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A model theoretic approach to simple groups of finite Morley rank with finitary groups of automorphisms

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Abstract

In [8], the author proved the following theorem:

An infinite simple group of finite Morley rank admitting a finitary group of automorphisms is a Chevalley group over an algebraically closed field of positive characteristic.

Proof of the theorem above was achieved by group theoretic methods, with a heavy use of methods (but not the statement of the Classification of Finite Simple Groups). In this paper we give an alternative model theoretic proof. In particular, this new approach is shorter and more transparent than the original proof.

1 Introduction

The motivation for classifying infinite simple groups of finite Morley rank [1, 3] arises from the famous Cherlin-Zilber Conjecture, made in the 1970's [5, 12]:

> *Simple infinite groups of finite Morley rank are algebraic groups over algebraically closed fields.*

A detailed discussion of the Cherlin-Zilber Conjecture, as well as development of relevant techniques and purely algebraic axiomatisation of the concept of (finite) Morley rank can be found in [1] and [3].

In [8], the author proved that the Cherlin-Zilber Conjecture holds in the specific case when a simple group of finite Morley rank admits a finitary group of automorphisms; the definitions are given in Section 2.

In this paper we show that model theory provides an alternative and much shorter way to prove the main theorem of [8]:

Theorem 1. *An infinite simple group of finite Morley rank admitting a finitary group of automorphisms is a Chevalley group over an algebraically closed field of a positive characteristic.*

This paper is organised as follows. In Section 2, we define a finitary group of automorphisms of a group G and the corresponding locally finite core G^*

of G. In Section 3, we recall the model theoretic preliminaries needed in this paper and apply the Tarski-Vaught Test thus showing that G^* is an elementary substructure of G. Then, in Section 5, we prove that G^* is simple. This allows us to apply to G^* the result by Thomas [11] about simple locally finite groups of finite Morley rank and conclude that G, being elementary equivalent to G^*, is indeed a Chevalley group over some algebraically closed field.

2 Finitary groups of automorphisms

The key definitions of the present paper follow [8] but are given here in a more general setup.

Definition 1. *Let $\mathcal{M}$ be an infinite algebraic structure with underlying set M. We say that an infinite group A of automorphisms of $\mathcal{M}$ is* finitary*, if*

(1) *For every $\alpha \in A \setminus \{1\}$, the substructure of fixed points $\mathrm{Fix}_{\mathcal{M}}(\alpha)$ is finite.*

(2) *If X is a definable subset in the direct product M^n of finitely many copies of M and X is invariant under the action of a non-trivial automorphism $\alpha \in A$, then there exists an element $x \in X$ with a finite orbit x^A (equivalently, the stabiliser $\mathrm{Stab}_A(x)$ has a finite index in A).*

Under assumptions (1) *and* (2) *above, we define*

(3) *the* locally finite core $\mathcal{M}^*$ *of $\mathcal{M}$ as the set*

$$M^* = \{\, m \in M \mid \text{ the orbit } m^A \text{ is finite.}\}.$$

Notice that assumption (2) of Definition 1, applied to $X = M$, ensures that M^* is non-empty. Notice further that M^* is infinite for otherwise A would have a finite orbit x^A on the definable A-invariant subset $X = M \setminus M^*$ with x not belonging to M^* – a contradiction.

Proposition 1. *M^* with operations and relations inherited from $\mathcal{M}$ is a locally finite substructure of $\mathcal{M}$.*

Proof. Let $x_1, \dots, x_n \in M^*$, and $\mathcal{X} = \langle x_1, \dots, x_n \rangle$ the substructure generated by elements $x_1, \dots, x_n$ in M, that is, the minimal subset of M which contains $x_1, \dots, x_n$ and is closed under all algebraic operations on $\mathcal{M}$. Then the stabilisers of $x_1, \dots, x_n$ in A have finite indices in A and their intersection

$$B = \mathrm{Stab}_A(x_1) \cap \dots \cap \mathrm{Stab}_A(x_n)$$

also has a finite index in A, hence is not trivial. Here B fixes every element in X, hence A-orbits of elements in X are finite and $X \subseteq M^*$, which also proves that M^* is closed under all operations from $\mathcal{M}$ and is therefore a substructure of $\mathcal{M}$. Moreover, $X \subseteq \mathrm{Fix}_M(B)$ is finite by the definition of finitary groups of automorphisms, hence $\mathcal{M}^*$ is locally finite. □

When applying Definition 1 to groups, we shall follow standard notation and refer to stabilisers of points as *centralisers* and write $C_A(x)$ instead of $\mathrm{Stab}_A(x)$ and $C_X(\alpha)$ instead of $\mathrm{Fix}_X(\alpha)$, etc.

