambda-2)} & 0 \\ 0 & \frac{\lambda^2}{\lambda-1} \end{bmatrix},$$

which is in Smith–McMillan form. According to Lemma 3.4, the Smith–McMillan form of

$$\begin{bmatrix} G(\lambda) & 0 \\ 0 & 1 \end{bmatrix}$$

is

$$\begin{bmatrix} \frac{1}{(\lambda-1)(\lambda-2)} & 0 & 0 \\ 0 & \frac{\lambda}{\lambda-1} & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}.$$

This can also be easily computed via the Smith normal form of the polynomial

$$(\lambda - 1)(\lambda - 2)\,\mathrm{Diag}(G(\lambda), 1).$$

A consequence of Definition 3.3 and Lemma 3.4 is that $\nu(G(\lambda)) = \nu(\widehat{G}(\lambda))$. Thus if $L(\lambda)$ of (8) is a linearization of $G(\lambda)$ then $n \geq \deg(\det(A_1\lambda + A_0)) = \nu(G(\lambda))$. Therefore, the minimum size of $A_1\lambda + A_0$ is obtained when A_1 is invertible. In fact, in this case and only in this case the order of a linearization of $G(\lambda)$ is equal to n, i.e., to the size of $A_1\lambda + A_0$. This observation motivates the following definition.

Definition 3.6 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$. A linearization*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)}$$

of $G(\lambda)$ is said to be a linearization with state matrix of minimum size if $n = \nu(G(\lambda))$ or, equivalently, if A_1 is nonsingular when $\nu(G(\lambda)) > 0$.

Linearizations of $G(\lambda)$ with state matrices of minimum size have obvious computational advantages. In addition, we will see in Section 6 that they are fundamental in the definition of strong linearizations of rational matrices. However, we emphasize that the results of this section remain valid for *any* linearization, i.e., independently of the size of $A_1\lambda + A_0$.

Definition 3.3 extends the usual definition of linearization of matrix polynomials. In fact, let $P(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$ with $\epsilon_1(\lambda), \dots, \epsilon_r(\lambda)$ as invariant polynomials and let $L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s)) \times (n+(m+s))}$ be a linearization of $P(\lambda)$ in the sense of Definition 3.3 (recall that we are assuming $s := s_1 \geq 0$ and $s_2 = 0$). Let $\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$. Then

- Since $\nu(P(\lambda)) = 0$, $n \geq \deg(\det(A_1\lambda + A_0)) = 0$. This implies $n = 0$ or $A_1\lambda + A_0$ unimodular. In both cases, $\widehat{G}(\lambda)$ is a matrix polynomial.

- From Definition 3.3 *(ii)* and Lemma 3.4 the invariant polynomials of $\widehat{G}(\lambda)$ are $1, \dots, 1, \epsilon_1(\lambda), \dots, \epsilon_r(\lambda)$ (with at least s invariant polynomials equal to 1).

- From Definition 3.3 *(i)* and Theorem 2.1 the invariant polynomials of $L(\lambda)$ are $1, \dots, 1, \epsilon_1(\lambda), \dots, \epsilon_r(\lambda)$ (with at least $n + s$ invariant polynomials equal to 1). Thus, there exist $E(\lambda) \in \mathbb{F}[\lambda]^{(n+p+s) \times (n+p+s)}$ and $F(\lambda) \in \mathbb{F}[\lambda]^{(n+m+s) \times (n+m+s)}$ both unimodular such that

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{n+s} \end{bmatrix}. \tag{10}$$

Therefore $L(\lambda)$ is a linearization of $P(\lambda)$ in the usual sense of matrix polynomials [22, 20, 9]. And conversely, if a matrix pencil $L(\lambda)$ is a linearization of the polynomial matrix $P(\lambda)$ in the usual sense then, taking $n = 0$ in Definition 3.3, $L(\lambda)$ is a linearization of $P(\lambda)$ in the sense of Definition 3.3.

The following definition is introduced in order to state concisely the spectral characterization of linearizations presented in Theorem 3.9.

Definition 3.7 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ and let $L(\lambda) \in \mathbb{F}[\lambda]^{(n+q) \times (n+r)}$ be a minimal linear polynomial system matrix as in (8). We will say that $L(\lambda)$ preserves the finite structure of poles and zeros of $G(\lambda)$ if the following condition holds true: For all $\lambda_0 \in \overline{\mathbb{F}}$, $(\lambda - \lambda_0)^w$, with $w > 0$, appears in the prime factorization of exactly k denominators (respectively numerators) $\psi_i(\lambda)$ (respectively $\epsilon_i(\lambda)$) in the Smith–McMillan form of $G(\lambda)$ if and only if $A_1\lambda + A_0$ (respectively $L(\lambda)$) has exactly k finite elementary divisors equal to $(\lambda - \lambda_0)^w$.*

Equivalently, $L(\lambda)$ preserves the finite structure of poles and zeros of $G(\lambda)$ if (and only if) the poles of $G(\lambda)$ are the zeros of $A_1\lambda + A_0$, with the same partial multiplicities in both matrices, and the zeros of $G(\lambda)$ are the zeros of $L(\lambda)$ with the same partial multiplicities in both matrices.

If $L(\lambda)$ is a minimal linear polynomial system matrix as in (8) and $\widehat{G}(\lambda)$ is its transfer function matrix then, by Theorem 2.1, $L(\lambda)$ preserves the finite structure of poles and zeros of $\widehat{G}(\lambda)$. Even more, it follows directly from Definition 3.3, Theorem 2.1 and Lemma 3.4 that all linearizations of a given rational matrix $G(\lambda)$ preserve the finite structure of zeros and poles of $G(\lambda)$. The converse however is not true in general, i.e., if $L(\lambda)$ is a minimal linear polynomial system matrix as in (8) which preserves the finite structure of zeros and poles of $G(\lambda)$ then $L(\lambda)$ may not be a linearization of $G(\lambda)$. In fact, if $\widehat{G}(\lambda)$ is the transfer function matrix of $L(\lambda)$ then, by Theorem 2.1 and Lemma 3.4, $\widehat{G}(\lambda)$ and $\mathrm{Diag}(G(\lambda), I_s)$ will have exactly the same numerators and denominators different from 1 in their invariant rational functions, but they are not equivalent if they do not have the same rank. Thus, in answering when a matrix pencil $L(\lambda)$ preserving the finite structure of poles and zeros of $G(\lambda)$ is a linearization, the null-spaces of $G(\lambda)$ and $\widehat{G}(\lambda)$ will play an important role. Once these spaces are taken into account, we will see (Theorem 3.9) that for linearizations of rational matrix functions a result similar to [9, Thm. 4.1] for matrix polynomials holds true.

Let us denote $\mathcal{N}_\ell(G(\lambda))$ and $\mathcal{N}_r(G(\lambda))$ the left and right null-spaces over $\mathbb{F}(\lambda)$ of $G(\lambda)$, respectively, i.e., if $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$,

$$\mathcal{N}_\ell(G(\lambda)) = \{x(\lambda) \in \mathbb{F}(\lambda)^{p \times 1} : x(\lambda)^T G(\lambda) = 0\},$$
$$\mathcal{N}_r(G(\lambda)) = \{x(\lambda) \in \mathbb{F}(\lambda)^{m \times 1} : G(\lambda)x(\lambda) = 0\}.$$

These sets are vector subspaces over the field of rational functions of $\mathbb{F}(\lambda)^p$ and $\mathbb{F}(\lambda)^m$, respectively. Recall the rank-nullity theorem: $\dim \mathcal{N}_\ell(G(\lambda)) = p - \mathrm{rank}\, G(\lambda)$ and $\dim \mathcal{N}_r(G(\lambda)) = m - \mathrm{rank}\, G(\lambda)$. Notice that for $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ and $T(\lambda) \in \mathbb{F}(\lambda)^{(p+q)\times(m+q)}$, $q \geq 0$, $\mathrm{rank}\, T(\lambda) = q + \mathrm{rank}\, G(\lambda)$ if and only if $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(T(\lambda))$. Also $\mathrm{rank}\, T(\lambda) = q + \mathrm{rank}\, G(\lambda)$ if and only if $\dim \mathcal{N}_\ell(G(\lambda)) = \dim \mathcal{N}_\ell(T(\lambda))$. In what follows we will bear in mind that $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(T(\lambda))$ and $\dim \mathcal{N}_\ell(G(\lambda)) = \dim \mathcal{N}_\ell(T(\lambda))$ are equivalent and, so, exchangeable conditions.

Also, in computing the partial multiplicities of the poles and zeros of $G(\lambda)$ only the numerators and denominators different from 1 in the Smith–McMillan form of $G(\lambda)$ must be taken into account. They will be called *nontrivial* or *non-unity* numerators and denominators of $G(\lambda)$, respectively. Similarly the non-unity or nontrivial invariant polynomials of a matrix polynomial are those different from 1.

Lemma 3.8 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ and let*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

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be a minimal polynomial system matrix giving rise to $\widehat{G}(\lambda)$. Then $L(\lambda)$ is a linearization of $G(\lambda)$ if and only if the following two conditions hold:

(a) $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(\widehat{G}(\lambda))$, and

(b) $G(\lambda)$ and $\widehat{G}(\lambda)$ have exactly the same nontrivial numerators and exactly the same nontrivial denominators in their Smith–McMillan forms.

Proof.- The necessity follows from Lemma 3.4, Theorem 2.1 and the definition of linearization. For the sufficiency, note that the rank-nullity theorem and condition (a) imply that $\operatorname{rank} \widehat{G}(\lambda) = s + \operatorname{rank} G(\lambda)$. Let $r = \operatorname{rank} G(\lambda)$ and let $\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}$ be the invariant rational functions of $G(\lambda)$. Assume that $\epsilon_{t+1}(\lambda)|\epsilon_{t+2}(\lambda)|\cdots|\epsilon_r(\lambda)$ and $\psi_q(\lambda)|\psi_{q-1}(\lambda)|\cdots|\psi_1(\lambda)$ are the nontrivial numerators and denominators, respectively, of $G(\lambda)$. By hypothesis these polynomials are also the nontrivial numerators and denominators of $\widehat{G}(\lambda)$. Then, the Smith–McMillan form of this matrix is

$$\left[\begin{array}{cc} \operatorname{Diag}\left(\frac{\widetilde{\epsilon}_1(\lambda)}{\widetilde{\psi}_1(\lambda)}, \ldots, \frac{\widetilde{\epsilon}_{r+s}(\lambda)}{\widetilde{\psi}_{r+s}(\lambda)} \right) & 0 \\ 0 & 0 \end{array} \right] \in \mathbb{F}(\lambda)^{(p+s)\times(m+s)},$$

where

$$\begin{array}{ll} \widetilde{\epsilon}_1(\lambda) = \cdots = \widetilde{\epsilon}_s(\lambda) = 1, & \widetilde{\epsilon}_{s+i}(\lambda) = \epsilon_i(\lambda), \ i = 1, \ldots, r, \\ \widetilde{\psi}_i(\lambda) = \psi_i(\lambda), \ i = 1, \ldots, r, & \widetilde{\psi}_{r+1}(\lambda) = \cdots = \widetilde{\psi}_{r+s}(\lambda) = 1. \end{array}$$

It follows from Lemma 3.4 that $\widehat{G}(\lambda)$ and $\operatorname{Diag}(G(\lambda), I_s)$ are equivalent. This completes the proof. ∎

Theorem 3.9 (Spectral characterization of linearizations) *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let*

$$L(\lambda) = \left[\begin{array}{cc} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{array} \right] \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

be a polynomial system matrix of least order. Then $L(\lambda)$ is a linearization of $G(\lambda)$ if and only if the following two conditions hold:

(a) $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$, and

(b) $L(\lambda)$ preserves the finite structure of poles and zeros of $G(\lambda)$.

Proof.- Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$. By Theorem 2.1, $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$. In addition, $L(\lambda)$ preserves the finite structure of poles and zeros of $\widehat{G}(\lambda)$. The result follows from Lemma 3.8. ∎

Remark 3.10 As explained before condition (a) in Theorem 3.9 is equivalent to $\dim \mathcal{N}_\ell(G(\lambda)) = \dim \mathcal{N}_\ell(L(\lambda))$. Therefore condition (a) in Theorem 3.9 can be equivalently stated as "$G(\lambda)$ and $L(\lambda)$ have the same number of left and the same number of right minimal indices", as it was done in [9, Thm. 4.1] for linearizations of polynomial matrices. Analogously, condition (a) of Lemma 3.8 can be equivalently stated as "$G(\lambda)$ and $\widehat{G}(\lambda)$ have the same number of left and the same number of right minimal indices".

4 Transfer system equivalence

We analyze deeper the relationship between rational matrices and linearizations. Let us recall at this point the notion of *strict system equivalence* (see [27, Ch. 2, Sec. 3.1]): Two polynomial system matrices

$$P_1(\lambda) = \begin{bmatrix} A_1(\lambda) & B_1(\lambda) \\ -C_1(\lambda) & D_1(\lambda) \end{bmatrix} \quad \text{and} \quad P_2(\lambda) = \begin{bmatrix} A_2(\lambda) & B_2(\lambda) \\ -C_2(\lambda) & D_2(\lambda) \end{bmatrix}$$

($A_i(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ nonsingular, $\deg(\det A_i(\lambda)) \leq n$, $B_i(\lambda) \in \mathbb{F}[\lambda]^{n \times m}$, $C_i(\lambda) \in \mathbb{F}[\lambda]^{p \times n}$, $D_i(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$, $i = 1, 2$) are said to be strictly system equivalent if there exist unimodular matrices $U(\lambda), V(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ and polynomial matrices $X(\lambda) \in \mathbb{F}[\lambda]^{p \times n}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{n \times m}$ such that

$$\begin{bmatrix} U(\lambda) & 0 \\ X(\lambda) & I_p \end{bmatrix} \begin{bmatrix} A_1(\lambda) & B_1(\lambda) \\ -C_1(\lambda) & D_1(\lambda) \end{bmatrix} \begin{bmatrix} V(\lambda) & Y(\lambda) \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} A_2(\lambda) & B_2(\lambda) \\ -C_2(\lambda) & D_2(\lambda) \end{bmatrix}. \quad (11)$$

An important feature of strict system equivalence is that any two strictly system equivalent polynomial system matrices have the same order and give rise to the same transfer function matrix ([27, Ch. 2, Thm. 3.1]). Bearing in mind Definition 3.3, we are interested in characterizing when two polynomial system matrices have equivalent transfer function matrices. We extend the definition of strict system equivalence in an obvious way to reach this goal.

Definition 4.1 *Two polynomial system matrices*

$$P_1(\lambda) = \begin{bmatrix} A_1(\lambda) & B_1(\lambda) \\ -C_1(\lambda) & D_1(\lambda) \end{bmatrix} \quad \text{and} \quad P_2(\lambda) = \begin{bmatrix} A_2(\lambda) & B_2(\lambda) \\ -C_2(\lambda) & D_2(\lambda) \end{bmatrix},$$

both of size $(n + p) \times (n + m)$, will be said to be transfer system equivalent if there exist unimodular matrices $U(\lambda), V(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$, $W(\lambda) \in \mathbb{F}[\lambda]^{p \times p}$, $T(\lambda) \in \mathbb{F}[\lambda]^{m \times m}$ and polynomial matrices $X(\lambda) \in \mathbb{F}[\lambda]^{p \times n}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{n \times m}$ such that

$$\begin{bmatrix} U(\lambda) & 0 \\ X(\lambda) & W(\lambda) \end{bmatrix} P_1(\lambda) \begin{bmatrix} V(\lambda) & Y(\lambda) \\ 0 & T(\lambda) \end{bmatrix} = P_2(\lambda).$$

Theorem 3.3 of [27, Ch. 2] shows that strict system equivalence reduces to system similarity when $P_1(\lambda)$ and $P_2(\lambda)$ are polynomial system matrices

in state-space form[2]. The same proof can be used to prove the following proposition.

Proposition 4.2 *Let*

$$P_1(\lambda) = \begin{bmatrix} \lambda E_1 - A_1 & B_1 \\ -C_1 & D_1(\lambda) \end{bmatrix} \quad and \quad P_2(\lambda) = \begin{bmatrix} \lambda E_2 - A_2 & B_2 \\ -C_2 & D_2(\lambda) \end{bmatrix}$$

be polynomial system matrices in state-space form with $\det E_i \neq 0$ *for* $i = 1, 2$. *Then* $P_1(\lambda)$ *and* $P_2(\lambda)$ *are transfer system equivalent if and only if there are invertible constant matrices* $T, S \in \mathbb{F}^{n \times n}$ *and unimodular matrices* $U(\lambda), V(\lambda)$ *such that*

$$\begin{bmatrix} T & 0 \\ 0 & U(\lambda) \end{bmatrix} P_1(\lambda) \begin{bmatrix} S & 0 \\ 0 & V(\lambda) \end{bmatrix} = P_2(\lambda).$$

Proposition 4.3 *If the polynomial system matrices*

$$P_1(\lambda) = \begin{bmatrix} A_1(\lambda) & B_1(\lambda) \\ -C_1(\lambda) & D_1(\lambda) \end{bmatrix} \quad and \quad P_2(\lambda) = \begin{bmatrix} A_2(\lambda) & B_2(\lambda) \\ -C_2(\lambda) & D_2(\lambda) \end{bmatrix}$$

are transfer system equivalent then their transfer function matrices $G_1(\lambda) = D_1(\lambda) + C_1(\lambda)A_1(\lambda)^{-1}B_1(\lambda)$ *and* $G_2(\lambda) = D_2(\lambda) + C_2(\lambda)A_2(\lambda)^{-1}B_2(\lambda)$ *are equivalent rational matrices. Moreover,* $A_1(\lambda)$ *and* $A_2(\lambda)$ *are equivalent polynomial matrices.*

Proof.- A straightforward computation shows that if

$$\begin{bmatrix} U(\lambda) & 0 \\ X(\lambda) & W(\lambda) \end{bmatrix} P_1(\lambda) \begin{bmatrix} V(\lambda) & Y(\lambda) \\ 0 & T(\lambda) \end{bmatrix} = P_2(\lambda)$$

then $G_2(\lambda) = W(\lambda)G_1(\lambda)T(\lambda)$ and $A_2(\lambda) = U(\lambda)A_1(\lambda)V(\lambda)$. $\blacksquare$

The converse of this result is not true in general, i.e., two polynomial system matrices of the same size that give rise to equivalent transfer function matrices are not necessarily transfer system equivalent. But, as we show in a moment, it does hold true when the polynomial system matrices are of least order. The following result in [27, Ch. 3, Sec. 3] is fundamental for minimal polynomial system matrices.

