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Khan, Arbaz and Upadhyay,
Chandra Shekhar and Gerritsma, Marc

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Spectral element method for parabolic interface problems: Regularity estimates, stability theorem and error estimate

Arbaz Khan[a,*], Chandra Shekhar Upadhyay[b], Marc Gerritsma[c]

[a]*School of Mathematics, The University of Manchester, M13 9PL Manchester, UK*
[b]*Department of Aerospace Engineering, Indian Institute of Technology Kanpur, kanpur India-208016*
[c]*Delft University of Technology, Faculty of Aerospace Engineering, Aerodynamics Group P.O. Box 5058, 2600 GB Delft, The Netherlands*

Abstract

In this paper, an h/p spectral element method with least-square formulation for parabolic interface problem will be presented. The regularity result of the parabolic interface problem is proven for non-homogeneous interface data. The differentiability estimates and the main stability estimate theorem, using non-conforming spectral element functions, are proven. Error estimates are derived for h and p versions of the proposed method.

Keywords: Least-squares method, nonconforming, spectral element method, Linear parabolic interface problems, Sobolev spaces of different orders in space and time

1. Introduction

In this paper, we consider a linear parabolic interface problem of the form

$$\begin{aligned} \mathcal{L}u = u_t - \nabla \cdot (\mathcal{A}\nabla u) = F \text{ in } (\Omega_1 \cup \Omega_2) \times I, \\ u = f \text{ on } \Omega \times \{0\} \qquad \text{(initial condition)} \\ u = g \text{ on } \Gamma \times I, \quad \text{(exterior boundary condition)} \end{aligned} \tag{1.1}$$

which satisfies the interface conditions

$$[u] = q_0 \quad \text{and} \quad [n \cdot \mathcal{A}\nabla u] = q_1 \text{ on } \Gamma_0 \times I,$$

where $n = (n_1, n_2)^T$ is a unit outward normal vector to the interface Γ_0 and $I = (0, T)$. Here Ω and Ω_1 ($\bar{\Omega}_1 \subset \Omega$) are open bounded domains in $\mathbb{R}^2$ with C^2 boundaries $\partial\Omega = \Gamma$ and $\partial\Omega_1 = \Gamma_0$, respectively (see Fig. 1). Further, $\Omega_2 = \Omega \setminus \bar{\Omega}_1$. The symbol $[v]$ denotes the jump of a quantity v across the interface Γ_0, i.e., $[v](x,t) = v_1(x,t) - v_2(x,t), (x,t) \in \Gamma_0 \times I$. Let

$$\mathcal{A} = \begin{cases} \mathcal{A}^1 \text{ in } \Omega_1 \times I, \\ \mathcal{A}^2 \text{ in } \Omega_2 \times I. \end{cases} . \tag{1.2}$$

Then the jump term $n \cdot \mathcal{A}\nabla u$ is defined as follows:

$$[n \cdot \mathcal{A}\nabla u] = n \cdot (\mathcal{A}^1 \nabla u_1 - \mathcal{A}^2 \nabla u_2) \quad \text{on } \Gamma_0 \times I,$$

where each 2×2 matrix $\mathcal{A}^k$ ($k = 1, 2$) is symmetric and positive definite, uniformly on $\Omega_k \times I$. The components $a^k_{i,j}(x,t)$ of $\mathcal{A}^k$ are smooth for each k. Here $n \cdot \mathcal{A}^k \nabla u_k$ denotes the conormal derivative on Γ_0, i.e.

$$n \cdot \mathcal{A}^k \nabla u_k = \sum_{i,j=1}^{2} a^k_{i,j} \frac{\partial u_k}{\partial x_i} n_j, \quad k = 1, 2.$$

In engineering and science, many problems can be formulated in terms of parabolic partial differential

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*Corresponding author
Email address: arbazkha@gmail.com, arbaz.khan@iwr.uni-heidelberg.de (Arbaz Khan)

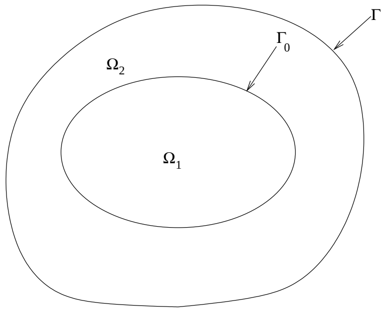


Figure 1: Domain Ω with boundary Γ and its subdomains Ω_1, Ω_2 with interface Γ_0

equations with discontinuous coefficients. Heat diffusion, electrostatics, multiphase and porous media flow problems are some examples from physics. A special case of parabolic equations with discontinuous coefficients consists of interface problems (1.1) which arise, for example, in heat conduction.

Several methods have been proposed and analyzed both theoretically and computationally for interface problems in [23, 24, 25, 28, 30, 31, 32, 33] (and references cited therein) and have been shown to be very effective.

If the given data, the boundary Γ and the interface Γ_0 of parabolic interface problem (1.1) are smooth then the solution of the problem is also very smooth in the individual regions, while the global regularity of solution becomes low because of non-homogeneous jump terms (see [15, 31, 30]). Many standard finite difference methods are not applicable to interface problems because of lack of this global regularity. The use of an immersed-interface method in the framework of finite difference methods has some disadvantages, which are discussed in [24]. Immersed-interface finite element methods for elliptic interface problems have been presented in [23, 24]. In an immersed-interface method, the jump conditions are enforced through the construction of special finite element basis functions which satisfy homogeneous interface conditions. Immersed-interface finite element methods can achieve optimal convergent rates with linear finite elements. Recently, Albright et al. [28] proposed a high-order accurate difference potential method for parabolic problems. In that paper, they presented two approaches which are second order and fourth order accurate.

Conforming finite element methods are the most used methods to solve interface problems. This requires the triangulation of different subregions to be geometrically conforming at the interface. Conforming methods, however impose serious restrictions on the computational domain when the physical solutions of the interface problems are of different scales in different subregions. Methods that allow relaxation of such conditions are the nonconforming methods like mortar finite element methods and discontinuous Galerkin finite element methods. Schötzau et al. [20] presented time discretization of parabolic problems by the *hp*-version of the discontinuous Galerkin finite element method. Dutt et al. [7] proposed $h-version$ and $p-version$ least-squares spectral element methods for parabolic partial differential equations (PDE) with smooth coefficients on bounded domains. Recently, we proposed the least-squares spectral element method for parabolic initial value problems with non-smooth data in [10, 11]. The method proposed in this paper is a nonconforming least-squares spectral element method (see [7, 10, 11, 12, 13, 14]). Sobolev spaces of different orders in space and time to formulate the results are given in [16].

Bochev and Gunzburger [2] have summarized the least-squares finite element method (LSFEM) for parabolic problems. The obvious advantage of this class of methods is that the discrete problems are positive definite and symmetric. Least-square spectral element methods (LSSEM) have been presented by Proot et al. [19] for the Stokes problem, and Pontaza et al. [18] for the Navier-Stokes equations, combining the least-square formulation with spectral element approximation. Maerschalck et al. [29] presented the use of Chebyshev polynomials in a space-time least-squares spectral element method. The advantage of LSSEM is that it has the generality of finite element methods with the accuracy of spectral methods.

Over the past three decades, spectral methods have been extensively used for solving partial differential equations because of high order of accuracy (see [3, 4, 5, 6, 9] and the references therein). Kumar et al. [25] proposed a least-square spectral element method for two-dimensional elliptic interface problem with a smooth interface, following the approach proposed in [26]. Recently, we proposed a least-squares spectral element method for three-dimensional elliptic interface problem with a smooth interface in [14]. In this method, the domain is divided into a finite number of subdomains such that the sub-divisions match along the interface. The interface is resolved exactly using blending elements [27].

In this paper, an h/p least-squares spectral element method is presented to solve the parabolic interface problem with smooth interface. Our method is based on minimizing the sum of the squares of a weighted squared norm of the residuals in the partial differential equation and the sum of the residuals in the boundary conditions in fractional Sobolev norms and the sum of the jumps in the value and its derivatives across the interface in appropriate fractional Sobolev norms. The continuity along the inter-element boundary is enforced by adding a term, which measures the sum of the squares of the jump in the function and its derivatives in fractional Sobolev norms.

The content of the paper is organized as follows: Section 2 is devoted to defining the parabolic interface problem and to proving regularity results. In Section 3, the discretization of the domain and stability result are presented. In Section 4, error estimates are given for h and p versions of the proposed method.

2. Preliminaries

Let r and s be two non-negative real numbers. As in [16], define

$$H^{r,s}\big(\Omega \times I\big) = H^0\big(I; H^r(\Omega)\big) \cap H^s\big(I; H^0(\Omega)\big), \tag{2.1}$$

which is a Hilbert space with norm

$$\left(\int_0^T \|u(t)\|^2{}_{H^r(\Omega)} dt + \|u\|^2{}_{H^s(I;H^0(\Omega))} \right)^{1/2},$$

where $H^r(\Omega)$ denotes the standard Sobolev space of order r. Here $H^0(\Omega) = L^2(\Omega)$ and $H^{0,0}(\Omega \times I) = L^2(\Omega \times I)$.

Let $u_1 = u|_{\Omega_1 \times I}$ and $u_2 = u|_{\Omega_2 \times I}$. Next, we define following spaces

$$H^r(\Omega_1 \cup \Omega_2) = \{u \in L^2(\Omega) \,|\, u|_{\Omega_i} \in H^r(\Omega_i) \text{ for } i = 1,2\},$$
$$H^{r,s}(\Omega_1 \cup \Omega_2 \times I) = \{u \in L^2(\Omega \times I) \,|\, u|_{\Omega_i \times I} \in H^{r,s}(\Omega_i \times I) \text{ for } i = 1,2\}.$$

Let

$$||u||^2_{r,\Omega_1 \cup \Omega_2} = ||u_1||^2_{H^r(\Omega_1)} + ||u_2||^2_{H^r(\Omega_2)}, \tag{2.2}$$
$$||u||^2_{(r,s),\Omega_1 \cup \Omega_2 \times I} = ||u_1||^2_{H^{r,s}(\Omega_1 \times I)} + ||u_2||^2_{H^{r,s}(\Omega_2 \times I)}. \tag{2.3}$$

We also use the following notations in throughout paper:

$$||(\cdot)||_{\Omega} = ||(\cdot)||_{L^2(\Omega)} \quad \text{and} \quad ||(\cdot)||_{\Omega \times I} = ||(\cdot)||_{L^2(\Omega \times I)}.$$

We now define some Gevrey Spaces [17] which are needed for our error analysis.

$$\mathscr{D}_1(\bar{\Omega}) = \{\Phi \in C^\infty(\overline{\Omega}) \,|\exists A_1, B_1 > 0 \ : \sup_{x \in \overline{\Omega}} |\ D_x^\alpha \Phi(x)| \leq A_1 (B_1)^i \, i! \ , \ |\alpha| = i, \, i = 0, 1, \cdots \} \ .$$

$$\mathscr{D}_{2,1}(\bar{\Omega} \times \bar{I}) = \Big\{ \psi \in C^\infty(\overline{\Omega} \times \bar{I}) \,|\exists A_1, B_1 > 0 \ : \sup_{(x,t) \in \overline{\Omega} \times \bar{I}} |\ D_x^\alpha D_t^j \psi(x,t)| \leq A_1 (B_1)^{i+j} \, i!(j!)^2 \ , \ |\alpha| = i, \forall \, i, j = 0, 1, \cdots \Big\} \ .$$

2.1. Regularity estimate

In general, the solution of problem (1.1) does not belong to $H^{2,1}(\Omega \times I)$ due to the presence of a discontinuity/reduced regularity in $\mathcal{A}$. Moreover, the solution does not belong to $H^{1,0}(\Omega \times I)$ unless the jump term at the interface $[u]$ is equal to zero. We can get better local regularity using local smoothness of the coefficients. An a-priori result for the problem (1.1) is given in Theorem 2.1 with appropriate assumptions on F, g, q_0, q_1 and f. First, we prove the following Lemma 2.1 which we use to obtain our main regularity result.

Lemma 2.1. *Consider the problem*

$$\mathcal{L}v = v_t - \nabla\cdot(\mathcal{A}\nabla v) = \tilde{F} \quad in \quad \Omega_1\cup\Omega_2\times I, \tag{2.4}$$
$$v = v_0 \quad on \quad \Omega_1\cup\Omega_2\times\{0\} \qquad (initial\ condition)$$
$$v = 0 \quad on \quad \Gamma\times I, \quad (exterior\ boundary\ condition)$$

along with the interface conditions

$$[v] = 0 \quad and \quad [n\cdot\mathcal{A}\nabla v] = 0 \quad on \quad \Gamma_0\times I. \tag{2.5}$$

Let $\tilde{F}\in H^{0,0}(\Omega_1\cup\Omega_2\times I)$ and $v_0\in H^1(\Omega_1\cup\Omega_2\times\{0\})$. If the interface Γ_0 and the boundary Γ are C^2 and the given data satisfy required compatibility condition (see [16]), then the solution $v\in H^{2,1}(\Omega_1\cup\Omega_2\times I)$ and

$$||v||^2_{(2,1),\Omega_1\cup\Omega_2\times I} \leq C\Big(||\tilde{F}||^2_{(0,0),\Omega_1\cup\Omega_2\times I} + ||v_0||^2_{1,\Omega_1\cup\Omega_2\times\{0\}}\Big). \tag{2.6}$$

Here C is a generic constant.

