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# THE NON-SMOOTH PITCHFORK BIFURCATION

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Abstract. The bifurcations of strange nonchaotic attractors in quasi-periodically forced systems are poorly understood. A simple two-parameter example is introduced which unifies previous observations of the non-smooth pitchfork bifurcation. There are two types of generalized pitchfork bifurcation which occur in this example, and the corresponding bifurcation curves can be calculated analytically. The example shows how these bifurcations are organized around a codimension two point in parameter space.

## 1. Introduction.

In the mid 1980s Grebogi, Ott, Yorke and co-workers published a series of papers [2, 4, 5, 11, 20, 21, 22] which established two fundamental results. First they showed that strange nonchaotic attractors, attractors with complicated geometry but on which the dynamics is not chaotic, can exist over ranges of parameter values with positive measure in parameter space [11] and second, that these attractors can be observed in a variety of quasi-periodically forced oscillators [2, 4, 20, 21]. The first result shows that these strange nonchaotic attractors are mathematically important, and the second that they are also physically important. Some experimental investigations have also been undertaken [6, 29]. Given that quasi-periodic forcing – forcing with more than one independent frequency – is the natural extension of periodic forcing which is a standard topic in physics and engineering, it is surprising how little is known about this case. In particular, it seems that the possibility of having new classes of attractors was overlooked for many years.

The analysis of these attractors is still in its infancy. The stroboscopic map (or Poincaré map) for quasi-periodically forced systems, obtained by observing the system after every period of one of the forcing terms, takes the form

$$x_{n+1} = x_n + \omega, \quad y_{n+1} = f(x_n, y_n) \tag{1}$$

where $x$ is an angular variable and may be taken *mod 1*, $y$ denotes the phase space of the oscillator, and $f$ is periodic in $x$: $f(x,y) = f(x+1,y)$. The parameter $\omega$ is irrational and represents the constant phase through which a second independent frequency advances over one period of the strobing frequency. To simplify analysis we shall restrict attention to equations of this form, see [1, 3, 5, 7, 9, 12, 18, 19, 22, 23, 28] for further details and examples. Since $\omega$ is irrational, the map (1) has no fixed points or periodic orbits; the simplest invariant objects are continuous curves.

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Having established that strange nonchaotic attractors arise in physically relevant models, it is natural to investigate how they are created and destroyed as parameters are varied. There is as yet no systematic bifurcation theory for these objects, but one remarkable feature has emerged from numerical and theoretical studies [11, 17, 26]: in many systems, strange nonchaotic attractors are created abruptly from the bifurcation of smooth invariant curves. A simple example of this is found in the original example of Grebogi *et al* [11]:

$$x_{n+1} = x_n + \omega, \quad y_{n+1} = 2\sigma\cos(2\pi x_n)\tanh y_n \tag{2}$$

with $x$ taken *mod 1* as before and $\sigma > 0$. The line $\mathcal{L} = \{(x,y) \mid y = 0\}$ is invariant and typical orbits on $\mathcal{L}$ are stable if $\sigma < 1$ and unstable if $\sigma > 1$ [11]. On the other hand, since all points with $x = \frac{1}{4}$ or with $x = \frac{3}{4}$ are mapped to $\mathcal{L}$, $\mathcal{L}$ forms part of the attractor for all values of $\sigma$. If $\sigma > 1$ then this combination of pinching and instability implies the existence of a strange nonchaotic attractor [8, 11, 13].

A second type of transition of strange nonchaotic attractors has been observed numerically: a strange nonchaotic attractor seems to lose stability creating a pair of stable invariant curves separated by an unstable invariant curve [17, 26]. Again, this may be thought of as a generalized pitchfork bifurcation: a stable object (the strange nonchaotic attractor) loses stability to create a pair of stable sets separated by an unstable set. There have been no examples in the literature for which the locus of such a bifurcation can be calculated explicitly, and so it has not been clear how much weight can be attached to these numerical observations as the bifurcation thresholds are not known, and questions about continuity or smoothness are hard to answer using numerical simulations.

