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Reduction and relative equilibria for the 2-body problem on spaces of constant curvature

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Abstract

We consider the two-body problem on surfaces of constant non-zero curvature and classify the relative equilibria and their stability. On the hyperbolic plane, for each $q > 0$ we show there are two relative equilibria where the masses are separated by a distance q. One of these is geometrically of elliptic type and the other of hyperbolic type. The hyperbolic ones are always unstable, while the elliptic ones are stable when sufficiently close, but unstable when far apart.

On the sphere of positive curvature, if the masses are different, there is a unique relative equilibrium (RE) for every angular separation except $\pi/2$. When the angle is acute, the RE is elliptic, and when it is obtuse the RE can be either elliptic or linearly unstable. We show using a KAM argument that the acute ones are almost always nonlinearly stable. If the masses are equal there are two families of relative equilibria: one where the masses are at equal angles with the axis of rotation ('isosceles RE') and the other when the two masses subtend a right angle at the centre of the sphere. The isosceles RE are elliptic if the angle subtended by the particles is acute and is unstable if it is obtuse. At $\pi/2$, the two families meet and a pitchfork bifurcation takes place. Right-angled RE are elliptic away from the bifurcation point.

In each of the two geometric settings, we use a global reduction to eliminate the group of symmetries and analyse the resulting reduced equations which live on a 5-dimensional phase space and possess one Casimir function.

Introduction

The study of the dynamics of material points and rigid bodies in spaces of constant curvature has been a popular subject of research in the past two decades. For recent advances in this area, see the review [5, 7] and the book by Diacu [11] (and its critique in [7]).

In this paper we focus on the 2-body problem on a (complete and simply connected) two dimensional space of constant non-zero curvature. Our interest in the problem is mathematical, although there is a possible physical motivation in that the background curvature of the universe may be non-zero . However the effect of the curvature would probably be negligible at the length scales involved in isolated 2-body problems. Contrary to the situation in flat space, the system is not equivalent to the corresponding generalisation of the Kepler problem: it is nonintegrable and exhibits chaotic behaviour. This is due to there being no analogue of the centre of mass frame in a curved space, and no Galilean invariance. Recent papers like [4, 7, 8, 13, 15, 26, 27] have considered the reduction by symmetries and some qualitative aspects of the problem. Libration points and choreographies are treated in [8, 15, 22] and the restricted two-body problem is considered in [6, 10].

In the opinion of the authors, there is no coherent, systematic, and complete treatment of the classification and stability of the relative equilibria (RE) of the problem, accessible to the community of celestial mechanics, and the principal aim of this paper is to fill this gap.

Our approach to studying the RE of the problem relies on the use of the explicit form of the reduced equations of motion. This allows us to recover previous results in a systematic and elementary way, and to extend them. Original results of our paper include the classification of RE for arbitrary attractive potentials, a detailed discussion of the qualitative features of the motion and the stability analysis of RE in the case of positive curvature, and the presentation of the energy-momentum bifurcation diagram in the case of negative curvature.

We describe the results of the paper in more detail below, but we begin by recalling the general notion of relative equilibrium, which goes back to Poincaré and applies to any dynamical system with a continuous group of symmetries.

Definition 1. Consider a dynamical system with a symmetry group G. A ***relative equilibrium*** is a trajectory of the dynamical system which coincides with the motion given by the action of a 1-parameter subgroup of the group G.

In other words, a trajectory $\gamma(t)$ is a relative equilibrium if there is a 1-parameter subgroup $g(t)$ of G such that $\gamma(t) = g(t)\cdot\gamma(0)$, and one can show that this is equivalent to the trajectory lying in a single group orbit in the phase space. In particular, relative equilibria correspond to equilibrium points of the reduced equations of motion. Note that if $\gamma(t)$ is a relative equilibrium, then so is $k\cdot\gamma(t)$ (for any $k\in G$), with corresponding 1-parameter subgroup $kg(t)k^{-1}$. In the literature, the term relative equilibrium can refer to the trajectory, to any point on the trajectory, or to the entire group orbit containing the trajectory. More details on relative equilibria in the Hamiltonian context can be found in many places, for example the Lecture Notes by Marsden [18]. The symmetry groups we use in this paper are $SO(3)$, the group of isometries of the sphere, and $SO(2,1)$ (or $SL(2,\mathbb{R})$), the isometries of the Lobachevsky plane (also called the hyperbolic plane), and we describe their 1-parameter subgroups in the relevant section. See also [20, 21] for details about relative equilibria for these groups.

In the problems we consider, the RE are always 'rigid motions'. On the sphere, these consist simply of uniform rotations about a fixed axis. However, in hyperbolic geometry there is more than one type of rigid motion: the so-called elliptic, hyperbolic and parabolic motions. The elliptic motions are periodic, while the others are unbounded; see the discussion below for more details.

Another important consideration in these systems is *time-reversal symmetry*, which holds whenever the Hamiltonian is given by the sum of a quadratic kinetic energy and a potential energy which depends only on the configuration. In canonical coordinates, this symmetry is given by the map on phase space $(q,p)\mapsto(q,-p)$. If $(q(t),p(t))$ is a trajectory of the system, then its 'reversal' $(q(-t),-p(-t))$ is another trajectory of the system. In particular, if such a trajectory is a relative equilibrium then so is its reversal.

In the statements of existence of relative equilibria, we do not distinguish between 2 trajectories that either lie in the same group orbit, or are related by time reversal. Thus we count RE 'up to all symmetries', including time reversal and exchange of the bodies when the masses are equal.

Reduction

As mentioned above, the reduction of the problem has been considered before [4, 8]. We have nevertheless included a self-contained presentation of the reduction for completeness.

For both the positive and negative curvature cases, the unreduced system is a four degree of freedom symplectic Hamiltonian system, the symmetry group is three dimensional and acts freely and properly. The reduced system is a five dimensional Poisson Hamiltonian system, whose generic symplectic leaves are the four dimensional level sets of a Casimir function.

We first deal with the case of positive curvature and consider the problem on the 2-dimensional sphere S^2. We perform the reduction of the problem by the action of $SO(3)$ that simultaneously rotates both masses. In our treatment we do not consider collisions nor antipodal configurations, which allows us to introduce a moving coordinate frame whose axes are aligned according to the configuration of the masses in a convenient way. The Hamiltonian of the system may then be written in terms of the angle q subtended by the masses at the centre of the sphere, its conjugate momentum p, and the vector of angular momentum $\boldsymbol{m}$ written in the moving frame. These quantities do not depend on the orientation of the fixed frame and may therefore be used as coordinates on the reduced space. This approach is inspired by the reduction of the free rigid body problem and, just as happens for that problem, the Euclidean squared norm of $\boldsymbol{m}$ passes down to the quotient space as a Casimir function whose level sets are the symplectic leaves of the reduced space.

We apply an analogous reduction scheme in the case of negative curvature by considering the action of $SO(2,1)$ (or equivalently of $SL(2,\mathbb{R})$) on the Lobachevsky plane L^2. This time, the Casimir function C on the reduced space is the squared norm of the momentum vector $\boldsymbol{m}$ with respect to the Minkowski metric.

Classification and stability of relative equilibria

After finding the reduced equations, in the final two sections we proceed to classify the RE of the problem by finding all the equilibria of the reduced equations. In this way we recover the results of [1, 12, 13] in a systematic and elementary fashion. Moreover, with this approach, we are able to conveniently analyse their stability. We now summarize these results.

The case of negative curvature

In hyperbolic geometry, there are 3 types of non-trivial isometry, known as elliptic, hyperbolic and parabolic transformations and correspondingly 3 types of 1-parameter subgroup of $SO(2,1)$. The elliptic transformations are characterised by having precisely one fixed point in the hyperbolic plane. The hyperbolic and parabolic transformations have no fixed points, but have respectively two and one fixed points 'at infinity'; see any text on hyperbolic geometry, for example [14].

As is known [12, 13], for the 2-body problem on the Lobachevsky plane L^2, there are two families of RE, known as the ***hyperbolic*** and ***elliptic*** families, according to the type of their 1-parameter subgroup. The former are unbounded solutions that do not have an analog in Euclidean space, while the latter are periodic solutions analogous to the RE in Euclidean space. This classification becomes very transparent in our treatment: hyperbolic RE correspond to equilibria of the reduced Hamiltonian system restricted to negative values of the Casimir function C, whereas elliptic RE are those for positive values of C (see Section 2.3 below). It is also known that parabolic transformations do not give rise to RE of the problem [12, 13]. In our treatment, this corresponds to the absence of equilibria of the reduced system when $C = 0$.

The (nonlinear) stability properties of the RE described above was first established in [13] by working on a symplectic slice for the unreduced system since the reduced equations were not available (in a journal in English!). With the reduced system at hand, we are able to recover these results in elementary terms by directly analysing the signature of the Hessian of the reduced Hamiltonian at the RE. The hyperbolic RE are always unstable, whereas the elliptic RE are stable if the masses are sufficiently close. However, as the distance between them grows, the family undergoes a saddle-node bifurcation and the elliptic RE become unstable.

All of the information of the RE of the system is illustrated in the Energy-Momentum diagram shown in Fig. 4. This kind of diagram, as well as the underlying topological considerations of the analysis, goes back to Smale and has been been developed in detail for integrable systems by Bolsinov, Borisov and Mamaev [1]. The application of this kind of analysis to nonintegrable systems, such as the one considered in this paper, is not very common.[1]

The case of positive curvature

In this case all RE are periodic solutions in which the masses rotate about an axis that passes through the shortest geodesic that joins them. As indicated first in [8], the classification of RE in this case is more intricate than for negative curvature since it depends on how the masses of the bodies compare to each other:

(i) If the masses are different, there are two disjoint families of RE that we term ***acute*** and ***obtuse***, according to the (constant) value of the angle between the masses during the motion. For acute RE, it is the heavier mass that is closer to the axis of rotation and hence these are a natural generalisation of the RE of the problem in Euclidean space. On the other hand, for obtuse RE it is the lighter mass which is closer to the axis of rotation, and these RE do not have an analog in Euclidean (or hyperbolic) space.

[1] We mention only the paper [2] on the Conley index where new isosceles vortex configurations were found and their stability was established using topological methods.

(ii) If the masses are equal, there are two families of RE. The family of ***isosceles*** RE are those for which the axis of rotation bisects the arc that joins the two masses, while in the family of ***right angled*** RE the angle between the masses is $\pi/2$ and the axis of rotation is located anywhere between them. These two families meet when the axis of rotation subtends an angle of $\pi/4$ with each mass, and a pitchfork bifurcation of the RE of the system takes place.

As for the case of negative curvature, we compute the signature of the Hessian of the reduced Hamiltonian at the RE in an attempt to establish stability results. Via this analysis it is possible to conclude the instability of certain RE of the system. However, contrary to the case of negative curvature, this is insufficient to prove any kind of nonlinear stability results of the RE since the Hessian matrix is not definite and hence the reduced Hamiltonian may not be used as a Lyapunov function of the system. This surprising feature of the problem was also found in [25] by working on a symplectic slice of the unreduced system.

In view of the above considerations, we take an analytical approach to the study of the nonlinear stability of certain RE of the problem. By using Birkhoff normal forms and applying KAM theory, we are able to show that, if the masses are different, the generic acute RE of the problem are stable.

Stability and reduced stability

Since we study here the reduced equations of motion, the stability we consider is ***reduced stability*** (whether nonlinear or linear), and it is important to appreciate the relationship between this reduced stability and the stability in the full unreduced equations of motion. This involves the 'geometry' of the momentum value for a given relative equilibrium.

Many of the relative equilibria are periodic motions, and for such motion there is a well-known concept of ***orbital stability***, where there is a tubular neighbourhood of the orbit such that any solution that intersects that neighbourhood stays close to the orbit.

If $\boldsymbol{M}$ is the momentum value at (any point of) the relative equilibrium, then there is a subgroup of the symmetry group G which fixes that value, which we denote $G_{\boldsymbol{M}}$. If this subgroup is isomorphic to $SO(2)$ (which it is in all the RE that enjoy reduced stability) then nonlinear reduced stability implies orbital stability. For a discussion concerning stability of RE in the Lobachevski plane, see for example [21].

Outline of the paper

The reduction of the problem on S^2 and L^2 is respectively presented in Sections 1 and 2. The case of positive curvature is presented first since the geometry is better known. We then proceed to classify the RE of the problem and study their stability. We first deal with the case of negative curvature in Section 3 and then with the case of positive curvature in Section 4. We have chosen to present first the negative curvature results since, as was discussed above, the analysis is more straightforward. Finally, some related open problems are described at the end.

1 Reduction for the sphere S^2

The 2-body problem on S^2 concerns the motion of two masses μ_1 and μ_2 on the unit sphere on $\mathbb{R}^3$, that are subject to the attractive force that only depends on the distance between the particles. The accepted generalization of the inverse squared law from the planar case to the problem on the sphere is defined by the potential

$$U_{\text{grav}}(q) = -\frac{G\mu_1\mu_2}{\tan q}, \tag{1.1}$$

where $q \in (0,\pi)$ is the angle subtended by the masses (their Riemannian distance) and $G > 0$ is the 'gravitational' constant. This form of the potential leads to Bertrand's property in Kepler's problem and to a natural generalization of Kepler's first law [9, 16].

Note that the potential U_{grav} is singular at configurations where $q = 0$ and $q = \pi$. Namely, at collisions and antipodal positions of the particles. In this paper we consider more general attractive, q-dependent potentials $U : (0,\pi) \to \mathbb{R}$, that have the same qualitative properties as U_{grav}:

$$U'(q) > 0, \qquad \lim_{q\to 0^+} U(q) = -\infty, \qquad \lim_{q\to \pi^-} U(q) = \infty.$$

The configuration space for the system is

$$Q = (S^2 \times S^2) \setminus \Delta, \tag{1.2}$$

where Δ is the set of collisions and antipodal configurations. Q is a four dimensional manifold which is an open dense subset of $S^2 \times S^2$. The momentum phase space of the system is the eight-dimensional manifold T^*Q. The dynamics of the system is Hamiltonian with respect to the canonical symplectic structure on T^*Q, and the Hamiltonian $H : T^*Q \to \mathbb{R}$ given by $H = T + U$ where T is the sum of the kinetic energy of the particles.