Theorem 4.4 *Let* $P_1(\lambda)$ *and* $P_2(\lambda)$ *be two* $(n + p) \times (n + m)$ *polynomial system matrices of least order. Then* $P_1(\lambda)$ *and* $P_2(\lambda)$ *are strictly system equivalent if and only if they have the same transfer function matrix.*

---

[2]According to the classical reference [27, p. 43], in Section 2 we have defined that a polynomial system matrix is in state-space form if $A(\lambda)$ is a monic linear polynomial matrix and $B(\lambda)$ and $C(\lambda)$ are constant matrices. In the sequel, with a slight abuse of notation, we will say that a polynomial system matrix is in state-space form if $A(\lambda) = \lambda E - A$ with E and A constant matrices, E is nonsingular, and $B(\lambda)$ and $C(\lambda)$ are constant matrices.

Remark 4.5 We must recall at this point that for

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

to be a polynomial system matrix, the condition $n \geq \deg(\det(A(\lambda))$ is imposed. In proving Theorem 4.4 it is where this condition plays a fundamental role. Specifically, if $A_1(\lambda), A_2(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ and

$$P_1(\lambda) = \begin{bmatrix} A_1(\lambda) & B_1(\lambda) \\ -C_1(\lambda) & D_1(\lambda) \end{bmatrix}, \quad P_2(\lambda) = \begin{bmatrix} A_2(\lambda) & B_2(\lambda) \\ -C_2(\lambda) & D_2(\lambda) \end{bmatrix}$$

are minimal polynomial system matrices of size $(n + p) \times (n + m)$ with the same transfer function matrix then condition $n \geq \deg(\det A_1(\lambda)) = \deg(\det A_2(\lambda)) = \nu(G(\lambda))$ is necessary for $P_1(\lambda)$ and $P_2(\lambda)$ to be strictly system equivalent. This fact is illustrated by the following two matrices [27, p. 108]:

$$P_1(\lambda) = \left[\begin{array}{c|c} (\lambda+2)^2 & \lambda+1 \\ \hline -1 & 0 \end{array} \right], \quad P_2(\lambda) = \left[\begin{array}{c|c} (\lambda+2)^2 & -1 \\ \hline \lambda+1 & 0 \end{array} \right].$$

Note that $P_1(\lambda)$ and $P_2(\lambda)$ are not polynomial system matrices, because $n = 1 < \deg(\det A_i(\lambda)) = 2$, for $i = 1, 2$. Nevertheless, it is immediate to check that $P_1(\lambda)$ and $P_2(\lambda)$ satisfy all other requirements of minimal polynomial system matrices of $g(\lambda) = \frac{\lambda+1}{(\lambda+2)^2}$. However, it is shown in [27, p. 108] that $P_1(\lambda)$ and $P_2(\lambda)$ are not strictly system equivalent.

We prove our main result in this section with the help of Theorem 4.4.

Theorem 4.6 *Let $P_1(\lambda)$ and $P_2(\lambda)$ be two $(n + p) \times (n + m)$ polynomial system matrices of least order. Then $P_1(\lambda)$ and $P_2(\lambda)$ are transfer system equivalent if and only if their transfer function matrices are equivalent.*

Proof.- By Proposition 4.3, if two polynomial system matrices are transfer system equivalent then their transfer function matrices are equivalent.

Assume now that $G_1(\lambda), G_2(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ are the respective transfer function matrices of the following polynomial system matrices of least order:

$$P_1(\lambda) = \begin{bmatrix} A_1(\lambda) & B_1(\lambda) \\ -C_1(\lambda) & D_1(\lambda) \end{bmatrix} \quad \text{and} \quad P_2(\lambda) = \begin{bmatrix} A_2(\lambda) & B_2(\lambda) \\ -C_2(\lambda) & D_2(\lambda) \end{bmatrix},$$

and $G_2(\lambda) = W(\lambda)G_1(\lambda)T(\lambda)$ for some unimodular matrices $W(\lambda)$ and $T(\lambda)$. Then

$$\begin{aligned} D_2(\lambda) + C_2(\lambda)A_2(\lambda)^{-1}B_2(\lambda) &= W(\lambda)(D_1(\lambda) + C_1(\lambda)A_1(\lambda)^{-1}B_1(\lambda))T(\lambda) \\ &= W(\lambda)D_1(\lambda)T(\lambda) + W(\lambda)C_1(\lambda)A_1(\lambda)^{-1}B_1(\lambda)T(\lambda). \end{aligned}$$

Now since $A_1(\lambda)$ and $C_1(\lambda)$ are right coprime, $A_1(\lambda)$ and $W(\lambda)C_1(\lambda)$ are also right coprime. In fact, if $X(\lambda)$ is a common right factor of $A_1(\lambda)$ and $W(\lambda)C_1(\lambda)$ then $A_1(\lambda) = \widehat{A}_1(\lambda)X(\lambda)$ and $W(\lambda)C_1(\lambda) = \widehat{C}_1(\lambda)X(\lambda)$, with $\widehat{A}_1(\lambda)$ and $\widehat{C}_1(\lambda)$ both matrix polynomials. But since $W(\lambda)$ is unimodular, $C_1(\lambda) = W(\lambda)^{-1}\widehat{C}_1(\lambda)X(\lambda)$. Hence $A_1(\lambda)$ and $C_1(\lambda)$ have also $X(\lambda)$ as a right common factor. It must be a unimodular matrix because $A_1(\lambda)$ and $C_1(\lambda)$ are right coprime. The proof that $A_1(\lambda)$ and $B_1(\lambda)T(\lambda)$ are left coprime is similar.

Now, we have two polynomial system matrices of $G_2(\lambda)$, both of least order: $P_2(\lambda)$ and

$$\widehat{P}_1(\lambda) = \begin{bmatrix} A_1(\lambda) & B_1(\lambda)T(\lambda) \\ -W(\lambda)C_1(\lambda) & W(\lambda)D_1(\lambda)T(\lambda) \end{bmatrix}.$$

By Theorem 4.4 these two polynomial system matrices are strictly system equivalent. Thus,

$$\begin{bmatrix} U(\lambda) & 0 \\ X(\lambda) & I_p \end{bmatrix} \widehat{P}_1(\lambda) \begin{bmatrix} V(\lambda) & Y(\lambda) \\ 0 & I_m \end{bmatrix} = P_2(\lambda)$$

for some unimodular matrices $U(\lambda)$ and $V(\lambda)$ and matrix polynomials $X(\lambda)$ and $Y(\lambda)$. Therefore

$$\begin{bmatrix} U(\lambda) & 0 \\ X(\lambda) & W(\lambda) \end{bmatrix} P_1(\lambda) \begin{bmatrix} V(\lambda) & Y(\lambda) \\ 0 & T(\lambda) \end{bmatrix} = P_2(\lambda),$$

and this means that $P_1(\lambda)$ and $P_2(\lambda)$ are transfer system equivalent, as desired. ∎

Theorem 4.6 allows us to obtain linearizations of rational matrices out of their minimal polynomial system matrices by means of elementary operations. This will be a consequence of Theorem 4.8 below. We will need the following technical result.

Lemma 4.7 *Let $P_1(\lambda), P_2(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be polynomial system matrices. If $P_1(\lambda)$ and $P_2(\lambda)$ are transfer system equivalent then $P_1(\lambda)$ is of least order if and only if $P_2(\lambda)$ is of least order.*

Proof.- Let

$$P_1(\lambda) = \begin{bmatrix} A_1(\lambda) & B_1(\lambda) \\ -C_1(\lambda) & D_1(\lambda) \end{bmatrix}, P_2(\lambda) = \begin{bmatrix} A_2(\lambda) & B_2(\lambda) \\ -C_2(\lambda) & D_2(\lambda) \end{bmatrix}.$$

We recall (see Section 2) that $P_1(\lambda)$ is of least order if and only if $A_1(\lambda)$ and $B_1(\lambda)$ are left coprime and $A_1(\lambda)$ and $C_1(\lambda)$ are right coprime. According to [27, Thm. 6.1, Ch. 2] two matrix polynomials $R(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ and $S(\lambda) \in \mathbb{F}[\lambda]^{m\times p}$ are left coprime if and only if the Smith normal form

of $\begin{bmatrix} R(\lambda) & S(\lambda) \end{bmatrix}$ is $\begin{bmatrix} I_m & 0 \end{bmatrix}$. It is easy to prove (see [27, p. 55]) that if $P_1(\lambda)$ and $P_2(\lambda)$ are transfer system equivalent then $\begin{bmatrix} A_1(\lambda) & B_1(\lambda) \end{bmatrix}$ and $\begin{bmatrix} A_2(\lambda) & B_2(\lambda) \end{bmatrix}$ have the same Smith normal form. Thus $A_1(\lambda)$ and $B_1(\lambda)$ are left coprime if and only if $A_2(\lambda)$ and $B_2(\lambda)$ are left coprime.

It can be proved in a similar way that $A_1(\lambda)$ and $C_1(\lambda)$ are right coprime if and only if $A_2(\lambda)$ and $C_2(\lambda)$ are right coprime. ∎

Theorem 4.8 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ and let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(q+p)\times(q+m)}$$

be a polynomial system matrix of least order of $G(\lambda)$. Let

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

such that $n, s \geq 0$. Define

$$\widehat{P}(\lambda) = \left[\begin{array}{cc|cc} I_{n-q} & 0 & 0 & 0 \\ 0 & A(\lambda) & B(\lambda) & 0 \\ \hline 0 & -C(\lambda) & D(\lambda) & 0 \\ 0 & 0 & 0 & I_s \end{array} \right] \quad and \quad \widehat{L}(\lambda) = L(\lambda)$$

or

$$\widehat{P}(\lambda) = \left[\begin{array}{c|cc} A(\lambda) & B(\lambda) & 0 \\ \hline -C(\lambda) & D(\lambda) & 0 \\ 0 & 0 & I_s \end{array} \right] \quad and \quad \widehat{L}(\lambda) = \left[\begin{array}{cc|c} I_{q-n} & 0 & 0 \\ 0 & A_1\lambda + A_0 & B_1\lambda + B_0 \\ \hline 0 & -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{array} \right]$$

according as $n \geq q$ or $q \geq n$. Then $L(\lambda)$ is a linearization of $G(\lambda)$ if and only if $\widehat{P}(\lambda)$ and $\widehat{L}(\lambda)$ are transfer system equivalent.

Proof.- Let us assume $q \geq n$. The proof in the other case is similar.

Put $\widehat{B}(\lambda) = \begin{bmatrix} B(\lambda) & 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{q\times(m+s)}$, $\widehat{C}(\lambda) = \begin{bmatrix} C(\lambda) \\ 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(p+s)\times q}$ and

$\widehat{D}(\lambda) = \begin{bmatrix} D(\lambda) & 0 \\ 0 & I_s \end{bmatrix} \in \mathbb{F}[\lambda]^{(p+s)\times(m+s)}$. Notice also that

$$\widehat{L}(\lambda) = \begin{bmatrix} \lambda\widehat{A}_1 + \widehat{A}_0 & \lambda\widehat{B}_1 + \widehat{B}_0 \\ -(\lambda\widehat{C}_1 + \widehat{C}_0) & \lambda D_1 + D_0 \end{bmatrix}$$

with

$$\widehat{A}_0 = \begin{bmatrix} I_{q-n} & 0 \\ 0 & A_0 \end{bmatrix}, \quad \widehat{A}_1 = \begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix},$$

and for $i = 0, 1$,

$$\widehat{B}_i = \begin{bmatrix} 0 \\ B_i \end{bmatrix}, \quad \widehat{C}_i = \begin{bmatrix} 0 & C_i \end{bmatrix}.$$

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So, $\widehat{P}(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_s \end{bmatrix} = \begin{bmatrix} A(\lambda) & \widehat{B}(\lambda) \\ -\widehat{C}(\lambda) & \widehat{D}(\lambda) \end{bmatrix}$ is a polynomial system matrix

of least order with $\begin{bmatrix} G(\lambda) & 0 \\ 0 & I_s \end{bmatrix}$ as transfer function matrix and $L(\lambda)$ and

$\widehat{L}(\lambda)$ have the same transfer function matrix, $\widehat{G}(\lambda)$ say.

Assume that $L(\lambda)$ is a linearization of $G(\lambda)$. This means that $L(\lambda)$ is a minimal polynomial system matrix of $\widehat{G}(\lambda)$ and this matrix is equivalent to $\mathrm{Diag}(G(\lambda), I_s)$. Since $\det(\lambda A_1 + A_0) = \det(\lambda \widehat{A}_1 + \widehat{A}_0)$, $\widehat{L}(\lambda)$ is also a minimal polynomial system matrix of $\widehat{G}(\lambda)$. Thus we have two minimal polynomial system matrices of the same size, $\widehat{L}(\lambda)$ and $\widehat{P}(\lambda)$, of least order with equivalent transfer functions matrices. By Theorem 4.6 they are transfer system equivalent.

Conversely, assume that $\widehat{P}(\lambda)$ and $\widehat{L}(\lambda)$ are transfer system equivalent and let $\widehat{G}(\lambda)$ be the transfer function matrix of $\widehat{L}(\lambda)$ (and $L(\lambda)$). As $\widehat{P}(\lambda)$ is of least order, it follows from Lemma 4.7 that $\widehat{L}(\lambda)$ (and so $L(\lambda)$) are of least order. By Theorem 4.6, $\widehat{G}(\lambda)$ and $\mathrm{Diag}(G(\lambda), I_s)$ are equivalent. In conclusion, $L(\lambda)$ is a minimal polynomial system matrix of $\widehat{G}(\lambda)$ and this matrix and $\mathrm{Diag}(G(\lambda), I_s)$ are equivalent. By definition, $L(\lambda)$ is a linearization of $G(\lambda)$. ∎

4.1 A general procedure to obtain linearizations

Theorem 4.8 can be used to give a very general procedure to obtain linearizations for a given rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ from any of its minimal polynomial system matrices. In fact, a particular case of this procedure will be used in Section 8 to construct a wide family of infinitely many linearizations of any rational matrix with the additional property of being strong linearizations according to Definition 6.2. Let

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(q+p) \times (q+m)}$$

be such a minimal polynomial system matrix of $G(\lambda)$. First we must choose the size n of the linear pencil $\lambda A_1 + A_0$ in the linearization

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s)) \times (n+(m+s))}$$

that we are searching for. According to Theorem 4.8 (and Proposition 4.3), if $L(\lambda)$ is a linearization of $G(\lambda)$ then there are unimodular matrices $U(\lambda)$ and $V(\lambda)$ such that either

$$U(\lambda)(\lambda A_1 + A_0)V(\lambda) = \mathrm{Diag}(I_{n-q}, A(\lambda))$$

or

$$U(\lambda)A(\lambda)V(\lambda) = \mathrm{Diag}(I_{q-n}, \lambda A_1 + A_0).$$

Thus, the matrices $A(\lambda)$ and $\lambda A_1 + A_0$ must satisfy the following constraints:

(c_1) $\nu(G(\lambda)) = \deg(\det A(\lambda)) \leq q$,

(c_2) $\nu(G(\lambda)) = \deg(\det(\lambda A_1 + A_0)) \leq n$, and

(c_3) $\lambda A_1 + A_0$ is a linearization of $A(\lambda)$.

It follows from (c_1) that $A(\lambda)$ admits linearizations of size $\nu(G(\lambda))$. In fact, if $\alpha_1(\lambda) \mid \cdots \mid \alpha_r(\lambda)$ are the invariant polynomials of $A(\lambda)$ different from 1 then $\deg(\alpha_1(\lambda)) + \cdots + \deg(\alpha_r(\lambda)) = \deg(\det A(\lambda)) = \nu(G(\lambda))$. Choosing $\widehat{A}_i$ as the companion matrix of $\alpha_i(\lambda)$ for $i = 1, \ldots, r$ and $\widehat{A} = \mathrm{Diag}(\widehat{A}_1, \ldots, \widehat{A}_r)$, it turns out that $\lambda I - \widehat{A}$ is a linearization of $A(\lambda)$ of size $\nu(G(\lambda)) \times \nu(G(\lambda))$. Thus, we take $n \geq \nu(G(\lambda))$, form matrix $\widehat{P}(\lambda)$ of Theorem 4.8 and the goal is to produce $\widehat{L}(\lambda)$ of that theorem by using transfer system equivalent transformations. If $q \geq n$ then the desired linearization $L(\lambda)$ is obtained by removing from $\widehat{L}(\lambda)$ the first $q-n$ rows and columns; otherwise, $L(\lambda) = \widehat{L}(\lambda)$.

As a consequence of Theorem 4.8, the procedure described above is the most general method one can imagine to construct linearizations of a rational matrix $G(\lambda)$ when one of its minimal polynomial system matrices is given. However, we emphasize that in all the explicit constructions of linearizations presented in Section 8 just a very particular instance of this procedure is used. The two key ingredients of this particular method are that it takes $\nu(G(\lambda)) = n = q$ and that both the starting minimal polynomial system matrix of $G(\lambda)$ and the target linearization are in state-space form. This fact restricts and simplifies considerably the class of transfer system equivalent transformations that can be performed in view of Proposition 4.2. The constructions in Section 8 are based on Corollary 4.9, whose proof is omitted since it is an immediate consequence of Theorem 4.8 and Proposition 4.2.

Corollary 4.9 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ and let*

$$P(\lambda) = \begin{bmatrix} \lambda E_P - A_P & B_P \\ -C_P & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p) \times (n+m)} \tag{12}$$

be a minimal polynomial system matrix of $G(\lambda)$ with $E_P \in \mathbb{F}^{n \times n}$ nonsingular. A pencil

$$L(\lambda) = \begin{bmatrix} \lambda E_L - A_L & B_L \\ -C_L & \lambda D_1 + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s)) \times (n+(m+s))}, \tag{13}$$

with $E_L \in \mathbb{F}^{n \times n}$ nonsingular, is a linearization of $G(\lambda)$ if there exist constant nonsingular matrices $T, S \in \mathbb{F}^{n \times n}$ and unimodular matrices $U(\lambda) \in \mathbb{F}[\lambda]^{(p+s) \times (p+s)}$, $V(\lambda) \in \mathbb{F}[\lambda]^{(m+s) \times (m+s)}$ such that

$$\begin{bmatrix} T & 0 \\ 0 & U(\lambda) \end{bmatrix} L(\lambda) \begin{bmatrix} S & 0 \\ 0 & V(\lambda) \end{bmatrix} = \left[\begin{array}{c|cc} \lambda E_P - A_P & B_P & 0 \\ \hline -C_P & D(\lambda) & 0 \\ 0 & 0 & I_s \end{array} \right]. \tag{14}$$

Note that equation (14) is equivalent to the four equations

$$T\left(\lambda E_L - A_L\right)S = \left(\lambda E_P - A_P\right), \quad U(\lambda)\left(\lambda D_1 + D_0\right)V(\lambda) = \mathrm{Diag}(D(\lambda), I_s),$$

$$U(\lambda)\, C_L\, S = \begin{bmatrix} C_P \\ 0 \end{bmatrix}, \qquad\qquad T\, B_L\, V(\lambda) = \begin{bmatrix} B_P & 0 \end{bmatrix},$$

which reveal that in order $L(\lambda)$ in (13) to be a linearization of $G(\lambda)$, $\lambda D_1 + D_0$ must necessarily be a linearization of the polynomial matrix $D(\lambda)$ in the usual sense of matrix polynomials [22, 20, 9]. Thus, Corollary 4.9 suggests the following symbolic algorithm for constructing linearizations in state-space form of $G(\lambda)$.