Proof. Our proof is a generalization of the approach of [31, 32]. For a.e. $t\in I$, $v = v(x,t)$ solves

$$-\nabla\cdot(\mathcal{A}\nabla v) = \tilde{F} - v_t \quad \text{in} \quad \Omega_1\cup\Omega_2\times I, \tag{2.7}$$
$$v = 0 \quad \text{on} \quad \Gamma\times I, \quad \text{(exterior boundary condition)}$$

along with the interface conditions

$$[v] = 0 \quad \text{and} \quad [n\cdot\mathcal{A}\nabla v] = 0 \quad \text{on} \quad \Gamma_0\times I. \tag{2.8}$$

Applying the regularity result for the elliptic interface problems of [30], it follows:

$$||v||^2_{2,\Omega_1\cup\Omega_2} \leq C||\tilde{F} - v_t||^2_{0,\Omega_1\cup\Omega_2}. \tag{2.9}$$

Multiplying v_t both side in equation (2.4) and integrating w. r. to x over $\Omega_1\cup\Omega_2$, we obtain

$$||v_t||^2_{0,\Omega_1\cup\Omega_2} - \int_{\Omega_1\cup\Omega_2}\nabla\cdot(\mathcal{A}\nabla v)v_t dx = \int_{\Omega_1\cup\Omega_2}\tilde{F}v_t dx. \tag{2.10}$$

Here $v\in H^{1,0}((\Omega_1\cup\Omega_2)\times I)$ and $[v]=0$ on Γ_0, it follows

$$[v_t] = 0 \quad \text{on } \Gamma_0. \tag{2.11}$$

Using integration by parts and the equation (2.11), we obtain

$$\begin{aligned}\int_{\Omega_1\cup\Omega_2}\nabla\cdot(\mathcal{A}\nabla v)v_t dx &= \int_{\Omega_1\cup\Omega_2}\sum_{i,j=1}^{2}(a_{i,j}v_{x_j})_{x_i}v_t dx\\ &= -\int_{\Omega_1\cup\Omega_2}\sum_{i,j=1}^{2}(a_{i,j}v_{x_j})(v_t)_{x_i}dx\\ &= -\frac{1}{2}\partial_t\int_{\Omega_1\cup\Omega_2}\sum_{i,j=1}^{2}(a_{i,j}v_{x_j})v_{x_i}dx + \frac{1}{2}\int_{\Omega_1\cup\Omega_2}\sum_{i,j=1}^{2}(a_{i,j})_t v_{x_j}v_{x_i}dx.\end{aligned} \tag{2.12}$$

Inserting the equation (2.12) into the equation (2.10), implies

$$||v_t||^2_{0,\Omega_1\cup\Omega_2} + \frac{1}{2}\partial_t\int_{\Omega_1\cup\Omega_2}\sum_{i,j=1}^{2}(a_{i,j}v_{x_j})v_{x_i}dx = \int_{\Omega_1\cup\Omega_2}\tilde{F}v_t dx + \frac{1}{2}\int_{\Omega_1\cup\Omega_2}\sum_{i,j=1}^{2}(a_{i,j})_t v_{x_j}v_{x_i}dx. \tag{2.13}$$

Integrating the equation (2.13) w. r. to t over I, it follows:

$$\int_I ||v_t||^2_{0,\Omega_1\cup\Omega_2}dt + \frac{1}{2}\int_{(\Omega_1\cup\Omega_2)\times\{T\}}\sum_{i,j=1}^{2}(a_{i,j}v_{x_j})v_{x_i}dx$$
$$= \int_{(\Omega_1\cup\Omega_2)\times I}\tilde{F}v_t dxdt + \frac{1}{2}\int_{(\Omega_1\cup\Omega_2)\times I}\sum_{i,j=1}^{2}(a_{i,j})_t v_{x_j}v_{x_i}dxdt$$
$$+\frac{1}{2}\int_{(\Omega_1\cup\Omega_2)\times\{0\}}\sum_{i,j=1}^{2}(a_{i,j}v_{x_j})v_{x_i}dx. \quad (2.14)$$

Using Cauchy-Schwarz inequality and applying a standard kickback argument, it holds:

$$||v_t||^2_{(0,0),(\Omega_1\cup\Omega_2)\times I} + ||v||^2_{1,(\Omega_1\cup\Omega_2)\times\{T\}} \leq C\Big(\int_I ||\tilde{F}||^2_{0,\Omega_1\cup\Omega_2} + ||v||^2_{1,(\Omega_1\cup\Omega_2)\times\{0\}}\Big)$$
$$+ C\int_I ||v||^2_{1,(\Omega_1\cup\Omega_2)}dt. \quad (2.15)$$

Applying an application of Gronwall's lemma, implies the desired result. □

We are now in a position to state the main regularity result.

Theorem 2.1. *Let $F \in H^{0,0}(\Omega_1\cup\Omega_2\times I), g \in H^{\frac{3}{2},\frac{3}{4}}(\Gamma\times I), q_0 \in H^{\frac{3}{2},\frac{3}{4}}(\Gamma_0\times I)$, $q_1 \in H^{\frac{1}{2},\frac{1}{4}}(\Gamma_0\times I)$ and $f \in H^1(\Omega_1\cup\Omega_2\times\{0\})$. If the interface Γ_0 and the boundary Γ are C^2 and the given data satisfy required compatibility condition (see [16, 34]), then the solution $u \in H^{2,1}(\Omega_1\cup\Omega_2\times I)$ and*

$$||u||^2_{(2,1),\Omega_1\cup\Omega_2\times I} \leq C\left(||F||^2_{(0,0),\Omega\times I} + ||g||^2_{(\frac{3}{2},\frac{3}{4}),\Gamma\times I} + ||q_0||^2_{(\frac{3}{2},\frac{3}{4}),\Gamma_0\times I}\right.$$
$$\left. + ||q_1||^2_{(\frac{1}{2},\frac{1}{4}),\Gamma_0\times I} + ||f||^2_{1,\Omega_1\cup\Omega_2\times\{0\}}\right). \quad (2.16)$$

Here C is a generic constant.

Proof. First, we define $\bar{u}_2 \in H^{2,1}(\Omega_2\times I)$, which satisfies

$$\bar{u}_2 = g \text{ on } \Gamma\times I, \quad \text{and} \quad n\cdot\mathcal{A}^2\nabla\bar{u}_2 = \bar{u}_2 = 0 \text{ on } \Gamma_0\times I. \quad (2.17)$$

If $g \in H^{\frac{3}{2},\frac{3}{4}}(\Gamma\times I)$ and $\bar{u}_2(x,0) \in H^1(\Omega_2\times\{0\})$, and satisfy the compatibility condition, then from Theorem 2.1 of [16], the following estimates hold:

$$||g||_{(\frac{3}{2},\frac{3}{4}),\Gamma\times I} \leq C||\bar{u}_2||_{(2,1),\Omega_2\times I},$$
$$||\bar{u}_2||_{1,\Omega_2\times\{0\}} \leq C||\bar{u}_2||_{(2,1),\Omega_2\times I}. \quad (2.18)$$

Further, using Theorem 2.4 of [34], the following estimate holds:

$$||\bar{u}_2||^2_{(2,1),\Omega_2\times I} \leq C||g||^2_{(\frac{3}{2},\frac{3}{4}),\Gamma\times I}. \quad (2.19)$$

Similarly, we define $\bar{u}_1 \in H^{2,1}(\Omega_1\times I)$, which satisfies

$$\bar{u}_1 = q_0 \text{ on } \Gamma_0\times I, \quad \text{and} \quad n\cdot\mathcal{A}^1\nabla\bar{u}_1 = q_1 \text{ on } \Gamma_0\times I. \quad (2.20)$$

If $q_0 \in H^{\frac{3}{2},\frac{3}{4}}(\Gamma_0\times I), q_1 \in H^{\frac{1}{2},\frac{1}{4}}(\Gamma_0\times I)$ and $\bar{u}_1(x,0) \in H^1(\Omega_1\times\{0\})$, and satisfy the compatibility condition, then from Theorem 2.3 of [16], the following estimate holds:

$$||q_0||_{(\frac{3}{2},\frac{3}{4}),\Gamma_0\times I} \leq C||\bar{u}_1||_{(2,1),\Omega_1\times I},$$
$$||q_1||_{(\frac{1}{2},\frac{1}{4}),\Gamma_0\times I} \leq C||\bar{u}_1||_{(2,1),\Omega_1\times I}, \quad (2.21)$$
$$||\bar{u}_1||_{1,\Omega_1\times\{0\}} \leq C||\bar{u}_1||_{(2,1),\Omega_1\times I}.$$

Similarly, using Theorem 2.4 of [34], the following estimate holds:

$$||\bar{u}_1||^2_{(2,1),\Omega_1\times I} \leq C\left(||q_1||^2_{(\frac{1}{2},\frac{1}{4}),\Gamma_0\times I} + ||q_0||^2_{(\frac{3}{2},\frac{3}{4}),\Gamma_0\times I}\right). \quad (2.22)$$

Now we define $\bar{u}$ as in $((\Omega_1\cup\Omega_2)\times I)$ which satisfies the following conditions

1. $\bar{u}|_{\Omega_1} = \bar{u}_1$ and $\bar{u}|_{\Omega_2} = \bar{u}_2$
2. $\bar{u} = g \quad \text{on} \quad \Gamma \times I.$
3. At interface, $\bar{u}$ is defined as

$$[\bar{u}] = \bar{u}_1 - \bar{u}_2 = q_0 \quad \text{and} \quad [n \cdot \mathcal{A}\nabla \bar{u}] = n \cdot (\mathcal{A}^1 \nabla \bar{u}_1 - \mathcal{A}^2 \nabla \bar{u}_2) = q_1 \text{ on } \Gamma_0 \times I.$$

Using the definition of the norm (2.3), we obtain

$$||\bar{u}||^2_{(2,1),(\Omega_1\cup\Omega_2)\times I} = ||\bar{u}_1||^2_{(2,1),\Omega_1\times I} + ||\bar{u}_2||^2_{(2,1),\Omega_2\times I}. \tag{2.23}$$

From equations (2.19) and (2.22), we establish the following estimate

$$||\bar{u}||^2_{(2,1),(\Omega_1\cup\Omega_2)\times I} \leq C\Big(||g||^2_{(\frac{3}{2},\frac{3}{4}),\Gamma\times I} + ||q_0||^2_{(\frac{3}{2},\frac{3}{4}),\Gamma_0\times I} + ||q_1||^2_{(\frac{1}{2},\frac{1}{4}),\Gamma_0\times I}\Big). \tag{2.24}$$

Finally, we define $v = u - \bar{u}$, where u solve the problem (1.1). Then v satisfies the following interface problem

$$\begin{aligned} \mathcal{L}v &= F - \mathcal{L}\bar{u} \quad \text{in} \quad \Omega_1 \cup \Omega_2 \times I, \\ v &= v(x,0) \quad \text{on} \quad \Omega_1 \cup \Omega_2 \times \{0\} \qquad \text{(initial condition)} \\ v &= 0 \quad \text{on} \quad \Gamma \times I, \quad \text{(exterior boundary condition)} \end{aligned} \tag{2.25}$$

along with the interface conditions

$$[v] = 0 \quad \text{and} \quad [n \cdot \mathcal{A}\nabla u] = 0 \quad \text{on} \quad \Gamma_0 \times I. \tag{2.26}$$

From Lemma 2.1, $v \in H^{2,1}((\Omega_1 \cup \Omega_2) \times I)$ and satisfies the following estimate:

$$||v||^2_{(2,1),\Omega_1\cup\Omega_2\times I} \leq C\Big(||F - \mathcal{L}\bar{u}||^2_{(0,0),\Omega_1\cup\Omega_2\times I} + ||v||^2_{1,\Omega_1\cup\Omega_2\times\{0\}}\Big). \tag{2.27}$$

Moreover, we get

$$||u||^2_{(2,1),(\Omega_1\cup\Omega_2)\times I} \leq ||u - \bar{u}||^2_{(2,1),\Omega_1\cup\Omega_2\times I} + ||\bar{u}||^2_{(2,1),\Omega_1\cup\Omega_2\times I}. \tag{2.28}$$