The aim of this paper is to show that the two types of generalized pitchfork bifurcation described above are really part of the same generalized bifurcation (the *non-smooth* pitchfork bifurcation) by which two stable invariant sets are created from one stable set. In the next section an example is constructed for which all the bifurcation curves of one invariant set (the equivalent of $\mathcal{L}$ above) can be calculated analytically, making it possible to provide much stronger numerical confirmation that the second type of bifurcation occurs as reported in [17, 26], rather than (for example) by a process of fractalization [16]. In fact, results of [13] can be used to prove mathematically that the bifurcations occur as observed numerically. The bifurcation curves are organized about a codimension two point so that in typical parameter paths passing on one side of this point there is a standard pitchfork bifurcation of invariant curves, whilst for parameter paths which pass on the other side of the codimension two point an invariant curve loses stability creating a strange nonchaotic attractor which is then destroyed in a bifurcation creating a pair of stable invariant curves. Thus the net effect of the pair of bifurcations involving the strange nonchaotic attractor is the same as a standard pitchfork bifurcation. We conjecture that this is a general phenomenon and that similar bifurcations will be observed in a variety of quasi-preiodically forced systems.

## 2. **The quasi-periodically forced map.**

The quasi-periodically forced map which will be the focus of the remainder of this paper is a natural generalization of (2):

$$x_{n+1} = x_n + \omega, \quad y_{n+1} = 2\sigma(\alpha + \cos 2\pi x_n)\tanh y_n \tag{3}$$

where, as usual, $x$ is taken *mod 1*, $\omega$ is irrational and the real parameters $\alpha$ and $\sigma$ are positive. If $\alpha = 0$ then the map reduces to the original model of [11]. As in (2), the line (or circle)

$$\mathcal{L} = \{(x,y) \mid y = 0\} \tag{4}$$

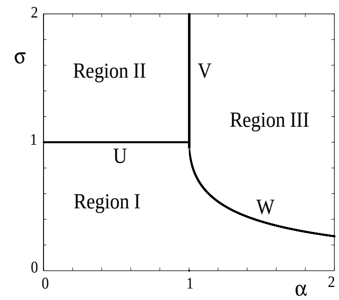


FIGURE 1. Schematic diagram of part of the positive quadrant of the $(\alpha, \sigma)$ parameter plane showing the three regions of behaviour separated by the bifurcation curves $U$, $V$ and $W$ defined in the main text.

is invariant, and the transverse Liapunov exponent of typical points on $\mathcal{L}$ is

$$\begin{aligned} \lambda &= \int_0^1 \ln|2\sigma(\alpha + \cos 2\pi x)|\ dx \\ &= \tfrac{1}{2}\int_0^1 \ln(2\sigma(\alpha+\cos 2\pi x))^2\ dx \end{aligned} \tag{5}$$

Fortunately, the integral on the second line of (5) is known (using 4.226(1) of [10]) and a simple calculation gives

$$\lambda = \begin{cases} \ln\sigma & \text{if } 0 \le \alpha \le 1 \\ \ln\sigma + \ln\left(\alpha + \sqrt{\alpha^2-1}\right) & \text{if } \alpha > 1 \end{cases} \tag{6}$$

As shown in Fig. 1, (6) defines two curves, $U$ and $W$, in the positive quadrant of the $(\alpha, \sigma)$ parameter plane on which the typical transverse Liapunov exponent of $\mathcal{L}$ is zero. In $\alpha < 1$ we have

$$U = \{(\alpha,\sigma) \mid \sigma = 1,\ 0 < \alpha \le 1\} \tag{7}$$

and in $\alpha > 1$

$$V = \{(\alpha,\sigma) \mid \sigma = (\alpha + \sqrt{\alpha^2-1})^{-1},\ \alpha \ge 1\} \tag{8}$$