Algorithm 4.10 Given a minimal polynomial system matrix $P(\lambda)$ in state-space form as in (12) of a rational matrix $G(\lambda)$, this algorithm constructs a linearization of $G(\lambda)$ in state-space form, when it ends.

Step 1. Choose any linearization $\lambda D_1 + D_0$ of the polynomial matrix $D(\lambda)$ together with unimodular matrices $U(\lambda), V(\lambda)$ such that $U(\lambda)\left(\lambda D_1 + D_0\right)V(\lambda) = \mathrm{Diag}(D(\lambda), I_s)$. We emphasize that there are infinitely many choices available in the literature for constructing linearizations of polynomial matrices (see, for instance, [2, 5, 6, 7, 8, 11, 17, 22, 25] and the references therein).

Step 2. Construct $U(\lambda)^{-1} \begin{bmatrix} C_P \\ 0_{s\times n} \end{bmatrix}$ and $\begin{bmatrix} B_P & 0_{n\times s} \end{bmatrix} V(\lambda)^{-1}$ and check whether these matrices are constant matrices. If true, continue; if false, stop.

Step 3. Choose any pair of $n \times n$ constant nonsingular matrices T, S and define

$$\left(\lambda E_L - A_L\right) := T^{-1}\left(\lambda E_P - A_P\right)S^{-1},$$

$$C_L := U(\lambda)^{-1} \begin{bmatrix} C_P \\ 0_{s\times n} \end{bmatrix} S^{-1}, \text{ and}$$

$$B_L := T^{-1} \begin{bmatrix} B_P & 0_{n\times s} \end{bmatrix} V(\lambda)^{-1}.$$

Step 4. The pencil $L(\lambda)$ constructed as in (13) with all the pencils specified in **Steps 1** and **3** is a linearization of $G(\lambda)$ by Corollary 4.9.

Algorithm 4.10 is the method we have used to construct in Section 8 infinitely many linearizations of any rational matrix and is the method we recommend to construct linearizations of rational matrices based on previously known linearizations of polynomial matrices. It is a rather general procedure that includes, as very particular cases, the constructions presented in [29, 1]. However, the reader should notice that Algorithm 4.10 is itself quite particular when compared with the general method described in the

first part of this section, which is based on Theorem 4.8, and that may produce, in general, linearizations where $D_1\lambda + D_0$ is not a linearization of the polynomial matrix $D(\lambda)$. In addition, we emphasize that Algorithm 4.10 may stop and fail in `Step 2`. However, note that the unimodular matrices $U(\lambda)$ and $V(\lambda)$ satisfying $U(\lambda)(\lambda D_1 + D_0)V(\lambda) = \mathrm{Diag}(D(\lambda), I_s)$ are by no means unique and that a considerable freedom is available in their choice, though it is an open problem to characterize such precise amount of freedom. Finally, note also that `Steps 2` and `3` of Algorithm 4.10 only need to use the first p columns of $U(\lambda)^{-1}$ and the first m rows of $V(\lambda)^{-1}$.

5 Comparison with Alam-Behera's definition of linearization

The definition of linearization in [1, Def. 5.3] relies on the fact that every rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ can be written as a right or left coprime matrix fraction description [19, Ch. 6]. This means that there are square and nonsingular polynomial matrices $D_R(\lambda) \in \mathbb{F}[\lambda]^{m \times m}$, $D_L(\lambda) \in \mathbb{F}[\lambda]^{p \times p}$ and polynomial matrices $N_R(\lambda), N_L(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$, such that

- $D_R(\lambda)$ and $N_R(\lambda)$ are right coprime,

- $D_L(\lambda)$ and $N_L(\lambda)$ are left coprime, and

- $G(\lambda) = N_R(\lambda)D_R(\lambda)^{-1} = D_L(\lambda)^{-1}N_L(\lambda)$.

$N_R(\lambda)D_R(\lambda)^{-1}$ (respectively $D_L(\lambda)^{-1}N_L(\lambda)$) is called a right coprime (respectively left coprime) *Matrix Fraction Description* (MFD) of $G(\lambda)$.

It turns out ([19, p. 447]) that the nontrivial invariant polynomials of $D_L(\lambda)$ and $D_R(\lambda)$ are the same and they coincide with the nontrivial denominators of the Smith–McMillan form of $G(\lambda)$. Similarly, the nontrivial invariant polynomials of $N_L(\lambda)$ and $N_R(\lambda)$ are the same and they coincide with the nontrivial numerators of the Smith–McMillan form of $G(\lambda)$. Bearing these facts in mind, we can adapt the definition of linearization in [1, Def. 5.3] (stated only for square rational matrices) to include rational matrices of any size.

Definition 5.1 ([1]) *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ and let $G(\lambda) = N(\lambda)D(\lambda)^{-1}$ be a right coprime MFD. Let $n = \deg(\det D(\lambda))$ and $r = \max(m, n)$. A linearization of $G(\lambda)$ is a linear pencil of the form*

$$L(\lambda) = \begin{bmatrix} \lambda E + A & B \\ C & \lambda Y + X \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+t)} \tag{15}$$

such that the following conditions hold:

(a) $\lambda E + A \in \mathbb{F}[\lambda]^{n \times n}$ and $\det E \neq 0$,

(b) Diag($I_{r-m}, D(\lambda)$) *and* Diag($I_{r-n}, \lambda E + A$) *are equivalent polynomial matrices,*

(c) *there are integers* $s_1, s_2 \geq 0$ *satisfying* $s_1 - s_2 = n + q - p = n + t - m$ *such that* Diag($I_{s_1}, N(\lambda)$) *and* Diag($I_{s_2}, L(\lambda)$) *are equivalent polynomial matrices.*

Conditions *(b)* and *(c)* are equivalent to saying that $\lambda E + A$ is a linearization of $D(\lambda)$ and that $L(\lambda)$ is a linearization of $N(\lambda)$, respectively.

In our opinion, Definition 5.1 has two drawbacks. First, q and t are positive integers that must be allowed to be big enough for $L(\lambda)$ to have the same nontrivial invariant factors as $N(\lambda)$. For example, if $G(\lambda) = $ Diag($(\lambda - 1)^{-1}, (\lambda - 3)^3(\lambda - 1)^{-1}$) and we define

$$N(\lambda) = \text{Diag}(1, (\lambda - 3)^3), \quad D(\lambda) = \text{Diag}((\lambda - 1), (\lambda - 1))$$

then $G(\lambda) = N(\lambda)D(\lambda)^{-1}$ is a right coprime MFD. All 2×2 linearizations of $D(\lambda)$ have $\alpha_1(\lambda) = \alpha_2(\lambda) = \lambda - 1$ as invariant polynomials and the 3×3 linearizations of $N(\lambda)$ have $\gamma_1(\lambda) = \gamma_2(\lambda) = 1$ and $\gamma_3(\lambda) = (\lambda - 3)^3$ as invariant polynomials. However there is no 3×3 linear pencil $L(\lambda)$ with these invariant polynomials and having a 2×2 linear pencil in the upper left hand side corner with $\alpha_1(\lambda) = \alpha_2(\lambda) = \lambda - 1$ as invariant polynomials. The reason is that the invariant polynomials of $L(\lambda)$ must satisfy the interlacing inequality (see [24, 31])

$$\gamma_1(\lambda) \mid \alpha_1(\lambda) \mid \gamma_3(\lambda).$$

The minimum possible value of q and t for $L(\lambda)$ in this example is $q = t = 2$; but then $L(\lambda)$ will necessarily have infinite elementary divisors. Are they related to the infinite poles and zeros of $G(\lambda)$? According to Lemma 2.3 the answer is in the affirmative provided that for some $s \geq 0$, Diag($G(\lambda), I_s$) and the transfer function of $L(\lambda)$ have the same structure at infinity, which is not guaranteed by Definition 5.1.

The second, and perhaps more important remark, about Definition 5.1 is that $\lambda E + A$ and B are not required to be left coprime and $\lambda E + A$ and C are not required to be right coprime. Thus $L(\lambda)$ can be seen as a polynomial system matrix in state-space form that may not be of least order. In other words, $L(\lambda)$ is allowed to have input or output decoupling zeros. This implies that the structure of poles and zeros of $G(\lambda)$ is not the same as that of the transfer function matrix of $L(\lambda)$ (compare with Lemma 3.4). The following is an extreme case. Let

$$G(\lambda) = \text{Diag}\left(\frac{\lambda + 1}{\lambda + 2}, \frac{\lambda + 2}{\lambda + 1}\right) = \begin{bmatrix} \lambda + 1 & 0 \\ 0 & \lambda + 2 \end{bmatrix} \begin{bmatrix} \lambda + 2 & 0 \\ 0 & \lambda + 1 \end{bmatrix}^{-1}.$$

This is a right coprime MFD and a linearization of $G(\lambda)$ that satisfies the conditions of Definition 5.1 is

$$L(\lambda) = \begin{bmatrix} \lambda + 2 & 0 & 1 & 0 \\ 0 & \lambda + 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Its transfer function matrix is $\widehat{G}(\lambda) = I_2$. No information about the poles and zeros of $G(\lambda)$ is conveyed to the transfer function matrix of $L(\lambda)$. While the REPs $G(\lambda)x = 0$ and $\widehat{G}(\lambda)\widehat{x} = 0$ are equivalent when we use Definition 3.3, this property may be lost when using Definition 5.1. Nevertheless whether $L(\lambda)$ is of least order or not, it reflects the finite structure of poles and zeros of $G(\lambda)$ and may be useful for computing them.

Now, if $L(\lambda)$ in (15) is required to be of least order then Definition 5.1 is a particular case of Definition 3.3. In fact, notice first that rank $G(\lambda) = $ rank $N(\lambda)$ and so, bearing in mind that $G(\lambda)$ and $N(\lambda)$ have the same size, $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(N(\lambda))$. Assume now that $L(\lambda)$ is a linearization in the sense of Definition 5.1. Then $\mathrm{Diag}(I_{s_1}, N(\lambda))$ and $\mathrm{Diag}(I_{s_2}, L(\lambda))$ are equivalent matrix polynomials for some nonnegative integers s_1 and s_2, i.e., they have the same size and rank. Hence, $\dim \mathcal{N}_r(N(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$ and so $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$. Also, the finite zeros of $G(\lambda)$, including partial multiplicities, are those of $L(\lambda)$ and the finite poles of $G(\lambda)$, including partial multiplicities, are the finite zeros of $\lambda E + A$. Therefore conditions (a) and (b) of Theorem 3.9 are fulfilled for $G(\lambda)$ and $L(\lambda)$. Since $L(\lambda)$ is a polynomial system matrix of least order, it follows from Theorem 3.9 that $L(\lambda)$ is a linearization in the sense of Definition 3.3.

It was shown in [1] that for square rational matrices, certain linear pencils (see, more precisely, [1, Def. 3.2 and Thm. 3.6]), which are generalizations of the Fiedler linearizations for matrix polynomials [12, 5, 8], are linearizations in the sense of Definiton 5.1. Thus linearizations in the sense of this definition always exist for square rational matrices. In constructing such linearizations for a given $G(\lambda) \in \mathbb{F}(\lambda)^{m \times m}$, the authors of [1] start with one of its minimal polynomial system matrix in state-space form

$$P(\lambda) = \begin{bmatrix} \lambda E - A & B \\ -C & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+m) \times (n+m)},$$

where $\det E \neq 0$, and show that for some $s > 0$ and any permutation σ, there are unimodular matrices $U(\lambda), V(\lambda) \in \mathbb{F}[\lambda]^{(m+s) \times (m+s)}$ such that

$$\begin{bmatrix} I_n & 0 \\ 0 & U(\lambda) \end{bmatrix} \left[\begin{array}{c|cc} \lambda E - A & B & 0 \\ \hline -C & D(\lambda) & 0 \\ 0 & 0 & I_s \end{array} \right] \begin{bmatrix} I_n & 0 \\ 0 & V(\lambda) \end{bmatrix} = L_\sigma(\lambda),$$

where $L_\sigma(\lambda)$ is the Fiedler-like linearization corresponding to σ. In addition, Theorem 3.6 in [1] proves that $L_\sigma(\lambda)$ is a minimal linear polynomial system

matrix in state-space form. By Proposition 4.2, $L_\sigma(\lambda)$ and $\mathrm{Diag}(P(\lambda), I_s)$ are transfer system equivalent. So, since $P(\lambda)$ is a polynomial system matrix of least order, by Theorem 4.8 (or more clearly by its Corollary 4.9), $L_\sigma(\lambda)$ is also a linearization in the sense of Definition 3.3.

In conclusion, although $L(\lambda)$ is not required to be of least order in Definition 5.1, the Fiedler-like linearizations constructed in [1] are of least order and so they are linearizations in the sense of Definition 3.3. It must be noticed that in constructing these linearizations the general procedure of Section 4.1 is implemented with $P(\lambda)$ in state-space form from the beginning and with a target linearization $L_\sigma(\lambda)$ which is also in state-space form, therefore the construction in [1] can be seen as a particular application of Algorithm 4.10. All these Fiedler-like linearizations preserve the finite structure of poles and zeros of square rational matrices $G(\lambda)$. We will see in Section 8 how to construct a very wide class of linearizations of arbitrary (possibly rectangular) rational matrices $G(\lambda)$ that include in particular (modulo permutations with structure $\mathrm{Diag}(I_n, \Pi)$) all the Fiedler-like linearizations from [1] (and so the companion Frobenius-like linearizations in [29]) and that have the fundamental additional property of being strong linearizations, i.e., they preserve both the finite and infinite structures of poles and zeros of $G(\lambda)$.

6 Strong linearizations of rational matrices

Our aim in this section is to provide a definition of strong linearization for any rational matrix. We want it to be a natural extension of the usual definition for matrix polynomials. We will rely primarily on that of [20] although we use a different notation. Alternative references are, for example, [22, 9]. Let $\mathbb{F}_\lambda(\lambda)$ be the local ring of $\mathbb{F}[\lambda]$ at λ; that is, the ring of rational functions with denominators prime with λ

$$\mathbb{F}_\lambda(\lambda) = \left\{ \frac{p(\lambda)}{q(\lambda)} : q(0) \neq 0 \right\}.$$

A matrix $U(\lambda)$ is invertible in $\mathbb{F}_\lambda(\lambda)$ if all its entries are in $\mathbb{F}_\lambda(\lambda)$ and both the numerator and denominator of its determinant are prime with λ. Let $P(\lambda)$ be a matrix polynomial. We denote by $\mathrm{rev}\, P(\lambda)$ the reversal of $P(\lambda)$, that is (see Section 2), $\mathrm{rev}\, P(\lambda) = \lambda^d P\left(\frac{1}{\lambda}\right)$, where $d = \deg(P(\lambda))$. A *strong linearization* of $P(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$ is any linear matrix polynomial $L(\lambda) \in \mathbb{F}[\lambda]^{q \times r}$ such that there are integers $s_1, s_2 \geq 0$, unimodular matrices $U(\lambda) \in \mathbb{F}[\lambda]^{(p+s_1) \times (p+s_1)}$, $V(\lambda) \in \mathbb{F}[\lambda]^{(m+s_1) \times (m+s_1)}$ and invertible matrices in $\mathbb{F}_\lambda(\lambda)$, $E(\lambda) \in \mathbb{F}_\lambda(\lambda)^{(p+s_1) \times (p+s_1)}$ and $F(\lambda) \in \mathbb{F}_\lambda(\lambda)^{(m+s_1) \times (m+s_1)}$ such that $s_1 - s_2 = q - p = r - m$ and

$$U(\lambda)\, \mathrm{Diag}(P(\lambda), I_{s_1}) V(\lambda) = \mathrm{Diag}(L(\lambda), I_{s_2}), \tag{16}$$

$$E(\lambda)\, \mathrm{Diag}(\mathrm{rev}\, P(\lambda), I_{s_1}) F(\lambda) = \mathrm{Diag}(\mathrm{rev}\, L(\lambda), I_{s_2}). \tag{17}$$

Remark 6.1 Note that, assuming that (16) holds, the condition (17) is equivalent to the existence of unimodular matrices $\widetilde{U}(\lambda)$ and $\widetilde{V}(\lambda)$ such that $\widetilde{U}(\lambda) \operatorname{Diag}(\operatorname{rev} P(\lambda), I_{s_1})\widetilde{V}(\lambda) = \operatorname{Diag}(\operatorname{rev} L(\lambda), I_{s_2})$, which is a condition used often in the definition of strong linearizations of matrix polynomials [9, 22]. We emphasize that such condition includes a high level of redundancy while (17) does not, because unimodular matrices are very particular instances of matrices invertible in $\mathbb{F}_\lambda(\lambda)$. The equivalence of these two conditions when (16) holds is a consequence of the effect of Möbius transformations on the elementary divisors of polynomial matrices [4, 23, 30].

Recall (see Section 2) that for every rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ there exist biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p \times p}$, $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m \times m}$ such that

$$B_1(\lambda)G(\lambda)B_2(\lambda) = \begin{bmatrix} \operatorname{Diag}\left(\frac{1}{\lambda^{q_1}}, \ldots, \frac{1}{\lambda^{q_r}}\right) & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{F}(\lambda)^{p \times m}$$

where $r = \operatorname{rank} G(\lambda)$ and $q_1 \leq \cdots \leq q_r$ are integers, called the invariant orders at infinity of $G(\lambda)$. These integers determine the zeros and poles at infinity of $G(\lambda)$, i.e., if the invariant orders at infinity are of the form $q_1 \leq \cdots \leq q_k < 0 = q_{k+1} = \cdots = q_{u-1} < q_u \leq \cdots \leq q_r$ then $G(\lambda)$ has $r - u + 1$ zeros at infinity each one of order $q_u, \ldots, q_r$ and k poles at infinity each one of order $-q_1, \ldots, -q_k$. Moreover, if $G(\lambda)$ is not strictly proper then $-q_1$ is the degree of its polynomial part (recall (4)). Furthermore, recall also that the least order of $G(\lambda)$, $\nu(G(\lambda))$, is the sum of the multiplicities of its finite poles.

Our definition for strong linearization of a rational matrix is the following.