From equation (2.27), it follows:

$$\begin{aligned} ||u||^2_{(2,1),(\Omega_1\cup\Omega_2)\times I} &\leq C\Big(||F - \mathcal{L}\bar{u}||^2_{(0,0),\Omega_1\cup\Omega_2\times I} + ||v||^2_{1,\Omega_1\cup\Omega_2\times\{0\}}\Big) + ||\bar{u}||^2_{(2,1),\Omega_1\cup\Omega_2\times I} \\ &\leq C\Big(||F||^2_{(0,0),\Omega_1\cup\Omega_2\times I} + ||\bar{u}||^2_{(2,1),\Omega_1\cup\Omega_2\times I} + ||u||^2_{1,\Omega_1\cup\Omega_2\times\{0\}}\Big). \end{aligned} \tag{2.29}$$

Combining equations (2.24) and (2.29), the final result follows. □

Theorem 2.2. *Let* $F \in H^{2r,r}(\Omega_1 \cup \Omega_2 \times I)), g \in H^{\frac{3}{2}+2r,\frac{3}{4}+r}(\Gamma \times I), q_0 \in H^{\frac{3}{2}+2r,\frac{3}{4}+r}(\Gamma_0 \times I), q_1 \in H^{\frac{1}{2}+2r,\frac{1}{4}+r}(\Gamma_0 \times I)$ *and* $f \in H^{2r+2,r+1}(\Omega_1 \cup \Omega_2 \times \{0\})$. *If the interface* Γ_0 *and boundary* Γ *is* C^{2r+2} *and the given data satisfy the required compatibility condition (see [16]), then the solution* $u \in H^{2r+2,r+1}(\Omega_1 \cup \Omega_2 \times I)$ *and*

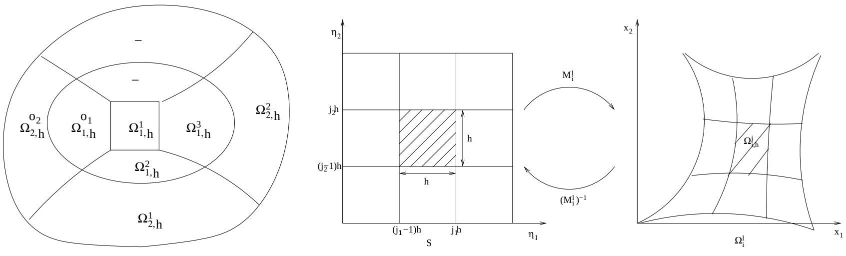
$$\begin{aligned} ||u||^2_{(2r+1,r+1),\Omega_1\cup\Omega_2\times I} \leq C_r\Big(&||F||^2_{(2r,r),\Omega_1\cup\Omega_2\times I} + ||g||^2_{(\frac{3}{2}+2r,\frac{3}{4}+r),\Gamma\times I} \\ &+ ||q_0||^2_{(\frac{3}{2}+2r,\frac{3}{4}+r),\Gamma_0\times I} + ||q_1||^2_{(\frac{1}{2}+2r,\frac{1}{4}+r),\Gamma_0\times I} + ||f||^2_{2r+1,\Omega_1\cup\Omega_2\times\{0\}}\Big). \end{aligned}$$

Proof. The idea of proof is the same as in Theorem 2.1. □

3. Discretization and Stability Estimate

First, the domains Ω_1 and Ω_2 are partitioned into quadrilaterals $\Omega_1^1, \Omega_1^2, \cdots, \Omega_1^{r_1}$ and $\Omega_2^1, \Omega_2^2, \cdots, \Omega_2^{r_2}$ such that the subdomain divisions match on the interface. We define a smooth function $M_i^l = (X_{1,i}^l, X_{2,i}^l)$ that maps the unit square S to $\Omega_i^l, i = 1, 2$ as in [1, 21] and is given by

$$x_{1,i}^l = X_{1,i}^l(\eta_1, \eta_2) \ \text{ and } \ x_{2,i}^l = X_{2,i}^l(\eta_1, \eta_2). \tag{3.1}$$



(a) (b)

Figure 2: (a) Discretization, (b) Mesh imposing on $\Omega_i, i = 1, 2$..

We now divide S into a mesh of squares of side h. Consequently, the image Ω_i^l, which is divided into a quasi-uniform mesh of curvilinear rectangles of side proportional to h, is the grid of squares S under the mapping M_i^l as shown in Fig 2(b). Moreover, the domains Ω_1 and Ω_2 are divided into curvilinear rectangles $\Omega_{1,h}^1, \Omega_{1,h}^2, \cdots, \Omega_{1,h}^{o_1}$ and $\Omega_{2,h}^1, \Omega_{2,h}^2, \cdots, \Omega_{2,h}^{o_2}$ of width proportional to h such that the subdomain divisions match on the interface as shown in Fig 2(a). Thus, $\Omega_{i,h}^l$ is the image of $((j_1 - 1)h \leq \eta_1 \leq j_1 h) \times ((j_2 - 1)h \leq \eta_1 \leq j_2 h)$ under the mapping M_i^l. We choose the time step k proportional to h^2. We introduce new coordinates $s = t/k$, $y_i = x_i/h$ and define $\widetilde{u}(y_1, y_2, s) = u(hy_1, hy_2, ks)$. In this new coordinate system the differential equation becomes

$$\mathscr{L}\widetilde{u} = k\widetilde{F}, \tag{3.2a}$$

where

$$\mathscr{L}\widetilde{u} = \widetilde{u}_s - \sum_{i,j=1}^{2} (\alpha_{ij}(y, s)\widetilde{u}_{y_j})_{y_i}. \tag{3.2b}$$

Clearly the coefficients satisfy the following condition:

$$|D_y^\mu D_s^\gamma \alpha_{i,j}| = O(h^{|\mu|} k^\gamma). \tag{3.3}$$

Let $\widetilde{\Omega}_i$ and $\widetilde{\Omega}_{i,h}^l$ be the images of Ω_i and $\Omega_{i,h}^l$ in the y-coordinates. Further, let $\widetilde{\gamma}_m$ be the image of the size γ_m common to $\Omega_{i,h}^l$ and $\Omega_{j,h}^l$. Now we define a map N_i^l where $N_i^l : S \to \widetilde{\Omega}_{i,h}^l$ for every l in $i = 1, 2$. The form of N_i^l is as follows:

$$N_i^l(\xi_1, \xi_2) = \frac{1}{h} M_i^{l_1}((l_2 - 1)h + h\xi_1, (l_3 - 1)h + h\xi_2).$$

Let J_i^l be the Jacobian of the map N_i^l. Then there exist two uniform constants V_1 and V_2, which depend on the decomposition of Ω_i $(i = 1, 2)$ into $\Omega_{i,h}^l$, and satisfy the following

$$V_1 \leq |J_i^l(\xi_1, \xi_2)| \leq V_2. \tag{3.4}$$

for all $l = 1, 2, \cdots, o_1$ with $i = 1$ and $l = 1, 2, \cdots, o_2$ with $i = 2$.

Furthermore, the step $nk \leq t < (n+1)k$ is mapped to $n \leq s < (n+1)$ by the transformation $s = t/k$.

3.1. Stability Estimate

Define the spectral element functions $\check{w}_\kappa^l(\xi_1, \xi_2, s), \kappa = 1, 2.$, which are polynomials of degree p in each of the space variables ξ_1 and ξ_2 and of degree q in the time variable s, i.e.

$$\check{w}_\kappa^l(\xi_1, \xi_2, s) = \sum_{i_1=0}^{p} \sum_{i_2=0}^{p} \sum_{i_3=0}^{q} \delta_{i_1,i_2,i_3}^{l,n,\kappa} \xi_1^{i_1} \xi_2^{i_2} (s - n)^{i_3}$$

for $(\xi_1, \xi_2) \in S$ and $n \leq s < n + 1$. Here $\delta_{i_1,i_2,i_3}^{l,n,\kappa}$ denote the coefficients. Then

$$\check{w}_\kappa^l(y_1, y_2, s) = \check{w}_\kappa^l((N_\kappa^l)^{-1}(y_1, y_2), s).$$

Choosing $\eta = Kh^2$ and $\tilde{v}^l_\kappa = \tilde{w}^l_\kappa e^{-\eta s}$, where K is a positive constant, then $(\mathscr{L}\tilde{w}^l_\kappa)e^{-\eta s} = (\mathscr{L}+\eta)\tilde{v}^l_\kappa$. Using the chain rule, we can write

$$\frac{\partial \tilde{w}^l_\kappa}{\partial y_1} = (\check{w}^l_\kappa)_{\xi_1}(\xi_1)_{y_1} + (\check{w}^l_\kappa)_{\xi_2}(\xi_2)_{y_1} \text{ and } \frac{\partial \tilde{w}^l_\kappa}{\partial y_2} = (\check{w}^l_\kappa)_{\xi_1}(\xi_1)_{y_2} + (\check{w}^l_\kappa)_{\xi_2}(\xi_2)_{y_2}.$$

Define $\xi = (\xi_1, \xi_2)$. Assume that $\widehat{(\xi_1)}_{y_1}(\xi), \widehat{(\xi_2)}_{y_1}(\xi), \widehat{(\xi_1)}_{y_2}(\xi)$ and $\widehat{(\xi_2)}_{y_2}(\xi)$ are the orthogonal projections of $(\xi_1)_{y_1}(\xi)$, $(\xi_2)_{y_1}(\xi)$, $(\xi_1)_{y_2}(\xi)$ and $(\xi_2)_{y_2}(\xi)$, respectively, into the space of polynomials of degree p with respect to the inner product in $H^2(S)$. Let

$$\left(\frac{\partial \tilde{v}^l_\kappa}{\partial y_1}\right)^a = (\check{v}^l_\kappa)_{\xi_1}\widehat{(\xi_1)}_{y_1} + (\check{v}^l_\kappa)_{\xi_2}\widehat{(\xi_2)}_{y_1} \text{ and } \left(\frac{\partial \tilde{v}^l_\kappa}{\partial y_2}\right)^a = (\check{v}^l_\kappa)_{\xi_1}\widehat{(\xi_1)}_{y_2} + (\check{v}^l_\kappa)_{\xi_2}\widehat{(\xi_2)}_{y_2}.$$

Let $\tilde{\gamma}_m$ be a side common to $\Omega^m_{\kappa,h}$ and $\Omega^l_{\kappa,h}$ which is the image of $\xi_1 = 1$ under the map N^m_κ and the image of $\xi_1 = 0$ under the map N^l_κ. Now, we define the jump term at the inter element boundary $\tilde{\gamma}_m$:

$$||[\tilde{v}]||^2_{(r,s),\tilde{\gamma}_m\times I_n} = ||\check{v}^m_\kappa(1,\xi_2,s) - \check{v}^l_\kappa(0,\xi_2,s)||^2_{(r,s),(0,1)\times I_n} \tag{3.5}$$

and the derivative of the jump term at the inter element boundary $\tilde{\gamma}_m$

$$\left|\left|\left[\left(\frac{\partial \tilde{v}}{\partial y_j}\right)^a\right]\right|\right|^2_{(r,s),\tilde{\gamma}_m\times I_n} = \left|\left|\left[\left(\frac{\partial \tilde{v}^m_\kappa}{\partial y_j}\right)^a(1,\xi_2,s) - \left(\frac{\partial \tilde{v}^l_\kappa}{\partial y_j}\right)^a(0,\xi_2,s)\right]\right|\right|^2_{(r,s),(0,1)\times I_n}$$

for $j = 1, 2$, where $I_n = (n, n+1)$. We then define

$$\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n} |\mathscr{L}\tilde{v}^l_\kappa|^2 dy_1 dy_2 ds = \int_{S\times I_n} |\mathscr{L}^l_\kappa \check{v}^l_\kappa|^2 d\xi_1 d\xi_2 ds = ||\mathscr{L}^l_\kappa \check{v}^l_\kappa||^2_{S\times I_n}, \tag{3.6}$$

where $\mathscr{L}^l_\kappa = \check{\mathscr{L}}\sqrt{\mathscr{J}^l_\kappa}$ and $\check{\mathscr{L}}$ is the differential operator $\mathscr{L}$ in ξ_1, ξ_2 and s coordinates. Here $\mathscr{J}^l_\kappa$ denotes the Jacobian of the map N^l_κ from S to $\tilde{\Omega}^l_{\kappa,h}$. Define a new differential operator $(\mathscr{L}^l_\kappa)^a$, so that its coefficients are polynomials of degree p in each of the space variables ξ_1 and ξ_2 and of degree q in the time variable s defined as the orthogonal projections of the coefficients of the corresponding differential operator $\mathscr{L}^l_\kappa$ into the space of polynomials with respect to the usual inner product in $H^{2,1}(S \times I_n)$. Moreover

$$\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n} |\mathscr{L}\tilde{v}^l_\kappa|^2 dy_1 dy_2 ds = \int_{S\times I_n} |(\mathscr{L}^l_\kappa)^a \check{v}^l_\kappa|^2 d\xi_1 d\xi_2 ds = ||(\mathscr{L}^l_\kappa)^a \check{v}^l_\kappa||^2_{S\times I_n}, \tag{3.7}$$

up to a negligible error term [7, 22].
Let $\mathcal{F}^{(n)}_{v_1,v_2}$ be the spectral element representation of the function v i.e.