The invariant line $\mathcal{L}$ is stable for values of $(\alpha, \sigma)$ which lie below the union of these two curves (region I of Fig. 1). To describe the full bifurcation picture a third curve is needed:

$$W = \{(\alpha,\sigma) \mid \alpha = 1,\ \sigma \ge 1\} \tag{9}$$

$W$ is the boundary of the region of parameter space in which the map (3) is invertible and on which $\mathcal{L}$ is unstable. Note that all three of these curves terminate at the point $(1, 1)$. The regions in parameter space separated by these curves may be labelled I-III as shown in Fig. 1. The associated dynamics which can be observed numerically is shown in Figs. 2 and 3, where the parameters are chosen close to one of the bifurcation curves $U$, $V$ or $W$. It is these observations which will be explained in the next section.

3. **Dynamics.** In Region I the results of [13] applied to $|y_{n+1}|$ can be used to prove that the attractor is just the straight line $\mathcal{L}$ on which the dynamics is an irrational rotation. This is relatively uninteresting, so details will only be given for the other two regions.

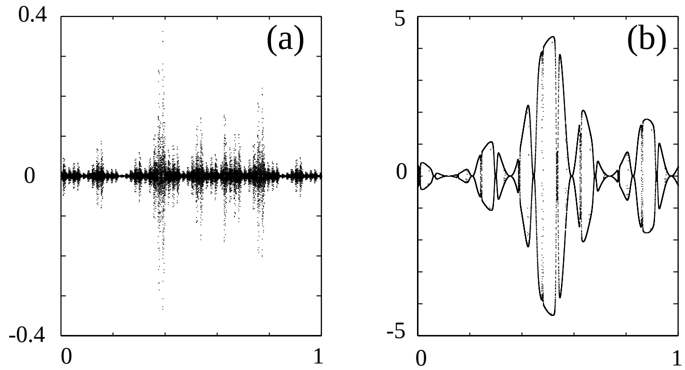


FIGURE 2. Strange nonchaotic attractors in Region II. In this and the next figure, the $x-$axis is horizontal and the $y-$axis is vertical. (a) $\alpha = 0.7$, $\sigma = 1.0001$; 40000 iterates of $(0.25, 0.3)$ are shown, having discarded the first 5000. (b) $\alpha = 0.9999$, $\sigma = 1.3$; 40000 iterates of $(0.25, 0.3)$ are shown, having discarded the first 5000.

3.1. **Region II.** Region II has $\alpha < 1$ and $\sigma > 1$. Fig. 2 shows examples of numerically calculated attractors in this region. These certainly give the impression of being strange nonchaotic attractors, and this can be proved.

As $\sigma > 1$, typical transverse Liapunov exponents of $\mathcal{L}$ are positive, so this set is unstable. Since $\alpha < 1$, the factor of $\alpha + \cos 2\pi x$ in the definition of $y_{n+1}$ in (3) is zero for some values of $x$. This leads to the pinching effect commented upon earlier, all points on the lines $\{(x, y) \mid x = \gamma\}$ where $\cos 2\pi\gamma = -\alpha^2$, map to $(\gamma+\omega, 0)$ on $\mathcal{L}$, which implies that some points on $\mathcal{L}$ have Liapunov exponents equal to $-\infty$. As in the original case of Grebogi *et al*, (2), it follows that there is a strange nonchaotic attractor [8, 11, 13]. More precisely, the attractor is bounded above and below by semi-continuous invariant curves. Recall that a real function $f(x)$ is *upper semi-continuous at* $x$ if, for any sequence $(x_k)$ tending to $x$, $f(x) \geq \limsup f(x_k)$. It is upper semi-continuous if it is upper semi-continuous for all $x$. A function $f$ is *lower semi-continuous* if $-f$ is upper semi-continuous. From [13] it is possible to show that the attractor contains two bounding graphs $y = \psi(x)$ and $y = -\psi(x)$ with the following properties:

(i) $\psi(x) \geq 0$ for all $x \in [0, 1)$;
(ii) the union of the upper and lower bounding graphs, $\Psi = \{(x, y) \mid |y| = \psi(x)\}$, is invariant;
(iii) typical transverse Liapounov exponents on $\Psi$ are negative;
(iv) $\psi$ is discontinuous at almost all $x \in [0, 1)$;
(v) $\psi$ is upper semi-continuous (see [**?**]);
(vi) $\psi(x) = 0$ on a dense, but measure zero, set of values of $x \in [0, 1)$.