The system is clearly invariant under the (cotangent lift) of the action of $SO(3)$ that simultaneously rotates both particles; this action is free and consequently the reduced orbit space $T^*Q/SO(3)$ is a manifold.

Theorem 1.1. *For Q given in (1.2), the reduced space $T^*Q/SO(3)$ is isomorphic as a Poisson manifold to $\mathbb{R}^3 \times (0,\pi) \times \mathbb{R} \ni (\boldsymbol{m}, q, p)$. The Poisson structure on this space is defined by the relations*

$$\{m_x, m_y\} = -m_z, \quad \{m_y, m_z\} = -m_x, \quad \{m_z, m_x\} = -m_y, \quad \{q, p\} = 1, \tag{1.3}$$

where $\boldsymbol{m} = (m_x, m_y, m_z)^T$.

For the 2-body problem on the sphere with interaction governed by potential energy $U(q)$, the reduced dynamics is Hamiltonian with respect to this Poisson structure with Hamiltonian

$$H(\boldsymbol{m}, q, p) = \frac{1}{2\mu_1}\left((\boldsymbol{m}, \mathbf{A}(q)\boldsymbol{m}) + 2m_x p + (1+\mu)p^2\right) + U(q), \tag{1.4}$$

where $(\cdot,\cdot)$ is the Euclidean scalar product in $\mathbb{R}^3$ and

$$\mathbf{A}(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \dfrac{\cos q}{\sin q} \\ 0 & \dfrac{\cos q}{\sin q} & \dfrac{(\mu + \cos^2 q)}{\sin^2 q} \end{pmatrix}, \qquad \mu = \frac{\mu_1}{\mu_2}. \tag{1.5}$$

As a consequence of the above theorem, the reduced equations of motion take the form

$$\dot{\boldsymbol{m}} = \boldsymbol{m} \times \frac{\partial H}{\partial \boldsymbol{m}}, \quad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \tag{1.6}$$

with H given by (1.2). Note that the Poisson bracket (1.3) is degenerate and possesses the Casimir function

$$C(\boldsymbol{m}) = (\boldsymbol{m}, \boldsymbol{m}) = m_x^2 + m_y^2 + m_z^2, \tag{1.7}$$

which is a first integral of (1.6).

The theorem is proved in the following 2 subsections.

1.1 Group parametrization of configurations

Let $OXYZ$ be a fixed coordinate system on $\mathbb{R}^3$ and let $\boldsymbol{R}_\alpha = (X_\alpha, Y_\alpha, Z_\alpha)^T$ be the Cartesian coordinates of the point mass μ_α, $\alpha = 1, 2$. The key idea behind the proof of Theorem 1.1 is the introduction of a moving orthogonal coordinate system $Oxyz$ determined by the following two conditions (see Fig. 1):

(i) the axis Oz passes through the mass μ_1,

(ii) the mass μ_2 is contained in the plane Oyz with coordinate $y > 0$.

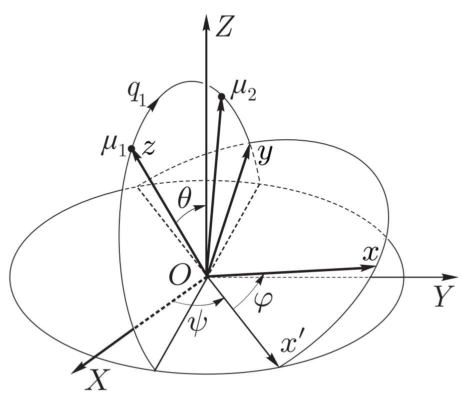


Figure 1: Euler angles for the 2-body configuration on S^2.

According to our convention, the coordinates of the masses on the moving frame are given by the vectors

$$\boldsymbol{r}_1 = (0,0,1)^T, \qquad \boldsymbol{r}_2 = (0, \sin q, \cos q)^T. \tag{1.8}$$

Note that associated to any configuration, we can determine the angle $q \in (0, \pi)$ subtended by the masses, and an element in $SO(3)$ via conditions (i) and (ii). This process may be inverted and shows that as a manifold

$$Q = SO(3) \times (0, \pi).$$

An element $g \in SO(3)$ changes coordinates from the body frame $Oxyz$ into the space frame. We introduce Euler angles (θ, φ, ψ) in $SO(3)$ according to the convention $g = R_\psi^Z R_\theta^X R_\varphi^Z$ (see Fig. 1). Then $0 < \theta < \pi$, $0 < \varphi < 2\pi$, $0 < \psi < 2\pi$, and

$$g(\theta, \varphi, \psi) = \begin{pmatrix} \cos\varphi\cos\psi - \cos\theta\sin\psi\sin\varphi & -\sin\varphi\cos\psi - \cos\theta\sin\psi\cos\varphi & \sin\theta\sin\psi \\ \cos\varphi\sin\psi + \cos\theta\cos\psi\sin\varphi & -\sin\varphi\sin\psi + \cos\theta\cos\psi\cos\varphi & -\sin\theta\cos\psi \\ \sin\theta\sin\varphi & \sin\theta\cos\varphi & \cos\theta \end{pmatrix}. \tag{1.9}$$

Therefore we may use (θ,φ,ψ,q) as generalized coordinates for Q. In particular, the position of the masses in the fixed frame in terms of our generalized coordinates are $\boldsymbol{R}_1 = g\boldsymbol{r}_1$ and $\boldsymbol{R}_2 = g\boldsymbol{r}_2$.

Using these expressions one can obtain an explicit expression for the Lagrangian of the system $L = T - U$, where the kinetic energy

$$T = \frac{1}{2}\left(\mu_1||\dot{\boldsymbol{R}}_1||^2 + \mu_2||\dot{\boldsymbol{R}}_2||^2\right), \tag{1.10}$$

and the potential energy $U = U(q)$. To exploit the symmetries we write T in terms of the (left invariant) body frame angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$, defined by $\widehat{\boldsymbol{\omega}} = g^{-1}\dot{g}$ where, as usual, $\widehat{\boldsymbol{\omega}}$ is the matrix

$$\widehat{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}. \tag{1.11}$$

Performing the algebra one finds

$$\omega_x = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, \quad \omega_y = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi, \quad \omega_z = \dot{\varphi} + \dot{\psi}\cos\theta, \tag{1.12}$$

and the following expression for the kinetic energy

$$T = \frac{\mu_1}{2}(\mathbf{A}(q)^{-1}\boldsymbol{\omega},\boldsymbol{\omega}) + \frac{\mu_2}{2}(\omega_x - \dot{q})^2, \tag{1.13}$$

where $\mathbf{A}(q)$ is given by (1.5). The independence of the above expression on the Euler angles is due to the $SO(3)$ invariance of the system (the invariance under rotations of the space frame).

1.2 Generalized momenta and reduction

We define the generalized momenta of the system in the standard way:

$$P_\theta = \frac{\partial L}{\partial\dot{\theta}}, \qquad P_\varphi = \frac{\partial L}{\partial\dot{\varphi}}, \qquad P_\psi = \frac{\partial L}{\partial\dot{\psi}}, \qquad p = \frac{\partial L}{\partial\dot{q}}, \tag{1.14}$$

where $L = T - U$. The canonical Poisson structure on T^*Q is determined by the canonical relations

$$\{\theta,P_\theta\} = 1, \qquad \{\varphi,P_\varphi\} = 1, \qquad \{\psi,P_\psi\} = 1, \qquad \{q,p\} = 1, \tag{1.15}$$

with all other brackets equal to zero.

To perform the reduction we introduce the body frame representation of the angular momentum $\boldsymbol{m} := \frac{\partial T}{\partial\boldsymbol{\omega}}$. Its expression in terms of the canonical coordinates is

$$\boldsymbol{m} = \frac{1}{\sin\theta}\begin{pmatrix} \sin\varphi\,(P_\psi - P_\varphi\cos\theta) + P_\theta\sin\theta\cos\varphi \\ \cos\varphi\,(P_\psi - P_\varphi\cos\theta) - P_\theta\sin\theta\sin\varphi \\ \sin\theta P_\varphi, \end{pmatrix}, \tag{1.16}$$

and the kinetic energy may be expressed as

$$T = \frac{1}{2\mu_1}\left((\boldsymbol{m},\mathbf{A}(q)\boldsymbol{m}) + 2m_x p + (1+\mu)p^2\right).$$

This establishes the validity of (1.4). The commutation relations (1.3) are directly obtained from (1.15) and (1.16), and Theorem 1.1 is proved.

Remark 1.2. Geometrically, the reduction process carried above consists of working out the left trivialization of $T^*SO(3)$ to arrive at the decomposition

$$T^*Q = T^*SO(3) \times T^*(0,\pi) = SO(3) \times so(3)^* \times (0,\pi) \times \mathbb{R},$$

and then eliminating the $SO(3)$ component by the symmetries. The resulting bracket is the product of the Lie-Poisson bracket on $so(3)^*$ and the canonical bracket on $T^*(0,\pi) = (0,\pi) \times \mathbb{R}$ as may be recognized in (1.3). A fuller description of this process and the reduction for the N-body problem in S^2 can be found in [7]. $\diamond$

1.3 Conserved quantities and reconstruction

Assume that we are given a solution $(\boldsymbol{m}(t), q(t), p(t))$ to the reduced system (1.6). We now explain how to determine the time dependence of the Euler angles.

Observe that by the rotational invariance of the problem, the angular momentum $\boldsymbol{M}$ of the system written in the fixed axes is constant along the motion:

$$\boldsymbol{M} := \sum_{\alpha=1}^{2} \mu_\alpha \boldsymbol{R}_\alpha \times \dot{\boldsymbol{R}}_\alpha.$$

We choose the fixed frame $OXYZ$ in such a way that $\boldsymbol{M} \parallel OZ$. Hence, $\boldsymbol{M} = (0,0,M_0)^T$ with $M_0 \geq 0$, and using (1.9) we get

$$\boldsymbol{m}(t) = g(t)^{-1}\boldsymbol{M} = M_0 \begin{pmatrix} \sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ \cos\theta \end{pmatrix}. \tag{1.17}$$

Therefore,

$$\cos\theta = \frac{m_z(t)}{M_0}, \quad \tan\varphi = \frac{m_x(t)}{m_y(t)}. \tag{1.18}$$

A quadrature for the evolution of ψ is obtained by using (1.12) to write

$$\dot{\psi} = \frac{\sin\varphi\,\omega_x + \cos\varphi\,\omega_y}{\sin\theta} = \frac{M_0\left(m_x(t)\omega_x(t) + m_y(t)\omega_y(t)\right)}{M_0^2 - m_z^2(t)}, \tag{1.19}$$

where

$$\omega_x = \frac{\partial H}{\partial m_x}, \quad \omega_y = \frac{\partial H}{\partial m_y}.$$

A particular case of reconstruction is from relative equilibria, which correspond to equilibria of the reduced system. Thus, $\boldsymbol{m}, q, p$ are constant, and hence so is $\boldsymbol{\omega}$. Consequently, θ and φ are also constant, and only ψ varies, and it varies uniformly. This shows (as expected) that the relative equilibrium consists of a uniform rotation about the axis containing $\boldsymbol{M}$.

2 Reduction for the Lobachevski plane L^2

We now consider the setting of two particles on the Lobachevsky plane L^2 (also called the pseudo-sphere, or the hyperbolic plane). The reduction procedure is very similar to the spherical setting above, so we give fewer details.

Consider the three-dimensional Minkowski space and attach the fixed coordinate system $OXYZ$ to it. The scalar product is given by

$$\langle \boldsymbol{R}, \boldsymbol{R} \rangle_{\mathrm{K}} = (\boldsymbol{R}, \mathrm{K}\boldsymbol{R}), \quad \mathrm{K} = \mathrm{diag}(1, 1, -1), \tag{2.1}$$

where again $(\cdot, \cdot)$ is the usual Euclidean scalar product. The Lobachevsky plane is defined as the Riemannian manifold that consists of the upper sheet of the hyperboloid

$$\langle \boldsymbol{R}, \boldsymbol{R} \rangle_{\mathrm{K}} = X^2 + Y^2 - Z^2 = -1, \tag{2.2}$$

equipped with the metric that it inherits from the ambient Minkowski space. The Gaussian curvature of L^2 equals -1. A standard expression for the metric is

$$ds^2 = d\theta^2 + \sinh^2\theta \, d\varphi^2,$$

and may be obtained by choosing local coordinates on (2.2) in the form

$$X = \sinh\theta \cos\varphi, \quad Y = \sinh\theta \sin\varphi, \quad Z = \cosh\theta,$$

and then restricting (2.1). The group of orientation preserving isometries of L^2 is $SO(2,1)$ (the set of 3×3 invertible matrices g satisfying $g^T \mathrm{K} g = \mathrm{K}$) acting naturally on $(X, Y, Z)^T \in L^2$.

The setup of the 2-body problem on L^2 is analogous to the one described for S^2. In this case, the generalization of the inverse square law is given by the potential

$$U_{\mathrm{grav}}(q) = -\frac{G\mu_1\mu_2}{\tanh q}, \tag{2.3}$$

where $q \in (0, \infty)$ is the Riemannian distance between the particles. As for the spherical case, this form of the potential leads to Bertrand's property in Kepler's problem and to a natural generalization of Kepler's first law [9, 16]. We consider more general attractive potentials $U = U(q)$ satisfying $U'(q) > 0$ for all $q \in (0, \infty)$ and $U(q) \to -\infty$ as $q \to 0^+$.

The configuration space of the problem is

$$Q = L^2 \times L^2 \setminus \Delta \tag{2.4}$$

where Δ is the collision set. The momentum phase space is the eight-dimensional manifold T^*Q, with Hamiltonian $H = T + U$. The kinetic energy T is the sum of the kinetic energies of the particles. Each of them is obtained as the product of the mass with the norm squared of the velocity vector, where the norm is computed with respect to the Riemannian metric.