Definition 6.2 Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$. Let q_1 be its first invariant order at infinity and $g = \min(0, q_1)$. Let $n = \nu(G(\lambda))$. A strong linearization of $G(\lambda)$ is a linear polynomial matrix

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q) \times (n+r)} \tag{18}$$

such that the following conditions hold:

(a) *if $n > 0$ then $\det(A_1\lambda + A_0) \neq 0$, and*

(b) *if $\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$, $\widehat{q}_1$ is its first invariant order at infinity and $\widehat{g} = \min(0, \widehat{q}_1)$ then:*

  (i) *there are integers $s_1, s_2 \geq 0$ and unimodular matrices $U_1(\lambda) \in \mathbb{F}[\lambda]^{(p+s_1) \times (p+s_1)}$ and $U_2(\lambda) \in \mathbb{F}[\lambda]^{(m+s_1) \times (m+s_1)}$ so that $s_1 - s_2 = q - p = r - m$ and*

$$U_1(\lambda) \operatorname{Diag}(G(\lambda), I_{s_1})U_2(\lambda) = \operatorname{Diag}(\widehat{G}(\lambda), I_{s_2}), \text{ and}$$

(ii) there are biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(p+s_1)\times(p+s_1)}$ and $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(m+s_1)\times(m+s_1)}$ such that

$$B_1(\lambda) \operatorname{Diag}(\lambda^g G(\lambda), I_{s_1}) B_2(\lambda) = \operatorname{Diag}(\lambda^{\widehat{g}}\widehat{G}(\lambda), I_{s_2}).$$

Notice that $n = 0$ if $\nu(G(\lambda)) = 0$, i.e., $G(\lambda)$ is a matrix polynomial. In this case $A_1\lambda + A_0$, $B_1\lambda + B_0$ and $C_1\lambda + C_0$ are not present and so $L(\lambda) = \widehat{G}(\lambda) = D_1\lambda + D_0$ and $D_1 + C_1 A_1^{-1} B_1 = D_1$. Furthermore, $g = 0$ (respectively, $\widehat{g} = 0$) if and only if $G(\lambda)$ (respectively, $\widehat{G}(\lambda)$) is proper; otherwise g (respectively, $\widehat{g}$) is minus the degree of the polynomial part of $G(\lambda)$ (respectively, $\widehat{G}(\lambda)$). In any case, $\lambda^g G(\lambda)$ and $\lambda^{\widehat{g}}\widehat{G}(\lambda)$ are both proper rational matrices.

Remark 6.3 Strong linearizations of polynomial matrices are often defined as those linearizations that satisfy in addition condition (17) (or its redundant version via unimodular matrices [9, 22]). However, Definition 6.2 does not follow that pattern because $L(\lambda)$ is not explicitly required to be a linearization of $G(\lambda)$. This is however a consequence of that definition. In fact, since $n = \nu(G(\lambda))$, by Definition 6.2 *(i)* and Lemma 3.4, $n = \nu(\widehat{G}(\lambda))$. Therefore, $L(\lambda)$ is a minimal polynomial system matrix of $\widehat{G}(\lambda)$ and so $L(\lambda)$ is a linearization of $G(\lambda)$. In summary, we can equivalently define a strong linearization of a rational matrix as a linearization with state matrix of minimum size (recall Definition 3.6) that satisfies in addition condition *(ii)* in Definition 6.2. In our opinion, the self-contained Definition 6.2 is more convenient.

Observe also that if $n > 0$, as $n = \deg(\det(A_1\lambda + A_0))$, then A_1 is invertible. However, A_1 invertible does not imply the minimality of $L(\lambda)$ (for instance matrix $\widehat{P}(\lambda)$ in Example 3.2).

Remark 6.4 It is a straightforward consequence of A_1 being invertible that $(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$ is proper and so is $(C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}$. These properties are important in view of Lemma 2.3.

Remark 6.5 The transfer function matrix $\widehat{G}(\lambda)$ of $L(\lambda)$ in (18) is $\widehat{G}(\lambda) = (D_1 + C_1 A_1^{-1} B_1)\lambda + \widehat{G}_{pr}(\lambda)$, where $\widehat{G}_{pr}(\lambda)$ is proper. Thus, $D_1 + C_1 A_1^{-1} B_1 \neq 0$ if and only if $\widehat{g} = -1$, and $D_1 + C_1 A_1^{-1} B_1 = 0$ if and only if $\widehat{g} = 0$.

Condition *(ii)* in Definition 6.2 is equivalent, by [4, Lem. 6.9 and Prop. 6.10], to:

(ii') there are invertible matrices in $\mathbb{F}_\lambda(\lambda)$, $\widetilde{U}_1(\lambda) \in \mathbb{F}_\lambda(\lambda)^{(p+s_1)\times(p+s_1)}$ and $\widetilde{U}_2(\lambda) \in \mathbb{F}_\lambda(\lambda)^{(m+s_1)\times(m+s_1)}$ such that

$$\widetilde{U}_1(\lambda) \operatorname{Diag}\left(\frac{1}{\lambda^g}G\left(\frac{1}{\lambda}\right), I_{s_1}\right) \widetilde{U}_2(\lambda) = \operatorname{Diag}\left(\frac{1}{\lambda^{\widehat{g}}}\widehat{G}\left(\frac{1}{\lambda}\right), I_{s_2}\right).$$

Notice that both $\frac{1}{\lambda^g}G\left(\frac{1}{\lambda}\right)$ and $\frac{1}{\lambda^{\widehat{g}}}\widehat{G}\left(\frac{1}{\lambda}\right)$ are matrices with elements in $\mathbb{F}_\lambda(\lambda)$.

Remark 6.6 Similarly to the discussion in Remark 6.1, if condition *(i)* in Definition 6.2 holds, then condition *(ii')* is equivalent to the existence of unimodular matrices $W_1(\lambda)$ and $W_2(\lambda)$ such that

$$W_1(\lambda)\operatorname{Diag}\left(\frac{1}{\lambda^g}G\left(\frac{1}{\lambda}\right), I_{s_1}\right)W_2(\lambda) = \operatorname{Diag}\left(\frac{1}{\lambda^{\widehat{g}}}\widehat{G}\left(\frac{1}{\lambda}\right), I_{s_2}\right). \quad (19)$$

This equivalence can be proved by using the results in [4, Sect. 6] that describe how the finite and infinite structures of a rational matrix are modified by a Möbius transformation. As in Remark 6.1, it is worth to emphasize that condition (19) is redundant with respect to *(ii')*.

In the particular case that $G(\lambda)$ is polynomial then $n = \nu(G(\lambda)) = 0$ and, therefore, any strong linearization is of the form $L(\lambda) = D_1\lambda + D_0$, with $\widehat{G}(\lambda) = L(\lambda)$ such that *(i)* and *(ii)* are satisfied. The first condition means that $L(\lambda)$ is a linearization of $G(\lambda)$ in the classical sense of matrix polynomials (see Section 3). If $L(\lambda)$ also satisfies the second condition then it is a strong linearization of $G(\lambda)$ in the usual sense of matrix polynomials. In fact, $g = q_1 = -\deg(G(\lambda))$, $\widehat{g} = \widehat{q}_1 = -\deg(\widehat{G}(\lambda))$, $\widehat{G}(\lambda) = L(\lambda)$ and condition *(ii')* yields

$$\widetilde{U}_1(\lambda)\operatorname{Diag}(\operatorname{rev}G(\lambda), I_{s_1})\widetilde{U}_2(\lambda) = \operatorname{Diag}(\operatorname{rev}L(\lambda), I_{s_2}),$$

showing that $\operatorname{Diag}(\operatorname{rev}G(\lambda), I_{s_1})$ is equivalent at λ to $\operatorname{Diag}(\operatorname{rev}L(\lambda), I_{s_2})$.

Since strong linearizations of matrix polynomials have been extensively and deeply analyzed, we will focus mainly on the case $n = \nu(G(\lambda)) > 0$.

While condition *(i)* in Definition 6.2 means that matrices $\operatorname{Diag}(G(\lambda), I_{s_1})$ and $\operatorname{Diag}(\widehat{G}(\lambda), I_{s_2})$ are equivalent, condition *(ii)* says that matrices $\operatorname{Diag}(\lambda^g G(\lambda), I_{s_1})$ and $\operatorname{Diag}(\lambda^{\widehat{g}}\widehat{G}(\lambda), I_{s_2})$ are equivalent at infinity. With this in mind the proof of the following lemma is straightforward.

As in Section 3, we can always take $s_1 = 0$ or $s_2 = 0$ according as $p \geq q$ and $m \geq r$ or $q \geq p$ and $r \geq m$, respectively. In what follows we will assume $s := s_1 \geq 0$ and $s_2 = 0$. The invariant orders at infinity different from zero will be called *nontrivial*.

Lemma 6.7 *Let* $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$. *Let* q_1 *be its first invariant order at infinity,* $g = \min(0, q_1)$ *and* $n = \nu(G(\lambda))$. *Let*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

be a polynomial system matrix of $\widehat{G}(\lambda)$, $\widehat{q}_1$ *be the first invariant order at infinity of* $\widehat{G}(\lambda)$ *and* $\widehat{g} = \min(0, \widehat{q}_1)$. *Then* $L(\lambda)$ *is a strong linearization of* $G(\lambda)$ *if and only if the following three conditions hold:*

(a) $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(\widehat{G}(\lambda))$,

(b) $G(\lambda)$ and $\widehat{G}(\lambda)$ have the same nontrivial numerators and the same nontrivial denominators in their (finite) Smith–McMillan forms, and

(c) $\lambda^g G(\lambda)$ and $\lambda^{\widehat{g}} \widehat{G}(\lambda)$ have the same nontrivial invariant orders at infinity.

Proof.- Notice that condition *(i)* of Definition 6.2 is equivalent to conditions (a) and (b) simultaneously. And condition *(ii)* of Definition 6.2 is satisfied if and only if conditions (a) and (c) are fulfilled . ∎

Lemma 6.8 *Let $n \geq 0$ and*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+\hat{p})\times(n+\hat{m})}$$

with A_1 invertible if $n > 0$. Let $\widehat{G}(\lambda)$ be its transfer function matrix. Then $L(\lambda)$ is equivalent at infinity to $\mathrm{Diag}(\lambda I_n, \widehat{G}(\lambda))$.

Proof.- If $n = 0$, $L(\lambda) = D_1\lambda + D_0$ and $L(\lambda) = \widehat{G}(\lambda)$. Otherwise, by Remark 6.4 and Lemma 2.3, $L(\lambda)$ is equivalent at infinity to $\mathrm{Diag}(A_1\lambda + A_0, \widehat{G}(\lambda))$. Moreover, since A_1 is invertible, $\frac{A_1\lambda + A_0}{\lambda} = A_1 + \frac{A_0}{\lambda}$ is biproper. Thus, $\frac{A_1\lambda + A_0}{\lambda}$ is equivalent at infinity to I_n and $A_1\lambda + A_0$ to λI_n. Therefore, $L(\lambda)$ is equivalent at infinity to $\mathrm{Diag}(\lambda I_n, \widehat{G}(\lambda))$. ∎

From now on we use symbol $\overset{ei}{\sim}$ for equivalence at infinity.

Lemma 6.9 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, let q_1 be its first invariant order at infinity, $g = \min(0, q_1)$ and $n = \nu(G(\lambda))$. Let*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

be a strong linearization of $G(\lambda)$. If $D_1 + C_1 A_1^{-1} B_1 \neq 0$ then $L(\lambda)$ is equivalent at infinity to $\mathrm{Diag}(\lambda I_{n+s}, \lambda^{g+1} G(\lambda))$; otherwise, $L(\lambda)$ is equivalent at infinity to $\mathrm{Diag}(\lambda I_n, I_s, \lambda^g G(\lambda))$.

Proof.- Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$. By the previous lemma,

$$L(\lambda) \overset{ei}{\sim} \mathrm{Diag}(\lambda I_n, \widehat{G}(\lambda)). \tag{20}$$

Moreover, condition *(ii)* of Definition 6.2 is equivalent to

$$\lambda^{\widehat{g}} \widehat{G}(\lambda) \overset{ei}{\sim} \mathrm{Diag}(\lambda^g G(\lambda), I_s).$$

Now, if $D_1 + C_1 A_1^{-1} B_1 \neq 0$, by Remark 6.5, $\widehat{g} = -1$ and so

$$\mathrm{Diag}(\lambda I_n, \widehat{G}(\lambda)) \overset{ei}{\sim} \mathrm{Diag}(\lambda I_{n+s}, \lambda^{g+1} G(\lambda)).$$

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However, if $D_1 + C_1 A_1^{-1} B_1 = 0$, by Remark 6.5, $\widehat{g} = 0$ and so

$$\text{Diag}(\lambda I_n, \widehat{G}(\lambda)) \overset{ei}{\sim} \text{Diag}(\lambda I_n, I_s, \lambda^g G(\lambda)).$$

Hence, by (20), in the first case $L(\lambda)$ is equivalent at infinity to $\text{Diag}(\lambda I_{n+s}, \lambda^{g+1} G(\lambda))$, and in the second case, $L(\lambda)$ is equivalent at infinity to $\text{Diag}(\lambda I_n, I_s, \lambda^g G(\lambda))$.
∎

The following definition is introduced with the purpose of stating concisely the spectral characterization of strong linearizations proved in Theorem 6.11. Observe that in Definition 6.10 the matrices $\lambda^{-1} L(\lambda), \lambda^g G(\lambda)$ and $\text{Diag}(\lambda^{-1} I_s, \lambda^{g-1} G(\lambda))$ are all proper.

Definition 6.10 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ with q_1 as first invariant order at infinity and $g = \min(0, q_1)$. Let $L(\lambda) \in \mathbb{F}[\lambda]^{(n+q) \times (n+r)}$ be a minimal linear polynomial system matrix as in (18). We will say that $L(\lambda)$ preserves the finite and infinite structures of poles and zeros of $G(\lambda)$ if the following conditions hold true:*

(a) *For any $\lambda_0 \in \overline{\mathbb{F}}$, $(\lambda - \lambda_0)^w$, with $w > 0$, appears in the prime factorization of exactly k denominators (respectively numerators) $\psi_i(\lambda)$ (respectively $\epsilon_i(\lambda)$) in the (finite) Smith–McMillan form of $G(\lambda)$ if and only if $A_1 \lambda + A_0$ (respectively $L(\lambda)$) has exactly k finite elementary divisors equal to $(\lambda - \lambda_0)^w$.*

(b) *A_1 is invertible if $n > 0$ and for any nonzero integer u, u is an invariant order at infinity with multiplicity k of $\lambda^{-1} L(\lambda)$ if and only if u is an invariant order at infinity with multiplicity k of $\lambda^g G(\lambda)$ if $D_1 + C_1 A_1^{-1} B_1 \neq 0$ or of $\text{Diag}(\lambda^{-1} I_s, \lambda^{g-1} G(\lambda))$ otherwise.*

Equivalently, $L(\lambda)$ preserves the finite and infinite structures of poles and zeros of $G(\lambda)$ if (and only if) the finite poles of $G(\lambda)$ are the finite zeros of $A_1 \lambda + A_0$, with the same partial multiplicities in both matrices, the finite zeros of $G(\lambda)$ are the finite zeros of $L(\lambda)$, with the same partial multiplicities, and the number and orders of the infinite zeros of $\lambda^{-1} L(\lambda)$ are the same as the number and orders of the infinite zeros of $\lambda^g G(\lambda)$ if $D_1 + C_1 A_1^{-1} B_1 \neq 0$ or of $\text{Diag}(\lambda^{-1} I_s, \lambda^{g-1} G(\lambda))$ otherwise.

Theorem 6.11 (Spectral characterization of strong linearizations)
Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ and $n = \nu(G(\lambda))$. Let

$$L(\lambda) = \begin{bmatrix} A_1 \lambda + A_0 & B_1 \lambda + B_0 \\ -(C_1 \lambda + C_0) & D_1 \lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s)) \times (n+(m+s))}.$$

Then $L(\lambda)$ is a strong linearization of $G(\lambda)$ if and only if the following two conditions hold:

(I) *$\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$, and*

(II) $L(\lambda)$ *preserves the finite and infinite structures of poles and zeros of $G(\lambda)$.*

Proof.- The necessity is a direct consequence of Theorem 3.9 and Lemma 6.9. For the sufficiency, notice that $n = \nu(G(\lambda))$ and conditions (I) and (II) imply, by Theorem 3.9, that $L(\lambda)$ is a linearization of $G(\lambda)$. Therefore, conditions *(a)* and *(i)* of Definition 6.2 are satisfied. Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$. It follows from Lemma 6.8 that $L(\lambda)$ is equivalent at infinity to $\mathrm{Diag}(\lambda I_n, \widehat{G}(\lambda))$. Therefore,

$$\lambda^{-1} L(\lambda) \overset{ei}{\sim} \mathrm{Diag}(I_n, \lambda^{-1}\widehat{G}(\lambda)). \tag{21}$$

Now, conditions (I) and (II) imply (recall that (I) is equivalent to rank $L(\lambda) = n + s + \mathrm{rank}\, G(\lambda)$)

$$\lambda^{-1} L(\lambda) \overset{ei}{\sim} \begin{cases} \mathrm{Diag}(I_{n+s}, \lambda^g G(\lambda)), & \text{if } D_1 + C_1 A_1^{-1} B_1 \neq 0 \\ \mathrm{Diag}(I_n, \lambda^{-1} I_s, \lambda^{g-1} G(\lambda)), & \text{if } D_1 + C_1 A_1^{-1} B_1 = 0 \end{cases}. \tag{22}$$

Recall (Remark 6.5) that if $D_1 + C_1 A_1^{-1} B_1 \neq 0$ then $\widehat{g} = -1$ and $\widehat{g} = 0$ otherwise. Thus, by (21) and (22),

$$\lambda^{-1} \widehat{G}(\lambda) \overset{ei}{\sim} \begin{cases} \mathrm{Diag}(I_s, \lambda^g G(\lambda)), & \text{if } D_1 + C_1 A_1^{-1} B_1 \neq 0 \\ \mathrm{Diag}(\lambda^{-1} I_s, \lambda^{g-1} G(\lambda)), & \text{if } D_1 + C_1 A_1^{-1} B_1 = 0 \end{cases}.$$

In any case condition *(ii)* of Definition 6.2 follows. ∎

Remark 6.12 As in Remark 3.10 condition (I) in Theorem 6.11 is equivalent to $\dim \mathcal{N}_\ell(G(\lambda)) = \dim \mathcal{N}_\ell(L(\lambda))$. Therefore condition (I) is equivalent to state that "$G(\lambda)$ and $L(\lambda)$ have the same number of left and the same number of right minimal indices".