In [8], the author proved Theorem 1 using the following strategy. First it was shown that G^* is a simple locally finite group of finite Morley rank, thus a Chevalley group over an algebraically closed field [1, 2, 10]. Then it was possible to expand a specific configuration of the Curtis-Phan-Tits-Lyons Theorem [6, Theorem 1.24.2] present in G^*, from G^* to G, and finally apply the Curtis-Phan-Tits-Lyons Theorem to G.

3 Model theoretic preliminaries and the Tarski-Vaught Test

In this section, we give definition of an *elementary substructure* and state the *Tarski-Vaught Test* for elementary substructures.

Definition 2 (Elementary substructure)**.** *Let $\mathcal{N}$ and $\mathcal{M}$ be $\mathcal{L}$-structures and $\mathcal{N}$ be a substructure of $\mathcal{M}$. Then $\mathcal{N}$ is called an* elementary substructure *of $\mathcal{M}$ and denoted by $\mathcal{N} \preceq \mathcal{M}$ if for every $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$, with free variables $x_1, \dots, x_n$, and tuple $\bar{a} = a_1, \dots, a_n$ of elements of N we have $\mathcal{N} \models \varphi(\bar{a})$ if and only if $\mathcal{M} \models \varphi(\bar{a})$.*

The well-known Tarski-Vaught Test below gives a criterion for a substructure to be an elementary substructure:

Fact 1 (Tarski-Vaught Test, [7], Theorem 2.5.1)**.** *Suppose that $\mathcal{N}$ is a substructure of $\mathcal{M}$. Then $\mathcal{N} \preceq \mathcal{M}$ if and only if for all $\varphi(x_1, \dots, x_n, y) \in \mathcal{L}$ and $a_1, \dots, a_n \in N$*

$$\text{if } \mathcal{M} \models (\exists y)\varphi(\bar{a}, y) \text{ then there is } b \in N \text{ with } \mathcal{M} \models \varphi(\bar{a}, b).$$

There is an obvious restatement of the Tarski-Vaught Test in terms of definable subsets:

> $\mathcal{N} \preceq \mathcal{M}$ if and only if for all non-empty definable subsets $S \subseteq M$ defined with parameters from N, $S \cap N$ is non-empty.

Theorem 2. *Let $\mathcal{M}$ be an algebraic structure and A a finitary group of automorphisms of $\mathcal{M}$. Then, $\mathcal{M}^*$ is an elementary substructure of $\mathcal{M}$.*

Proof. We shall show that every $\mathcal{M}^*$-definable non-empty subset $X \subseteq M$ has a point in M^*. This will prove the theorem by invoking the Tarski-Vaught Test.

Indeed since X is a M^*-definable set it is of the form

$$X = \{(x_1, \dots, x_n) \in M^n \mid \varphi(\overline{x}, \overline{m})\},$$

where $\overline{m} = (m_1, \dots, m_k) \in M^*$. Every m_i has a finite A-orbit and therefore its stabiliser $B_i = \mathrm{Stab}_A(m_i)$ has finite index in A, hence the intersection

$$B = \mathrm{Stab}_A(m_1) \cap \cdots \cap \mathrm{Stab}_A(m_k)$$

of all B_i's also has a finite index in A. Clearly, for all $i = 1, \dots, k$, B fixes all m_i's. Now the set X is B-invariant and B has a finite orbit O in X such that the point-wise stabiliser $\mathrm{Stab}_A(O)$ has a finite index in A; hence A also has a finite orbit in X. Elements in this orbit O belong to M^* so $X \cap M^* \neq \emptyset$ as is claimed.

□

4 Groups of finite Morley rank

Similarly as in [8], we consider groups of finite Morley via a *ranked definable universe* following the notations of [1,3]. This way we define a *definable universe* $\mathcal{U}$ to be a set of sets closed under taking singletons (one-elements sets), Boolean operations, direct products, and projections. Further, we call definable universe $\mathcal{U}$ *ranked* if it carries a rank function as defined by Borovik and Poizat [9]. This means that we can assign to every set $S \in \mathcal{U}$ a natural number which behaves as a "dimension" of the corresponding set. A group G is *ranked* if its base set and graphs of multiplication and inversion belong to a ranked universe. It is well-known [3, 9] that a simple ranked group is ω-stable of finite Morley rank.

Chevalley groups are defined by generators and relations and provide a description of important classes of simple algebraic groups (including those over algebraically closed fields) as *abstract* groups, ignoring the underlying algebraic variety. In this paper, we won't present the definition of Chevalley groups but a reader unfamiliar with the construction may find all necessary details in [4].

The classification of simple algebraic groups over algebraically closed fields amounts to the statement that they are Chevalley groups over corresponding fields; the reverse statements is also true.

We list a few properties of groups of finite Morley rank.