The problem is invariant under the action of $SO(2,1)$ which simultaneously "rotates" the particles. In analogy with Theorem 1.1 we have:

Theorem 2.1. *For Q given by (2.4), the reduced space $T^*Q/SO(2,1)$ is isomorphic as a Poisson manifold to $\mathbb{R}^3 \times (0, \infty) \times \mathbb{R} \ni (\boldsymbol{m}, q, p)$. The Poisson structure on this space is defined by the relations,*

$$\{m_x, m_y\} = m_z, \quad \{m_y, m_z\} = -m_x, \quad \{m_z, m_x\} = -m_y, \quad \{q, p\} = 1, \tag{2.5}$$

where $\boldsymbol{m} = (m_x, m_y, m_z)^T$.

For the 2-body problem on the Lobachevsky plane with interaction governed by potential energy $U(q)$*, the reduced dynamics is Hamiltonian with respect to this Poisson structure with Hamiltonian*

$$H(\boldsymbol{m}, q, p) = \frac{1}{2\mu_1}\left((\boldsymbol{m}, \mathbf{A}(q)\boldsymbol{m}) - 2m_x p + (1+\mu)p^2\right) + U(q), \tag{2.6}$$

where $(\cdot,\cdot)$ *is the Euclidean scalar product and*

$$\mathbf{A}(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\dfrac{\cosh q}{\sinh q} \\ 0 & -\dfrac{\cosh q}{\sinh q} & \dfrac{(\mu + \cosh^2 q)}{\sinh^2 q} \end{pmatrix}, \qquad \mu = \frac{\mu_1}{\mu_2}. \tag{2.7}$$

As a consequence of the theorem, the reduced equations of motion are

$$\dot{\boldsymbol{m}} = (\mathrm{K}\boldsymbol{m}) \times \frac{\partial H}{\partial \boldsymbol{m}}, \quad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{2.8}$$

The Poisson bracket (2.5) has generic rank 4 (its rank drops to 2 when $\boldsymbol{m} = 0$). Its 4-dimensional symplectic leaves are the regular level sets of the Casimir function

$$C(\boldsymbol{m}) = -\langle \boldsymbol{m}, \boldsymbol{m}\rangle_{\mathrm{K}} = -m_x^2 - m_y^2 + m_z^2, \tag{2.9}$$

which is a first integral of the reduced equations of motion (2.8).

The following 2 subsections give a proof of Theorem 2.1.

2.1 Group parametrization of configurations

The proof of Theorem 2.1 proceeds as in the spherical case. We introduce a moving frame $Oxyz$ which is orthogonal in the metric (2.1), such that the point masses have body coordinates

$$\boldsymbol{r}_1 = (0,0,1)^T, \qquad \boldsymbol{r}_2 = (0, \sinh q, \cosh q)^T,$$

where $q \in (0,\infty)$ is the hyperbolic distance between the masses. As for the spherical case, the moving frame is completely determined by the requirement that μ_1 is on the z-axis and μ_2 lies on the positive Oyz plane. Hence, as a manifold, $Q = SO(2,1) \times (0,\infty)$.

We introduce local coordinates for the group $SO(2,1)$ by adapting the Euler angles for $SO(3)$ used in Section 1.1 (see Fig. 2). Consider an element $g \in SO(2,1)$ that relates the moving and the fixed frame as a sequence of 3 rotations $g = R_\psi^Z R_\theta^X R_\varphi^Z$, with the second rotation R_θ^X being hyperbolic. This leads to

$$g(\varphi,\theta,\psi) = \begin{pmatrix} \cos\varphi\cos\psi - \cosh\theta\sin\varphi\sin\psi & -\sin\varphi\cos\psi - \cosh\theta\cos\varphi\sin\psi & -\sinh\theta\sin\psi \\ \cos\varphi\sin\psi + \cosh\theta\sin\varphi\cos\psi & -\sin\varphi\sin\psi + \cosh\theta\cos\varphi\cos\psi & \sinh\theta\cos\psi \\ \sinh\theta\sin\varphi & \sinh\theta\cos\varphi & \cosh\theta \end{pmatrix}, \tag{2.10}$$

with $0 \le \varphi, \psi < 2\pi$, $\theta \in [0,\infty)$. In this way, (θ,φ,ψ,q) are generalized coordinates for the problem. The position of the masses with respect to the fixed coordinate system are given by $\boldsymbol{R}_1 = g\boldsymbol{r}_1$ and $\boldsymbol{R}_2 = g\boldsymbol{r}_2$, and the kinetic energy of the system is

$$T = \frac{1}{2}\left(\mu_1\langle \dot{\boldsymbol{R}}_1, \dot{\boldsymbol{R}}_1\rangle_{\mathrm{K}} + \mu_2\langle \dot{\boldsymbol{R}}_2, \dot{\boldsymbol{R}}_2\rangle_{\mathrm{K}}\right). \tag{2.11}$$

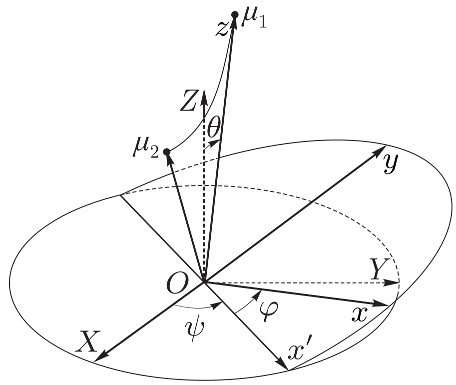


Figure 2: Generalized Euler angles for the 2-body configuration on L^2.

Analogous to (1.12), the left-invariant angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ is defined by $\widetilde{\boldsymbol{\omega}} = g^{-1}\dot{g}$. Here $\widetilde{\boldsymbol{\omega}} := \widehat{\boldsymbol{\omega}}\mathrm{K}$ where, as usual, $\widehat{\boldsymbol{\omega}}$ is given by (1.11). Performing the algebra we find

$$\omega_x = \dot{\theta}\cos\varphi + \dot{\psi}\sinh\theta\sin\varphi, \quad \omega_y = -\dot{\theta}\sin\varphi + \dot{\psi}\sinh\theta\cos\varphi, \quad \omega_z = \dot{\varphi} + \dot{\psi}\cosh\theta. \tag{2.12}$$

The kinetic energy T is written in terms of $(\boldsymbol{\omega}, \dot{q}, q)$ as

$$T = \frac{\mu_1}{2}(\mathbf{A}(q)^{-1}\boldsymbol{\omega}, \boldsymbol{\omega}) + \frac{\mu_2}{2}(\omega_x + \dot{q})^2,$$

with $\mathbf{A}(q)$ given by (2.7).

2.2 Generalized momenta and reduction

The procedure now is exactly as in Section 1.2. After the introduction of the angular momentum vector in the moving frame $\boldsymbol{m} := \frac{\partial T}{\partial \boldsymbol{\omega}}$ and the canonical momenta $(P_\theta, P_\varphi, P_\psi, p)$ that are conjugate to $(\theta, \varphi, \psi, q)$, the relevant formulae to complete the proof of Theorem 2.1 are

$$\boldsymbol{m} = \frac{1}{\sinh\theta}\begin{pmatrix} \sin\varphi\,(P_\psi - P_\varphi\cosh\theta) + P_\theta\sinh\theta\cos\varphi \\ \cos\varphi\,(P_\psi - P_\varphi\cosh\theta) - P_\theta\sinh\theta\sin\varphi \\ \sinh\theta P_\varphi, \end{pmatrix}, \tag{2.13}$$

and

$$T = \frac{1}{2\mu_1}\left((\boldsymbol{m}, \mathbf{A}(q)\boldsymbol{m}) - 2m_x p + (1+\mu)p^2\right).$$

Remark 2.2. The geometric interpretation is also analogous to the spherical case. We have worked out the left trivialization of $T^*SO(2,1)$ to arrive at the decomposition

$$T^*Q = T^*SO(2,1) \times T^*(0,\infty) = SO(2,1) \times so(2,1)^* \times (0,\infty) \times \mathbb{R}.$$

We then eliminated the $SO(2,1)$ component to obtain a bracket on $T^*Q/SO(2,1)$ that is the product of the Lie-Poisson bracket on $so(2,1)^*$ and the canonical bracket on $T^*(0,\infty) = (0,\infty) \times \mathbb{R}$. $\diamond$

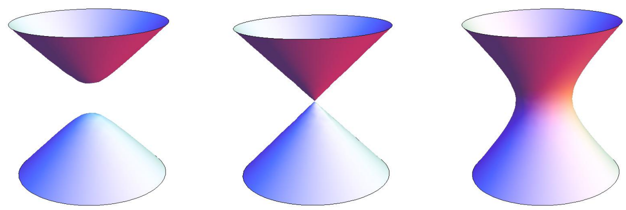


Figure 3: The three possible types of non-zero momentum, from left to right: elliptic, parabolic and hyperbolic, determined by the sign of the Casimir (2.9).

2.3 Conserved quantities and reconstruction

As for the spherical case, we indicate how to determine the time evolution of (θ, φ, ψ) assuming that a solution $(\boldsymbol{m}(t), q(t), p(t))$ to the reduced system (2.8) is known.

By the $SO(2,1)$ invariance of the problem, the angular momentum $\boldsymbol{M}$ of the system written in the fixed axes is constant along the motion. According to our previous definitions we may write

$$\boldsymbol{M} := \sum_{\alpha=1}^{2} \mu_\alpha (\mathrm{K}\boldsymbol{R}_\alpha) \times (\mathrm{K}\dot{\boldsymbol{R}}_\alpha),$$

and we have $\boldsymbol{m}(t) = g(t)^{-1}\boldsymbol{M}$.

In contrast to the spherical setting, here the geometry is more subtle and for example we cannot simply choose $\boldsymbol{M} = (0,0,M_0)^T$ as this is insufficiently general. For a given $\boldsymbol{m}(0)$ the momentum vector $\boldsymbol{m}(t)$ evolves on a surface with different possible geometries depending on the sign of the Casimir function C given by (2.9) . See Fig. 3. If $\boldsymbol{m} \neq 0$ there are three types:

- ***Elliptic momentum*** Here $C(\boldsymbol{m}) > 0$, and the momentum evolves on one sheet of a two-sheeted hyperboloid (the sheets are distinguished by the sign of m_z). For a motion with elliptic momentum, one can choose a frame so that $\boldsymbol{M} = (0,0,M_0)^T$, with $M_0 \in \mathbb{R}$. Since such an $\boldsymbol{M}$ defines a point on the hyperbolic plane L^2, the subgroup of transformations in $SO(2,1)$ that preserve an elliptic momentum fixes that point in L^2 and hence is an elliptic subgroup; it is the group of rotations about $\boldsymbol{M}$ and is isomorphic to $SO(2)$.

- ***Hyperbolic momentum*** Here $C(\boldsymbol{m}) < 0$, and the momentum evolves on a one-sheeted hyperboloid. Here one can choose $\boldsymbol{M} = (0, M_0, 0)^T$ with $M_0 > 0$. The subgroup of transformations in $SO(2,1)$ that preserve a hyperbolic momentum is a hyperbolic subgroup, and is the group of hyperbolic rotations about $\boldsymbol{M}$ which is isomorphic to $\mathbb{R}$.

- ***Parabolic momentum*** Here $C(\boldsymbol{m}) = 0$, and the momentum evolves on one sheet of the cone with the origin removed (again, the two sheets are distinguished by the sign of m_z). Here one can chose $\boldsymbol{M} = \pm(1,0,1)^T$, and the subgroup preserving such a vector is a parabolic subgroup and is also isomorphic to $\mathbb{R}$.

Note that in Minkowski geometry, elliptic, hyperbolic and parabolic vectors are often called timelike, spacelike and lightlike vectors, repectively.

Elliptic momentum. Consider a solution $(\boldsymbol{m}(t), q(t), p(t))$ of the reduced system (2.8) having $m_z(t) > 0$. In this case, by a choice of inertial frame, we can assume $\boldsymbol{M} = (0,0,M_0)^T$ with $M_0 > 0$, and using

(2.10) we get

$$\boldsymbol{m}(t) = g(t)^{-1}\boldsymbol{M} = M_0 \begin{pmatrix} -\sinh\theta\sin\varphi \\ -\sinh\theta\cos\varphi \\ \cosh\theta \end{pmatrix}. \tag{2.14}$$

Therefore,

$$\cosh\theta = \frac{m_z(t)}{M_0}, \quad \tan\varphi = \frac{m_x(t)}{m_y(t)}. \tag{2.15}$$

The quadrature for the evolution of ψ is in this case obtained by using (2.12) to write

$$\dot{\psi} = \frac{\sin\varphi\,\omega_x + \cos\varphi\,\omega_y}{\sinh\theta} = -\frac{M_0\left(m_x(t)\omega_x(t) + m_y(t)\omega_y(t)\right)}{M_0^2 - m_z^2(t)}, \tag{2.16}$$

where

$$\omega_x = \frac{\partial H}{\partial m_x}, \quad \omega_y = \frac{\partial H}{\partial m_y}.$$

At a relative equilibrium of elliptic type, since $\boldsymbol{m}$, q, p are constant, we deduce that φ, θ are constant, while ψ varies uniformly. Thus an elliptic relative equilibrium consists of a uniform rotation about the $\boldsymbol{M}$-axis, and hence is a periodic orbit.