The last four properties more than justify calling $\Psi$ a strange nonchaotic attractor.

The two strange nonchaotic attractors pictured here are at parameter values close to the boundary of Region II. In particular, Fig. 2(a) shows the attractor very close to $U$ (Equ. (7)), so the line $\mathcal{L}$ has only just lost stability. This is reflected in the fact that the attractor is fairly concentrated about $\mathcal{L}$. In Fig. 2(b), the parameters are chosen close to the boundary $V$ (Equ. (8)) beyond which the map

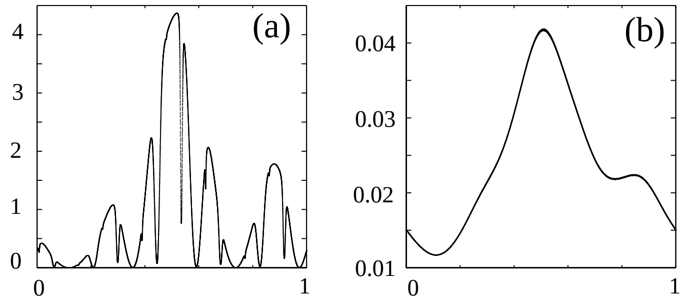


FIGURE 3. Two attractors which exist in Region III with parameters chosen close to the boundary $V$. (a) $\alpha = 1.0001$, $\sigma = 1.3$; 40000 iterates of $(0.25, 0.3)$ are shown, having discarded the first 1000. (b) $\alpha = 0.7$, $\sigma = (\alpha + \sqrt{\alpha^2 - 1})^{-1} + 0.0001$; 20000 iterates of $(0.25, 0.3)$ are shown, having discarded the first 10000.

is invertible and the pinching effect no longer occurs. This attractor is very similar in character to the attractor of Grebogi *et al* [11]. Figure 3(a) shows one of the attractors at parameter values on the other side of $V$: it is clearly a smooth curve as expected.

3.2. **Region III.** In Region III the map is invertible ($\alpha > 1$) and the line $\mathcal{L}$ is unstable: $\sigma > (\alpha + \sqrt{\alpha^2 - 1})^{-1}$. Fig. 3(a) shows one of the attractors of the system close to the boundary $V$ (Equ. (8)), and Fig. 3(b) shows one of the attractors in Region III close to the boundary $W$ (Equ. (9)). In both cases there is a second attractor below $\mathcal{L}$ which is the image of the attractor shown under the symmetry $y \to -y$.

The curve in Fig. 3(a) is very close to the upper boundary of the strange non-chaotic attractor of Fig. 2(b), as should be expected since the parameter values differ by only 0.0002, or 0.02%, and the invariant curve of Fig. 3(b) has much smaller amplitude and bifurcates from the line $\mathcal{L}$.

In Region III the map is invertible and $y_n > 0$ implies that $y_{n+1} > 0$ (and $y_n < 0$ implies that $y_{n+1} < 0$) so the results of Keller [13] apply with only minor reinterpretation: there is a continuous (in fact, differentiable [24, 25, 27]) attracting invariant curve $y = \phi(x)$ in $y > 0$ and its symmetric image in $y < 0$, establishing the dynamics observed numerically.