Fact 2. *A definably simple group of finite Morley rank is simple.*

Fact 3 ([3], Theorem 7.8)**.** *Let G be an infinite group of finite Morley rank without non-trivial abelian normal subgroups. Then*

- *the product $L(G)$ of minimal definable connected subnormal subgroups of G is is a direct product*

$$L(G) = L_1 \times L_2 \times \cdots \times L_k$$

  of such subgroups.

- *$L(G)$ is a definable characteristic subgroups of G.*

- $C_G(L(G)) = 1$.

Fact 4. *A locally finite simple group of finite Morley rank is a Chevalley group over an algebraically closed field of positive characteristic.*

Proof. This fact was first proven by Thomas [10] who used the Classification of Finite Simple Groups (CSFG). Later results by Borovik [2], and by Altınel, Borovik and Cherlin [1] removed the reference to CFSG. □

Fact 5 ([1], Fact II.2.25)**.** *Let G be a group of finite Morley rank, and H a definable group of automorphisms acting (faithfully) on G. Assume that G is an infinite simple Chevalley group over an algebraically closed field F. Then $H \leqslant G \rtimes \Gamma$, where G is identified with the group of inner automorphisms of G, and Γ is the group of graph automorphisms of G relative to a choice of maximal torus and Borel subgroup in G. In particular, Γ is finite.*

5 Proof of Theorem 1

From now on G always refers to an infinite simple group of finite Morley rank and A to a finitary group of automorphisms of G. In view of Theorem 2, $G^* \preceq G$ and is therefore also a group of finite Morley rank.

Proposition 2. *$Z(G^*) = 1$ and G^* contains a unique normal infinite definable simple subgroup $L^\diamond = L(G^*)$.*

Proof. Observe first that if G^* has a non-trivial abelian normal subgroup Z and $1 \neq z \in Z$ then the statement

$$(\forall x) z \cdot z^x = z^x \cdot z$$

holds in G^*, and, in view or elementary embedding $G^* \preceq G$, in G. But then $\langle z^G \rangle$ is a nontrivial abelian subgroup of G; a contradiction. In particular, $Z(G^*) = 1$.

Therefore we can work under assumptions of Fact 3 and consider the subgroup $L^\diamond = L(G^*)$,

$$L^\diamond = L_1^\diamond \times \cdots \times L_m^\diamond,$$

where $L_i^\diamond$ are simple definable subgroups. Assume that $m > 1$. We use a simple fact from group theory:

Lemma 1. *For any subgroup F of an arbitrary group H,*

$$F \leqslant C_H(C_H(F)),$$

and if $F_1 \times F_2 \times \cdots \times F_m \leqslant H$ where F_i, $i = 1, \ldots m$, have trivial centers, $Z(F_i) = 1$, then the double centralisers $C_H(C_H(F_i))$ form a direct product and

$$F_1 \times \cdots \times F_m \leqslant C_H(C_H(F_1)) \times \cdots \times C_H(C_H(F_m)).$$

It follows that

$$L^\diamond \leqslant C_G(C_G(L_1^\diamond)) \times \cdots \times C_G(C_G(L_m^\diamond)) < G,$$

because, in the case of equality, G would not be simple.

Denote $M = C_G(C_G(L_1^\diamond)) \times \cdots \times C_G(C_G(L_m^\diamond))$. The crucial observation of our proof is that M and $N_G(M)$ are definable A-invariant subgroups in G and $N_G(M) < G$ since G is simple. Observe further that G^* normalises M, that is, $G^* \leqslant N_G(M)$. But then $G \setminus N_G(M)$ is a definable A-invariant subgroup, and, by definition of the finitary group of automorphisms, A has on $G \setminus N_G(M)$ a finite orbit which has to belong to G^* by construction of the latter. This contradiction shows that $L^\diamond$ is the unique minimal normal subgroup of G^*. □

Now we can complete the proof of Theorem 1.

In view of Fact 4, $L^\diamond$ is a Chevalley group over an algebraically closed field of positive characteristic. By Fact 3, G^* is a definable group of automorphisms of $L^\diamond$, and by Fact 5, $G^*/L^\diamond$ is finite, hence $L^\diamond$ is the connected component of G^* with the index $|G^* : L^\diamond|$ being the Morley degree of G^*. Since $G^* \preceq G$, the group G has the same Morley degree, and since G is simple, $|G^* : L^\diamond| = 1$ and $G^* = L^\diamond$ is a Chevalley group over an algebraically closed field.

The main result follows from the elementary embedding $G^* \preceq G$ together with the following fact:

Fact 6 (Thomas [11]). *Being a Chevalley group over an algebraically closed field is a first order axiomatisable property.*

That is, we have shown that G^* is a Chevalley group over an algebraically closed field. By Fact 6, G^* passes this property to G, thus Theorem 1 holds.

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