Hyperbolic momentum In this case, by a choice of inertial frame, we can assume $\boldsymbol{M} = (M_0, 0, 0)^T$ with $M_0 > 0$. For the reconstruction it is convenient to use an alternative set of hyperbolic Euler angles defined by writing $g = R^X_\kappa R^Z_\psi R^X_\theta$, where the rotations with respect to κ and θ are hyperbolic. This leads to:

$$g = \begin{pmatrix} \cos\psi & -\sin\psi\cosh\theta & -\sin\psi\sinh\theta \\ \cosh\kappa\sin\psi & \cosh\kappa\cos\psi\cosh\theta + \sinh\kappa\sinh\theta & \cosh\kappa\cos\psi\sinh\theta + \sinh\kappa\cosh\theta \\ \sinh\kappa\sin\psi & \sinh\kappa\cos\psi\cosh\theta + \cosh\kappa\sinh\theta & \sinh\kappa\cos\psi\sinh\theta + \cosh\kappa\cosh\theta \end{pmatrix}, \tag{2.17}$$

in place of (2.10) (note that, unlike (2.10), this representation is not global). Putting $\boldsymbol{m}(t) = g^{-1}(t)\boldsymbol{M}$ with $\boldsymbol{M} = (M_0, 0, 0)$ one finds

$$\boldsymbol{m}(t) = g^{-1}(t)\boldsymbol{M} = M_0 \begin{pmatrix} \cos\psi \\ -\sin\psi\cosh\theta \\ \sin\psi\sinh\theta \end{pmatrix}. \tag{2.18}$$

This choice of Euler angles only allows reconstruction for $|m_x(t)| \le M_0$, but that will suffice for reconstructing the relative equilibria we find for this system. Therefore,

$$\cos\psi = \frac{m_x(t)}{M_0}, \quad \tanh\theta = -\frac{m_z(t)}{m_y(t)}.$$

The quadrature for the evolution of κ is in this case obtained by noting that the angular velocity $\boldsymbol{\omega}$ (following the same construction leading to (2.12)) is given by

$$\boldsymbol{\omega} = \left(\dot{\theta} + \dot{\kappa}\cos\psi,\, \dot{\psi}\sinh\theta - \dot{\kappa}\cosh\theta\sin\psi,\, \dot{\psi}\cosh\theta - \dot{\kappa}\sinh\theta\sin\psi\right),$$

and therefore,

$$\dot{\kappa} = \frac{-\cosh\theta\,\omega_y + \sinh\theta\,\omega_z}{\sin\psi} = \frac{M_0\left(m_y(t)\omega_y(t) + m_z(t)\omega_z(t)\right)}{M_0^2 - m_x^2(t)}, \tag{2.19}$$

where

$$\omega_y = \frac{\partial H}{\partial m_x}, \quad \omega_z = \frac{\partial H}{\partial m_y}.$$

3 Relative equilibria for the 2-body problem on the Lobachevsky plane

Consider two masses μ_1 and μ_2 on the Lobachevsky plane L^2, interacting via an attracting conservative force with potential energy $U(q)$ as introduced in Section 2 where q is the Riemannian distance between the masses and where by attracting we mean $U'(q) > 0$ for all $q \in (0,\infty)$. Recall that the mass ratio is denoted by $\mu = \mu_1/\mu_2$.

3.1 Classification of relative equilibria

The symmetry group of this problem is $SO(2,1)$ (or equivalently $SL(2,\mathbb{R})$) and has three types of non-trivial 1-parameter subgroup (as is well known from hyperbolic geometry), and correspondingly 3 types of relative equilibrium (see Definition 1):

- ***elliptic subgroup***: here the subgroup is compact and isomorphic to $SO(2)$, and hence the motion is periodic; REs of this type can only occur if $C > 0$.
- ***hyperbolic subgroup***: here the subgroup is not compact, consists of semisimple matrices, and is isomorphic simply to $\mathbb{R}$ and hence the motion is unbounded; REs of this type can only occur if $C < 0$.
- ***parabolic subgroup***: this is also non-compact, so unbounded trajectories, but unlike the first two, the elements are not semisimple; and an RE of parabolic type would require $C = 0$ (although we show below there are none for the 2-body system).

The relation between the type of subgroup and the sign of the Casimir C is described in Section 2.3.

Theorem 3.1. *In the 2-body problem on the Lobachevsky plane, governed by an attractive force, for each value of $q > 0$ there are precisely two relative equilibria where the particles are a distance q apart, one of elliptic and one of hyperbolic type.*

Elliptic and hyperbolic RE are illustrated as bifurcation curves on the energy-momentum diagram in Fig. 4. Elliptic RE correspond the curve with a cusp on the half-plane $C > 0$ whereas hyperbolic RE correspond to the smooth bifurcation curve on the half-plane $C < 0$. These curves meet each other at the punctured point when $C = 0$ which corresponds to the non-existence of parabolic RE.

The qualitative properties of the motion along these RE is explained in subsection 3.2. The results of Theorem 3.1 were given before in [12, 13] for the gravitational potential (2.3). Here we extend the classification to arbitrary attractive potentials.

Proof of Theorem 3.1

Relative equilibria correspond to equilibrium points of the reduced system (2.8), so they are solutions of the following set of equations:

$$\frac{\partial H}{\partial p} = 0, \tag{3.1a}$$

$$(\mathrm{K}\boldsymbol{m}) \times \frac{\partial H}{\partial \boldsymbol{m}} = \boldsymbol{0}, \tag{3.1b}$$

$$\frac{\partial H}{\partial q} = 0, \tag{3.1c}$$

where the Hamiltonian H is given by (2.6), the matrix K is defined in (2.1) and $\times$ is the vector product in $\mathbb{R}^3$. The condition (3.1a) yields

$$p = \frac{m_x}{1+\mu}. \tag{3.2}$$

Substituting this expression into the last two components of (3.1b) yields two possibilities:

1) $m_y = m_z = 0$. In this case (3.1c) implies $U'(q) = 0$, and there is no solution for q by our assumption that U is attractive.

2) $m_x = 0$. We analyse this case in what follows assuming that m_y and m_z do not vanish simultaneously, since otherwise we are back in case (i).

Elliptic relative equilibria. Recall that we are considering the case $m_x = 0$. We parametrise the open region of the reduced phase space having $C(\boldsymbol{m}) > 0$ with the parameters $M \neq 0$ and $\alpha \in \mathbb{R}$, by putting

$$m_y = M\sinh\alpha, \qquad m_z = M\cosh\alpha. \tag{3.3}$$

Then $C(\boldsymbol{m}) = M^2$ and (3.1b) is satisfied provided that

$$\sinh 2(q-\alpha) = \mu \sinh 2\alpha. \tag{3.4}$$

Equation (3.4) admits the unique solution for α

$$\alpha = \frac{q}{2} + \frac{1}{4}\ln\left(\frac{\mu + e^{2q}}{1+\mu e^{2q}}\right). \tag{3.5}$$

With the above value of α, equation (3.1c) is satisfied provided that M is such that

$$M^2 = \frac{\mu_1 \sinh^3 q\, U'(q)}{\mu\cosh^2\alpha\cosh q + \cosh\alpha\cosh(q-\alpha)}. \tag{3.6}$$

For an attractive potential, the right hand side of this expression is strictly positive. The choice of sign for M corresponds to two solutions related by the time reversing symmetry of the system, and we do not distinguish them in our classification.

Hyperbolic relative equilibria. The analysis is analogous to the above. This time we put

$$m_y = M\cosh\alpha, \qquad m_z = M\sinh\alpha, \tag{3.7}$$

so $C(\boldsymbol{m}) = -M^2 < 0$. Taking into account that $m_x = 0$, then (3.1b) is satisfied provided that (3.4), and hence also (3.5), hold. The condition for M in this case is

$$M^2 = \frac{\mu_1 \sinh^3 q U'(q)}{\mu\sinh^2\alpha\cosh q - \sinh\alpha\sinh(q-\alpha)}. \tag{3.8}$$

Using (3.4) to eliminate μ, one can write the denominator of the above expression as

$$\frac{\sinh^2\alpha\sinh(q-\alpha)}{\sinh 2\alpha}\big(2\sinh^2(q-\alpha)\cosh\alpha + \sinh\alpha\sinh(2(q-\alpha))\big).$$

Considering that (3.5) implies $0 < \alpha < q$, it is immediate to check that all of the terms on the right hand side of this expression are positive. Therefore, the right hand side of (3.8) is also positive and there is a unique solution for M (modulo the time reversibility symmetry of the problem).

Parabolic relative equilibria. Finally we show that there are no solutions of (3.1b) having $C(\boldsymbol{m}) = 0$. Substituting $m_x = 0$ and $m_y = \pm m_z$ into the first component of (3.1b) yields, after a simple calculation,

$$\cosh 2q \pm \sinh 2q + \mu = 0,$$

which clearly has no solutions for q since $\mu > 0$.

3.2 Reconstruction

Here we relate the (relative) equilibria described above in the reduced space to the corresponding motion in the original unreduced space. The qualitative properties of these unreduced RE depend on the sign that C takes along them.

(i) Fix $q > 0$ and consider the corresponding elliptic RE. From the discussion above, the angular momentum $\boldsymbol{m}$ satisfies $m_x = 0$ and m_y and m_z are given by (3.3) where $\alpha > 0$ is determined by (3.5) and M by (3.6). We use the generalised Euler angle convention given by (2.10) for the reconstruction. The equations (2.14) are satisfied for the constant value of the angles $\varphi = \pi$ and $\theta = \alpha$ (the other possibility, namely that $\varphi = 0$ and $\theta = -\alpha$, leads to a solution in the same group orbit of the RE - see the discussion after Definition 1). The angle $\psi = \omega t$ with constant $\omega := \dot\psi$ given by (2.16). Using (3.6) and (3.4) it is possible to simplify

$$\omega^2 = \zeta^{-1} U'(q),$$

where $\zeta = \frac{\mu_1 \sinh 2\alpha}{2} = \frac{\mu_2 \sinh 2(q-\alpha)}{2}$. According to the conventions of section 2.1, the position of the particles in the fixed frame along the RE is

$$\boldsymbol{R}_1(t) = \begin{pmatrix} -\sinh\alpha \sin\omega t \\ \sinh\alpha\cos\omega t \\ \cosh\alpha \end{pmatrix}, \qquad \boldsymbol{R}_2(t) = \begin{pmatrix} \sinh(q-\alpha)\sin\omega t \\ -\sinh(q-\alpha)\cos\omega t \\ \cosh(q-\alpha) \end{pmatrix}.$$

The motion is then periodic with constant angular speed ω. We note that throughout the motion, the particle μ_1 (respectively μ_2) has constant distance α (respectively $q-\alpha$) to the point with space coordinates $(0,0,1)$. This point corresponds to the *centre of mass* as defined in [13]. Using (3.5) one may check that this point is closer to the heavier mass. Hence, elliptic RE in L^2 generalise the RE in flat space where the centrifugal forces are balanced by gravitational attraction.

In the next section we show that these motions are stable if $0 < q < q^*$ and unstable if $q > q^*$. The critical distance q^* depends on the mass ratio μ as indicated in the statement of Theorem 3.3 below.

(ii) Now consider the hyperbolic RE corresponding to $q > 0$. The angular momentum $\boldsymbol{m}$ satisfies $m_x = 0$ and m_y and m_z are given by (3.7) where $\alpha > 0$ is determined by (3.5) and M by (3.8). This time we use the generalised Euler angle convention given by (2.17) for the reconstruction. Equation (2.18) implies that ψ and θ have the constant values $\theta = -\alpha$, $\psi = 3\pi/2$ along the motion. On the other hand $\kappa = \omega t$ where $\omega := \dot\kappa$ is defined by (2.19). Using (3.8) and (3.4), one may simplify

$$\omega^2 = \zeta^{-1} U'(q),$$

where, as before, $\zeta = \frac{\mu_1 \sinh 2\alpha}{2} = \frac{\mu_2 \sinh 2(q-\alpha)}{2}$. This time, the position of the particles in the fixed frame along the RE is

$$\boldsymbol{R}_1(t) = \begin{pmatrix} \sinh\alpha \\ \cosh\alpha \sinh\omega t \\ \cosh\alpha\cosh\omega t \end{pmatrix}, \qquad \boldsymbol{R}_2(t) = \begin{pmatrix} \sinh(q+\alpha) \\ \cosh(q+\alpha)\sinh\omega t \\ \cosh(q+\alpha)\cosh\omega t \end{pmatrix}.$$

Note that the motion along these RE is unbounded. This type of relative equilibrium does not exist in the positive or zero curvature case. As explained in [13] their existence is due to the property of the Lovachevsky space that makes parallel geodesics "separate". This separating effect is balanced by the attractive forces in a very delicate manner. We show below that these RE are all unstable.

3.3 Stability analysis of the relative equilibria on the Lobachevsky plane

The stability of the relative equilibria depends on the form of the potential $U(q)$. We restrict attention here to the gravitational potential given in (2.3). We assume that $\mu_1 = 1$ and $G\mu_1\mu_2 = 1$,[2] so the only external parameter left in the problem is the mass ratio μ.

Theorem 3.2. *Of the relative equilibria described in Theorem 3.1 above, and assuming a potential of the form (2.3):*

(i) All hyperbolic RE are (linearly) unstable.

(ii) There is a critical distance $q^* > 0$ depending on the mass ratio μ such that when the distance between the particles is less than q^* the elliptic RE is nonlinearly stable, and when they are more than q^* apart it is unstable.
The critical value q^* is given by

$$q^* = \alpha^* + \frac{1}{2}\operatorname{arcsinh}(\mu(\sinh 2\alpha^*))$$

where α^* is the unique positive solution of the equation

$$2\sinh^2\alpha + 1 = 2\sinh^2\alpha\sqrt{1+\mu^2\sinh^2 2\alpha}.$$

We prove this theorem[3] by analyzing the signature of the Hessian matrix of the Hamiltonian function restricted to to the symplectic leaves. According to Lyapunov's Theorem, if a RE is a local maximum or minimum of the Hamiltonian, then it is nonlinearly stable. This happens in particular if the corresponding Hessian matrix at the equilibrium is positive or negative definite. On the other hand, if the Hessian matrix has an odd number of negative eigenvalues, then the linearised system has at least one eigenvalue with positive real part and the corresponding RE is unstable. The proof of Theorem 3.2 follows at once from these observations and the following proposition.

Proposition 3.3. *Let q^* defined as in the statement of Theorem 3.2. The Hessian matrix of the reduced Hamiltonian (restricted to the corresponding symplectic leaf) at the RE of the problem described in Theorem 3.1, has the following signature:*

(i) All hyperbolic RE have signature $(+++-)$.

(ii) Elliptic RE with $0 < q < q^*$ have signature $(++++)$.

(iii) Elliptic RE with $q^* < q$ have signature $(+++-)$.

Moreover, the Casimir function C restricted to the branch of elliptic RE has a maximum at $q = q^*$.