$$\mathcal{F}^{(n)}_{v_1,v_2} = \left\{\{\check{v}^l_1(\xi_1,\xi_2,s)\}_{1\le l\le o_1}, \{\check{v}^l_2(\xi_1,\xi_2,s)\}_{1\le l\le o_2}\right\}^{\mathcal{M}-1}_{n=0}, \text{ where } \mathcal{M}k = T.$$

By $\mathcal{S}^{(n)}_{p,q}(\mathcal{F}^{(n)}_{v_1,v_2})$, we denote the space of spectral element functions.
Define $F^l_\kappa(\xi,s) = (\mathscr{J}^l_\kappa)^{\frac{1}{2}}\tilde{F}^l_\kappa(N^l_\kappa(\xi_1,\xi_2)h, sk)$ and assume $\hat{F}^l_\kappa(\xi,s)$ to be the orthogonal projection of $F^l_\kappa(\xi,s)$ into the space of polynomials of degree $2p$ in each of the space variables ξ_1 and ξ_2 and of degree $2q$ in the time variable s with respect to the usual inner product in $L^2(S\times I_n)$. Similarly, we define $f^l_\kappa(\xi) = f^l_\kappa(N^l_\kappa(\xi_1,\xi_2)h)$ and let $\hat{f}^l_\kappa(\xi)$ be the orthogonal projection of $f^l_\kappa(\xi)$ into the space of polynomials of degree p in ξ_1 and ξ_2 with respect to the usual inner product in $H^1(S)$. For the boundary and interface terms, let $\tilde{\gamma}_m$ belong to either Γ or Γ_0 and assume that $\tilde{\gamma}_m$ is the image of $\xi_1 = 1$ under the mapping $N^l_\kappa : S \to \tilde{\Omega}^l_{\kappa,h}$. Define $g^l(\xi_2,s) = g(N^l_\kappa(1,\xi_2)h, sk), q^l_0(\xi_2,s) = q^l_0(N^l_\kappa(1,\xi_2)h, sk)$ and $q^l_1(\xi_2,s) = q_1(N^l_\kappa(1,\xi_2)h, sk)$. Let $\hat{g}^l(\xi_2,s), \hat{q}^l_0(\xi_2,s)$ and $\hat{q}^l_1(\xi_2,s)$ denote the orthogonal projection of $g^l(\xi_2,s), q^l_0(\xi_2,s)$ and $q^l_1(\xi_2,s)$ into the space of polynomials of degree p in ξ_2 and q in s.
To initialize the scheme, we define

$$\check{w}^l_\kappa(\xi, s = 0^-) = f^l_\kappa(\xi).$$

To obtain the solution for $0 \leq s < n$, where n is an integer, we define our approximate solution for $n \leq s < n+1$ to be the unique $\mathcal{F}^{(n)}_{w_1,w_2}$ which minimizes the functional

$$\begin{aligned}\mathscr{R}^{(n)}(\mathcal{F}^{(n)}_{v_1,v_2}) =&\mathscr{R}_{Initial}(\mathcal{F}^{(n)}_{v_1,v_2},f) + \frac{1}{h^2}\Big(\mathscr{R}_{PDE}(\mathcal{F}^{(n)}_{v_1,v_2},F) + \mathscr{R}_{Jump}(\mathcal{F}^{(n)}_{v_1,v_2}) \\ &+ \mathscr{R}_{Boundary}(\mathcal{F}^{(n)}_{v_1,v_2},g) + \mathscr{R}_{Interface}(\mathcal{F}^{(n)}_{v_1,v_2},q)\Big) \end{aligned} \tag{3.8}$$

over all $\mathcal{S}^{(n)}_{p,q}(\mathcal{F}^{(n)}_{v_1,v_2})$, where

$$\begin{aligned}\mathscr{R}_{Initial}(\mathcal{F}^{(n)}_{v_1,v_2},f) =&\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Big(\int_{S\times\{n^+\}}|\check{v}^l_\kappa - \check{w}^l_\kappa(\xi,n^-)|^2\,\hat{\mathscr{J}}^l_i d\xi \\ &+\sum_{i,j=1}^{2}\int_{S\times\{n^+\}}(\check{v}^l_\kappa - \check{w}^l_\kappa(\xi,n^-))_{\xi_i}(\hat{\mathscr{A}}^l_\kappa)_{i,j}(\check{v}^l_\kappa - \check{w}^l_\kappa(\xi,n^-))_{\xi_j}d\xi\Big),\\ \mathscr{R}_{PDE}(\mathcal{F}^{(n)}_{v_1,v_2},F) =&\sum_{l=1}^{o_1}||(\mathscr{L}^l_1)^a\check{v}^l_1 - \hat{F}^l_1||^2_{S\times I_n} + \sum_{l=1}^{o_2}||(\mathscr{L}^l_2)^a\check{v}^l_2 - \hat{F}^l_2||^2_{S\times I_n},\\ \mathscr{R}_{Jump}(\mathcal{F}^{(n)}_{v_1,v_2}) =&\sum_{\kappa=1}^{2}\sum_{\tilde{\gamma}_m\subseteq\tilde{\Omega}_\kappa}\Big(|||[\check{v}]|||_{(0,3/4),\tilde{\gamma}_m\times I_n} + \sum_{j=1}^{2}|||[(\check{v}_{y_j})^a]|||_{(1/2,1/4),\tilde{\gamma}_m\times I_n}\Big),\\ \mathscr{R}_{Boundary}(\mathcal{F}^{(n)}_{v_1,v_2},g) =& \sum_{\tilde{\gamma}_m\subseteq\Gamma}\Big(||\check{v} - \hat{g}^m||_{(0,3/4),\tilde{\gamma}_m\times I_n} + ||(\check{v})^a_\tau - (\hat{g}^m)^a_\tau||_{(1/2,1/4),\tilde{\gamma}_m\times I_n}\Big),\\ \mathscr{R}_{Interface}(\mathcal{F}^{(n)}_{v_1,v_2},q) =& \sum_{\tilde{\gamma}_m\subseteq\Gamma_0}\Big(|||[\check{v}] - \hat{q}_0||_{(\frac{3}{2},\frac{3}{4}),\tilde{\gamma}_m\times I_n} + \Big|\Big|\Big[\Big(\frac{\partial\check{v}}{\partial\nu}\Big)_\alpha\Big] - \hat{q}_1\Big|\Big|_{(\frac{1}{2},\frac{1}{4}),\tilde{\gamma}_m\times I_n}\Big).\end{aligned}$$

Here, $(\check{v})_\tau$ and $\frac{\partial\check{v}}{\partial\nu}$ denote the tangential and normal derivatives on $\tilde{\gamma}_m$ same as defined in [7, 10, 11]. As defined in (3.8), we choose our approximate solution to be the unique $\mathcal{F}^{(n)}_{w_1,w_2}$ which minimizes the functional $\mathscr{R}^{(n)}(\mathcal{F}^{(n)}_{v_1,v_2})$ over all $\mathcal{S}^{(n)}_{p,q}(\mathcal{F}^{(n)}_{v_1,v_2})$. Now, we define the functional

$$\begin{aligned}\mathscr{W}^{(n)}(\mathcal{F}^{(n)}_{w_1,w_2}) =&\mathscr{R}_{Initial}(\mathcal{F}^{(n)}_{w_1,w_2},0) + \frac{1}{h^2}\Big(\mathscr{R}_{PDE}(\mathcal{F}^{(n)}_{w_1,w_2},0) + \mathscr{R}_{Jump}(\mathcal{F}^{(n)}_{w_1,w_2}) \\ &+ \mathscr{R}_{Boundary}(\mathcal{F}^{(n)}_{w_1,w_2},0) + \mathscr{R}_{Interface}(\mathcal{F}^{(n)}_{w_1,w_2},0)\Big).\end{aligned} \tag{3.9}$$

From equations (3.8) and (3.9), it is clear that $\mathscr{W}^{(n)}(\mathcal{F}^{(n)}_{v_1,v_2})$ is the functional $\mathscr{R}^{(n)}(\mathcal{F}^{(n)}_{v_1,v_2})$ with zero data. We are now in a position to state the main stability theorem.

Theorem 3.1 (Stability theorem)**.** *The estimate*

$$\begin{aligned}g_4\sum_{\kappa=1}^{2}\Big(\sum_{l=1}^{o_\kappa}\Big(h^2||\check{w}^l_\kappa||^2_{S\times I_n} + ||\partial_s\check{w}^l_\kappa||^2_{S\times I_n} + \sum_{1\leq|\alpha|\leq 2}||D^\alpha_y\check{w}^l_\kappa||^2_{S\times I_n} + ||\check{w}^l_\kappa||^2_{S\times(n+1)^-}\\ + \sum_{i,j=1}^{2}\int_{S\times(n+1)^-}(\check{w}^l_\kappa)_{\xi_i}(\hat{\mathscr{A}}^l_\kappa)_{i,j}(\check{w}^l_\kappa)_{\xi_j}d\xi_1 d\xi_2\Big) \leq (1+c_1h^2)\mathscr{W}^{(n)}(\mathcal{F}^{(n)}_{w_1,w_2})\end{aligned}$$

holds for large enough $\frac{1}{h}$ and p with $\ln p = o(1/h)$. Here g_4 and c_1 are constants.

3.2. Proof of the stability theorem

To calculate the estimate of higher order derivatives of $\tilde{v}$ as in [22], we decompose the problem, which is as follows:

$$\mathscr{L}\tilde{v} = \tilde{v}_s - \mathscr{E}\tilde{v}, \quad \text{where } \mathscr{E}\tilde{v} = \sum_{i,j=1}^{2}(\alpha_{i,j}\tilde{v}_{y_j})_{y_i}. \tag{3.10}$$

Assume $\nu = (\nu_1,\nu_2)$ to be the outward normal to the curve $\tilde{\gamma}_m$ at the point ξ. Now, we define $\left(\frac{\partial\tilde{v}}{\partial\nu}\right)_\alpha(\xi) = \sum_{i,j=1}^{2}\nu_i\alpha_{i,j}\left(\frac{\partial\tilde{v}}{\partial y_j}\right)(\xi)$ which denotes the conormal derivative at a point on $\tilde{\gamma}_m$. Furthermore, let $d\mu$ be an element of arc length along $\tilde{\gamma}_m$.

Lemma 3.1. *The estimate*

$$\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Big(||\tilde v_\kappa^l||^2_{\tilde\Omega^l_{\kappa,h}\times(n+1)^-}+b_1\sum_{i=1}^{2}||(\tilde v_\kappa^l)_{y_i}||^2_{\tilde\Omega^l_{\kappa,h}\times I_n}+\frac{3}{2}Kh^2||(\tilde v_\kappa^l)||^2_{\tilde\Omega^l_{\kappa,h}\times I_n}\Big)$$
$$-\sum_{\kappa=1}^{2}\sum_{\tilde\gamma_m\subseteq\tilde\Omega_\kappa}\int_{\tilde\gamma_m\times I_n}2\Big[\tilde v\Big(\frac{\partial\tilde v}{\partial\nu}\Big)_\alpha\Big]d\mu ds-\sum_{\tilde\gamma_m\subseteq\Gamma_0}\int_{\tilde\gamma_m\times I_n}2\Big[\tilde v\Big(\frac{\partial\tilde v}{\partial\nu}\Big)_\alpha\Big]d\mu ds$$
$$-\sum_{\tilde\gamma_m\subseteq\Gamma}\int_{\tilde\gamma_m\times I_n}2\tilde v\Big(\frac{\partial\tilde v}{\partial\nu}\Big)_\alpha d\mu ds\le\frac{1}{h^2}\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Big(||(\mathscr{L}+\eta)\tilde v_l^\kappa||^2_{\tilde\Omega^l_{\kappa,h}\times I_n}+||\tilde v_l^\kappa||^2_{\tilde\Omega^l_{\kappa,h}\times\{n^+\}}\Big)$$

holds for large enough K, where $\eta=Kh^2$. Here m^- and m^+ denote respectively $\lim_{t\uparrow m}t$ and $\lim_{t\downarrow m}t$. $b_1>0$ is a constant.