This theorem allows us to obtain the infinite structure of a rational matrix from the elementary divisors at infinity of any of its strong linearizations in a very simple form. Namely, let $G(\lambda)$ be a $p \times m$ rational matrix of rank r, let $q_1 \leq \cdots \leq q_r$ be its invariant orders at infinity and let $L(\lambda)$ be a strong linearization of $G(\lambda)$. Define $g = \min(0, q_1)$. By (I), rank $L(\lambda) = n + s + r$. Let $\lambda^{e_1}, \ldots, \lambda^{e_{n+s+r}}$ be the infinite elementary divisors (including possible exponents equal to zero) of $L(\lambda)$. Thus, $0 \leq e_1 \leq \cdots \leq e_{n+s+r}$. We want to get the invariant orders at infinity of $G(\lambda)$ out of $e_1, \ldots, e_{n+s+r}$. Recall that the degree of the polynomial part of $G(\lambda)$ (if present) is $-q_1$. Suppose that the degree of $L(\lambda)$ is d (d can only be equal to 1 or 0). By (7), the invariant orders at infinity of $L(\lambda)$ are $e_i - d$, $i = 1, \ldots, n + s + r$. We distinguish three cases:

- $n \geq 0$ and $D_1 + C_1 A_1^{-1} B_1 \neq 0$. In this case $d = 1$ and, by Lemma 6.9, $e_i = 0$ for $i = 1, \ldots, n + s$ and $e_{n+s+1} - 1, \ldots, e_{n+s+r} - 1$ are the invariant orders at infinity of $\lambda^{g+1} G(\lambda)$. Thus,

$$q_i = e_{n+s+i} + g, \ 1 \leq i \leq r.$$

- $n > 0$ and $D_1 + C_1 A_1^{-1} B_1 = 0$. Then $d = 1$ and, by Lemma 6.9, $e_i = 0$ for $i = 1, \ldots, n$, $e_{n+i} = 1$ for $i = 1, \ldots, s$, and $e_{n+s+1} - 1, \ldots, e_{n+s+r} - 1$ are the invariant orders at infinity of $\lambda^g G(\lambda)$. Thus,

$$q_i = e_{n+s+i} + g - 1, \ 1 \leq i \leq r.$$

- $n = 0$ and $D_1 = 0$. In this case $G(\lambda)$ is a matrix polynomial, $L(\lambda) = \widehat{G}(\lambda) = D_0$, $\mathrm{Diag}(G(\lambda), I_s)$ is equivalent to D_0 (i.e., all its invariant factors are equal to 1) and $\mathrm{Diag}(\lambda^{q_1} G(\lambda), I_s)$ is equivalent at infinity to D_0. Since the invariant orders at infinity of D_0 are 0, $q_i = q_1$ for $i = 1, \ldots, r$.

7 Transfer system equivalence at infinity

In this section we define the transfer system equivalence at infinity in a similar way as we defined the transfer system equivalence. We will work with rational matrices, instead of matrix polynomials, of the following form:

$$R_1(\lambda) = \begin{bmatrix} E_1(\lambda) & F_1(\lambda) \\ -J_1(\lambda) & K_1(\lambda) \end{bmatrix} \quad \text{and} \quad R_2(\lambda) = \begin{bmatrix} E_2(\lambda) & F_2(\lambda) \\ -J_2(\lambda) & K_2(\lambda) \end{bmatrix} \qquad (23)$$

with $E_i(\lambda) \in \mathbb{F}(\lambda)^{n \times n}$ nonsingular, $F_i(\lambda) \in \mathbb{F}(\lambda)^{n \times m}$, $J_i(\lambda) \in \mathbb{F}(\lambda)^{p \times n}$, $K_i(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$, $i = 1, 2$. $R_1(\lambda)$ and $R_2(\lambda)$ are said to be in *rational form* in [27].

Definition 7.1 $R_1(\lambda)$ and $R_2(\lambda)$ as in (23) are said to be strictly system equivalent at infinity if there exist biproper matrices $B_1(\lambda), B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n \times n}$ and proper matrices $W(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p \times n}$, $Z(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n \times m}$ such that

$$\begin{bmatrix} B_1(\lambda) & 0 \\ W(\lambda) & I_p \end{bmatrix} R_1(\lambda) \begin{bmatrix} B_2(\lambda) & Z(\lambda) \\ 0 & I_m \end{bmatrix} = R_2(\lambda). \qquad (24)$$

This is an equivalence relation since the inverse and product of the block triangular biproper matrices in (24) are biproper matrices with the same block triangular structures (including the identity blocks). Moreover, if two matrices are strictly system equivalent at infinity then they are equivalent at infinity.

Let $G_i(\lambda) = K_i(\lambda) + J_i(\lambda) E_i(\lambda)^{-1} F_i(\lambda)$ be the transfer function matrix of $R_i(\lambda)$, $i = 1, 2$. The next result can be proved straightforwardly.

Proposition 7.2 *If $R_1(\lambda)$ and $R_2(\lambda)$ as in (23) are strictly system equivalent at infinity then they give rise to the same transfer function matrix. Moreover, $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity.*

Definition 7.3 *$R_1(\lambda)$ and $R_2(\lambda)$ as in (23) will be said to be transfer system equivalent at infinity if there exist biproper matrices $B_1(\lambda), B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n \times n}$, $B_3(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p \times p}$, $B_4(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m \times m}$ and proper matrices $W(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p \times n}$, $Z(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n \times m}$ such that*

$$\begin{bmatrix} B_1(\lambda) & 0 \\ W(\lambda) & B_3(\lambda) \end{bmatrix} R_1(\lambda) \begin{bmatrix} B_2(\lambda) & Z(\lambda) \\ 0 & B_4(\lambda) \end{bmatrix} = R_2(\lambda). \tag{25}$$

This is again an equivalence relation. Furthermore, if two matrices are transfer system equivalent at infinity then they are equivalent at infinity as well.

Proposition 7.4 *If $R_1(\lambda)$ and $R_2(\lambda)$ as in (23) are transfer system equivalent at infinity then their transfer function matrices are equivalent at infinity. Moreover, $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity.*

Proof.- If

$$\begin{bmatrix} B_1(\lambda) & 0 \\ W(\lambda) & B_3(\lambda) \end{bmatrix} R_1(\lambda) \begin{bmatrix} B_2(\lambda) & Z(\lambda) \\ 0 & B_4(\lambda) \end{bmatrix} = R_2(\lambda)$$

then $E_2(\lambda) = B_1(\lambda)E_1(\lambda)B_2(\lambda)$ and $G_2(\lambda) = B_3(\lambda)G_1(\lambda)B_4(\lambda)$. ∎

The converse of this result is not true in general, i.e., two matrices of the form (23) that give rise to equivalent transfer function matrices at infinity are not necessarily transfer system equivalent at infinity. However, it does hold true when $J_i(\lambda)E_i(\lambda)^{-1}$ and $E_i(\lambda)^{-1}F_i(\lambda)$ are proper for $i = 1, 2$.

The proof of the following lemma is the same as that of Lemma 2.3.

Lemma 7.5 *Let*

$$R(\lambda) = \begin{bmatrix} E(\lambda) & F(\lambda) \\ -J(\lambda) & K(\lambda) \end{bmatrix} \in \mathbb{F}(\lambda)^{(n+p) \times (n+m)}$$

with $J(\lambda)E(\lambda)^{-1}$ and $E(\lambda)^{-1}F(\lambda)$ proper matrices. Let $G(\lambda) = K(\lambda) + J(\lambda)E(\lambda)^{-1}F(\lambda)$. Then $R(\lambda)$ and $\mathrm{Diag}(E(\lambda), G(\lambda))$ are strictly system equivalent at infinity.

Theorem 7.6 *Let $R_i(\lambda)$ and $G_i(\lambda)$ be as in the previous lemma with $J_i(\lambda)E_i(\lambda)^{-1}$ and $E_i(\lambda)^{-1}F_i(\lambda)$ proper, $i = 1, 2$.*

1. *$R_1(\lambda)$ and $R_2(\lambda)$ are strictly system equivalent at infinity if and only if $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity and $G_1(\lambda) = G_2(\lambda)$.*

2. $R_1(\lambda)$ and $R_2(\lambda)$ are transfer system equivalent at infinity if and only if $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity and $G_1(\lambda)$ and $G_2(\lambda)$ are equivalent at infinity.

Proof.- The necessity follows from Propositions 7.2 and 7.4. Suppose that $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity and $G_1(\lambda) = G_2(\lambda)$. Then there exist biproper matrices $B_1(\lambda), B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n \times n}$ such that

$$
\begin{bmatrix} B_1(\lambda) & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} E_1(\lambda) & 0 \\ 0 & G_1(\lambda) \end{bmatrix} \begin{bmatrix} B_2(\lambda) & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} E_2(\lambda) & 0 \\ 0 & G_2(\lambda) \end{bmatrix}.
$$

This means that $\mathrm{Diag}(E_1(\lambda), G_1(\lambda))$ and $\mathrm{Diag}(E_2(\lambda), G_2(\lambda))$ are strictly system equivalent at infinity. By Lemma 7.5, $R_1(\lambda)$ and $R_2(\lambda)$ are strictly system equivalent at infinity. Analogously, if $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity and $G_1(\lambda)$ and $G_2(\lambda)$ are equivalent at infinity, there exist biproper matrices $B_1(\lambda), B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n \times n}$, $B_3(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p \times p}$, $B_4(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m \times m}$ such that

$$
\begin{bmatrix} B_1(\lambda) & 0 \\ 0 & B_3(\lambda) \end{bmatrix} \begin{bmatrix} E_1(\lambda) & 0 \\ 0 & G_1(\lambda) \end{bmatrix} \begin{bmatrix} B_2(\lambda) & 0 \\ 0 & B_4(\lambda) \end{bmatrix} = \begin{bmatrix} E_2(\lambda) & 0 \\ 0 & G_2(\lambda) \end{bmatrix}.
$$

Then $\mathrm{Diag}(E_1(\lambda), G_1(\lambda))$ and $\mathrm{Diag}(E_2(\lambda), G_2(\lambda))$ are transfer system equivalent at infinity. By Lemma 7.5, $R_1(\lambda)$ and $R_2(\lambda)$ are transfer system equivalent at infinity. ∎

If $A_{11}, A_{21} \in \mathbb{F}^{n \times n}$ are invertible and $A_{10}, A_{20} \in \mathbb{F}^{n \times n}$ then $A_{11}\lambda + A_{10}$ and $A_{21}\lambda + A_{20}$ are equivalent at infinity (see the proof of Lemma 6.8). The next result is a straightforward consequence of Theorem 7.6.

Corollary 7.7 *Let*

$$
R_i(\lambda) = \begin{bmatrix} A_{i1}\lambda + A_{i0} & F_i(\lambda) \\ -J_i(\lambda) & K_i(\lambda) \end{bmatrix}
$$

with $A_{i1} \in \mathbb{F}^{n \times n}$ invertible, $A_{i0} \in \mathbb{F}^{n \times n}$, $F_i(\lambda) \in \mathbb{F}(\lambda)^{n \times m}$, $J_i(\lambda) \in \mathbb{F}(\lambda)^{p \times n}$, $K_i(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ such that $J_i(\lambda)(A_{i1}\lambda + A_{i0})^{-1}$ and $(A_{i1}\lambda + A_{i0})^{-1}F_i(\lambda)$ are proper matrices, $i = 1, 2$. Let $G_i(\lambda) = K_i(\lambda) + J_i(\lambda)(A_{i1}\lambda + A_{i0})^{-1}F_i(\lambda)$, $i = 1, 2$.

1. *$R_1(\lambda)$ and $R_2(\lambda)$ are strictly system equivalent at infinity if and only if $G_1(\lambda) = G_2(\lambda)$.*

2. *$R_1(\lambda)$ and $R_2(\lambda)$ are transfer system equivalent at infinity if and only if $G_1(\lambda)$ and $G_2(\lambda)$ are equivalent at infinity.*

The next corollaries provide means to obtain strong linearizations of rational matrices from their minimal polynomial system matrices by performing elementary transformations that preserve both the transfer system equivalence and the transfer system equivalence at infinity.

Corollary 7.8 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$, let q_1 be its first invariant order at infinity, $g = \min(0, q_1)$ and $n = \nu(G(\lambda))$. Let*

$$P(\lambda) = \begin{bmatrix} A_{P1}\lambda + A_{P0} & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p) \times (n+m)}$$

be a polynomial system matrix of least order giving rise to $G(\lambda)$ such that both $C(\lambda)(A_{P1}\lambda + A_{P0})^{-1}$ and $\lambda^g(A_{P1}\lambda + A_{P0})^{-1}B(\lambda)$ are proper. Let

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s)) \times (n+(m+s))} \quad (s \geq 0)$$

with A_1 invertible if $n > 0$. Let $\widehat{g} = -1$ if $D_1 + C_1 A_1^{-1} B_1 \neq 0$ and $\widehat{g} = 0$ otherwise. Then $L(\lambda)$ is a strong linearization of $G(\lambda)$ if and only if

(i) $\mathrm{Diag}(P(\lambda), I_s)$ and $L(\lambda)$ are transfer system equivalent, and

(ii) $\begin{bmatrix} A_{P1}\lambda + A_{P0} & \lambda^g B(\lambda) & 0 \\ -C(\lambda) & \lambda^g D(\lambda) & 0 \\ 0 & 0 & I_s \end{bmatrix}$ *and* $\begin{bmatrix} A_1\lambda + A_0 & \lambda^{\widehat{g}}(B_1\lambda + B_0) \\ -(C_1\lambda + C_0) & \lambda^{\widehat{g}}(D_1\lambda + D_0) \end{bmatrix}$ *are transfer system equivalent at infinity.*

Proof.- By Theorem 4.8, $L(\lambda)$ is a linearization of $G(\lambda)$ if and only if condition *(i)* is satisfied. Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$. We prove now that $\lambda^{\widehat{g}}\widehat{G}(\lambda)$ and $\mathrm{Diag}(\lambda^g G(\lambda), I_s)$ are equivalent at infinity if and only if condition *(ii)* holds true. Notice that $\mathrm{Diag}(\lambda^g G(\lambda), I_s)$ is the transfer function matrix of

$$\begin{bmatrix} A_{P1}\lambda + A_{P0} & \lambda^g B(\lambda) & 0 \\ -C(\lambda) & \lambda^g D(\lambda) & 0 \\ 0 & 0 & I_s \end{bmatrix},$$

while $\lambda^{\widehat{g}}\widehat{G}(\lambda)$ is that of

$$\begin{bmatrix} A_1\lambda + A_0 & \lambda^{\widehat{g}}(B_1\lambda + B_0) \\ -(C_1\lambda + C_0) & \lambda^{\widehat{g}}(D_1\lambda + D_0) \end{bmatrix}.$$

The result follows from Corollary 7.7, by taking into account that A_{P1} is invertible, since $P(\lambda)$ is of least order and $n = \nu(G(\lambda))$, and that $(C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}$ and $\lambda^{\widehat{g}}(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$ are proper by Remark 6.4. ∎

Analogously, we can prove the following result.

Corollary 7.9 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$, let q_1 be its first invariant order at infinity, $g = \min(0, q_1)$ and $n = \nu(G(\lambda))$. Let*

$$P(\lambda) = \begin{bmatrix} A_{P1}\lambda + A_{P0} & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p) \times (n+m)}$$

be a polynomial system matrix of least order giving rise to $G(\lambda)$ such that both $\lambda^g C(\lambda)(A_{P1}\lambda + A_{P0})^{-1}$ and $(A_{P1}\lambda + A_{P0})^{-1}B(\lambda)$ are proper. Let

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))} \quad (s \geq 0)$$

with A_1 invertible if $n > 0$. Let $\widehat{g} = -1$ if $D_1 + C_1 A_1^{-1} B_1 \neq 0$ and $\widehat{g} = 0$ otherwise. Then $L(\lambda)$ is a strong linearization of $G(\lambda)$ if and only if

(i) $\mathrm{Diag}(P(\lambda), I_s)$ and $L(\lambda)$ are transfer system equivalent, and

(ii) $\begin{bmatrix} A_{P1}\lambda + A_{P0} & B(\lambda) & 0 \\ -\lambda^g C(\lambda) & \lambda^g D(\lambda) & 0 \\ 0 & 0 & I_s \end{bmatrix}$ *and* $\begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -\lambda^{\widehat{g}}(C_1\lambda + C_0) & \lambda^{\widehat{g}}(D_1\lambda + D_0) \end{bmatrix}$

are transfer system equivalent at infinity.

8 Construction of strong linearizations of rational matrices

We show in this section that strong linearizations always exist for every rational matrix by constructing explicitly infinitely many examples. The proposed construction is based on Algorithm 4.10 in Section 4.1 and the formal proof that the constructed pencils are indeed strong linearizations relies on Corollary 7.8. The new class of strong linearizations contains, as very particular cases, the Fiedler-like linearizations (modulo permutations) introduced in [1] only for square rational matrices, and so the extension to rational matrices of the classical Frobenius companion pencils ([1, Prop. 3.7] and [29]). We emphasize once again that the strong linearizations introduced in this section are much more general than those in [1] from several important points of view: (1) they are strong linearizations, while [1] does not guarantee that the structure at infinity of the original rational matrix is preserved; (2) they are valid for rational matrices of arbitrary sizes, while [1] only considers square rational matrices, as it also happens in [29]; and (3) the class of linearizations presented here is much wider.

Algorithm 4.10 needs two key ingredients: a minimal polynomial system matrix in state-space form of the given rational matrix $G(\lambda)$ and a linearization of its polynomial part $D(\lambda)$ (see (4)) when $\deg(D(\lambda)) > 1$, together with the unimodular matrices that transform the linearization of $D(\lambda)$ into $\mathrm{Diag}(D(\lambda), I_s)$. As linearizations of polynomial matrices, we will use the very recently introduced class of *strong block minimal bases pencils* [11, Section 3], which includes Fiedler-type linearizations, among many others, and has already been used in a number of applications [21, 26]. Strong block minimal bases linearizations of matrix polynomials and the corresponding unimodular transformations will be revised in Section 8.1. Next, we pay

attention to the construction of the starting minimal polynomial system matrix in state-space form via the following two-step approach.

Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ be any rational matrix.

1. Compute the unique decomposition $G(\lambda) = D(\lambda) + G_{sp}(\lambda)$ with $D(\lambda)$ polynomial and $G_{sp}(\lambda)$ strictly proper. In many applications [29], this decomposition can be obtained (or guessed) without any computational effort.

2. Compute a minimal order state-space realization (A, B, C) of $G_{sp}(\lambda)$

$$G_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B, \tag{26}$$

where $n = \nu(G(\lambda)) = \nu(G_{sp}(\lambda))$. A summary of stable algorithms for constructing minimal state-space realizations for $G_{sp}(\lambda)$ can be found in [28]. In addition, in many applications [29], this realization can be obtained (or guessed) without any computational effort.