[2] This assumption is done without loss of generality since one may eliminate these quantities from the equations of motion (2.8) by rescaling time $t \to \frac{\sqrt{\kappa}t}{\mu_1}$, and the momenta $p \to \frac{p}{\sqrt{\kappa}}$, $\boldsymbol{m} \to \frac{\boldsymbol{m}}{\sqrt{\kappa}}$, where $\kappa = \mu_1\sqrt{G\mu_2}$.

[3] An equivalent proof of Theorem 3.2 was given before in [13] by working on a symplectic slice in the unreduced system, since the reduced equations of motion were not known at that time. Although equivalent, the approach that we follow in this paper is elementary.

The results described above are illustrated in the energy-momentum diagram Fig. 4 where we have indicated the signature of the Hessian along the branches of RE. The critical value q^* at which the elliptic RE undergo a saddle-node bifurcation corresponds to the cusp where C has a maximum. Fig. 5 shows the stability region on the q-μ plane for elliptic RE. Such region is delimited by the curve $q = q^*(\mu)$.

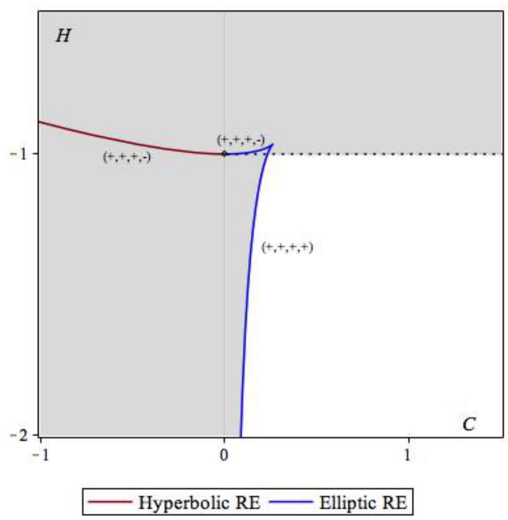


Figure 4: Energy-Momentum bifurcation diagram of relative equilibria on L^2 for the gravitational potential with $\mu = 1/2$. The shaded area on the C-H plane shows all possible values of (C, H). We also indicate the signature of the Hessian matrix of the Hamiltonian along each branch of relative equilibria. Notice the change in signature at the cusp of the elliptic relative equilibria where a saddle-node bifurcation takes place.

Proof of Proposition 3.3

(i): Fix $M \neq 0$ and consider the restriction of the reduced system to the symplectic leaf $\mathcal{M}_{-M^2}$ defined by $C(\boldsymbol{m}, \boldsymbol{m}) = -M^2$. Denote by H_{-M^2} the restriction of the reduced Hamiltonian (2.6) to $\mathcal{M}_{-M^2}$. The RE with $C = -M^2$ correspond to critical points of H_{-M^2}.

By the time-reversibility of the problem it is sufficient to consider the stability of hyperbolic RE for which $m_y > 0$. We introduce coordinates[4] (α, q, z, p) on $\mathcal{M}_{-M^2}$ by setting:

$$m_x = z, \quad m_y = \sqrt{M^2 - z^2}\cosh\alpha, \quad m_z = \sqrt{M^2 - z^2}\sinh\alpha.$$

In view of (2.6) and (2.3), we have

$$H_{-M^2}(\alpha, q, z, p) = \frac{1}{2}\Bigg[(1+\mu)p^2 - 2pz + \frac{\cosh 2q - \cosh 2(q-\alpha) - 2\mu\sinh^2\alpha}{2\sinh^2 q}z^2 - \frac{1}{\sinh^2 q}\Big(\sinh 2q - \frac{M^2}{2}\big(\cosh(2(q-\alpha)) - 1 + \mu(\cosh 2\alpha - 1)\big)\Big)\Bigg].$$

[4] In fact (α, q, z, p) as defined are Darboux coordinates that generalize the Andoyer variables on $T^*SO(3)$.

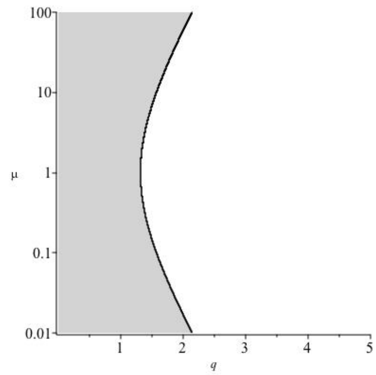


Figure 5: The shaded region corresponds to stable elliptic RE on the Lobachevsky plane, as a function of the distance q (horizontally) and mass ratio $\mu = \mu_1/\mu_2$ (vertically). The curve delimiting the two regions is $q = q^*(\mu)$ as defined in the statement of Theorem 3.2. Note that the vertical μ-axis on the figure above has a logarithmic scale.

From the proof of Theorem 3.1, we know that the critical points of H_{-M^2} occur at points where $z = p = 0$, and q and α are such that (3.4) and (3.8) hold. With our assumptions on the potential, (3.8) simplifies to

$$M^2 = \frac{\sinh q}{\sinh\alpha(\mu\sinh\alpha\cosh q - \sinh(q-\alpha)}. \tag{3.9}$$

The Hessian matrix of H_{-M^2} at such points is given by

$$D^2 H_{-M^2}(\alpha, q, 0, 0) =: \mathbf{N} = \begin{pmatrix} \mathbf{N}^{(1)} & 0 \\ 0 & \mathbf{N}^{(2)} \end{pmatrix},$$

where $\mathbf{N}^{(1)}$ and $\mathbf{N}^{(2)}$ are symmetric 2×2 matrices. The entries of $\mathbf{N}^{(1)}$ may be written as

$$\begin{aligned} \mathbf{N}^{(1)}_{11} &= \frac{M^2(\cosh(2(q-\alpha)) + \mu\cosh 2\alpha)}{\sinh^2 q}, \qquad \mathbf{N}^{(1)}_{22} = \frac{M^2(1+\mu)\sinh^2\alpha}{\sinh^4 q}, \\ \mathbf{N}^{(1)}_{12} &= \mathbf{N}^{(1)}_{21} = -\frac{M^2(-\sinh q\cosh 2\alpha + (1+\mu)\cosh q\sinh 2\alpha)}{\sinh^3 q}, \end{aligned} \tag{3.10}$$

where we have used (3.9) to simplify $\mathbf{N}^{(1)}_{22}$. On the other hand

$$\mathbf{N}^{(2)} = \begin{pmatrix} \dfrac{-\cosh(2(q-\alpha)) + \cosh 2q - 2\mu\sinh^2\alpha}{\cosh 2q - 1} & -1 \\ -1 & 1+\mu \end{pmatrix}. \tag{3.11}$$

A long but straightforward calculation using (3.4) shows that

$$\det(\mathbf{N}^{(1)}) = -\frac{M^4\sinh^2(q-\alpha)}{\cosh^2\alpha\sinh^4\alpha}\left(4\cosh^2\alpha\cosh^2(q-\alpha) - 1\right),$$

that is clearly negative. Hence $\mathbf{N}^{(1)}$ has one positive and one negative eigenvalue. On the other hand, by a calculation that uses again (3.4) we obtain

$$\det(\mathbf{N}^{(2)}) = \frac{\sinh^2(q-\alpha)}{\cosh^2\alpha} > 0.$$

Considering that $\mathbf{N}^{(2)}_{22} > 0$ we conclude that $\mathbf{N}^{(2)}$ is positive definite. Therefore, the signature of $\mathbf{N}$ is $(+++-)$ as stated.

(ii) and *(iii)*: We proceed in an analogous fashion. Let H_{M^2} be the restriction of the reduced Hamiltonian (2.6) to the symplectic leaf $\mathcal{M}_{M^2}$ defined by $C = M^2$. By the time-reversibility of the problem it is sufficient to consider the stability of hyperbolic RE for which $m_z > 0$. Introduce coordinates[5] (α, q, z, p) on $\mathcal{M}_{M^2}$ by:

$$m_x = z, \quad m_y = \sqrt{M^2 + z^2}\sinh\alpha, \quad m_z = \sqrt{M^2+z^2}\cosh\alpha.$$

Then, in view of (2.6) and (2.3), we have

$$\begin{aligned} H_{M^2}(\alpha, q, z, p) = \frac{1}{2}\Bigg[&(1+\mu)p^2 - 2pz + \frac{\cosh 2q + \cosh 2(q-\alpha) + \mu(1+\cosh 2\alpha)}{2\sinh^2 q} z^2 \\ &- \frac{1}{\sinh^2 q}\left(\sinh 2q - \frac{M^2}{2}\big(1+\cosh(2(q-\alpha)) + \mu(1+\cosh 2\alpha)\big)\right)\Bigg]. \end{aligned}$$

From the proof of Theorem 3.1, we know that the critical points of H_{M^2} occur at points where $z = p = 0$, and q and α are such that (3.4) and (3.6) hold. With our assumptions on the potential, (3.6) simplifies to

$$M^2 = \frac{\sinh q}{\cosh\alpha(\mu\cosh\alpha\cosh q + \cosh(q-\alpha))}. \tag{3.12}$$

The Hessian matrix of H_{M^2} at such points has again block diagonal form

$$D^2 H_{M^2}(\alpha, q, 0, 0) =: \mathbf{L} = \begin{pmatrix} \mathbf{L}^{(1)} & 0 \\ 0 & \mathbf{L}^{(2)} \end{pmatrix}.$$

The elements of the matrices $\mathbf{L}^{(i)}$, $i = 1, 2$, coincide respectively with those of $\mathbf{N}^{(i)}$ given by (3.10) and (3.11) except for the entries $\mathbf{L}^{(1)}_{22}$ and $\mathbf{L}^{(2)}_{11}$ that may be written as

$$\mathbf{L}^{(1)}_{22} = \frac{M^2(1+\mu)\cosh^2\alpha}{\sinh^4 q}, \qquad \mathbf{L}^{(2)}_{11} = \frac{\cosh\alpha(\cosh(2q-\alpha) + \mu\cosh\alpha)}{\sinh^2 q}.$$

The simplification of $\mathbf{L}^{(1)}_{22}$ given above is obtained with the help of (3.12).

We have

$$\det(\mathbf{L}^{(2)}) = \frac{\mu^2\cosh^2\alpha + 2\mu\cosh\alpha\cosh(q-\alpha) + \cosh^2(q-\alpha)}{\sinh^2 q}$$

which is clearly positive. Considering that $\mathbf{L}^{(2)}_{22} > 0$ it follows that $\mathbf{L}^{(2)}$ is positive definite.

Next, given that $\mathbf{L}^{(1)}_{22} > 0$ it follows that $\mathbf{L}^{(1)}$ has one positive eigenvalue and the signature of the other one coincides with the signature of its determinant. On the other hand, using (3.4) to eliminate μ, we can factorize

$$\det(\mathbf{L}^{(1)}) = \frac{4M^4\cosh^2\alpha\cosh^2(q-\alpha)}{\sinh^4 q\sinh^2 2\alpha} f(q, \alpha),$$

with

$$f(q, \alpha) := 1 - 4\sinh^2\alpha\sinh^2(q-\alpha).$$

[5] As before, (α, q, z, p) are Darboux coordinates that generalize the Andoyer variables on $T^*SO(3)$.

The signature of f along the RE can be determined by using (3.4) to write

$$q = q(\alpha) = \alpha + \frac{1}{2}\operatorname{arcsinh}(\mu(\sinh 2\alpha)). \tag{3.13}$$

This expression leads to the identity $4\sinh^2(q-\alpha) = 2\sqrt{1+\mu^2\sinh^2\alpha} - 1$. Setting $u = \sinh^2\alpha$ and using this identity allows one to write

$$f(q(\alpha),\alpha) = \tilde{f}(u) := 2u\left(1 - \sqrt{1+4\mu^2(1+u)u}\right) + 1.$$

It is now elementary to check that $\tilde{f}$ is decreasing for $u > 0$, and satisfies $\tilde{f}(0) = 1$ and $\lim_{u\to\infty}\tilde{f}(u) = -\infty$. Therefore, there exists a unique $u^* > 0$ such that $\tilde{f}(u)$ is positive for $0 < u < u^*$ and negative for $u^* < u$. Considering that u is an increasing function of α, we conclude that $f(q(\alpha),\alpha)$ is positive for $0 < \alpha < \alpha^*$ and negative for $\alpha^* < \alpha$ where α^* satisfies $\tilde{f}(\sinh^2\alpha^*) = 0$. The proof of items *(ii)* and *(iii)* in the proposition is completed by noting that (3.13) defines q as an increasing function of α.

It only remains to show that M^2 restricted to the branch of elliptic RE has a maximum at $q = q^*$. To show this we parametrize the branch by $\alpha > 0$ using (3.13). Differentiating (3.4) implicitly with respect to α leads to

$$\frac{dq}{d\alpha} = \frac{\sinh(2(q-\alpha))\cosh 2\alpha + \cosh(2(q-\alpha))\sinh 2\alpha}{\cosh(2(q-\alpha))\sinh 2\alpha}.$$

Starting from (3.12), using the above expression, and then (3.4) to eliminate μ, one may simplify

$$\frac{dM^2}{d\alpha} = \frac{f(q,\alpha)}{\cosh^2\alpha\cosh^2(q-\alpha)\cosh(2(q-\alpha))}.$$

Therefore, $dM^2/d\alpha$ same signature as $f(q,\alpha)$ (i.e. the same signature as $\det(\mathbf{L}^{(1)})$ and $\det(\mathbf{L})$). The same is true about dM^2/dq since (3.13) defines q as an increasing function of α. □

3.4 Topology of the energy-momentum level surfaces

Denote by $\mathcal{Z} \cong \mathbb{R}^4 \times \mathbb{R}^+$ the reduced space of the system with global coordinates $\boldsymbol{m}, p, q$. In this section we consider the topology of the fibres of the energy-momentum map $(C,H) : \mathcal{Z} \to \mathbb{R}^2$. We continue working with the gravitational potential (2.3) under the assumption that $\mu_1 = 1$ and $G\mu_1\mu_2 = 1$. In particular note that

$$\lim_{q\to\infty} U_{\text{grav}}(q) = -1.$$

We shall prove that the energy-momentum fibre over the point (c_0,h_0) is compact only for $c_0 > 0$ and $h_0 < -1$, and that in this case it is homeomorphic to the disjoint union of two 3-spheres. We start by noticing that the energy-momentum map $(C,H) : \mathcal{Z} \to \mathbb{R}^2$ is not proper. It has the following properties:

Proposition 3.4. *Let $\boldsymbol{\zeta}_n = (\boldsymbol{m}_n, p_n, q_n)$ be a sequence of points in $\mathcal{Z}$ and suppose that $(C,H)(\boldsymbol{\zeta}_n)$ converges to $(c_0,h_0) \in \mathbb{R}^2$. Then*

(i) *If $q_n \to \infty$ then $h_0 \geqslant -1$.*

(ii) *If $q_n \to 0$ then $c_0 \leqslant 0$.*

(iii) If $\varepsilon \leqslant q_n \leqslant \frac{1}{\varepsilon}$ for some $\varepsilon > 0$, then $\boldsymbol{m}_n$ and p_n are bounded.