Proof. From the equation (3.10), we have

$$\int_{\tilde\Omega^l_{\kappa,h}\times I_n}\tilde v_\kappa^l\big((\tilde v_\kappa^l)_s-\mathscr{E}\tilde v_\kappa^l+\eta\tilde v_\kappa^l\big)dyds=\int_{\tilde\Omega^l_{\kappa,h}\times I_n}\tilde v_\kappa^l((\mathscr{L}+\eta)\tilde v_\kappa^l)dyds. \tag{3.11}$$

Using integration by parts, it follows:

$$2\int_{\tilde\Omega^l_{\kappa,h}\times I_n}\tilde v_\kappa^l(\tilde v_\kappa^l)_s dyds=\int_{\tilde\Omega^l_{\kappa,h}\times\{(n+1)^-\}}|\tilde v_\kappa^l|^2dy-\int_{\tilde\Omega^l_{\kappa,h}\times\{n^+\}}|\tilde v_\kappa^l|^2dy, \tag{3.12}$$

$$-\int_{\tilde\Omega^l_{\kappa,h}\times I_n}\tilde v_\kappa^l(\mathscr{E}\tilde v_\kappa^l)dyds=\int_{\tilde\Omega^l_{\kappa,h}\times I_n}\sum_{i,j=1}^{2}(\tilde v_\kappa^l)_{y_i}\alpha_{ij}(\tilde v_\kappa^l)_{y_j}dyds$$
$$-\int_{\partial\tilde\Omega^l_{\kappa,h}\times I_n}\tilde v_\kappa^l\Big(\frac{\partial\tilde v_\kappa^l}{\partial\nu}\Big)_\alpha d\mu ds. \tag{3.13}$$

Inserting the equations (3.12)- (3.13) into (3.11), we obtain

$$\frac{1}{2}\int_{\tilde\Omega^l_{\kappa,h}\times(n+1)^-}|\tilde v_\kappa^l|^2dy+\int_{\tilde\Omega^l_{\kappa,h}\times I_n}\Big(\sum_{i,j=1}^{2}(\tilde v_\kappa^l)_{y_i}\alpha_{ij}(\tilde v_\kappa^l)_{y_j}+\eta|\tilde v_\kappa^l|^2\Big)dyds$$
$$-\int_{\partial\tilde\Omega^l_{\kappa,h}\times I_n}\tilde v_\kappa^l\Big(\frac{\partial\tilde v_\kappa^l}{\partial\nu}\Big)_\alpha d\mu ds=\int_{\tilde\Omega^l_{\kappa,h}\times I_n}\tilde v_\kappa^l((\mathscr{L}+\eta)\tilde v_\kappa^l)dyds+\int_{\tilde\Omega^l_{\kappa,h}\times\{n^+\}}|\tilde v_\kappa^l|^2\frac{dy}{2}.$$

Summing over l for each $\Omega^l_{\kappa,h},\kappa=1,2$, gives

$$\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Big(\int_{\tilde\Omega^l_{\kappa,h}\times(n+1)^-}|\tilde v_\kappa^l|^2\frac{dy}{2}+\int_{\tilde\Omega^l_{\kappa,h}\times I_n}\Big(\sum_{i,j=1}^{2}(\tilde v_\kappa^l)_{y_i}\alpha_{ij}(\tilde v_\kappa^l)_{y_j}+\eta|\tilde v_\kappa^l|^2\Big)dyds\Big)$$
$$-\sum_{\kappa=1}^{2}\sum_{\tilde\gamma_m\subseteq\tilde\Omega_\kappa}\int_{\tilde\gamma_m\times I_n}\Big[\tilde v\Big(\frac{\partial\tilde v}{\partial\nu}\Big)_\alpha\Big]d\mu ds-\sum_{\tilde\gamma_m\subseteq\Gamma_0}\int_{\tilde\gamma_m\times I_n}\Big[\tilde v\Big(\frac{\partial\tilde v}{\partial\nu}\Big)_\alpha\Big]d\mu ds$$
$$-\sum_{\tilde\gamma_m\subseteq\Gamma}\int_{\tilde\gamma_m\times I_n}\tilde v\Big(\frac{\partial\tilde v}{\partial\nu}\Big)_\alpha d\mu ds=\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Big(\int_{\tilde\Omega^l_{\kappa,h}\times I_n}\tilde v_\kappa^l((\mathscr{L}+\eta)\tilde v_\kappa^l)dyds+\int_{\tilde\Omega^l_{\kappa,h}\times\{n^+\}}|\tilde v_\kappa^l|^2\frac{dy}{2}\Big).$$

From (3.3) and choosing K large enough, where $\eta=Kh^2$, the result holds. □

In the following Lemma 3.2, we obtain estimates for higher order derivatives of $\tilde v_\kappa^l$.

Lemma 3.2. *The estimate*

$$e_1\Big(\sum_{l=1}^{o_1}\Big(\int_{\tilde\Omega^l_{1,h}\times I_n}\sum_{|\alpha|=2}|D_y^\alpha\tilde v_1^l|^2dyds\Big)+\sum_{l=1}^{o_2}\Big(\int_{\tilde\Omega^l_{2,h}\times I_n}\sum_{|\alpha|=2}|D_y^\alpha\tilde v_2^l|^2dyds\Big)\Big)\le\mathcal{E}_1.$$

holds, where

$$\mathcal{E}_1 = \sum_{\kappa=1}^{2}\Big(\sum_{l=1}^{o_\kappa}\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}|\mathscr{E}\tilde{v}^l_\kappa|^2 dyds + \sum_{\tilde{\gamma}_m\subseteq\tilde{\Omega}_\kappa\cup\Gamma_0}\Big(\int_{\tilde{\gamma}_m\times I_n}[\Phi(\tilde{v})]d\mu ds + \int_{I_n}[H(\tilde{v})]ds\big|_{\partial\tilde{\gamma}_m}\Big)$$
$$+ f_1 h^2\sum_{l=1}^{o_\kappa}\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{|\alpha|=1}|D^\alpha_y\tilde{v}^l_\kappa|^2 dyds + f_1 h\sum_{\tilde{\gamma}_m\subseteq\Gamma\cup\Gamma_0\cup\tilde{\Omega}_\kappa}\int_{\tilde{\gamma}_m\times I_n}\sum_{|\alpha|=1}|D^\alpha_y\tilde{v}^l_\kappa|^2 d\mu ds\Big)$$
$$+ \sum_{\tilde{\gamma}_m\subseteq\Gamma}\int_{\tilde{\gamma}_m\times I_n}\Phi(\tilde{v})d\mu ds + \sum_{\tilde{\gamma}_m\subseteq\Gamma_0}\int_{I_n}H(\tilde{v})ds\big|_{\partial\tilde{\gamma}_m},$$

$H(\tilde{v}) = \frac{d_1}{2}\tilde{v}_\nu(-2D\tilde{v}_t - E\tilde{v}_\nu)$ *and* $\Phi(\tilde{v}) = d_1\frac{\partial\tilde{v}}{\partial\tau}\frac{d}{d\mu}\left(E\frac{\partial\tilde{v}}{\partial\tau} + G\frac{\partial\tilde{v}}{\partial\nu}\right)$. *Here* d_1, e_1 *and* f_1 *are positive constants and* D, E *and* F *are defined as:*

$$\begin{bmatrix} D & E \\ E & F\end{bmatrix} = \begin{bmatrix}\tau_1 & \tau_2\\ \nu_1 & \nu_2\end{bmatrix}\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}\begin{bmatrix}\tau_1 & \tau_2\\ \nu_1 & \nu_2\end{bmatrix}^{-1}$$

and the matrix $\begin{bmatrix}\tau_1 & \tau_2\\ \nu_1 & \nu_2\end{bmatrix}$ *is orthogonal matrix and* $\alpha_{ij} = \alpha_{ji}$ *for each* $\Omega_\kappa, \kappa = 1, 2$.

Proof. To prove the above lemma, we use the result of equation (3.25) from [7], which is as follows:

$$\frac{c}{4}\int_{\tilde{\Omega}^l_{\kappa,h}}\sum_{|\alpha|=2}|D^\alpha_y\tilde{v}^l_\kappa|^2 dy \le \frac{1}{c}\int_{\tilde{\Omega}^l_{\kappa,h}}|\mathscr{E}\tilde{v}^l_\kappa|^2 dy + Ch^2\left(\int_{\tilde{\Omega}^l_{\kappa,h}}\sum_{|\alpha|=1}|D^\alpha_y\tilde{v}^l_\kappa|^2 dy\right)$$
$$+ Ch\sum_{j=1}^{4}\int_{\tilde{\gamma}_j}\sum_{|\alpha|=1}|D^\alpha_y\tilde{v}^l_\kappa|^2 d\mu + \sum_{j=1}^{4}\int_{\tilde{\gamma}_j}(\tilde{v}^l_\kappa)_\tau\frac{d}{d\mu}(E(\tilde{v}^l_\kappa)_\tau + G(\tilde{v}^l_\kappa)_\nu)d\mu$$
$$+ \sum_{j=1}^{4}\frac{1}{2}(\tilde{v}^l_\kappa)_\nu(-2D(\tilde{v}^l_\kappa)_\tau - E(\tilde{v}^l_\kappa)_\nu)\Big|_{\partial\tilde{\gamma}_j} \tag{3.14}$$

by choosing a small enough $c > 0$. Here $G = F + D$, and C is a generic constant.
Integrating the equation (3.14) w.r. to s over I_n and summing over l for $\tilde{\Omega}^l_{\kappa,h}, \kappa = 1, 2$, the desired result follows. ☐

Next, we prove the following Lemma 3.3, which we directly use to obtain the main stability result.

Lemma 3.3. *The estimate holds*

$$\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Bigg(\Big(Kh^2||\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n} + e_2\Big(||\partial_s\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n} + \sum_{1\le|\alpha|\le 2}||D^\alpha_y\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\Big)\Big)$$
$$+\Big(||\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times(n+1)^-} + \sum_{i,j=1}^{2}\int_{\tilde{\Omega}^l_{\kappa,h}\times(n+1)^-}(\tilde{v}^l_\kappa)_{y_i}\alpha_{i,j}(\tilde{v}^l_\kappa)_{y_j}dy\Big)\Bigg) \le \mathcal{E}_2 + \mathcal{E}_3$$

for small enough h. *Where*

$$\mathcal{E}_2 = \sum_{\kappa=1}^{2}\Big(\sum_{l=1}^{o_\kappa}\Big(||\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times(n)^+} + \sum_{i,j=1}^{2}\int_{\tilde{\Omega}^l_{\kappa,h}\times(n)^+}(\tilde{v}^l_\kappa)_{y_i}\alpha_{i,j}(\tilde{v}^l_\kappa)_{y_j}dy$$
$$+ \frac{1}{h^2}(1 + 2h^2)\sum_{l=1}^{o_\kappa}||(\mathscr{L} + \eta)\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\Big)$$

and

$$\begin{aligned}\mathcal{E}_3 =&\sum_{\kappa=1}^{2}\sum_{\tilde{\gamma}_m\subseteq\Omega_\kappa}\Big(\int_{\tilde{\gamma}_m\times I_n}\Big[J(\tilde{v})+2\tilde{v}\Big(\frac{\partial\tilde{v}}{\partial\nu}\Big)_\alpha\Big]d\mu ds+\int_{I_n}[H(\tilde{v})]ds\Big|_{\partial\tilde{\gamma}_m}\Big)\Big)\\&+\sum_{\tilde{\gamma}_m\subseteq\Gamma_0}\Big(\int_{\tilde{\gamma}_m\times I_n}\Big[J(\tilde{v})+2\tilde{v}\Big(\frac{\partial\tilde{v}}{\partial\nu}\Big)_\alpha\Big]d\mu ds+\int_{I_n}[H(\tilde{v})]ds\Big|_{\partial\tilde{\gamma}_m}\Big)\Big)\\&+\sum_{\tilde{\gamma}_m\subseteq\Gamma}\left(\int_{\tilde{\gamma}_m\times I_n}J(\tilde{v})+2\tilde{v}\Big(\frac{\partial\tilde{v}}{\partial\nu}\Big)_\alpha d\mu ds+\int_{I_n}H(\tilde{v})ds\Big|_{\partial\tilde{\gamma}_m}\right).\end{aligned}\tag{3.15}$$

Here $J(\tilde{v})=2\tilde{v}_s\left(\frac{\partial\tilde{v}}{\partial\nu}\right)_\alpha+d_1\left(\tilde{v}_\tau\frac{d}{d\mu}(E\tilde{v}_\mu+G\tilde{v}_\nu)\right)=2\tilde{v}_s\left(\frac{\partial\tilde{v}}{\partial\nu}\right)_\alpha+\Phi(\tilde{v})$ *and* $H(\tilde{v})=\frac{d}{2}\tilde{v}_\nu(-2D\tilde{v}_t-E\tilde{v}_\nu)$.