The fact that (A, B, C) in (26) is a minimal realization is equivalent to the facts that (A, B) is controllable and that (A, C) is observable [27]. That is to say:

$$\operatorname{rank} \begin{bmatrix} B & AB & A^2B & \cdots A^{n-1}B \end{bmatrix} = n, \quad \operatorname{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

Under these conditions

$$P(\lambda) = \begin{bmatrix} \lambda I_n - A & B \\ -C & D(\lambda) \end{bmatrix} \tag{27}$$

is a polynomial system matrix in state-space form of least order n whose transfer function matrix is $G(\lambda)$. A key observation on (27) is that if $D(\lambda) = 0$ or $\deg(D(\lambda)) \leq 1$, then $P(\lambda)$ is itself a strong linearization of $G(\lambda)$ according to Definition 6.2, since $\widehat{G}(\lambda) = G(\lambda)$ in that definition. Therefore, in Section 8.2 we will assume $\deg(D(\lambda)) > 1$.

8.1 Strong block minimal bases linearizations of polynomial matrices

In this section we review the definition and key properties of strong block minimal bases linearizations of polynomial matrices and related unimodular transformations. More information on this topic can be found in [11, Secs. 3, 4, 5]. In addition, some results from [11] are refined in order to use strong block minimal bases linearizations of polynomial matrices in the construction

of strong linearizations of rational matrices in Section 8.2. Classical concepts on minimal bases of rational vector spaces are often used in this section. For brevity, we do not review such concepts here and refer the reader to the original paper [13] or to the standard reference [19, Ch. 6]. The summaries presented in [10, Sec. 2] and [11, Sec. 2] may be of interest since they use exactly the nomenclature employed here. For instance, the definitions of minimal bases and indices can be found in [10, Defs. 2.1 and 2.2], the classical characterization of a minimal basis in terms of ranks of constant matrices appears in [10, Thm. 2.4], the definition of dual minimal bases is given in [10, Def. 2.10], etc. For brevity, we say that a polynomial matrix with more columns than rows is a minimal basis when its rows are a minimal basis of the rational subspace they span. The Kronecker product of two matrices, denoted by $A \otimes B$, is also used in this section. Its definition and properties are studied in [18, Ch. 4].

The following polynomial matrices[3]

$$L_k(\lambda) := \begin{bmatrix} -1 & \lambda & & \\ & -1 & \lambda & \\ & & \ddots & \ddots \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{k \times (k+1)}, \tag{28}$$

and

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{1 \times (k+1)}, \tag{29}$$

and their Kronecker products with an identity matrix, i.e., $L_k(\lambda) \otimes I_t$ and $\Lambda_k(\lambda)^T \otimes I_t$, are important in this section. Note that $L_k(\lambda)$ and $\Lambda_k(\lambda)^T$ are a pair of dual minimal bases, as well as $L_k(\lambda) \otimes I_t$ and $\Lambda_k(\lambda)^T \otimes I_t$ [11, Ex. 2.6]. With these matrices and the last column of I_{k+1}, denoted by e_{k+1}, we define the unimodular matrix

$$V_k(\lambda) := \begin{bmatrix} L_k(\lambda) \\ \hline e_{k+1}^T \end{bmatrix} = \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \\ \hline 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{(k+1) \times (k+1)}, \tag{30}$$

whose inverse is

$$V_k(\lambda)^{-1} = \begin{bmatrix} -1 & -\lambda & -\lambda^2 & \cdots & -\lambda^{k-1} & \lambda^k \\ & -1 & -\lambda & \ddots & \vdots & \lambda^{k-1} \\ & & -1 & \ddots & -\lambda^2 & \vdots \\ & & & \ddots & -\lambda & \lambda^2 \\ & & & & -1 & \lambda \\ & & & & & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{(k+1) \times (k+1)}. \tag{31}$$

---

[3]In the rest of the paper, we often omit some or all of the zero entries of a matrix.

Note that the last column of $V_k(\lambda)^{-1}$ is $\Lambda_k(\lambda)$.

The next definition is taken from [11, Def. 3.1 and Thm. 3.3].

Definition 8.1 *Let $D(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$ be a polynomial matrix. A strong block minimal bases pencil associated to $D(\lambda)$ is a linear polynomial matrix with the following structure*

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{array} \right] \begin{array}{l} \} \, p+\widehat{p} \\ \} \, \widehat{m} \end{array} \quad , \qquad (32)$$
$$\underbrace{}_{m+\widehat{m}} \underbrace{}_{\widehat{p}}$$

where $K_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m} \times (m+\widehat{m})}$ (respectively $K_2(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p} \times (p+\widehat{p})}$) is a minimal basis with all its row degrees equal to 1 and with the row degrees of a minimal basis $N_1(\lambda) \in \mathbb{F}[\lambda]^{m \times (m+\widehat{m})}$ (respectively $N_2(\lambda) \in \mathbb{F}[\lambda]^{p \times (p+\widehat{p})}$) dual to $K_1(\lambda)$ (respectively $K_2(\lambda)$) all equal, and such that

$$D(\lambda) = N_2(\lambda) M(\lambda) N_1(\lambda)^T. \qquad (33)$$

If, in addition, $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$ then $\mathcal{L}(\lambda)$ is said to be a strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree.

The most important property of any strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree is that it is a strong linearization of $D(\lambda)$ [11, Thm. 3.3]. Note that, for $i = 1, 2$, the row degrees of any minimal basis dual to $K_i(\lambda)$ are the right minimal indices of $K_i(\lambda)$ and so they are independent of the considered particular dual basis $N_i(\lambda)$. Another important remark (see [11, Rem. 3.4]) is that there are infinitely many minimal bases dual to $K_1(\lambda)$ and also infinitely many dual to $K_2(\lambda)$, but once two of them, say $N_1(\lambda)$ and $N_2(\lambda)$, are fixed, it is always possible to choose the pencil $M(\lambda)$ in such a way that (33) is satisfied. The choice of $M(\lambda)$ is, in general, not unique. We emphasize that in [11, Sec. 3], strong block minimal bases pencils were defined without an explicit reference to the polynomial matrix $D(\lambda)$, while here we take $D(\lambda)$ as the basic starting point. Taking into account [11, Rem. 3.4], another way to look at strong block minimal bases pencils is as follows: fixed $N_1(\lambda)$ and $N_2(\lambda)$ with the properties mentioned in Definition 8.1, then for any polynomial matrix $D(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$ with $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$, it is possible to choose $M(\lambda)$ in such a way that $\mathcal{L}(\lambda)$ in (32) is a strong linearization of $D(\lambda)$ and (33) is satisfied.

The sizes of the submatrices in (32) are related to the degrees of the dual minimal bases $N_1(\lambda)$ and $N_2(\lambda)$ via [13, Corollary p. 503] (see [10, Thm. 2.12] for a more explicit statement) as follows. Set

$$\varepsilon := \deg(N_1(\lambda)) \quad \text{and} \quad \eta := \deg(N_2(\lambda)). \qquad (34)$$

Since the row degrees of $N_1(\lambda)$ (respectively $N_2(\lambda)$) are all equal, they must be all equal to ε (respectively η) and [13, Corollary p. 503] implies

$$\widehat{m} = m\,\varepsilon \quad \text{and} \quad \widehat{p} = p\,\eta. \tag{35}$$

Definition 8.1 includes the "degenerate" cases $\widehat{m} = 0$, when the second block row in (32) is not present (or is an empty matrix), or $\widehat{p} = 0$, when the second block column in (32) is not present (or is an empty matrix). If $\widehat{m} = 0$ (respectively $\widehat{p} = 0$) then $N_1(\lambda)$ (respectively $N_2(\lambda)$) is taken to be a nonsingular constant matrix of size $m \times m$ (respectively $p \times p$) and the simplest choice is just $N_1(\lambda) = I_m$ (respectively $N_2(\lambda) = I_p$).

A final remark on Definition 8.1 is that in [11, Thm. 3.3] the condition $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$ defining the strong block minimal bases pencils with sharp degree is not mentioned at all. The reason is that the reversal of $D(\lambda)$ is defined in [11] with respect to the "grade" (see [11, Sec. 2]) $\deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$, while here reversals of polynomial matrices are always defined in an intrinsic way with respect to the degree (recall Section 2). We emphasize that $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$ is used in the proof of Theorem 8.11 on the construction of strong linearizations of rational matrices in Section 8.2 and that this condition together with (33) implies $\deg(M(\lambda)) = 1$.

Strong block minimal bases linearizations of polynomial matrices are a very wide set of linearizations which include different types of linearizations (see [11, Secs. 4 and 5], [21], [26]). In the next example, we present a particular class of strong block minimal bases linearizations introduced in [11, Sec. 5], which were called block Kronecker linearizations. They correspond to particular choices of $K_1(\lambda)$ and $K_2(\lambda)$ in (32). Even with this particular choice, there are infinitely many block Kronecker linearizations for any polynomial matrix $D(\lambda)$. They correspond to infinitely many choices of $M(\lambda)$ and can be very easily obtained out of the coefficients of $D(\lambda)$ (see [11, Thm. 5.4] and the comments after that theorem).

Example 8.2 Consider $D(\lambda) = D_q\lambda^q + D_{q-1}\lambda^{q-1} + \cdots + D_0 \in \mathbb{F}[\lambda]^{p \times m}$, with $q > 1$ and $D_q \neq 0$, and the matrices in (28) and (29). Then, a block Kronecker linearization of $D(\lambda)$ is a pencil

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_p \\ \hline L_\varepsilon(\lambda) \otimes I_m & 0 \end{array}\right] \begin{array}{l} \} (\eta+1)p \\ \} \varepsilon m \end{array} \quad , \tag{36}$$
$$\underbrace{}_{(\varepsilon+1)m} \underbrace{}_{\eta p}$$

such that

$$D(\lambda) = (\Lambda_\eta(\lambda)^T \otimes I_p)\,M(\lambda)\,(\Lambda_\varepsilon(\lambda) \otimes I_m).$$

Theorem 5.4 in [11] explains how to construct all possible $M(\lambda)$ that satisfy the previous equation (there are infinitely many). Observe that in the

notation of Definition 8.1, we are taking $N_1(\lambda) = \Lambda_\varepsilon(\lambda)^T \otimes I_m$ and $N_2(\lambda) = \Lambda_\eta(\lambda)^T \otimes I_p$. Particular examples of block Kronecker linearizations of $D(\lambda)$ are the first Frobenius companion form, which corresponds to $M(\lambda) = \begin{bmatrix} D_q\lambda + D_{q-1} & D_{q-2} & \cdots & D_0 \end{bmatrix}$, $\varepsilon = q-1$ and $\eta = 0$, and the second Frobenius companion form, with $M(\lambda) = \begin{bmatrix} D_q^T\lambda + D_{q-1}^T & D_{q-2}^T & \cdots & D_0^T \end{bmatrix}^T$, $\eta = q-1$ and $\varepsilon = 0$. The block Kronecker linearizations corresponding to the remaining (permuted) Fiedler pencils are extremely easy to construct as is discussed in [11, Thm. 4.5]. An interesting block Kronecker linearization for polynomial matrices with odd degrees $q = 2k+1$ is constructed by taking $L_\varepsilon(\lambda) = L_\eta(\lambda) = L_k(\lambda)$ and

$$
M(\lambda) = \begin{bmatrix} D_{2k+1}\lambda + D_{2k} & & & \\ & D_{2k-1}\lambda + D_{2k-2} & & \\ & & \ddots & \\ & & & D_1\lambda + D_0 \end{bmatrix}.
$$

Such linearization is very simple and in the case $D(\lambda)$ is symmetric or Hermitian the linearization is also symmetric or Hermitian. Note that the condition $D_q \neq 0$ guarantees that (36) is a strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree [11, Thm. 5.4].

Lemma 8.4 refines [11, Thm. 2.10] for the dual minimal bases as those appearing in Definition 8.1. The refinement comes from the fact that $\widehat{K}_1$ and $\widehat{K}_2$ in Lemma 8.4 are constant matrices, a property not guaranteed in [11] and that is essential in Section 8.2. In order to prove Lemma 8.4 we need to prove first Lemma 8.3.

Lemma 8.3 *For $i = 1, 2$ let $K_i(\lambda)$ be a linear pencil as in Definition 8.1 and let $N_i(\lambda)$ be a minimal basis dual to $K_i(\lambda)$. If $Q_i(\lambda)$ is another minimal basis dual to $K_i(\lambda)$, then there exists a nonsingular constant matrix H_i such that $Q_i(\lambda) = H_i\,N_i(\lambda)$.*

Proof.- The columns of $Q_i(\lambda)^T$ form a basis of the right null-space $\mathcal{N}_r(K_i(\lambda))$ over $\mathbb{F}(\lambda)$ defined in Section 3 and the columns of $N_i(\lambda)^T$ form another basis of $\mathcal{N}_r(K_i(\lambda))$. Therefore, there exists a nonsingular rational matrix $H_i(\lambda)$ such that $Q_i(\lambda) = H_i(\lambda)\,N_i(\lambda)$. Since $N_i(\lambda)$ and $Q_i(\lambda)$ are both minimal bases, the row degrees of $N_i(\lambda)$ are all equal, the row degrees of $Q_i(\lambda)$ are all equal, and the row degrees of $N_i(\lambda)$ are equal to those of $Q_i(\lambda)$, we get that $H_i(\lambda)$ must be a constant matrix. ∎

Lemma 8.4 *Let $K_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m} \times (m+\widehat{m})}$ be a linear pencil as in Definition 8.1 and $N_1(\lambda) \in \mathbb{F}[\lambda]^{m \times (m+\widehat{m})}$ be any of its minimal dual bases. Then there exist both $\widehat{N}_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m} \times (m+\widehat{m})}$ and a constant matrix $\widehat{K}_1 \in \mathbb{F}^{m \times (m+\widehat{m})}$ such that*

(a) $U_1(\lambda) = \begin{bmatrix} K_1(\lambda) \\ \widehat{K}_1 \end{bmatrix} \in \mathbb{F}[\lambda]^{(\widehat{m}+m)\times(m+\widehat{m})}$ *is a unimodular polynomial matrix, and*

(b) $U_1(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_1(\lambda)^T & N_1(\lambda)^T \end{bmatrix} \in \mathbb{F}[\lambda]^{(m+\widehat{m})\times(\widehat{m}+m)}.$

An analogous result holds for $K_2(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times(p+\widehat{p})}$ as in Definition 8.1 just by replacing 1 by 2, $\widehat{m}$ by $\widehat{p}$, and m by p.

Proof.- We only prove the result for $K_1(\lambda)$. In the proof, the matrices in (28), (29), (30) and (31) are frequently used. In addition, $V_k(\lambda)^{-1}$ is partitioned as $V_k(\lambda)^{-1} = \begin{bmatrix} W_k(\lambda) & \Lambda_k(\lambda) \end{bmatrix}$. We take $\varepsilon = \deg(N_1(\lambda))$ as in (34) and so $\widehat{m} = m\varepsilon$ as in (35). Since all the row degrees of $K_1(\lambda) = \lambda K_1^{(1)} + K_1^{(0)}$ are equal to 1 and $K_1(\lambda)$ is a minimal basis, $K_1^{(1)}$ has full row rank. This fact and [10, Thm. 2.4] implies that $K_1(\lambda)$ has neither infinite nor finite eigenvalues and has no left minimal indices. Therefore the Kronecker canonical form [14, Ch. XII] of $K_1(\lambda)$ has only right singular blocks of size $\varepsilon \times (\varepsilon + 1)$ (because the row degrees of $N_1(\lambda)$ are all equal to ε), i.e., there exist nonsingular constant matrices $R \in \mathbb{F}^{m\varepsilon \times m\varepsilon}$ and $S \in \mathbb{F}^{m(\varepsilon+1)\times m(\varepsilon+1)}$ such that

$$K_1(\lambda) = R^{-1}\left(I_m \otimes L_\varepsilon(\lambda)\right)S^{-1}.$$

Let $e_{\varepsilon+1}$ be the last column of $I_{\varepsilon+1}$. Define the constant matrix

$$\widetilde{K}_1 := \left(I_m \otimes e_{\varepsilon+1}^T\right)S^{-1} \in \mathbb{F}^{m\times m(\varepsilon+1)}$$

and the linear polynomial matrix

$$\widetilde{U}_1(\lambda) := \begin{bmatrix} K_1(\lambda) \\ \widetilde{K}_1 \end{bmatrix} = \begin{bmatrix} R^{-1} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_m \otimes L_\varepsilon(\lambda) \\ I_m \otimes e_{\varepsilon+1}^T \end{bmatrix} S^{-1} \in \mathbb{F}[\lambda]^{(m(\varepsilon+1))\times(m(\varepsilon+1))}.$$

Observe that $\widetilde{U}_1(\lambda)$ is unimodular because via an obvious row permutation Π we obtain

$$\Pi \begin{bmatrix} I_m \otimes L_\varepsilon(\lambda) \\ I_m \otimes e_{\varepsilon+1}^T \end{bmatrix} = I_m \otimes \begin{bmatrix} L_\varepsilon(\lambda) \\ e_{\varepsilon+1}^T \end{bmatrix} = I_m \otimes V_\epsilon(\lambda),$$

whose determinant is $(\det(V_\epsilon(\lambda)))^m$ and so it is constant. Observe also that

$$Q_1(\lambda) = (I_m \otimes \Lambda_\varepsilon(\lambda)^T)S^T \in \mathbb{F}[\lambda]^{m\times m(\varepsilon+1)}$$

is a minimal basis dual to $K_1(\lambda)$ because it is a minimal basis by [10, Thm. 2.4], $K_1(\lambda)Q_1(\lambda)^T = 0$. From Lemma 8.3, we know that $N_1(\lambda) = H_1^{-T}Q_1(\lambda)$ for some nonsingular constant matrix H_1. With this matrix, we finally define

$$U_1(\lambda) := \begin{bmatrix} K_1(\lambda) \\ \widehat{K}_1 \end{bmatrix} = \begin{bmatrix} I_{m\varepsilon} & 0 \\ 0 & H_1 \end{bmatrix} \widetilde{U}_1(\lambda),$$

48

which is unimodular since $\widetilde{U}_1(\lambda)$ is. In addition, note that

$$U_1(\lambda)^{-1} = \begin{bmatrix} S(I_m \otimes W_\varepsilon(\lambda))R & N_1(\lambda)^T \end{bmatrix},$$

since a direct multiplication proves that $U_1(\lambda)\,U_1(\lambda)^{-1} = I_{m(\varepsilon+1)}$. To perform this last matrix multiplication, take into account that (30) and the partition $V_k(\lambda)^{-1} = \begin{bmatrix} W_k(\lambda) & \Lambda_k(\lambda) \end{bmatrix}$ imply $L_\varepsilon(\lambda)W_\varepsilon(\lambda) = I_\varepsilon$, $e_{\varepsilon+1}^T W_\varepsilon(\lambda) = 0$, $L_\varepsilon(\lambda)\Lambda_\varepsilon(\lambda) = 0$ and $e_{\varepsilon+1}^T \Lambda_\varepsilon(\lambda) = 1$. ∎

Example 8.5 In the case of block Kronecker linearizations as those in (36) (which include the standard Frobenius companion forms), if we take $N_1(\lambda) = \Lambda_\varepsilon(\lambda)^T \otimes I_m$ and $N_2(\lambda) = \Lambda_\eta(\lambda)^T \otimes I_p$, then the constant matrices $\widehat{K}_1$ and $\widehat{K}_2$ in Lemma 8.4 can be taken simply to be $\widehat{K}_1 = e_{\varepsilon+1}^T \otimes I_m$ and $\widehat{K}_2 = e_{\eta+1}^T \otimes I_p$. This follows immediately from (30) and (31).