Proof. (i) Since the kinetic energy is positive, we have $H(\boldsymbol{\zeta}_n) \geqslant U_{\text{grav}}(q_n)$, and the result follows by letting $n \to \infty$.

(ii) Suppose $C(\boldsymbol{m}_n) = -(m_x)_n^2 - (m_y)_n^2 + (m_z)_n^2 \to c_0 > 0$. Then, for n large enough, $(m_z)_n^2 \geqslant \dfrac{c_0}{2}$ and we have

$$H(\boldsymbol{\zeta}_n) \geqslant \frac{c_0\mu}{4\mu_1 \sinh^2 q_n} + U_{\text{grav}}(q_n) = \frac{c_0\mu - 4\mu_1 \cosh q_n \sinh q_n}{4\mu_1 \sinh^2 q_n},$$

that grows without bound as $q_n \to 0$. This contradicts our hypothesis that $H(\boldsymbol{\zeta}_n)$ converges to h_0.

(iii) For $\varepsilon \leqslant q \leqslant \frac{1}{\varepsilon}$ it is possible to bound H from below by a constant, positive definite quadratic form on $\boldsymbol{m}_n$ and p_n with constant coefficients, plus a constant value (the minimum of U for $\varepsilon \leqslant q \leqslant \frac{1}{\varepsilon}$). Hence, the only way in which H can remain bounded if q is bounded is if $\boldsymbol{m}_n$ and p_n are also bounded.

$\square$

Based on these observations we describe the topology of the energy-momentum fibre over the point (c_0, h_0) with $c_0 > 0$.

Theorem 3.5. *Assume $c_0 > 0$. The energy-momentum fibre over the point (c_0, h_0) is:*

- *empty , 2 points, or homeomorphic to the disjoint union of two 3-spheres if $h_0 < -1$,*
- *for $h_0 > -1$ and (c_0, h_0) inside the small cusp region in Fig. 4, it is homeomorphic to the disjoint union of two 3-spheres and two 3-dimensional open balls,*
- *for $h_0 > -1$ and (c_0, h_0) outside the small cusp region in Fig. 4, it is homeomorphic to the disjoint union of two 3-dimensional open balls.*

On the other hand, all of the fibres having $c_0 < 0$ are unbounded but we were unable to determine more information about their topology. The transition from $c_0 > 0$ to $c_0 < 0$ is surprisingly complicated.

Discussion/proof: We refer to Fig. 4. If $h_0 < -1$ and c_0 is sufficiently large then the fibre is empty. Now assume $0 < c_0 < c_*$ where $c_* > 0$ is the value of the Casimir where the saddle-node bifurcation takes place. For $h_0 << -1$ the fibre is empty, and there is a local minimum of H on the level set of C (since the signature of the critical point is $(+,+,+,+)$), and in fact two local minima related by the time-reversal symmetry. In the region where this local minimum is less than -1, the minima are in fact global minima, and the fibre consists of just two isolated points. These critical points are non-degenerate so by elementary Morse theory the h_0-level sets of H on $C^{-1}(c_0)$ is the union of two 3-spheres, for $H_{\min} < h_0 < -1$.

As h_0 becomes larger than -1, there are two open sets arising 'from infinity' (large q). This gives a contribution to the fibre of two open balls. As (c_0, h_0) leaves the small cusp region by increasing h_0, there is a critical point of H of signature $(+,+,+,-)$. This corresponds to the spheres meeting the interior of the open ball and then coalescing, giving rise to just two open balls.

4 Relative equilibria for the 2-body problem on the sphere

Consider two masses μ_1 and μ_2 on the unit sphere, interacting via an attracting conservative force with potential energy $U(q)$, where q is the angular separation of the masses, where by attracting we mean $U'(q) > 0$ for all $q \in (0,\pi)$.

4.1 Classification of relative equilibria

The symmetry group of this problem is $SO(3)$, and every 1-parameter subgroup consists of rotations about a fixed axis. Thus, all of the relative equilibria of the problem for $\omega \neq 0$ are periodic solutions, and the 2 masses simultaneously rotate about a fixed axis of rotation at a steady angular speed ω. Given any configuration with $q \in (0,\pi)$ there is a uniquely defined shortest geodesic between the two masses. Any point lying on this arc is said to lie ***between*** the masses.

Recall from the introduction that in counting relative equilibria, we identify any relative equilibria that merely differ by a symmetry (including time-reversing symmetry).

Theorem 4.1. *In the 2-body problem on the sphere, governed by an attractive force, the set of relative equilibria (RE) depends on whether the masses are equal or not as follows. In every case, the axis of rotation lies between the masses in the sense described above.*

$\mu_2 \neq \mu_1$**:** *For each $q \in (0,\pi)$, $q \neq \pi/2$ there is a unique RE where the masses are separated by an angle q. The axis of rotation subtends an angle $\theta_j \in (0,\pi/2)$ with the mass μ_j (so $q = \theta_1 + \theta_2$) which are related by*

$$\mu_1 \sin(2\theta_1) = \mu_2 \sin(2\theta_2). \tag{4.1}$$

*We call these **acute** and **obtuse RE**, accordingly as $q < \pi/2$ or $q > \pi/2$. There is no RE for $q = \pi/2$. In the acute RE, the larger mass is closer to the axis of rotation, while in the obtuse RE, the smaller mass is closer. See Fig. 6.*

$\mu_2 = \mu_1$**:** *In this case there are two classes of RE, isosceles and right-angled (see Fig. 7):*

- *(i) Given any $q \in (0,\pi)$, $q \neq \pi/2$, there is a unique RE where the masses are separated by an angle q. In this case the axis of rotation passes through the sphere midway between the masses; these we call **isosceles RE**.*
- *(ii) Given any $\theta \in (0,\pi/2)$ there is a unique RE with angular separation $q = \pi/2$, called a **right-angled RE**, where θ is the smaller of the angles between the axis of rotation and the masses.*

Note that when $q = \pi/2$ and $\theta = \pi/4$ these two families meet, giving just one RE.

For all the RE, the speed of rotation ω is given by

$$\omega^2 = \zeta^{-1} U'(q), \tag{4.2}$$

where $\zeta = \frac{1}{2}\mu_1 \sin(2\theta_1) = \frac{1}{2}\mu_2 \sin(2\theta_2)$ and of course $q = \theta_1 + \theta_2$.

Previous existence results for the RE of the problem for the gravitational potential (1.1) were given before in [8], [24]. The theorem above completes their classification for arbitrary attractive potentials.

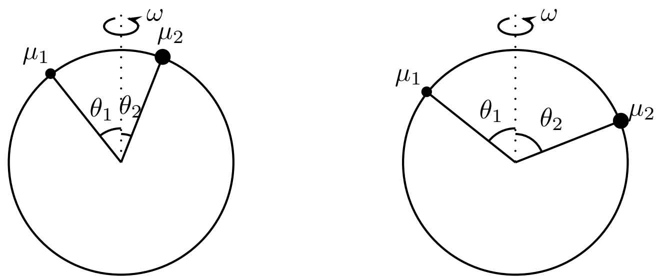


Figure 6: Acute and obtuse relative equilibria for distinct masses (here shown for $\mu = 0.7$, and $q = \pi/3$ and $2\pi/3$ respectively).

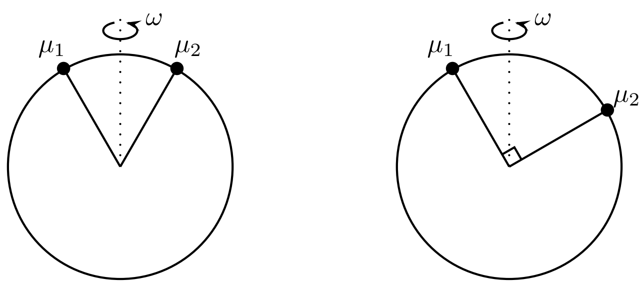


Figure 7: Isosceles and right-angled relative equilibria respectively, for a pair of equal masses.

Proof of Theorem 4.1

Relative equilibria are equilibrium points of the reduced system (1.6) so they correspond to solutions of the following equations:

$$\frac{\partial H}{\partial p} = 0, \tag{4.3a}$$

$$\frac{\partial H}{\partial \boldsymbol{m}} \times \boldsymbol{m} = \boldsymbol{0}, \tag{4.3b}$$

$$\frac{\partial H}{\partial q} = 0, \tag{4.3c}$$

where the reduced Hamiltonian H is given in (1.4).

The condition (4.3a) is equivalent to

$$p = -\frac{m_x}{1+\mu}. \tag{4.4}$$

Substituting (4.4) into the last two components of (4.3b) yields two possibilities:

(i) $m_y = m_z = 0$. In this case (4.3c) implies $U'(q) = 0$. Since the potential is attractive there is no solution of this form.

(ii) $m_x = 0$. We focus on this case in what follows. Note that (4.4) implies that $p = 0$.

Introduce polar coordinates for non-zero points in the m_y–m_z plane by putting

$$m_z = M_0 \cos\alpha, \qquad m_y = M_0 \sin\alpha, \tag{4.5}$$

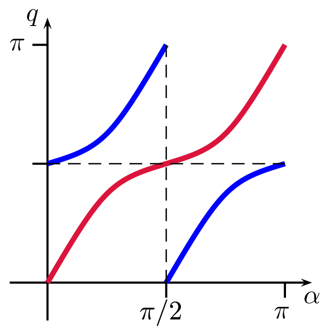


Figure 8: The relation between q and α given by (4.6a) for $\mu \neq 1$ (here $\mu = 0.7$). The red curve represents $q = q_-(\alpha)$ while the blue one represents $q = q_+(\alpha)$.

for $\alpha \in [0, 2\pi)$ and $M_0 > 0$ (we interpret α in terms of the configuration geometry below). The first component of (4.3b) and (4.3c) may be rewritten in these coordinates as:

$$\mu \sin(2\alpha) - \sin(2(q-\alpha)) = 0, \tag{4.6a}$$

$$M_0^2 = \frac{\mu_1 \sin^3 q\, U'(q)}{F_\mu(q,\alpha)}. \tag{4.6b}$$

where

$$F_\mu(q,\alpha) := \cos\alpha(\mu\cos q \cos\alpha + \cos(q-\alpha)). \tag{4.7}$$

It follows from (4.6b) that M_0 has a real value if and only if $F_\mu(q,\alpha) > 0$. Note also that if (q, α, M_0) is a solution to (4.6), then so is $(q, \alpha+\pi, M_0)$. This corresponds to changing the sign of $\boldsymbol{m}$, which is the time-reversing symmetry described in the introduction. Since we do not count such pairs of solutions separately, we restrict attention from now on to $\alpha \in [0,\pi)$.

The analysis of the solutions of (4.6) depends on whether or not the masses are equal.

Equal masses: In this case $\mu = 1$ and it follows from (4.6a) that either $q = \pi/2$ or $q = 2\alpha \pmod{\pi}$.

Firstly, if $q = \pi/2$ then $F_1(\pi/2, \alpha) = \cos\alpha \sin\alpha$ and this is strictly positive if and only if $\alpha \in (0, \pi/2)$.

Now suppose $q = 2\alpha \pmod{\pi}$. Then again, $F_1(q,\alpha) > 0$ if and only if $\alpha \in (0,\pi/2)$; in particular, if $\alpha > \pi/2$ then $q = 2\alpha - \pi$ and $F_1(q,\alpha) = -\cos^2\alpha(\cos 2\alpha + 1) < 0$.

For equal masses, the equations therefore have a solution if and only if $\alpha \in (0,\pi/2)$, with either $q = \pi/2$ or $q = 2\alpha$. We return to this after the analogous discussion for distinct masses.

Distinct masses: Without loss of generality we suppose $0 < \mu < 1$. The relation between q and α from (4.6a) is shown in Fig. 8. For each value of $\alpha \in (0,\pi)$ it is clear that there are precisely two values of $q \in [0,\pi)$ satisfying equation (4.6), namely

$$\begin{aligned} q_-(\alpha) &:= \alpha + \frac{1}{2}\arcsin(\mu(\sin 2\alpha)) \bmod \pi, \\ q_+(\alpha) &:= \alpha + \frac{\pi}{2} - \frac{1}{2}\arcsin(\mu(\sin 2\alpha)) \bmod \pi. \end{aligned} \tag{4.8}$$

The following lemma is proved at the end of the section.

Lemma 4.2. *Let $g_\pm : [0,\pi) \to \mathbb{R}$ be the functions defined by*

$$g_\pm(\alpha) = F_\mu(q_\pm(\alpha), \alpha),$$

Then, for any value of $\mu \in (0,1)$, each function $g_\pm$ is continuous, and moreover it is strictly positive if and only if $\alpha \in (0,\pi/2)$, and in this case $q_\pm(\alpha) > \alpha$.

If $q = \pi/2$ then $\alpha = 0 \mod \pi/2$ and $g_\pm(\alpha) = 0$. Consequently, M_0^2 in (4.6b) is undefined, showing there are no solutions with $q = \pi/2$.

Now for any $q \in (0,\pi/2)$, there is just one value of $\alpha \in (0,\pi/2)$ (as required by the lemma above) which is related by $q = q_-(\alpha)$. These correspond to acute relative equilibria.