Proof. Firstly, we calculate

$$\begin{aligned}\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}|\mathscr{L}\tilde{v}^l_\kappa|^2dyds&=\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}|(\tilde{v}^l_\kappa)_s-\mathscr{E}\tilde{v}^l_\kappa|^2dyds,\\&=\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}(|(\tilde{v}^l_\kappa)_s|^2-2(\tilde{v}^l_\kappa)_s\mathscr{E}\tilde{v}^l_\kappa+|\mathscr{E}\tilde{v}^l_\kappa|^2)dyds.\end{aligned}\tag{3.16}$$

Using integration by parts, we rewrite the following term

$$\begin{aligned}-2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}(\tilde{v}^l_\kappa)_s(\mathscr{E}\tilde{v}^l_\kappa)dyds=&\,2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_is}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dyds\\&-\sum_{\tilde{\gamma}_m\subseteq\partial\tilde{\Omega}^l_{\kappa,h}}2\int_{\tilde{\gamma}_m\times I_n}(\tilde{v}^l_\kappa)_s\left(\frac{\partial\tilde{v}^\kappa_l}{\partial\nu}\right)_\alpha d\mu ds.\end{aligned}\tag{3.17}$$

Again using integration by parts the first term of R.H.S. in (3.17) w. r. to s, gives:

$$\begin{aligned}2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_is}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dyds=&\int_{\tilde{\Omega}^l_{\kappa,h}\times\{(n+1)^-\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy\\&-\int_{\tilde{\Omega}^l_{\kappa,h}\times\{n^+\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy-\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}(\alpha_{ij})_s(\tilde{v}^l_\kappa)_{y_j}dyds.\end{aligned}\tag{3.18}$$

Inserting the equation (3.18) into (3.17), it follows

$$\begin{aligned}-2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}(\tilde{v}^l_\kappa)_s(\mathscr{E}\tilde{v}^l_\kappa)dyds=&\int_{\tilde{\Omega}^l_{\kappa,h}\times\{(n+1)^-\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy\\&-\int_{\tilde{\Omega}^l_{\kappa,h}\times\{n^+\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy-2\sum_{\tilde{\gamma}_m\subseteq\partial\tilde{\Omega}^l_{\kappa,h}}\int_{\tilde{\gamma}_m\times I_n}(\tilde{v}^l_\kappa)_s\left(\frac{\partial\tilde{v}^\kappa}{\partial\nu}\right)_\alpha d\mu ds\\&-\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}(\alpha_{ij})_s(\tilde{v}^l_\kappa)_{y_j}dyds.\end{aligned}\tag{3.19}$$

From equation (3.3), we have

$$\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}(\alpha_{ij})_s(\tilde{v}^l_\kappa)_{y_j}dyds\le Ch^2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{|\alpha|=1}|D^\alpha_y\tilde{v}^l_\kappa|^2dyds..\tag{3.20}$$

Substituting the result of the equation (3.20) into (3.19), the estimate is as follows:

$$-2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}(\tilde{v}^l_\kappa)_s(\mathscr{E}\tilde{v}^l_\kappa)dyds \geq \int_{\tilde{\Omega}^l_{\kappa,h}\times\{(n+1)^-\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy$$
$$-\int_{\tilde{\Omega}^l_{\kappa,h}\times\{n^+\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy - 2\sum_{\tilde{\gamma}_m\subseteq\partial\tilde{\Omega}^l_{\kappa,h}}\int_{\tilde{\gamma}_m\times I_n}(\tilde{v}^l_\kappa)_s\left(\frac{\partial\tilde{v}^\kappa}{\partial\nu}\right)_\alpha d\mu ds$$
$$-Ch^2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{|\alpha|=1}|D^\alpha_y\tilde{v}^l_\kappa|^2dyds. \tag{3.21}$$

Inserting the equation (3.21) in (3.16), the estimate holds:

$$\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}|\mathscr{L}\tilde{v}^l_\kappa|^2dyds \geq \int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}(|(\tilde{v}^l_\kappa)_s|^2+|\mathscr{E}\tilde{v}^l_\kappa|^2)dyds$$
$$+\int_{\tilde{\Omega}^l_{\kappa,h}\times\{(n+1)^-\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy - \int_{\tilde{\Omega}^l_{\kappa,h}\times\{n^+\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy$$
$$-2\sum_{\tilde{\gamma}_m\subseteq\partial\tilde{\Omega}^l_{\kappa,h}}\int_{\tilde{\gamma}_m\times I_n}(\tilde{v}^l_\kappa)_s\left(\frac{\partial\tilde{v}^l_\kappa}{\partial\nu}\right)_\alpha d\mu ds - Ch^2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{|\alpha|=1}|D^\alpha_y\tilde{v}^l_\kappa|^2dyds. \tag{3.22}$$

After rearranging the equation (3.22), we obtain

$$\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}(|(\tilde{v}^l_\kappa)_s|^2+|\mathscr{E}\tilde{v}^l_\kappa|^2)dyds + \int_{\tilde{\Omega}^l_{\kappa,h}\times\{(n+1)^-\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy$$
$$\leq \int_{\tilde{\Omega}^l_{\kappa,h}\times\{n^+\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy + \int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}|\mathscr{L}\tilde{v}^l_\kappa|^2dyds$$
$$+2\sum_{\tilde{\gamma}_m\subseteq\partial\tilde{\Omega}^l_{\kappa,h}}\int_{\tilde{\gamma}_m\times I_n}(\tilde{v}^l_\kappa)_s\left(\frac{\partial\tilde{v}^l_\kappa}{\partial\nu}\right)_\alpha d\mu ds + Ch^2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{|\alpha|=1}|D^\alpha_y\tilde{v}^l_\kappa|^2dyds. \tag{3.23}$$

Combining the equation (3.14) and (3.23), implies

$$c_1\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\left(\sum_{|\alpha|=2}|D^\alpha_y\tilde{v}^l_\kappa|^2+|(\tilde{v}^l_\kappa)_s|^2\right)dyds + \int_{\tilde{\Omega}^l_{\kappa,h}\times\{(n+1)^-\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy,$$
$$\leq \int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}|\mathscr{L}\tilde{v}^l_\kappa|^2dyds + \int_{\tilde{\Omega}^l_{\kappa,h}\times\{n^+\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy,$$
$$+Ch^2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{|\alpha|=1}|D^\alpha_y\tilde{v}^l_\kappa|^2dyds + g_1h\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{1\leq|\alpha|\leq2}|D^\alpha_y\tilde{v}^l_\kappa|^2dyds$$
$$+\sum_{\tilde{\gamma}_m\subseteq\partial\tilde{\Omega}^l_{\kappa,h}}\left(\int_{\tilde{\gamma}_m\times I_n}J(\tilde{v})d\mu ds + \int_{I_n}H(\tilde{v})ds\bigg|_{\partial\tilde{\gamma}_m}\right). \tag{3.24}$$

with the following estimate

$$\sum_{\tilde{\gamma}_m\subseteq\partial\tilde{\Omega}^l_{\kappa,h}}\int_{\tilde{\gamma}_m\times I_n}\sum_{|\alpha|=1}|D^\alpha_y\tilde{v}^l_\kappa|^2d\mu ds \leq g_1\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{1\leq|\alpha|\leq2}|D^\alpha_y\tilde{v}^l_\kappa|^2dyds,$$

where g_1 is a uniform constant and c_1 is a positive constant.
From equation (3.10), we obtain:

$$\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}|\mathscr{L}\tilde{v}^l_\kappa|^2dyds = \int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}|(\mathscr{L}+\eta)\tilde{v}^l_\kappa-\eta\tilde{v}^l_\kappa|^2dyds$$
$$\leq 2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}|(\mathscr{L}+\eta)\tilde{v}^l_\kappa|^2dyds \tag{3.25}$$

with the following estimate

$$\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n} |\eta\tilde{v}^l_\kappa|^2 dyds \leq \int_{\tilde{\Omega}^l_{\kappa,h}\times I_n} |(\mathscr{L}+\eta)\tilde{v}^l_\kappa|^2 dyds. \tag{3.26}$$

Inserting equation (3.25) in (3.24) and summing over l on $\tilde{\Omega}^l_{\kappa,h}$, $\kappa=1,2$, the estimate is as follows:

$$\begin{aligned}
&\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Big(\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{|\alpha|=2} c_1|(D^\alpha_y\tilde{v}^l_\kappa|^2+|(\tilde{v}^l_\kappa)_s|^2)dyds+\int_{\tilde{\Omega}^l_{\kappa,h}\times\{(n+1)^-\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy\Big)\\
&\quad\leq\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Big(2\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}|(\mathscr{L}+\eta)\tilde{v}^l_\kappa|^2dyds+\int_{\tilde{\Omega}^l_{\kappa,h}\times\{n^+\}}\sum_{i,j=1}^{2}(\tilde{v}^l_\kappa)_{y_i}\alpha_{ij}(\tilde{v}^l_\kappa)_{y_j}dy\\
&\qquad+Ch\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\sum_{1\leq|\alpha|\leq 2}|D^\alpha_y\tilde{v}^l_\kappa|^2dyds+Ch^4\int_{\tilde{\Omega}^l_{\kappa,h}\times I_n}|\tilde{v}^l_\kappa|^2dyds\Big)+\mathscr{I}(v)\\
&\qquad+\sum_{\kappa=1}^{2}\sum_{\tilde{\gamma}_m\subseteq\partial\tilde{\Omega}_\kappa}\bigg(\int_{\tilde{\gamma}_m\times I_n}[J(\tilde{v})]d\mu ds+\int_{I_n}[H(\tilde{v})]ds\Big|_{\partial\tilde{\gamma}_m}\bigg)\\
&\qquad+\sum_{\tilde{\gamma}_m\subseteq\Gamma_0}\bigg(\int_{\tilde{\gamma}_m\times I_n}[J(\tilde{v})]d\mu ds+\int_{I_n}[H(\tilde{v})]ds\Big|_{\partial\tilde{\gamma}_m}\bigg)\\
&\qquad+\sum_{\tilde{\gamma}_m\subseteq\Gamma}\bigg(\int_{\tilde{\gamma}_m\times I_n}J(\tilde{v})d\mu ds+\int_{I_n}H(\tilde{v})ds\Big|_{\partial\tilde{\gamma}_m}\bigg).
\end{aligned}\tag{3.27}$$

Combining Lemma 3.1 with (3.27), the desired result follows. ☐

Now, we estimate the bound for $\mathcal{E}_3$, which is defined in equation (3.15).

Lemma 3.4. *The estimate*

$$\mathcal{E}_3 \leq \mathcal{E}_4+\mathcal{E}_5 \tag{3.28}$$

holds for a constant K, such that, $\frac{1}{h}$ and p large enough and $\ln p = o(\frac{1}{h})$. Where

$$\mathcal{E}_4=\sum_{\kappa=1}^{2}\Big(\frac{7}{8}\sum_{l=1}^{o_\kappa}\Big(Kh^2||\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}+e_1\Big(||\partial_s\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}+\sum_{1\leq\alpha\leq 2}||D^\alpha_y\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\Big)\Big)$$

and

$$\mathcal{E}_5=\frac{1}{h^2}\Big(\mathscr{R}_{Jump}(\mathcal{F}^{(n)}_{v_1,v_2})+\mathscr{R}_{Boundary}(\mathcal{F}^{(n)}_{v_1,v_2},0)+\mathscr{R}_{Interface}(\mathcal{F}^{(n)}_{v_1,v_2},0)\Big).$$

Proof. Using the equaion (3.32) from [7], we conclude

$$\begin{aligned}
&\bigg|\sum_{\tilde{\gamma}_m\subseteq\tilde{\Omega}_\kappa}\Big(\int_{\tilde{\gamma}_m\times I_n}[\Phi(\tilde{v})]d\mu ds+\int_{I_n}[H(\tilde{v})]ds\Big|_{\partial\tilde{\gamma}_m}\Big)\bigg| \qquad (3.29)\\
&\quad\leq\frac{e}{16}\sum_{l=1}^{o_\kappa}\sum_{1\leq|\alpha_1|\leq 2}||D^{\alpha_1}_y\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}+C(\ln p)^2\sum_{\tilde{\gamma}_m\subseteq\tilde{\Omega}_\kappa}\Big(\sum_{i=1}^{2}|||[(\check{v})^a_{y_i}]|||^2_{(1/2,0),\tilde{\gamma}_m\times I_n}\Big)
\end{aligned}$$

for each $\kappa=1,2$, and

$$\begin{aligned}
\bigg|\sum_{\tilde{\gamma}_m\subseteq\Gamma}\Big(\int_{\tilde{\gamma}_m\times I_n}\Phi(\tilde{v})d\mu ds+\int_{I_n}H(\tilde{v})ds\Big|_{\partial\tilde{\gamma}_m}\Big)\bigg|&\leq\frac{e}{16}\sum_{l=1}^{o_2}\sum_{1\leq|\alpha_1|\leq 2}||D^{\alpha_1}_y\tilde{v}^l_2||^2_{\tilde{\Omega}^l_{2,h}\times I_n}\\
&+C(\ln p)^2\Big(\sum_{\tilde{\gamma}_m\subseteq\Gamma}||(\check{v})^a_\tau||^2_{(1/2,0),\tilde{\gamma}_m\times I_n}+\sum_{\tilde{\gamma}_m\subseteq\tilde{\Omega}_2}\sum_{i=1}^{2}|||[(\check{v})^a_{y_i}]|||^2_{(1/2,0),\tilde{\gamma}_m\times I_n}\Big).
\end{aligned}\tag{3.30}$$