Following the discussion of degenerate cases, if $\widehat{m} = 0$ (respectively $\widehat{p} = 0$) in Lemma 8.4, then $U_1(\lambda) = \widehat{K}_1 = N_1(\lambda)^{-T} \in \mathbb{F}^{m\times m}$ (respectively $U_2(\lambda) = \widehat{K}_2 = N_2(\lambda)^{-T} \in \mathbb{F}^{p\times p}$) is any nonsingular constant matrix and the simplest choice is just $\widehat{K}_1 = I_m$ (respectively $\widehat{K}_2 = I_p$).

Lemma 8.4 allows us to construct in Theorem 8.6 unimodular matrices $U(\lambda)$ and $V(\lambda)$ that will be used in Section 8.2 to develop (strong) linearizations of rational matrices via Algorithm 4.10. The proof of Theorem 8.6 is only briefly sketched, since it can be essentially found in [11, Proofs of Thm. 3.3 and Lem. 2.14].

Theorem 8.6 *Let $\mathcal{L}(\lambda)$ as in (32) be a strong block minimal bases pencil associated to $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$, let $N_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ and $N_2(\lambda) \in \mathbb{F}[\lambda]^{p\times(p+\widehat{p})}$ be minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively, and for $i = 1,2$ let*

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i \end{bmatrix} \quad and \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix}$$

be unimodular matrices with $\widehat{K}_i$ a constant matrix and $\widehat{N}_i(\lambda)$ as in Lemma 8.4. There are matrices $X(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times m}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{p\times\widehat{m}}$ and $Z(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times\widehat{m}}$ such that

$$V(\lambda) := \begin{bmatrix} \widehat{N}_1(\lambda)^T & N_1(\lambda)^T & 0 \\ 0 & 0 & I_{\widehat{p}} \end{bmatrix} \begin{bmatrix} 0 & I_{\widehat{m}} & 0 \\ I_m & 0 & 0 \\ -X(\lambda) & 0 & I_{\widehat{p}} \end{bmatrix},$$

$$U(\lambda) := \begin{bmatrix} 0 & I_p & -Y(\lambda) \\ 0 & 0 & I_{\widehat{m}} \\ I_{\widehat{p}} & 0 & -Z(\lambda) \end{bmatrix} \begin{bmatrix} \widehat{N}_2(\lambda) & 0 \\ N_2(\lambda) & 0 \\ 0 & I_{\widehat{m}} \end{bmatrix},$$

are unimodular matrices and

$$U(\lambda)\,\mathcal{L}(\lambda)\,V(\lambda) = \mathrm{Diag}(D(\lambda), I_{\widehat{m}+\widehat{p}}). \tag{37}$$

Remark 8.7 We emphasize that Theorem 8.6 does not assume that $\mathcal{L}(\lambda)$ is a strong block minimal bases pencil with sharp degree. This remark will be important to obtain Corollary 8.8 below.

Proof of Theorem 8.6.- Since the two factors defining $V(\lambda)$ and $U(\lambda)$ are unimodular matrices so are $V(\lambda)$ and $U(\lambda)$. Proving (37) is a matter of multiplication. However it is easier to split the proof into two steps. In the first one, we use $K_1(\lambda)\widehat{N}_1(\lambda)^T = I_{\widehat{m}}$, $K_1(\lambda)N_1(\lambda)^T = 0$, $\widehat{K}_1\widehat{N}_1(\lambda)^T = 0$, $\widehat{K}_1 N_1(\lambda)^T = I_m$, the analogous expressions obtained by replacing 1 by 2, m by p, and $\widehat{m}$ by $\widehat{p}$, and (33) to prove that

$$
\begin{bmatrix} \widehat{N}_2(\lambda) & 0 \\ N_2(\lambda) & 0 \\ 0 & I_{\widehat{m}} \end{bmatrix}
\begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}
\begin{bmatrix} \widehat{N}_1(\lambda)^T & N_1(\lambda)^T & 0 \\ 0 & 0 & I_{\widehat{p}} \end{bmatrix}
$$
$$
= \begin{bmatrix} Z(\lambda) & X(\lambda) & I_{\widehat{p}} \\ Y(\lambda) & D(\lambda) & 0 \\ I_{\widehat{m}} & 0 & 0 \end{bmatrix}. \tag{38}
$$

In the second step, the matrix in (38) is multiplied by

$$
\begin{bmatrix} 0 & I_p & -Y(\lambda) \\ 0 & 0 & I_{\widehat{m}} \\ I_{\widehat{p}} & 0 & -Z(\lambda) \end{bmatrix} \quad \text{and} \quad
\begin{bmatrix} 0 & I_{\widehat{m}} & 0 \\ I_m & 0 & 0 \\ -X(\lambda) & 0 & I_{\widehat{p}} \end{bmatrix}
$$

on the left and on the right, respectively, to get $\mathrm{Diag}(D(\lambda), I_{\widehat{m}+\widehat{p}})$. ∎

The final goal of the rest of this subsection is to construct certain biproper matrices related to strong block minimal bases pencils with sharp degree that will be used in Section 8.2 to prove by means of Corollary 7.8 that the new class of linearizations of rational matrices introduced in Theorem 8.11 are strong. To this purpose, let us revise first some properties of the reversal of a strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree as in (32)-(33). This pencil is

$$
\mathrm{rev}\,\mathcal{L}(\lambda) = \begin{bmatrix} \mathrm{rev}\,M(\lambda) & \mathrm{rev}\,K_2(\lambda)^T \\ \mathrm{rev}\,K_1(\lambda) & 0 \end{bmatrix},
$$

since $\deg(M(\lambda)) = 1$ by (33) and the condition $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$ in Definition 8.1. In [11, Thm. 2.7 and Proof of Thm. 3.3], it is proved that $\mathrm{rev}\,\mathcal{L}(\lambda)$ is a strong block minimal bases pencil associated to

$$
\mathrm{rev}\,D(\lambda) = (\mathrm{rev}\,N_2(\lambda))\,(\mathrm{rev}\,M(\lambda))\,(\mathrm{rev}\,N_1(\lambda)^T),
$$

although that it has sharp degree cannot be guaranteed. In particular, it is proved in [11, Thm. 2.7], that for $i = 1, 2$, $\mathrm{rev}\,N_i(\lambda)$ is a minimal basis dual to $\mathrm{rev}\,K_i(\lambda)$ with $\deg(\mathrm{rev}\,N_i(\lambda)) = \deg(N_i(\lambda))$. These observations allow us to apply Theorem 8.6 to $\mathrm{rev}\,\mathcal{L}(\lambda)$ and $\mathrm{rev}\,D(\lambda)$ and to obtain directly Corollary 8.8.

Corollary 8.8 *Let $\mathcal{L}(\lambda)$ be a strong block minimal bases pencil associated to $D(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$ with sharp degree as in Definition 8.1 and let $N_1(\lambda) \in \mathbb{F}[\lambda]^{m \times (m+\widehat{m})}$ and $N_2(\lambda) \in \mathbb{F}[\lambda]^{p \times (p+\widehat{p})}$ be minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively. Then there exist $\widetilde{N}_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m} \times (m+\widehat{m})}$, $\widetilde{N}_2(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p} \times (p+\widehat{p})}$, $\widetilde{X}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p} \times m}$, $\widetilde{Y}(\lambda) \in \mathbb{F}[\lambda]^{p \times \widehat{m}}$ and $\widetilde{Z}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p} \times \widehat{m}}$ such that*

$$
\widetilde{V}(\lambda) := \begin{bmatrix} \widetilde{N}_1(\lambda)^T & \mathrm{rev}\, N_1(\lambda)^T & 0 \\ 0 & 0 & I_{\widehat{p}} \end{bmatrix} \begin{bmatrix} 0 & I_{\widehat{m}} & 0 \\ I_m & 0 & 0 \\ -\widetilde{X}(\lambda) & 0 & I_{\widehat{p}} \end{bmatrix},
$$

$$
\widetilde{U}(\lambda) := \begin{bmatrix} 0 & I_p & -\widetilde{Y}(\lambda) \\ 0 & 0 & I_{\widehat{m}} \\ I_{\widehat{p}} & 0 & -\widetilde{Z}(\lambda) \end{bmatrix} \begin{bmatrix} \widetilde{N}_2(\lambda) & 0 \\ \mathrm{rev}\, N_2(\lambda) & 0 \\ 0 & I_{\widehat{m}} \end{bmatrix},
$$

are unimodular matrices and

$$
\widetilde{U}(\lambda)\,(\mathrm{rev}\,\mathcal{L}(\lambda))\,\widetilde{V}(\lambda) = \mathrm{Diag}(\mathrm{rev}\, D(\lambda), I_{\widehat{m}+\widehat{p}}). \tag{39}
$$

In addition, each of the factors defining $\widetilde{V}(\lambda)$ and $\widetilde{U}(\lambda)$ is unimodular.

A particular class of biproper rational matrices can be obtained from unimodular matrices as follows (see [3, Lem. 4.1]).

Lemma 8.9 *If $U(\lambda) \in \mathbb{F}[\lambda]^{m \times m}$ is a unimodular matrix, then $U(1/\lambda) \in \mathbb{F}_{pr}(\lambda)^{m \times m}$ is a biproper matrix.*

Combining Corollary 8.8 and Lemma 8.9, we obtain the last result of this section.

Corollary 8.10 *With the same assumptions and notation as in Corollary 8.8, let $\widetilde{V}(\lambda) \in \mathbb{F}[\lambda]^{(m+\widehat{m}+\widehat{p}) \times (m+\widehat{m}+\widehat{p})}$ and $\widetilde{U}(\lambda) \in \mathbb{F}[\lambda]^{(p+\widehat{m}+\widehat{p}) \times (p+\widehat{m}+\widehat{p})}$ be the unimodular matrices introduced in Corollary 8.8 and define from them the biproper matrices*

$$
\widetilde{V}(1/\lambda) := \begin{bmatrix} \widetilde{N}_1(1/\lambda)^T & \mathrm{rev}\, N_1(1/\lambda)^T & 0 \\ 0 & 0 & I_{\widehat{p}} \end{bmatrix} \begin{bmatrix} 0 & I_{\widehat{m}} & 0 \\ I_m & 0 & 0 \\ -\widetilde{X}(1/\lambda) & 0 & I_{\widehat{p}} \end{bmatrix},
$$

$$
\widetilde{U}(1/\lambda) := \begin{bmatrix} 0 & I_p & -\widetilde{Y}(1/\lambda) \\ 0 & 0 & I_{\widehat{m}} \\ I_{\widehat{p}} & 0 & -\widetilde{Z}(1/\lambda) \end{bmatrix} \begin{bmatrix} \widetilde{N}_2(1/\lambda) & 0 \\ \mathrm{rev}\, N_2(1/\lambda) & 0 \\ 0 & I_{\widehat{m}} \end{bmatrix}.
$$

Then

$$
\widetilde{U}(1/\lambda)\,(\lambda^{-1}\mathcal{L}(\lambda))\,\widetilde{V}(1/\lambda) = \mathrm{Diag}(\lambda^{-q} D(\lambda), I_{\widehat{m}+\widehat{p}}), \tag{40}
$$

where $q = \deg(D(\lambda))$. In addition, each of the factors defining $\widetilde{V}(1/\lambda)$ and $\widetilde{U}(1/\lambda)$ is biproper and any submatrix of these factors is a proper matrix.

Proof.- The properties of $\widetilde{U}(1/\lambda)$, $\widetilde{V}(1/\lambda)$, their factors, and the submatrices of them follow from Lemma 8.9. The equality (40) follows from (39) by replacing λ by $1/\lambda$ and taking into account that $\deg(\mathcal{L}(\lambda)) = 1$ since $\deg(M(\lambda)) = 1$. ∎

8.2 Strong block minimal bases linearizations of rational matrices

The goal of this section is to state and prove Theorem 8.11, which is the main theorem of this paper on the existence and explicit construction of strong linearizations of any rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$. For future reference, Theorem 8.11 includes all the needed assumptions and notations. The constructed strong linearizations are presented in equation (41) and the formal proof relies on Corollary 7.8. However, before getting into more details, we emphasize that such linearizations have been constructed via Algorithm 4.10 with input the minimal polynomial system matrix of $G(\lambda)$ in (27) and choosing in $\texttt{Step 1}$ any strong block minimal bases pencil $\mathcal{L}(\lambda)$ as in (32) associated to the polynomial part $D(\lambda)$ of $G(\lambda)$, the unimodular matrices $U(\lambda)$ and $V(\lambda)$ in Theorem 8.6, and taking $s = \widehat{m} + \widehat{p}$. To check this, note that

$$
V(\lambda)^{-1} = \begin{bmatrix} 0 & I_m & 0 \\ I_{\widehat{m}} & 0 & 0 \\ 0 & X(\lambda) & I_{\widehat{p}} \end{bmatrix} \begin{bmatrix} K_1(\lambda) & 0 \\ \widehat{K}_1 & 0 \\ 0 & I_{\widehat{p}} \end{bmatrix},
$$

$$
U(\lambda)^{-1} = \begin{bmatrix} K_2(\lambda)^T & \widehat{K}_2^T & 0 \\ 0 & 0 & I_{\widehat{m}} \end{bmatrix} \begin{bmatrix} 0 & Z(\lambda) & I_{\widehat{p}} \\ I_p & Y(\lambda) & 0 \\ 0 & I_{\widehat{m}} & 0 \end{bmatrix},
$$

which in $\texttt{Step 2}$ in Algorithm 4.10 yields

$$
U(\lambda)^{-1} \begin{bmatrix} C \\ 0_{\widehat{m} \times n} \\ 0_{\widehat{p} \times n} \end{bmatrix} = \begin{bmatrix} \widehat{K}_2^T C \\ 0_{\widehat{m} \times n} \end{bmatrix}, \qquad \begin{bmatrix} B & 0_{n \times \widehat{m}} & 0_{n \times \widehat{p}} \end{bmatrix} V(\lambda)^{-1} = \begin{bmatrix} B \widehat{K}_1 & 0_{n \times \widehat{p}} \end{bmatrix}.
$$

Since these two matrices are constant, Algorithm 4.10 does not stop. Then taking arbitrary nonsingular matrices T and S in $\texttt{Step 3}$, the linearization in (41) is obtained where matrices X and Y in that expression are T^{-1} and S^{-1} respectively. For obvious reasons, these linearizations are called *strong block minimal bases linearizations of rational matrices*.

Theorem 8.11 *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ be a rational matrix, let $G(\lambda) = D(\lambda) + G_{sp}(\lambda)$ be its unique decomposition into its polynomial part $D(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$ and its strictly proper part $G_{sp}(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$, and let $G_{sp}(\lambda) = C(\lambda I_n - A)^{-1} B$ be a minimal order state-space realization of $G_{sp}(\lambda)$, where*

$n = \nu(G(\lambda)) = \nu(G_{sp}(\lambda))$. *Assume that* $\deg(D(\lambda)) > 1$ *and let*

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{array} \right] \begin{array}{l} \} p+\widehat{p} \\ \} \widehat{m} \end{array} ,$$
$$\underbrace{}_{m+\widehat{m}} \underbrace{}_{\widehat{p}}$$

be a strong block minimal bases pencil associated to $D(\lambda)$ *with sharp degree, with* $N_1(\lambda) \in \mathbb{F}[\lambda]^{m \times (m+\widehat{m})}$ *and* $N_2(\lambda) \in \mathbb{F}[\lambda]^{p \times (p+\widehat{p})}$ *minimal bases dual to* $K_1(\lambda)$ *and* $K_2(\lambda)$, *respectively, such that* $D(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$. *Let* $\widehat{K}_1 \in \mathbb{F}^{m \times (m+\widehat{m})}$ *and* $\widehat{K}_2 \in \mathbb{F}^{p \times (p+\widehat{p})}$ *be constant matrices such that, for* $i = 1, 2$, *the matrices*

$$U_i(\lambda) = \left[\begin{array}{c} K_i(\lambda) \\ \widehat{K}_i \end{array} \right] \quad and \quad U_i(\lambda)^{-1} = \left[\widehat{N}_i(\lambda)^T \quad N_i(\lambda)^T \right]$$

are unimodular. Then, for any nonsingular constant matrices $X, Y \in \mathbb{F}^{n \times n}$ *the linear polynomial matrix*

$$L(\lambda) = \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & XB\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T CY & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array} \right] \tag{41}$$

is a strong linearization of $G(\lambda)$.

Remark 8.12 If $\mathcal{L}(\lambda)$ in Theorem 8.11 is a block Kronecker linearization of $D(\lambda)$ as in (36), Example 8.5 implies that

$$XB\widehat{K}_1 = e_{\varepsilon+1}^T \otimes XB = \left[0_{n \times m} \quad \cdots \quad 0_{n \times m} \quad XB \right]$$

and $\widehat{K}_2^T CY = e_{\eta+1} \otimes CY$, which has a similar zero column block structure. Recall also that in the degenerate case $\widehat{m} = 0$ (respectively $\widehat{p} = 0$) $\widehat{K}_1 \in \mathbb{F}^{m \times m}$ (respectively $\widehat{K}_2 \in \mathbb{F}^{p \times p}$) can be any nonsingular matrix with I_m (respectively I_p) as the simplest choice. There are infinitely many strong block minimal bases pencils $\mathcal{L}(\lambda)$ associated to $D(\lambda)$ with sharp degree and, so, infinitely many strong linearizations of $G(\lambda)$ inside the framework of Theorem 8.11. A subset of these infinitely many can be constructed very easily in the case we restrict ourselves to block Kronecker linearizations $\mathcal{L}(\lambda)$ of $D(\lambda)$.