Similarly, for any $q \in (\pi/2,\pi)$, there is again a single value of $\alpha \in (0,\pi/2)$ satisfying (4.6a), but now $q = q_+(\alpha)$. These are obtuse relative equilibria.

There now remains to relate (q,α) to the angles θ_1, θ_2 of the theorem. Note that for all the solutions described above (with $\alpha \in (0,\pi/2)$) we have $q > \alpha$ (for equal masses this is clear, for distinct masses it follows from the lemma). From (4.5) and the definition of the moving frame it follows that $\alpha = 0$ corresponds to the axis containing the mass μ_1, and then α increases in the direction towards μ_2. Thus α represents the angle between μ_1 and the axis of rotation (the axis containing $\boldsymbol{m}$), so $\alpha = \theta_1$. Moreover, since $q > \alpha$, the axis of rotation lies between the masses, and the angle between the axis of rotation and μ_2 is $\theta_2 = q - \alpha$. This shows that (4.1) is equivalent to (4.6a).

In view of the above discussion, and of the inequality

$$q_-(\alpha) < 2\alpha < q_+(\alpha), \qquad \alpha \in (0,\pi/2),$$

that is easily established from (4.8), it follows that $\theta_2 < \theta_1$ for acute RE, with the opposite inequality, $\theta_1 < \theta_2$, holding for obtuse RE. This proves the claim about which mass is closer to the axis of rotation.

Finally, the equation (4.2) for ω is obtained by starting with (1.19) and using $m_x = 0$, (4.5) and (4.6) to simplify the resulting expression.

Proof of Lemma 4.2. We begin by showing the continuity of $g_\pm$. Since $F_\mu(q,\alpha)$ is continuous in (q,α) the only candidate for discontinuities of $g_\pm$ is where $q_\pm(\alpha)$ is discontinuous in α, and this can only occur where $q_\pm = 0,\pi$. For q_- this only occurs for $\alpha = 0,\pi$ so does not give rise to a discontinuity. For $q_+(\alpha)$, which is increasing on each subinterval $(0,\pi/2)$ and $(\pi/2,\pi)$, this can only occur when $\alpha = \pi/2$, at which point $F_\mu(q,\pi/2) = 0$ and the discontinuity of q_+ has no effect on g_+.

Next, for $\alpha \in (0,\pi/2)$ it is clear that $0 < q_-(\alpha) < \pi/2$ and $0 < q_-(\alpha) - \alpha < \pi/2$. Therefore $g_-(\alpha) > 0$ since all terms in its expression are positive.

Now let $\alpha \in (0,\pi/2)$ and let us show that $g_+(\alpha) > 0$. Let $\beta = \frac{1}{2}\arcsin(\mu \sin 2\alpha)$, then $\beta \in (0,\pi/4)$ and we have $\sin^2\beta = \frac{1}{2} - \frac{1}{2}\sqrt{1-\mu^2\sin^2 2\alpha}$. Hence,

$$0 < \sin^4\beta = \sin^2\beta - \mu^2 \sin^2\alpha \cos^2\alpha,$$

which implies

$$\mu \sin\alpha \cos\alpha < \sin\beta. \tag{4.9}$$

Given that $q_+(\alpha) = \alpha - \beta + \pi/2$ we have

$$\frac{g_+(\alpha)}{\cos\alpha} = \mu \sin\beta \cos^2\alpha - \mu\cos\beta \sin\alpha\cos\alpha + \sin\beta \geq -\mu\cos\beta\sin\alpha\cos\alpha + \sin\beta,$$

and one can easily prove that the quantity on the right is positive using (4.9).

Now note that for $\alpha \in (0, \pi/2)$ one has

$$q_\pm(\pi/2+\alpha) = \pi - q_\pm(\pi/2-\alpha).$$

Using this relation in the definition of $g_\pm$ shows that $g_\pm$ is "odd" with respect to $\alpha = \pi/2$. Namely,

$$g_\pm(\pi/2+\alpha) = -g_\pm(\pi/2-\alpha).$$

This last expression shows that $g_\pm(\alpha) \leq 0$ for values of $\alpha \in [0,\pi)$ that do not lie on $(0,\pi/2)$.

The final inequality in the statement of the lemma follows immediately from (4.6a) given that $\alpha \in (0,\pi/2)$. □

4.2 Stability of the relative equilibria on the sphere

As for the Lobachevsky plane, the stability of the relative equilibria found above depends on the precise form of the potential, and in this section we restrict attention to the gravitational potential (1.1). Similar to our treatment in L^2, in our analysis we assume that the constants μ_1 and $G\mu_1\mu_2$ equal one. In this way, the gravitational potential $U(q) = -\cot(q)$, and the Hamiltonian depends on the parameters of the problem only through the mass ratio μ that we will continue to assume to be $0 < \mu < 1$.

The following theorem synthesises the results of the linear stability analysis of the problem. In its statement ***elliptic*** means that the linearisation of the reduced equations restricted to the symplectic leaf only has non-zero, purely imaginary eigenvalues.

Theorem 4.3. *For the relative equilibria described in Theorem 4.1, the linear stability analysis of the RE depends on whether the masses are equal or not as follows.*

$\mu_2 \neq \mu_1$: *(i) All acute RE are elliptic.*

(ii) There exists a critical obtuse angle $q^* \in (\pi/2,\pi)$, which depends on the mass ratio μ, such that obtuse RE are elliptic for $\pi/2 < q < q^*$, and are unstable for $q^* < q < \pi$. The critical angle q^* is given by $q^* = q_+(\alpha^*)$ where α^* is defined implicitly as the unique solution in $(0,\pi/2)$ of the equation

$$\cos 2\alpha = 2\sin^2\alpha\sqrt{1-\mu^2\sin^2 2\alpha}.$$

Moreover, along the branch of obtuse RE, the momentum M^2 has a minimum at q^*.

$\mu_2 = \mu_1$: *(i) All right-angled RE with $\theta \neq \pi/4$ are elliptic.*

(ii) All isosceles RE subtending an acute angle $q \in (0,\pi/2)$ are elliptic.

(iii) All isosceles RE subtending an obtuse angle $q \in (\pi/2,\pi)$ are unstable.

Contrary to the case in L^2 the Hamiltonian function cannot be used as a Lyapunov function to guarantee the nonlinear stability of the elliptic RE of the problem. This is due to the non-definiteness of the Hamiltonian at these points. More precisely we have

Proposition 4.4. *The Hessian matrix of the reduced Hamiltonian (restricted to the corresponding symplectic leaf) at the RE of the problem described in Theorem 4.1, has the following signature:*

$\mu_1 \neq \mu_2$. *(i) Acute RE have signature $(++--)$.*

(ii) *Obtuse RE with $\pi/2 < q < q^*$ have signature* $(++--)$.

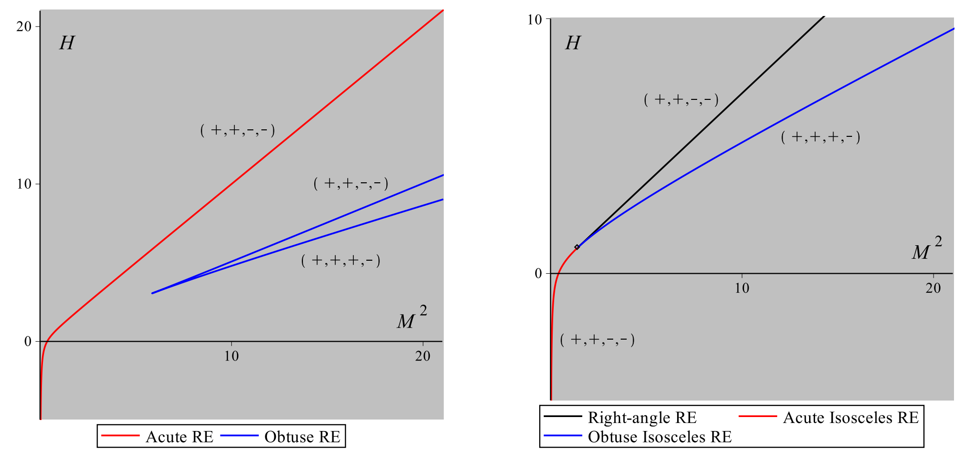
(iii) *Obtuse RE with $q^* < q < \pi$ have signature* $(+++-)$.

$\mu_1 \neq \mu_2$. (i) *Right-angled RE which are not isosceles have signature* $(++--)$.

(ii) *Isosceles RE subtending an acute angle have signature* $(++--)$.

(iii) *Isosceles RE subtending an obtuse angle have signature* $(+++-)$.

A proof of the results of Theorem 4.3 and Proposition 4.4 is given in Section 4.2.1 below.[6] The results are conveniently summarised in the energy-momentum diagram of the system given in Fig. 9.



(a) The case of different masses, $\mu_2 = 2\mu_1$. Notice the change in signature at the cusp of the obtuse relative equilibria corresponding to $q = q^*$, where M^2 has a minimum.

(b) The case of equal masses, $\mu_1 = \mu_2$. There is a change of signature passing from acute to obtuse isosceles RE when the two families of RE intersect.

Figure 9: Energy-Momentum bifurcation diagram of relative equilibria on the sphere for the gravitational potential. The shaded area on the M^2-H plane show all possible values of (M^2, H). All RE with signature $(++--)$ are elliptic.

Remark 4.5. Considering that the gravitational potential U satisfies $\lim_{q\to 0} U(q) = -\infty$, it is easy to construct a sequence q_n, p_n with $q_n \to 0$, and $p_n \to \infty$, such that H evaluated at $(\boldsymbol{m}, q, p) = (0, M_0, 0, q_n, p_n)$ converges to any arbitrary value $h_0 \in \mathbb{R}$. This shows that the energy-momentum map (M^2, H) is not proper and that all of its fibres are non-compact. ⋄

The results presented above are insufficient to show the nonlinear stability of the RE of the problem. By nonlinear stability we mean the stability in the sense of Lyapunov in the reduced space of the corresponding equilibrium. Since the restriction of the reduced system to the symplectic leaves defines a 2-degree of freedom Hamiltonian system, a nonlinear stability analysis may be performed using Birkhoff normal forms and applying KAM theory. We do this in Section 4.2.2, but we restrict our attention to the acute RE. These are parametrised by the mass ratio $\mu \in (0,1)$ (an 'external' parameter) and the arc $q \in (0, \pi/2)$ (an 'internal' parameter). In our treatment in

[6] An alternative proof is given in [25] by working on a symplectic slice of the unreduced system.

Section 4.2.2 we give numerical evidence of the validity of the following (which we are calling a 'theorem' although we have not carried out an analytic proof).

Theorem 4.6. *Acute RE are nonlinearly stable in an open dense set of the parameter space* $(\mu, q) \in (0,1) \times (0, \pi/2)$.

4.2.1 Linear analysis

Fix $M \neq 0$ and consider the restriction of the reduced system to the symplectic leaf $\mathcal{M}_{M^2}$ defined by $C(\boldsymbol{m}, \boldsymbol{m}) = M^2$. Denote by H_{M^2} the restriction of the reduced Hamiltonian (1.4) to $\mathcal{M}_{M^2}$. The RE with $C = M^2$ correspond to critical points of H_{M^2}.

Introduce canonical (Andoyer) coordinates on $\mathcal{M}_{M^2}$ by setting:

$$m_x = z, \quad m_y = \sqrt{M^2 - z^2}\sin\alpha, \quad m_z = \sqrt{M^2 - z^2}\cos\alpha.$$

Then (α, q, z, p) are Darboux coordinates on $\mathcal{M}_{M^2}$ and the restriction of the reduced equations of motion to $\mathcal{M}_{M^2}$ takes the canonical form

$$\dot{\alpha} = \frac{\partial H_{M^2}}{\partial z}, \quad \dot{q} = \frac{\partial H_{M^2}}{\partial p}, \quad \dot{z} = -\frac{\partial H_{M^2}}{\partial \alpha}, \quad \dot{p} = -\frac{\partial H_{M^2}}{\partial q}.$$

where, in view of (1.4), we have

$$\begin{aligned} H_{M^2}(\alpha, q, z, p) = \frac{1}{2}\Bigg[&(1+\mu)p^2 + 2pz - \frac{\cos 2q + \cos 2(q-\alpha) + \mu(1+\cos 2\alpha)}{2\sin^2 q} z^2 \\ &- \frac{1}{\sin^2 q}\left(\sin 2q - \frac{M^2}{2}\big(1 + \cos 2(q-\alpha) + \mu(1+\cos 2\alpha)\big)\right)\Bigg]. \end{aligned}$$

From our discussion in Section 4.1, we know that the critical points of H_{M^2} occur at points where $z = p = 0$, and q and α are such that (4.6) holds. Note that under our assumptions on the potential U, equation (4.6b) simplifies to

$$M^2 = \frac{\sin q}{\cos\alpha(\cos(q-\alpha) + \mu\cos\alpha\cos q)}. \tag{4.10}$$

The Hessian matrix of H_{M^2} along the equilibrium points is given by

$$D^2 H_{M^2}(\alpha, q, 0, 0) =: \mathbf{N} = \begin{pmatrix} \mathbf{N}^{(1)} & 0 \\ 0 & \mathbf{N}^{(2)} \end{pmatrix},$$

where $\mathbf{N}^{(1)}$ and $\mathbf{N}^{(2)}$ are symmetric 2×2 matrices. The entries of $\mathbf{N}^{(1)}$ may be written as

$$\begin{aligned} &\mathbf{N}^{(1)}_{11} = -\frac{M^2(\cos(2(q-\alpha)) + \mu\cos 2\alpha)}{\sin^2 q}, \qquad \mathbf{N}^{(1)}_{22} = \frac{M^2(1+\mu)\cos^2\alpha}{\sin^4 q}, \\ &\mathbf{N}^{(1)}_{12} = \mathbf{N}^{(1)}_{21} = \frac{M^2(-\sin q\cos 2\alpha + (1+\mu)\cos q\sin 2\alpha)}{\sin^3 q}, \end{aligned} \tag{4.11}$$

where we have used (4.10) to simplify $\mathbf{N}^{(1)}_{22}$. On the other hand

$$\mathbf{N}^{(2)} = \begin{pmatrix} -\dfrac{\cos\alpha\big(\cos(2q-\alpha) + \mu\cos\alpha\big)}{\sin^2 q} & 1 \\ 1 & 1+\mu \end{pmatrix}. \tag{4.12}$$

Lemma 4.7. *The matrix* $\mathbf{N}$ *is indefinite for all RE of the problem. It has at least 2 positive eigenvalues and at least 1 negative eigenvalue.*

Proof. That $\mathbf{N}$ has at least two positive eigenvalues follows immediately from its block diagonal form and the inequalities

$$\mathbf{N}^{(1)}_{22} > 0, \qquad \mathbf{N}^{(2)}_{22} > 0.$$

To complete the proof we show that $\mathbf{N}^{(2)}$ has a negative eigenvalue. This follows directly from the expression

$$\det(\mathbf{N}^{(2)}) = -\frac{\cos^2(q-\alpha)}{\sin^2\alpha} < 0$$

that is obtained by eliminating μ using (4.6a). □

Let us now introduce the quantity

$$a := \mathbf{N}^{(1)}_{11}\mathbf{N}^{(2)}_{11} + 2\mathbf{N}^{(1)}_{12}\mathbf{N}^{(2)}_{12} + \mathbf{N}^{(1)}_{22}\mathbf{N}^{(2)}_{22} = \frac{M^2(1+\cos(2(q-\alpha))}{\sin^2 q \sin^2\alpha}, \tag{4.13}$$

that will be relevant in what follows. To obtain the simplified expression on the right, one needs to use (4.10) and then (4.6a) to eliminate μ. Using again these conditions, one may show by a lengthy but straightforward calculation that

$$b := \det \mathbf{N} = \frac{a^2 f(q,\alpha)}{4}. \tag{4.14}$$

where

$$f(q,\alpha) := 1 - 4\sin^2\alpha \sin^2(q-\alpha). \tag{4.15}$$

Lemma 4.8. *The determinant of the matrix* $\mathbf{N}$ *along the RE is*

$\mu_1 \neq \mu_2$ *(i) positive for all acute RE and for the obtuse RE with* $\pi/2 < q < q^*$,

(ii) negative for obtuse RE with $q^* < q < \pi$.