4. **Interpretation.** In the previous section it was shown that the behaviour in each of the regions of Fig. 1 is theoretically understood and is as illustrated in the figures. Now consider any simple path through parameter space from the bottom left hand corner of Fig. 1 to the upper right hand corner, for example, from $(\frac{1}{4}, \frac{1}{4})$ to $(2, 2)$, which does not pass through the codimension two point $(1, 1)$. Such a path will either pass above $(1, 1)$ or below $(1, 1)$, and the net effect of changing the parameter on this path is that $\mathcal{L}$ loses stability and a pair of stable invariant curves are created. This is the effect of a standard pitchfork bifurcation of invariant curves. and indeed, if the path is chosen to pass below $(1, 1)$, the path will cross $W$, on which just such a standard pitchfork bifurcation of invariant curves occurs.

On the other hand, if the path passes above $(1, 1)$ then it must cross two bifurcation curves: $U$ and $V$. Crossing $U$, the line $\mathcal{L}$ loses stability, but there is still a pinching effect ($\alpha + \cos 2\pi x = 0$ has real solutions), so a strange nonchaotic attractor is created in Region II. This strange nonchaotic attractor is bounded by two semi-continuous graphs which intersect the line $\mathcal{L}$ at a dense set of points. This bifurcation can be thought of as a generalized pitchfork bifurcation in the sense that an object ($\mathcal{L}$) loses stability and a new attracting invariant set with two components, the bounding curves $\pm\psi$, is created. The complication is that these three sets are entangled to create a single strange nonchaotic attractor.

When the parameter path crosses $V$ the pinching effect no longer occurs and $\mathcal{L}$ becomes uniformly unstable: all transverse Liapunov exponents are positive on $\mathcal{L}$. Two smooth invariant curves are created: one in $y < 0$ and the other in $y > 0$, and these tend to the lower and upper bounding curves of the strange nonchaotic attractor as the parameter tends to $V$ from Region III as suggested by Fig. 2(b) and Fig. 3(a). Again, the bifurcation can be seen as a generalized pitchfork bifurcation in the sense that an attractor (the strange nonchaotic attractor) is destroyed to give two attractors separated by a repeller (the line $\mathcal{L}$).

The two parameter paths described above both have the net effect of creating two stable curves and one unstable curve from one stable curve. In this sense, the two parameter diagram of Fig. 1 unifies the different generalized pitchfork bifurcations and shows how the different mechanisms of producing this change fit together into one coherent picture.

Although the example presented is a natural extension of examples which have been studied elsewhere [11], it shares one important drawback with the earlier examples: over an important range of parameter values ($\alpha < 1$) it is not invertible, and hence cannot be the Poincaré map of a differential equation. This is not an unusual circumstance in the first examples of novel phenomena (think, for example, of the use of non-invertible circle maps to model the break up of invariant tori [14, 15] or the use of the logistic map to understand period-doubling cascades), and we expect the gross features of the generalized pitchfork bifurcations presented here to apply to differential equations even if the fine details differ somewhat. In differential equations, we expect to see regions of parameter space in which complicated dynamics can be observed associated with pitchfork bifurcations, but it may be that the clean transition: *(attracting invariant curve)* $\to$ *(strange nonchaotic attractor)* $\to$ *(two attracting invariant curves)* is less clear cut in differential equations, with regions of parameter space in which strange nonchaotic attractors exist for a positive measure set of parameter values rather than for all parameter values as in Region II.

5. **Conclusion.** Although the transition from a strange nonchaotic attractor to a pair of stable invariant curves has been observed numerically, the evidence has always been unclear since the precise bifurcation point has been unknown. The example presented here provides a test case in which the bifurcation can be located precisely (the bifurcation curve $V$ in Fig. 1) and can be confirmed both numerically and theoretically. Moreover, the analytic derivation of the bifurcation curves together with the mathematical results of Keller [13] leads to a new bifurcation diagram associated with a codimension two point, and this diagram shows how the standard pitchfork bifurcation of invariant curves fits in with the more novel bifurcations involving strange nonchaotic attractors. We expect features associated with this bifurcation to be observed in a variety of quasi-periodically forced systems.

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