From equations (3.29) and (3.30), the following estimate holds for interface (Γ_0)

$$\Big|\sum_{\tilde{\gamma}_m\subseteq\Gamma_0}\Big(\int_{\tilde{\gamma}_m\times I_n}[\Phi(\tilde{v})]d\mu ds+\int_{I_n}[H(\tilde{v})]ds\Big|_{\partial\tilde{\gamma}_m}\Big)\Big|$$
$$\leq\frac{e}{16}\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\sum_{1\leq|\alpha_1|\leq 2}||D_y^{\alpha_1}\tilde{v}_\kappa^l||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n} \tag{3.31}$$
$$+C(\ln p)^2\Big(\sum_{\tilde{\gamma}_m\subseteq\Gamma_0}\Big|\Big|\Big[\Big(\frac{\partial\check{v}}{\partial\nu}\Big)^a_\alpha\Big]\Big|\Big|^2_{(1/2,0),\tilde{\gamma}_m\times I_n}+\sum_{\kappa=1}^{2}\sum_{\tilde{\gamma}_m\subseteq\tilde{\Omega}_\kappa}\sum_{i=1}^{2}|||[(\check{v})^a_{y_i}]|||^2_{(1/2,0),\tilde{\gamma}_m\times I_n}\Big).$$

Using the equation (3.33) of [7], it follows that

$$\sum_{\tilde{\gamma}_m\subseteq\tilde{\Omega}_\kappa}\int_{\tilde{\gamma}_m\times I_n}\Big[2\tilde{v}\Big(\frac{\partial\tilde{v}}{\partial\nu}\Big)_\alpha\Big]d\mu ds$$
$$\leq\frac{1}{8}\sum_{l=1}^{o_\kappa}\Big(Kh^2||\tilde{v}_\kappa^l||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}+e_1\sum_{1\leq|\alpha_1|\leq 2}||D_y^{\alpha_1}\tilde{v}_\kappa^l||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\Big)$$
$$+\sum_{\tilde{\gamma}_m\subseteq\tilde{\Omega}_\kappa}\Big(\frac{1}{2h^2}\sum_{i=1}^{2}|||[(\check{v})^a_{y_i}]|||^2_{(1/2,0),\tilde{\gamma}_m\times I_n}+C||[\check{v}]||^2_{(0,0),\tilde{\gamma}_m\times I_n}\Big) \tag{3.32}$$

for each $\kappa=1,2$. Moreover

$$\sum_{\tilde{\gamma}_m\subseteq\Gamma}\int_{\tilde{\gamma}_m\times I_n}2\tilde{v}\Big(\frac{\partial\tilde{v}}{\partial\nu}\Big)_\alpha d\mu ds$$
$$\leq\sum_{l=1}^{o_2}\frac{e_1}{8}\Big(\sum_{1\leq|\alpha_1|\leq 2}||D_y^{\alpha_1}\tilde{v}_2^l||^2_{\tilde{\Omega}^l_{2,h}\times I_n}\Big)+C\sum_{\tilde{\gamma}_m\subseteq\Gamma}||[\check{v}]||^2_{(0,0),\tilde{\gamma}_m\times I_n}. \tag{3.33}$$

Similarly, the following estimate holds for the interface (Γ_0)

$$\sum_{\tilde{\gamma}_m\subseteq\Gamma_0}\int_{\tilde{\gamma}_m\times I_n}\Big[2\tilde{v}\Big(\frac{\partial\tilde{v}}{\partial\nu}\Big)_\alpha\Big]d\mu ds$$
$$\leq\frac{1}{8}\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Bigg(Kh^2||\tilde{v}_\kappa^l||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}+e_1\sum_{1\leq|\alpha_1|\leq 2}||D_y^{\alpha_1}\tilde{v}_\kappa^l||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\Bigg)$$
$$+\sum_{\tilde{\gamma}_m\subseteq\Gamma_0}\Bigg(\frac{1}{2h^2}\Big|\Big|\Big[\Big(\frac{\partial\check{v}}{\partial\nu}\Big)^a_\alpha\Big]\Big|\Big|^2_{(1/2,0),\tilde{\gamma}_m\times I_n}+C||[\check{v}]||^2_{(0,0),\tilde{\gamma}_m\times I_n}\Bigg). \tag{3.34}$$

Using equations (3.36) and (3.38) from [7], we obtain

$$\sum_{\tilde{\gamma}_m\subseteq\Omega_\kappa}\int_{\tilde{\gamma}_m\times I_n}\Big[2\tilde{v}_s\Big(\frac{\partial\tilde{v}}{\partial\nu}\Big)_\alpha\Big]d\mu ds$$
$$\leq\frac{1}{2h^2}\sum_{\tilde{\gamma}_m\subseteq\Omega_\kappa}\Bigg(||[\check{v}]||^2_{(0,3/4),\tilde{\gamma}_m\times I_n}+\sum_{j=1}^{2}||(\check{v}_{y_j})^a||^2_{(0,1/4),\tilde{\gamma}_m\times I_n}\Bigg)$$
$$+\frac{e_1}{8}\sum_{l=1}^{o_\kappa}\Bigg(||\partial_s\tilde{v}_\kappa^l||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}+\sum_{1\leq|\alpha_1|\leq 2}||D_y^{\alpha_1}\tilde{v}_\kappa^l||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\Bigg) \tag{3.35}$$

for each $\kappa=1,2$. Moreover

$$\sum_{\tilde{\gamma}_m\subseteq\Gamma}\int_{\tilde{\gamma}_m\times I_n}2\tilde{v}_s\left(\frac{\partial\tilde{v}}{\partial\nu}\right)_\alpha d\mu ds\leq\frac{1}{2h^2}\sum_{\tilde{\gamma}_m\subseteq\Omega_2}||\check{v}||^2_{(0,3/4),\tilde{\gamma}_m\times I_n}$$
$$+\frac{e_1}{8}\sum_{l=1}^{o_2}\Bigg(||\partial_s\tilde{v}_2^l||^2_{\tilde{\Omega}^l_{2,h}\times I_n}+\sum_{1\leq|\alpha_1|\leq 2}||D_y^{\alpha_1}\tilde{v}_2^l||^2_{\tilde{\Omega}^l_{2,h}\times I_n}\Bigg). \tag{3.36}$$

In same way, it follows that

$$\sum_{\tilde{\gamma}_m\subseteq\Gamma_0}\int_{\tilde{\gamma}_m\times I_n}\Big[2\tilde{v}_s\Big(\frac{\partial\tilde{v}}{\partial\nu}\Big)_\alpha\Big]d\mu ds$$

$$\leq \frac{1}{2h^2}\sum_{\tilde{\gamma}_m\subseteq\Gamma_0}\left(||[\check{v}]||^2_{(0,3/4),\tilde{\gamma}_m\times I_n}+\left|\left|\left[\left(\frac{\partial\check{v}}{\partial\nu}\right)^a_\alpha\right]\right|\right|^2_{(0,1/4),\tilde{\gamma}_m\times I_n}\right)$$

$$+\frac{e_1}{8}\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\left(||\partial_s\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}+\sum_{1\leq|\alpha_1|\leq 2}||D^{\alpha_1}_y\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}\right). \tag{3.37}$$

Combining the equations (3.29) − (3.37), imply the desired result. □

Combining the results of Lemma 3.3 and 3.4, implies that

$$g_4\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Big(h^2||\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}+||\partial_s\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}+\sum_{1\leq|\alpha|\leq 2}||D^\alpha_y\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times I_n}+||\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times(n+1)^-}$$

$$+\sum_{i,j=1}^{2}\int_{\tilde{\Omega}^l_{\kappa,h}\times(n+1)^-}(\tilde{v}^l_\kappa)_{y_i}\alpha_{i,j}(\tilde{v}^l_\kappa)_{y_j}dy\Big)\leq\mathcal{W}^{(n)}(\mathcal{F}^{(n)}_{v_1,v_2}) \tag{3.38}$$

holds for large enough $\frac{1}{h}$ and p with $\ln p = o(1/h)$. Here g_4 is a constant independent of h, p and q, and

$$\mathcal{W}^{(n)}(\mathcal{F}^{(n)}_{v_1,v_2})=\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\left(||\tilde{v}^l_\kappa||^2_{\tilde{\Omega}^l_{\kappa,h}\times(n+1)^-}+\sum_{i,j=1}^{2}\int_{\tilde{\Omega}^l_{\kappa,h}\times\{(n+1)^-\}}(\tilde{v}^l_\kappa)_{y_i}\alpha_{i,j}(\tilde{v}^l_\kappa)_{y_j}dy\right)$$

$$+\frac{1}{h^2}(1+2h^2)\left(\sum_{l=1}^{o_1}||(\mathscr{L}+\eta)\tilde{v}^l_1||^2_{\tilde{\Omega}^l_{1,h}\times I_n}+\sum_{l=1}^{o_2}||(\mathscr{L}+\eta)\tilde{v}^l_2||^2_{\tilde{\Omega}^l_{2,h}\times I_n}\right)$$

$$+\frac{1}{h^2}\Big(\mathscr{R}_{Jump}(\mathcal{F}^{(n)}_{v_1,v_2})+\mathscr{R}_{Boundary}(\mathcal{F}^{(n)}_{v_1,v_2},0)+\mathscr{R}_{Interface}(\mathcal{F}^{(n)}_{v_1,v_2},0)\Big).$$

Let $\mathscr{J}^l_\kappa$ be the Jacobian of the map N^l_κ from S to $\tilde{\Omega}^l_{\kappa,h}$ in each $\tilde{\Omega}_\kappa, \kappa = 1, 2$, then there exist matrices $\{(\mathscr{A}^l_\kappa)_{i,j}\}$ such that

$$\sum_{i,j=1}^{2}\int_{\tilde{\Omega}^l_{\kappa,h}\times\{s\}}(\tilde{v}^l_\kappa)_{y_i}\alpha_{i,j}(\tilde{v}^l_\kappa)_{y_j}dy=\sum_{i,j=1}^{2}\int_{S\times\{s\}}(\check{v}^l_\kappa)_{\xi_i}(\mathscr{A}^l_\kappa)_{i,j}(\check{v}^l_\kappa)_{\xi_j}d\xi_1 d\xi_2.$$

Now we define $\hat{\mathscr{J}}^l_\kappa$ and $(\hat{\mathscr{A}}^l_\kappa)_{i,j}$ which are orthogonal projection of $\mathscr{J}^l_\kappa$ and $(\mathscr{A}^l_\kappa)_{i,j}$ into the space of polynomial as before. Recall that $\eta = Kh^2$ and $\tilde{w}^l_\kappa = \tilde{v}^l_\kappa e^{\eta s}$. Using these arguments in equation (3.38), we obtain the final result.

4. Error estimate

In this section, we prove a priori error estimate for parabolic interface problems. Let $u^l_\kappa(\xi, s) = u(N^l_\kappa(\xi_1, \xi_2), s)$, where $l = 1, 2, \cdots, o_1$ for $\kappa = 1$ and $l = 1, 2, \cdots, o_2$ for $\kappa = 2$. Now we prove the following approximation result.

Lemma 4.1. *For each $\kappa = 1, 2$, let u_κ be a smooth function which is defined on $\bar{\Omega}_\kappa \times [0, T]$. Then there exist functions $\psi^l_\kappa(\xi, s)$ defined on $S \times [0, \mathcal{M}]$ (where $\mathcal{M}k = T$). Moreover, $\psi^l_\kappa(\xi, s)$ is continuous function of s and is a polynomial in ξ_1 and ξ_2 of degree p separately and in s of degree q for $s \in I_n$ with $n = 0, 1, \cdots, \mathcal{M} - 1$. Then the following error estimate*

$$\left(\sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\sum_{n=0}^{\mathcal{M}-1}||u^l_\kappa-\psi^l_\kappa||^2_{(2,1),S\times I_n}\right)^{\frac{1}{2}}\leq C_q h^{2q}||u||_{(2q+6,q+3),\Omega_1\cup\Omega_2\times(0,T)} \tag{4.1}$$

holds, provided $p = 2q+1$ and k is proportional to h^2 as $h \to 0$.

If $u_\kappa \in \mathscr{D}_{2,1}(\bar{\Omega}_\kappa \times [0,T])$ for each $\kappa = 1, 2$, then

$$\left(\sum_{\kappa=1}^{2} \sum_{l=1}^{o_\kappa} \sum_{n=0}^{\mathcal{M}-1} ||u_\kappa^l - \psi_\kappa^l||^2_{(2,1),S\times I_n} \right)^{\frac{1}{2}} \leq K e^{-\rho_1 p} h^{\rho_3 p} \tag{4.2}$$

provided q is proportional to p^2, as p tends to infinity and $\ln p = o(1/h)$. Where K, ρ_1 and ρ_3 are positive constants.

Proof. Let $\pi^{p,q}_{\xi,s} v(\xi, s) = \pi^p_\xi \pi^q_s v(\xi, s)$ be an operator from

$$H^{2q+6,q+3}(S \times I_0) \to (\mathcal{P}^p \times \mathcal{P}^p \times \mathcal{P}^q)(S \times I_0)$$

defined as [7, 21]. Now, we define $\psi^l_\kappa(\xi, s+n) = \pi^p_\xi \pi^q_s u^l_\kappa(\xi, s)$ for $0 \leq s < 1$. Thus $\psi^l_\kappa(\xi, s)$ is a continuous function of s for $0 \leq s < \mathcal{M}$ and separately for $\kappa = 1, 2$.