In items (i) and (ii) above, q^* *is defined in the statement of Theorem 4.3.*

$\mu_1 = \mu_2$ *(i) positive for right angled RE with* $\alpha \neq \pi/4$ *and for isosceles RE subtending acute angle,*

(ii) negative for isosceles RE subtending an obtuse angle.

Proof. It is clear from (4.14) that the sign of $\det\mathbf{N}$ coincides with the sign of $f(q,\alpha)$. We analyse the latter along the different RE of the problem.

$\mu_1 \neq \mu_2$. (i) For acute RE we have $q = q_-(\alpha)$ and therefore

$$f = 1 - 2\sin^2\alpha\left(1 - \sqrt{1-\mu^2\sin^2 2\alpha}\right).$$

Since $\mu^2\sin^2 2\alpha < \sin^2 2\alpha$ for all $\mu \in (0,1)$, we find that

$$f > 1 - 2\sin^2\alpha(1-|\cos 2\alpha|) = \begin{cases} 1-4\sin^4\alpha & 0<\alpha\leqslant \pi/4, \\ \cos^2 2\alpha & \pi/4 \leqslant \alpha < \pi/2. \end{cases}$$

The above function is everywhere greater than zero except at the point $\alpha = \pi/4$, but, as can be verified,

$$f|_{\alpha=\pi/4} = \sqrt{1-\mu^2} > 0.$$

(ii) For obtuse RE we have $q = q_+(\alpha)$ and the corresponding expression for f is $f = 1 - 2\sin^2\alpha\left(1+\sqrt{1-\mu^2\sin^2 2\alpha}\right)$. This is a strictly decreasing function of α on the interval $(0,\pi/2)$, that is positive for $0 < \alpha < \alpha_*$ and negative for $\alpha_* < \alpha < \pi/2$, with α_* defined in the statement of Theorem 4.3

$\mu_1 = \mu_2$. Suppose now that $\mu = 1$.

(i) Along the right-angled RE we have $q = \pi/2$ and

$$f = (2\cos^2\alpha - 1)^2.$$

(ii) Along the isosceles RE we have $q = 2\alpha$ and

$$f = (3-2\cos^2\alpha)(2\cos^2\alpha - 1).$$

The statement in the case $\mu_1 = \mu_2$ follows immediately from the above two equalities.

□

Combining the above lemmas gives a proof of Proposition 4.4. The statements about instability in Theorem 4.3 follow from this proposition. Now we show the elliptic nature of the other RE.

Lemma 4.9. *All RE of the problem having signature* $(++--)$ *are elliptic.*

Proof. The matrix of the linearisation of the system at RE is

$$\mathbf{A} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\mathbf{N} = \begin{pmatrix} 0 & \mathbf{N}^{(2)} \\ -\mathbf{N}^{(1)} & 0 \end{pmatrix}$$

with characteristic polynomial

$$P(\lambda) = \lambda^4 + a\lambda^2 + b,$$

where a is given by (4.13) and $b = \det\mathbf{N}$. Since both a and b are positive, the ellipticity condition is equivalent to $R_1 > 0$ where

$$R_1 := \frac{1}{4}a^2 - b. \tag{4.16}$$

However, using (4.14) we may write

$$R_1 = a^2\sin^2\alpha\sin^2(q-\alpha),$$

which is positive along all RE considered in the statement of the lemma. □

Finally, to complete the proof of Theorem 4.3, we show that along the branch of obtuse RE, the momentum M^2 has a minimum at $q = q^*$. We parametrise the branch by $\alpha \in (0,\pi/2)$ by writing $q = q_+(\alpha)$. Differentiating (4.6a) implicitly with respect to α leads to

$$\frac{dq}{d\alpha} = 1 + \tan(2(q-\alpha))\cot(2\alpha).$$

Now, by differentiating (4.10) with respect to α, using the above expression for $\frac{dq}{d\alpha}$, and (4.6a) to eliminate μ, we find

$$\frac{dM^2}{d\alpha} = \frac{f(\alpha,q)}{\cos^2(q-\alpha)\cos(2(q-\alpha))\cos^2\alpha},$$

where $f(\alpha,q)$ is given by (4.15). It is easy to show, from the definition of $q_+(\alpha)$ in (4.8), that $\pi/2 < 2(q_+(\alpha)-\alpha) < \pi$, for $\alpha \in (0,\pi/2)$. Therefore, $\cos(2(q-\alpha)) < 0$ in the above formula, and $\frac{dM^2}{d\alpha}$ has the opposite sign of $f(q,\alpha)$. Hence, in view of (4.14), we conclude that $\frac{dM^2}{d\alpha}$ has the opposite sign of $\det\mathbf{N}$. The result now follows from Lemma 4.8 and the fact that $\frac{dq_+}{d\alpha} > 0$.

4.2.2 Nonlinear stability of acute relative equilibria in the case of different masses

In this section we give numerical evidence for the validity of Theorem 4.6.

Consider an elliptic RE of the problem that projects to an (isolated) equilibrium point on $\mathcal{M}_{M^2}$. By Theorem 4.3 the eigenvalues of the linearized system are purely imaginary

$$\lambda_1 = i\Omega_1, \quad \lambda_2 = -i\Omega_1, \quad \lambda_3 = i\Omega_2, \quad \lambda_4 = -i\Omega_2, \quad 0 < \Omega_1 < \Omega_2.$$

Our investigation of its nonlinear stability will proceed by checking that the following two conditions are satisfied:

1°. there are no second or third-order resonances:

$$\Omega_2 \neq 2\Omega_1, \quad \Omega_2 \neq 3\Omega_1.$$

Under this condition one may put the Hamiltonian (restricted to $\mathcal{M}_{M^2}$) in Birkhoff normal form

$$H = \frac{1}{2}\sum_{j=1}^{2} \alpha_j I_j + \frac{1}{4}\sum_{j,k=1}^{2} \beta_{jk} I_j I_k + O_5, \quad I_j = x_j^2 + y_j^2, \quad |\alpha_j| = \Omega_j. \tag{4.17}$$

Here, x_j and y_j are suitable canonical coordinates on a neighbourhood of the equilibrium on $\mathcal{M}_{M^2}$ (i.e., $\{x_j, y_k\} = \delta_{jk}$) with the equilibrium located at $x_j = y_j = 0$, β_{jk} are constants, and O_5 denotes a power series containing terms of order no less that 5 in x_j, y_j.

If conditions 1° and 2° are satisfied, a sufficient condition for nonlinear stability (under perturbations within $\mathcal{M}_{M^2}$) may be given in terms of the nonlinear terms in (4.17). Specifically, one requires that

2°. the *Arnold determinant* is different from zero

$$D := \det\begin{pmatrix} \beta_{11} & \beta_{12} & \alpha_1 \\ \beta_{12} & \beta_{22} & \alpha_2 \\ \alpha_1 & \alpha_2 & 0 \end{pmatrix} = 2\beta_{12}\alpha_1\alpha_2 - \beta_{11}\alpha_2^2 - \beta_{22}\alpha_1^2 \neq 0.$$

This nonlinear condition allows one to apply the KAM theorem in such a way that the invariant tori act as boundaries for the flow on each constant energy surface, leading to Lyapunov stability of the equilibrium (see e.g. [23], §35 in [28], or Section 13 in [19] for proofs and details).

Remark 4.10. If the Arnold determinant $D = 0$, one may still obtain sufficient conditions for stability by considering higher order terms in the normal form expansion (4.17) (see e.g. [19]). On the other hand, the presence of second or third-order resonances may lead to instability. We shall not consider any of these possibilities here. ⋄

Remark 4.11. We emphasise that the above analysis ensures nonlinear stability of RE only with respect to perturbations on the initial conditions that lie on the momentum surface $\mathcal{M}_{M^2}$. ⋄

Resonances

Recall that the characteristic polynomial of the linearised system is

$$P(\lambda) = \lambda^4 + a\lambda^2 + b.$$

Condition 1°, which requires that there are no second or third-order resonances, is written in terms of the coefficients a and b as

$$R_2 := \frac{4}{25}a^2 - b \neq 0, \quad R_3 := \frac{9}{100}a^2 - b \neq 0. \tag{4.18}$$

Fig. 10 below illustrates the plane a-b of coefficients of the characteristic polynomial. The curves σ_1 and σ_2 respectively correspond to the values of (a, b) attained at the acute and obtuse RE of the problem for the fixed value of $\mu = 0.95$. These RE are conveniently parametrised by $\alpha \in (0, \pi/2)$ by setting $q = q_-(\alpha)$ (acute) and $q = q_+(\alpha)$ (obtuse). The figure also illustrates the parabolae corresponding to the zero loci of R_1 defined by (4.16), and of the second and third order resonance polynomials R_2 and R_3 defined in (4.18).

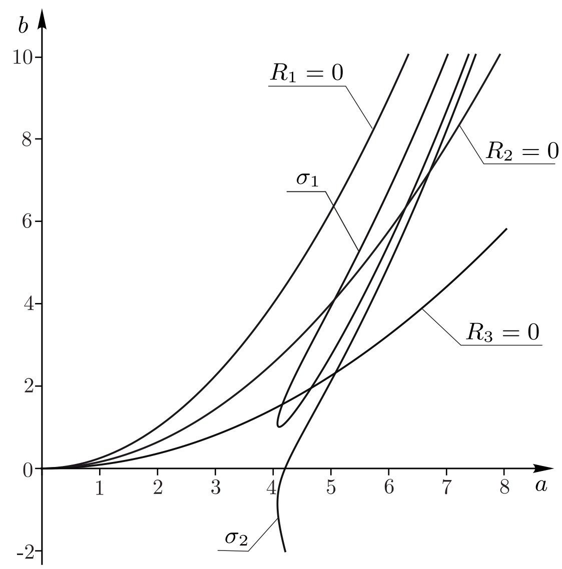


Figure 10: Curves σ_1, σ_2, corresponding to acute and obtuse RE on the coefficient plane (a, b) for $\mu = 0.95$.

An analytic investigation of condition (4.18) for a general $0 < \mu < 1$ involves very heavy calculations so we present numerical results. We restrict our attention to the acute RE that may be parametrised by $\alpha \in (0, \pi/2)$ by putting $q = q_-(\alpha)$. Our results are then presented in terms of the parameters $(\mu, \alpha) \in (0, 1) \times (0, \pi/2)$.

One can express R_2 and R_3 as functions of (μ, α) by substituting $q = q_-(\alpha)$ into (4.13) and (4.14). The zero loci of R_2 and R_3 on the α-μ-plane are the two curves illustrated in Fig. 11.

Analysis of the Arnold determinant

As for the resonance condition, we only present numerical results for our investigation of condition 2° for the acute RE. By using (4.6b) and setting $q = q_-(\alpha)$, we express $D = D(\mu, \alpha)$.

The zero locus of D on the plane α-μ consists of the two curves illustrated in Fig. 11 that provides numerical evidence for the validity of Theorem 4.6.

4.2.3 Open problems

To conclude, we point out a number of open problems concerning the stability of the two-body problem in the sphere:

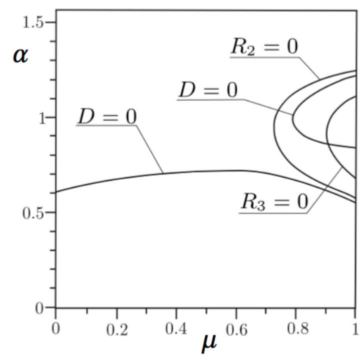


Figure 11: Curves on the plane μ-α plane corresponding to RE with second and third order resonances (respectively $R_2 = 0$ and $R_3 = 0$) and where the Arnold determinant vanishes ($D = 0$).

– For different masses, investigate the nonlinear stability of acute RE for which there are resonances ($R_2 = 0$, $R_3 = 0$) and/or the Arnold determinant vanishes ($D = 0$).

– Again for different masses in S^2, investigate the nonlinear stability of obtuse RE that are elliptic.

– In the case of equal masses, investigate the nonlinear stability of acute-isosceles RE and right-angled RE.

– Classify and investigate the stability of all RE for the spatial two-body problem on S^3 and L^3.

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