Proof of Theorem 8.11.- The proof is based on Corollary 7.8. In this proof, we adopt the notation in (34) for the degrees of $N_1(\lambda)$ and $N_2(\lambda)$ and $q := \deg(D(\lambda)) > 1$. Therefore, $q = \varepsilon + \eta + 1$ according to the condition on the degrees stated in Definition 8.1. Note that this condition and (33) imply that $\deg(M(\lambda)) = 1$, therefore the parameter $\widehat{g}$ in Corollary 7.8 is in this

case $\widehat{g} = -1$ for $L(\lambda)$ in (41), since in the notation of that corollary $D_1 \neq 0$, $C_1 = 0$, and $B_1 = 0$. In addition, the parameter g in Corollary 7.8 is $g = -q$ here. A key ingredient in this proof is the minimal polynomial system matrix in state-space form $P(\lambda)$ in (27) giving rise to $G(\lambda)$. Obviously, for this $P(\lambda)$ the matrices $C(\lambda I_n - A)^{-1}$ and $\lambda^{-q}(\lambda I_n - A)^{-1}B$ are both proper, and we are in the scenario of Corollary 7.8 in this respect.

According to Corollary 7.8, Theorem 8.11 is proved if

$$\mathbb{L}(\lambda) = \left[\begin{array}{c|cc} \lambda I_n - A & B\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T C & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right] \tag{42}$$

is proved to be a strong linearization of $G(\lambda)$. The reason is that

$$\mathrm{Diag}(X, I_{p+\widehat{m}+\widehat{p}})\, \mathbb{L}(\lambda)\, \mathrm{Diag}(Y, I_{m+\widehat{m}+\widehat{p}}) = L(\lambda),$$

which means that $\mathbb{L}(\lambda)$ and $L(\lambda)$ are transfer system equivalent, and that

$$\mathrm{Diag}(X, I_{p+\widehat{m}+\widehat{p}})\,(\mathbb{L}(\lambda)\, \mathrm{Diag}(I_n, \lambda^{-1}I_{m+\widehat{m}+\widehat{p}}))\, \mathrm{Diag}(Y, I_{m+\widehat{m}+\widehat{p}})$$
$$= L(\lambda)\, \mathrm{Diag}(I_n, \lambda^{-1}I_{m+\widehat{m}+\widehat{p}}),$$

which means that $\mathbb{L}(\lambda)\, \mathrm{Diag}(I_n, \lambda^{-1}I_{m+\widehat{m}+\widehat{p}})$ and $L(\lambda)\, \mathrm{Diag}(I_n, \lambda^{-1}I_{m+\widehat{m}+\widehat{p}})$ are transfer system equivalent at infinity. Therefore, in the rest of the proof we focus only on $\mathbb{L}(\lambda)$.

We prove first that $\mathbb{L}(\lambda)$ is transfer system equivalent to $\mathrm{Diag}(P(\lambda), I_{\widehat{m}+\widehat{p}})$, i.e., we prove first (i) in Corollary 7.8. To this purpose, the unimodular matrices $U(\lambda)$ and $V(\lambda)$ in Theorem 8.6 and (37) are used to prove that the transfer system equivalence

$$\mathrm{Diag}(I_n, U(\lambda))\, \mathbb{L}(\lambda)\, \mathrm{Diag}(I_n, V(\lambda)) = \mathrm{Diag}(P(\lambda), I_{\widehat{m}+\widehat{p}})$$

holds. In order to get the previous equation recall that $\widehat{K}_1 \widehat{N}_1(\lambda)^T = 0$, $\widehat{K}_1 N_1(\lambda)^T = I_m$, $\widehat{K}_2 \widehat{N}_2(\lambda)^T = 0$, $\widehat{K}_2 N_2(\lambda)^T = I_p$. Therefore,

$$\begin{bmatrix} B\widehat{K}_1 & 0_{n\times\widehat{p}} \end{bmatrix} V(\lambda)$$
$$= \begin{bmatrix} B\widehat{K}_1 \widehat{N}_1(\lambda)^T & B\widehat{K}_1 N_1(\lambda)^T & 0_{n\times\widehat{p}} \end{bmatrix} \begin{bmatrix} 0 & I_{\widehat{m}} & 0 \\ I_m & 0 & 0 \\ -X(\lambda) & 0 & I_{\widehat{p}} \end{bmatrix} = \begin{bmatrix} B & 0_{n\times(\widehat{m}+\widehat{p})} \end{bmatrix}$$

and

$$U(\lambda) \begin{bmatrix} -\widehat{K}_2^T C \\ 0_{\widehat{m}\times n} \end{bmatrix} = \begin{bmatrix} -C \\ 0_{(\widehat{m}+\widehat{p})\times n} \end{bmatrix}.$$

Next, we proceed to prove that (ii) in Corollary 7.8 holds for $\mathbb{L}(\lambda)$ in (42) and $P(\lambda)$ in (27), which requires considerable more effort. The proof

of this fact is split into two steps. The first step uses the biproper matrices $\widetilde{U}(1/\lambda)$ and $\widetilde{V}(1/\lambda)$ introduced in Corollary 8.10 and the submatrices of their factors to define the following proper matrices

$$\mathbb{W}(\lambda) := \begin{bmatrix} (\lambda^{-\eta} - 1)I_p \\ 0_{\widehat{m} \times p} \\ \widetilde{N}_2(1/\lambda)\,\widehat{K}_2^T \end{bmatrix} C(\lambda I_n - A)^{-1},$$

$$\mathbb{Z}(\lambda) := \lambda^{-1}(\lambda I_n - A)^{-1}B \begin{bmatrix} (\lambda^{-q+1} - \lambda^{-\varepsilon})I_m & -\widehat{K}_1\,\widetilde{N}_1(1/\lambda)^T & 0_{m \times \widehat{p}} \end{bmatrix}.$$

We have first that (40) holds, $\widehat{K}_1 \operatorname{rev} N_1(1/\lambda)^T = \lambda^{-\varepsilon}I_m$ and $\operatorname{rev} N_2(1/\lambda)\widehat{K}_2^T = \lambda^{-\eta}I_p$. With all these properties and $q = \varepsilon + \eta + 1$ in mind, one can prove, after somewhat long but direct algebraic manipulations, the following transfer system equivalence transformation at infinity

$$\begin{bmatrix} I_n & 0 \\ \mathbb{W}(\lambda) & \widetilde{U}(1/\lambda) \end{bmatrix} \left[\begin{array}{c|cc} \lambda I_n - A & \lambda^{-1}B\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T C & \lambda^{-1}M(\lambda) & \lambda^{-1}K_2(\lambda)^T \\ 0 & \lambda^{-1}K_1(\lambda) & 0 \end{array} \right] \begin{bmatrix} I_n & \mathbb{Z}(\lambda) \\ 0 & \widetilde{V}(1/\lambda) \end{bmatrix}$$

$$= \left[\begin{array}{c|ccc} \lambda I_n - A & \lambda^{-q}B & 0 & 0 \\ \hline -C & \lambda^{-q}D(\lambda) & H_{23}(\lambda) & 0 \\ 0 & 0 & I_{\widehat{m}} & 0 \\ 0 & H_{42}(\lambda) & H_{43}(\lambda) & I_{\widehat{p}} \end{array} \right] =: F(\lambda),$$

where

$$H_{42}(\lambda) = \lambda^{-(\varepsilon+1)}\,\widetilde{N}_2(1/\lambda)\widehat{K}_2^T G_{sp}(\lambda),$$
$$H_{23}(\lambda) = \lambda^{-(\eta+1)}\,G_{sp}(\lambda)\widehat{K}_1\widetilde{N}_1(1/\lambda)^T,$$
$$H_{43}(\lambda) = \lambda^{-1}\,\widetilde{N}_2(1/\lambda)\widehat{K}_2^T G_{sp}(\lambda)\widehat{K}_1\widetilde{N}_1(1/\lambda)^T$$

are strictly proper rational matrices. The second step of our proof of *(ii)* in Corollary 7.8 consists of the following transfer system equivalence transformation at infinity

$$\left[\begin{array}{c|ccc} I_n & 0 & 0 & 0 \\ \hline 0 & I_p & -H_{23}(\lambda) & 0 \\ 0 & 0 & I_{\widehat{m}} & 0 \\ 0 & 0 & 0 & I_{\widehat{p}} \end{array} \right] F(\lambda) \left[\begin{array}{c|ccc} I_n & 0 & 0 & 0 \\ \hline 0 & I_m & 0 & 0 \\ 0 & 0 & I_{\widehat{m}} & 0 \\ 0 & -H_{42}(\lambda) & -H_{43}(\lambda) & I_{\widehat{p}} \end{array} \right]$$

$$= \left[\begin{array}{c|ccc} \lambda I_n - A & \lambda^{-q}B & 0 & 0 \\ \hline -C & \lambda^{-q}D(\lambda) & 0 & 0 \\ 0 & 0 & I_{\widehat{m}} & 0 \\ 0 & 0 & 0 & I_{\widehat{p}} \end{array} \right],$$

which completes the proof of Theorem 8.11. ∎

8.3 Examples of strong linearizations of symmetric rational matrices

In the following two examples we implement the schemes developed in this paper to obtain strong linearizations for two symmetric rational matrices discussed in [29, Sec. 4.3 and 4.4] and that appear in applications. In addition, the corresponding strong linearizations will preserve the symmetric structure of the problems.

Example 8.13 Vibration of a fluid-solid structure. Let

$$G(\lambda) = A - \lambda B + \sum_{i=1}^{k} \frac{\lambda}{\lambda - \sigma_i} E_i,$$

with A and B $n \times n$ real nonzero symmetric positive semidefinite matrices, $\sigma_i > 0$, and $E_i = C_i C_i^T$, $C_i \in \mathbb{R}^{n \times r_i}$ and rank $C_i = r_i$, $i = 1, \ldots, k$. First, we separate the polynomial and strictly proper parts of $G(\lambda)$:

$$G(\lambda) = A + \sum_{i=1}^{k} C_i C_i^T - \lambda B + G_{sp}(\lambda)$$

where

$$G_{sp}(\lambda) = \sum_{i=1}^{k} \frac{\sigma_i}{\lambda - \sigma_i} C_i C_i^T.$$

The realization of $G_{sp}(\lambda)$ proposed in [29] is

$$G_{sp}(\lambda) = C(\lambda I_r - \Sigma)^{-1} \Sigma C^T \tag{43}$$

where $\Sigma = \text{Diag}(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_k I_{r_k})$, $C = \begin{bmatrix} C_1 & C_2 & \cdots & C_k \end{bmatrix}$ and $r = r_1 + \cdots + r_k$. Without further assumptions on the matrices C_i, we cannot conclude that this realization is minimal and so, some σ_i may not be poles of $G(\lambda)$. Henceforth the result about the number of eigenvalues of $G(\lambda)$ in a given interval (α, β) given at the end of Section 4.3 of [29] may not be correct. It turns out, however, that under very mild conditions the realization of (43) is controllable and observable. If, for example, rank $C = r$ and we put $H = \Sigma C^T$ then (Σ, H) is controllable, (Σ, C) is observable and $G_{sp}(\lambda) = C(\lambda I_r - \Sigma)^{-1} H$ is a minimal realization of $G_{sp}(\lambda)$. Hence, if rank $C = r$,

$$L_1(\lambda) = \begin{bmatrix} \lambda I_r - \Sigma & \Sigma C^T \\ -C & A + CC^T - \lambda B \end{bmatrix}$$

is a strong linearization of $G(\lambda)$, according to the observation in the paragraph below (27). In addition, if

$$\begin{aligned} L(\lambda) &= \begin{bmatrix} -\Sigma^{-1} & 0 \\ 0 & I_n \end{bmatrix} L_1(\lambda) = \begin{bmatrix} -\lambda \Sigma^{-1} + I_r & -C^T \\ -C & -\lambda B + (A + CC^T) \end{bmatrix} \\ &= \lambda \begin{bmatrix} -\Sigma^{-1} & 0 \\ 0 & -B \end{bmatrix} + \begin{bmatrix} I_r & -C^T \\ -C & A + CC^T \end{bmatrix} \end{aligned}$$

then $L_1(\lambda)$ and $L(\lambda)$ are strictly system equivalent. Also,

$$\begin{bmatrix} \lambda I_r - \Sigma & \lambda^{-1}\Sigma C^T \\ -C & \lambda^{-1}(A + CC^T - \lambda B) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\lambda\Sigma^{-1} + I_r & -\lambda^{-1}C^T \\ -C & \lambda^{-1}(-\lambda B + (A + CC^T)) \end{bmatrix}$$

are strictly system equivalent at infinity. Thus, by Corollary 7.8, $L(\lambda)$ is also a strong linearization of $G(\lambda)$. $L(\lambda)$ is a symmetric positive semidefinite (see Proposition 4.1 of [29]) strong linearization of $G(\lambda)$. The eigenvalues (finite and at infinity) of $G(\lambda)$ can be computed via the linear polynomial eigenvalue problem $L(\lambda)z = 0$.

Example 8.14 Damped vibration of a structure. Let

$$G(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i\lambda}\Delta G_i,$$

where M and K are $n \times n$ real symmetric positive definite matrices, $b_i > 0$ and $\Delta G_i = L_iL_i^T$ with $L_i \in \mathbb{R}^{n \times r_i}$ and rank $L_i = r_i$, $i = 1, \ldots, k$. The goal of this example is to use Theorem 8.11 to construct a strong linearization of $G(\lambda)$ that preserves the symmetric structure of the problem. Let us define $\sigma_i = \frac{1}{b_i}$. Then, the decomposition of $G(\lambda)$ into its polynomial and strictly proper parts is

$$G(\lambda) = \lambda^2 M + K + G_{sp}(\lambda),$$

where

$$G_{sp}(\lambda) = -\sum_{i=1}^{k} \frac{\sigma_i}{\lambda + \sigma_i}L_iL_i^T.$$

Let us denote

$$L = \begin{bmatrix} L_1 & L_2 & \cdots & L_k \end{bmatrix}, \quad \Sigma = \text{Diag}(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_k I_{r_k}),$$

and assume that rank $L = r = r_1 + r_2 + \cdots + r_k$. Again, if $C = -L$ and $B = \Sigma L^T$ then $G_{sp}(\lambda) = C(\lambda I_r + \Sigma)^{-1}B$ is a minimal state-space realization of $G_{sp}(\lambda)$. In addition,

$$\mathcal{L}(\lambda) = \begin{bmatrix} K & \lambda M \\ \lambda M & -M \end{bmatrix}$$

is a strong block minimal bases pencil associated to the polynomial part $D(\lambda) = \lambda^2 M + K$ with sharp degree. To check this, note that in the notation of Definition 8.1, $M(\lambda) = \begin{bmatrix} K & \lambda M \end{bmatrix}$, $K_1(\lambda) = \begin{bmatrix} \lambda M & -M \end{bmatrix}$, which is a minimal basis by [10, Thm. 2.4], $N_1(\lambda) = \begin{bmatrix} I_n & \lambda I_n \end{bmatrix}$ is a minimal bases dual to $K_1(\lambda)$ with all its row degrees equal to 1, and $\hat{p} = 0$, which allows us to take $N_2(\lambda) = I_n$. So, $D(\lambda) = M(\lambda)N_1(\lambda)^T$ and $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$. Now, the use of Theorem 8.11 to construct

strong linearizations of $G(\lambda)$ requires to know $\widehat{K}_1$, since $\widehat{K}_2^T$ can be taken to be any nonsingular matrix, and in particular I_n. To this purpose, note that

$$U_1(\lambda) = \begin{bmatrix} \lambda M & -M \\ I_n & 0 \end{bmatrix} \quad \text{and} \quad U_1(\lambda)^{-1} = \begin{bmatrix} 0 & I_n \\ -M^{-1} & \lambda I_n \end{bmatrix}$$

are unimodular, which means that we can choose $\widehat{K}_1 = \begin{bmatrix} I_n & 0 \end{bmatrix}$. With all this information, we can take $X = \Sigma^{-1}$ and $Y = I_r$ in (41) to obtain, as a consequence of Theorem 8.11, that

$$L_2(\lambda) = \begin{bmatrix} \lambda \Sigma^{-1} + I_r & L^T & 0 \\ L & K & \lambda M \\ 0 & \lambda M & -M \end{bmatrix}$$

is a strong linearization of $G(\lambda)$. In addition, $L_2(\lambda)$ preserves the symmetric structure of the problem.

We remark that the linearizations constructed in the two examples of this subsection not only preserve the (finite and infinite) poles and zeros of the original rational matrices but also its symmetric structure. Clearly, preserving the symmetry is not always possible, since it is well known [9, Sec. 7] that there are real symmetric polynomial matrices with even degree which do not have symmetric strong linearizations and we have proved that the Definition 6.2 of strong linearizations of rational matrices reduces to the standard one in the polynomial case. It remains as an open problem to study if in the non-polynomial rational case it is always possible to construct strong linearizations that preserve the symmetry or if restrictions similar to those in the polynomial case hold or if additional constraints arise coming from the interaction between the polynomial and strictly proper parts of the rational matrix.

9   Conclusions and future work

This paper develops for the first time in the literature a theory of strong linearizations of arbitrary rational matrices. In this process, a new definition of linearization (not necessarily strong) of rational matrices is introduced and compared with other definitions available in the literature. This comparison reveals a number of advantages of the new definition. A key feature of the definitions in this work is that they generalize smoothly the corresponding ones for polynomial matrices. The developed theory includes detailed necessary and sufficient spectral characterizations of linearizations and strong linearizations of rational matrices that show that the defined linear objects have precisely the expected properties. In addition, the concepts of transfer system equivalence and transfer system equivalence at infinity are defined and used to characterize the most general class of transformations that allow

us to construct strong linearizations of rational matrices. These equivalence relations are used to obtain infinitely many explicit examples of strong linearizations of rational matrices. Any of these explicit linearizations can be used to compute reliably the complete set of finite and infinite poles and zeros of an arbitrary rational matrix via well established algorithms for linear pencils as a consequence of the theory established in this paper.

In the last years considerable work has been devoted by many prestigious research groups all over the world to study different properties of linearizations and strong linearizations of polynomial matrices that are essential for understanding their behavior in the numerical solution of polynomial eigenvalue problems. Therefore, it can be expected that the complete theory presented in this paper will foster further research on strong linearizations of rational matrices as, for instance, the study of the preservation of different structures, the comparison of the conditioning of the eigenvalues of the linearizations with the conditioning of the zeros and poles of the original rational matrix, the analysis of the backward errors introduced in the original problem by those introduced by a backward stable eigenvalue algorithm applied on the linearization, the development of recovery procedures for the minimal indices and bases and eigenvectors, etc. We emphasize that many of these problems are of a different nature than the corresponding ones for polynomial matrices, since a minimal realization of the strictly proper part of a rational matrix can be chosen in many different ways.

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