Using the approximation results from equations (5.6) and (5.7) in [7], we obtain

$$\begin{aligned} ||u^l_\kappa - \psi^l_\kappa||^2_{(0,1),S\times I_0} \leq & C 2^{-2\sigma} \frac{(q-\sigma)!}{(q+\sigma)!} ||\partial_s^{\sigma+1} u^l_\kappa||^2_{(0,0),S\times I_0} + C 2^{-2\lambda} \frac{(p-\lambda)!}{(p+\lambda+2)!} \\ & \Big(\sum_{j=0}^{2} (||\partial^{\lambda+1}_{\xi_1} \partial^j_{\xi_2} u^l_\kappa||^2_{(0,1),S\times I_0} + ||\partial^j_{\xi_1} \partial^{\lambda+1}_{\xi_2} u^l_\kappa||^2_{(0,1),S\times I_0}) \Big) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} ||D^{\alpha_1}_\xi (u^l_\kappa - \psi^l_\kappa)||^2_{(0,0),S\times I_0} \leq & C \Big(2^{-2\nu} \frac{(p-\nu)!}{(p+\nu-2)!} \Big(\sum_{j=0}^{2} (||\partial^{\nu+1}_{\xi_1} \partial^j_{\xi_2} u^l_\kappa||^2_{(0,0),S\times I_0} \\ & + ||\partial^j_{\xi_1} \partial^{\nu+1}_{\xi_2} u^l_\kappa||^2_{(0,0),S\times I_0}) \Big) + \frac{2^{-2\mu}}{q(q+1)} \frac{(q-\mu)!}{(q+\mu)!} ||D^{\alpha_1}_\xi \pi^p_\xi \partial^{\mu+1}_s u^l_\kappa||^2_{(0,0),S\times I_0} \Big) \end{aligned} \tag{4.4}$$

for $0 \leq |\alpha_1| \leq 2$ and separately for $\kappa = 1, 2$.

For proving the first estimate, where u_κ is smooth in $\Omega_\kappa \times (0, T)$ and h tends to zero (p and q are fixed), we choose $p = 2q+1$, $\lambda = 2q+1$, $\sigma = q$, $\nu = 2q+1$ and $\mu = q$ in equations (4.3)-(4.4) as in [21]. Adding equations (4.3)-(4.4) and summing over l for $\Omega^l_\kappa, \kappa = 1, 2$, the desired result holds.

For proving the second estimate, where $u_\kappa \in \mathscr{D}_{2,1}(\bar{\Omega}_\kappa \times [0,T])$ and the map M^l_κ are analytic, we obtain

$$\sup_{(\xi,s)\in S\times(0,\mathcal{M})} |D^\alpha_\xi D^{\beta_1}_s u^l_\kappa(\xi, s)| \leq A_2 (B_2)^{j+\beta_1} j! (\beta_1!)^2 h^{2\beta_1 + j},$$

for $|\alpha| = j$. Here A_2 and B_2 are constants.

Now, we choose $q \propto p^2$, $\lambda = d_1 p$, $\sigma = d_2 p$, $\nu = d_3 p$ and $\mu = d_4 p$ in equations (4.3)-(4.4) as in [21], where $0 < d_\iota < 1$ for $\iota = 1, \cdots, 4$. Adding equations (4.3)-(4.4) and summing over l for $\Omega^l_\kappa, \kappa = 1, 2$, the desired result holds. □

Finally, we prove our main result of this section.

Theorem 4.1. *Let $\mathcal{F}^{(n)}_{w_1,w_2} \in \mathcal{S}^{p,q}_{(n)}$ minimize the functional $\mathscr{R}^{(n)}(\mathcal{F}^{(n)}_{v_1,v_2})$ over all $\mathcal{F}^{(n)}_{v_1,v_2} \in \mathcal{S}^{p,q}_{(n)}$. If u_κ is smooth in $\bar{\Omega}_\kappa \times [0,T]$ for each $\kappa = 1, 2$, then there exist a constant C_q such that the estimate*

$$\Big(\sum_{\kappa=1}^{2} \sum_{l=1}^{o_\kappa} \sum_{n=0}^{\mathcal{M}-1} ||u^l_\kappa - w^l_\kappa||^2_{(2,1),\Omega^l_{\kappa,h}\times I_n} \Big)^{\frac{1}{2}} \leq C_q h^{2q-1} ||u||_{(2q+6,q+3),\Omega_1\cup\Omega_2\times(0,T)} \tag{4.5}$$

holds, provided $p = 2q+1$ and k is proportional to h^2 as $h \to 0$.

If $u_k \in \mathscr{D}_{2,1}(\bar{\Omega}_\kappa \times [0,T])$ for each $\kappa = 1, 2$, then

$$\Big(\sum_{\kappa=1}^{2} \sum_{l=1}^{o_\kappa} \sum_{n=0}^{\mathcal{M}-1} ||u^\kappa_l - w^\kappa_l||^2_{(2,1),\Omega^l_{\kappa,h}\times I_n} \Big)^{\frac{1}{2}} \leq K e^{-\rho_1 p} h^{\rho_3 p} \tag{4.6}$$

provided q is proportional to p^2, as p tends to infinity and $\ln p = o(1/h)$. Where K, ρ_1 and ρ_3 are positive constants.

Proof. First, we divide the error into the following terms:

$$||u_\kappa^l - w_\kappa^l||^2_{\Omega^l_{\kappa,h}\times I_n} \leq C(||u_\kappa^l - \psi_\kappa^l||^2_{\Omega^l_{\kappa,h}\times I_n} + ||\psi_\kappa^l - w_\kappa^l||^2_{\Omega^l_{\kappa,h}\times I_n}),$$

for some positive constant C. Here the first term of R.H.S. is already estimated from the previous Lemma 4.1. Now, we estimate the second term of R.H.S. Let $\mathcal{F}^{(0)}_{w_1,w_2}$ minimizes $\mathscr{R}^{(0)}(\mathcal{F}^{(0)}_{v_1,v_2})$. Then we have

$$\mathscr{R}^{(0)}(\mathcal{F}^{(0)}_{\psi_1,\psi_2}) = \mathscr{R}^{(0)}(\mathcal{F}^{(0)}_{w_1,w_2}) + \mathscr{W}^{(0)}(\mathcal{F}^{(0)}_{\psi_1-w_1,\psi_2-w_2}). \tag{4.7}$$

Therefore, we conclude that

$$\mathscr{W}^{(0)}(\mathcal{F}^{(0)}_{\psi_1-w_1,\psi_2-w_2}) \leq \mathscr{R}^{(0)}(\mathcal{F}^{(0)}_{\psi_1,\psi_2}). \tag{4.8}$$

Replacing the approximate solution $\mathcal{F}^0_{w_1,w_2}$ by exact solution $\mathcal{F}^0_{u_1,u_2}$ in the equation (4.7) then we obtain

$$\mathscr{R}^{(0)}(\mathcal{F}^{(0)}_{\psi_1,\psi_2}) \equiv \mathscr{W}^{(0)}(\mathcal{F}^{(0)}_{\psi_1-u_1,\psi_2-u_2}), \tag{4.9}$$

using $\mathscr{R}^{(0)}(\mathcal{F}^{(0)}_{u_1,u_2}) \approx 0$.
Define

$$\mathcal{T}_n = \sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Big(h^2||\check{w}_\kappa^l - \psi_\kappa^l||^2_{S\times I_n} + ||\partial_s(\check{w}_\kappa^l - \psi_\kappa^l)||^2_{S\times I_n} + \sum_{1\leq|\alpha|\leq 2}||D^\alpha_\xi(\check{w}_\kappa^l - \psi_\kappa^l)||^2_{S\times I_n}\Big)$$

and

$$\Upsilon_n = \sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Big(||(\check{w}_\kappa^l - \psi_\kappa^l)||^2_{S\times n^-} + \sum_{i,j=1}^{2}\int_{S\times n^-}(\check{w}_\kappa^l - \psi_\kappa^l)_{\xi_i}(\hat{\mathscr{A}}_\kappa^l)_{i,j}(\check{w}_\kappa^l - \psi_\kappa^l)_{\xi_j}\,d\xi_1 d\xi_2\Big).$$

Using Theorem 3.1, the following estimate holds:

$$g_4(\mathcal{T}_0 + \Upsilon_1) \leq e^{\lambda k}\mathscr{W}^{(0)}(\mathcal{F}^{(0)}_{\psi_1-w_1,\psi_2-w_2}) \leq e^{\lambda k}\mathscr{R}^{(0)}(\mathcal{F}^{(0)}_{\psi_1,\psi_2}) \tag{4.10}$$

for choosing λ such that $1 + ch^2 = e^{\lambda k}$. Now we define

$$\widetilde{\mathscr{R}}^{(n)}(\mathcal{F}^{(n)}_{v_1,v_2}) = \mathscr{R}^{(n)}(\mathcal{F}^{(n)}_{v_1,v_2}) - \mathcal{I}_n,$$

where

$$\mathcal{I}_n = \sum_{\kappa=1}^{2}\sum_{l=1}^{o_\kappa}\Big(||\check{v}_\kappa^l||^2_{S\times n^+} + \sum_{i,j=1}^{2}\int_{S\times\{n^+\}}(\check{v}_\kappa^l)_{\xi_i}(\hat{\mathscr{A}}_\kappa^l)_{i,j}(\check{v}_\kappa^l)_{\xi_j}\,d\xi_1 d\xi_2\Big).$$

From equation (4.8), it follows:

$$\mathscr{W}^{(1)}(\mathcal{F}^{(1)}_{\psi_1-w_1,\psi_2-w_2}) \leq \mathscr{R}^{(1)}(\mathcal{F}^{(1)}_{\psi_1,\psi_2}). \tag{4.11}$$

Again using Theorem 3.1, the following estimate holds as in (4.10):

$$g_4(\mathcal{T}_1 + \Upsilon_2) \leq e^{\lambda k}\mathscr{W}^{(1)}(\mathcal{F}^{(1)}_{\psi_1-w_1,\psi_2-w_2}) \leq e^{\lambda k}\mathscr{R}^{(1)}(\mathcal{F}^{(1)}_{\psi_1,\psi_2}) \leq e^{\lambda k}\Big(\widetilde{\mathscr{R}}^{(1)}(\mathcal{F}^{(1)}_{\psi_1,\psi_2}) + \mathcal{I}_1\Big). \tag{4.12}$$

Here $\psi_\kappa^l(\xi, s)$ is continuous in s. Multiplying by $e^{\lambda k}$ in equation (4.10) and adding equations (4.10) & (4.12), imply:

$$g_4(e^{\lambda k}\mathcal{T}_0 + \mathcal{T}_1 + \Upsilon_2) \leq e^{2\lambda k}\mathscr{R}^0(\mathcal{F}_{\psi_1,\psi_2}) + e^{\lambda k}\widetilde{\mathscr{R}}^{(1)}(\mathcal{F}_{\psi_1,\psi_2}). \tag{4.13}$$

Continuing this process upto $\mathcal{M} - 1$ times, the final result is as follows:

$$g_4\sum_{n=0}^{\mathcal{M}-1}\mathcal{T}_n \leq e^{\lambda T}\Big(\mathscr{R}^0(\mathcal{F}_{\psi_1,\psi_2}) + \sum_{n=1}^{\mathcal{M}-1}\widetilde{\mathscr{R}}^{(n)}(\mathcal{F}_{\psi_1,\psi_2})\Big). \tag{4.14}$$

Combining the equations (4.9) and (4.14), we obtain the following result

$$g_4 \sum_{n=0}^{\mathcal{M}-1} \mathcal{T}_n \leq e^{\lambda T} \sum_{n=0}^{\mathcal{M}-1} \mathscr{W}^n(\mathcal{F}_{\psi_1-u_1,\psi_2-u_2}). \tag{4.15}$$

Using trace theorem from [16], the following result holds

$$\mathscr{W}^{(n)}(\mathcal{F}^{(n)}_{v_1,v_2}) \leq \frac{K}{h^2} \sum_{\kappa=1}^{2} \sum_{l=1}^{o_\kappa} ||\check{v}^l_\kappa||^2_{(2,1),S\times I_n} \ , \tag{4.16}$$

where K is a constant. Inserting the equation (4.16) in (4.15), implies

$$g_4 \sum_{n=0}^{\mathcal{M}-1} \mathcal{T}_n \leq e^{\lambda T} \frac{C}{h^2} \sum_{\kappa=1}^{2} \Big(\sum_{l=1}^{o_\kappa} \sum_{n=0}^{\mathcal{M}-1} ||u^l_\kappa - \psi^l_\kappa||^2_{(2,1),S\times I_n} \Big). \tag{4.17}$$

Applying Lemma 4.1 in equation (4.17), implies the estimates (4.5) and (4.6). □

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