A Continuum of Inductive Methods arising from a Generalized Principle of Instantial Relevance

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Abstract

In this paper we consider a natural generalization of the Principle of Instantial Relevance and give a complete characterization of the probabilistic belief functions satisfying this principle as a family of discrete probability functions parameterized by a single real $\delta \in [0, 1)$.

Introduction

The task of Inductive Reasoning could be briefly stated as that of how we, or any agent natural or artificial, should employ information presented to us from the world around to pre-empt that world and respond accordingly. As Quine puts it (in [15]) it is a problem of how we "in a world we never made, should stand better than random or coin-tossing chances of coming out right when we predict by inductions which are based on our innate, scientifically unjustified similarity standard".

The question we ask of ourselves is how evidence (or observation reports) bears upon hypotheses about objects possibly not contained in the evidence. In particular we will consider the case of what is commonly known as the 'Singular Predictive Inference', the case whereby we hypothesize about the properties of a single individual not previously observed. For example, no pigs encountered up to this time by the authors can fly. From this evidence the prediction that the next pig to be observed will also be land-bound seems reasonable.

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In addition to the singular predictive inference, we will also include the question of how we hypothesize about properties of a single individual that *has* previously been observed, but has not been observed to satisfy the properties (or the negations of the properties) in question. For example, suppose it has been observed that many flying objects also have wings, so a prediction that the property of flying is accompanied by the occurrence of wings may be reasonable. Suppose also that our evidence contains an individual, Porky, who has been observed not to have wings. So again the prediction that Porky cannot fly seems somewhat justified.

For this paper we formulate the task of inductive reasoning as that of defining a degree of belief, equivalently subjective probability, on the sentences of some fixed language.

Definition 1 *Let L be the language for the predicate calculus containing a countably infinite sequence $a_1, a_2, \ldots$ of distinct constant symbols*[1] *and a countably infinite number of distinct unary predicate symbols $P_1, P_2, \ldots$. We shall call a sublanguage L' of L a finite sublanguage if it contains only finitely many of the P_i, but still all of the $a_1, a_2, \ldots$. Let L^n denote the finite sublanguage of L with just n unary predicate symbols $P_1, P_2, ..., P_n$.*

We state the following definitions in terms of the language L although they should be taken to also carry over, mutatis mutandis, to any sublanguage of L.

Definition 2 *For a language L, define FL to be the formulae over L built using the connectives $\wedge, \vee, \neg$ and quantifiers $\forall, \exists$. Let SL be the subset of FL containing sentences (or closed formulae) of L . Let $QFSL$ be the subset of SL containing the quantifier free sentences of L.*

Definition 3 *A function $w : SL \to [0, 1]$ is a probability function if for all $\theta, \phi \in SL$ and $\psi(x) \in FL$*

(P1) If $\vDash \theta$ then $w(\theta) = 1$,
(P2) If $\vDash \neg(\theta \wedge \phi)$ then $w(\theta \vee \phi) = w(\theta) + w(\phi)$,
(P3) $w(\exists x \psi(x)) = \lim_{m\to\infty} w(\bigvee_{i=1}^{m} \psi(a_i))$.

There are a number of well known and straightforward consequences of (P1-3) that we shall assume throughout this paper, see for example [13], page 10. In particular we recall that for w a probability function, if θ and ϕ are logically equivalent then $w(\theta) = w(\phi)$.

For this paper we shall be concerned with probability functions which measure an agent's degree of belief, as subjective probabilities. For that reason we may

[1]The intention being that the distinct individuals named by these constants exhaust the universe.

equally refer to them as *belief functions*. Basically, the underlying question we are interested in is 'what belief function, w, is it rational for an agent to adopt *in the absence of any prior knowledge concerning the* P_i, a_j?'. One way to attempt to understand the term 'rational', and hence approach this question, is to specify some properties, or more grandly called principles, that one feels that such a 'rational' w should possess and see how available choices are restricted. This is the path we shall start on in the next section once we have recalled a further definition.

Definition 4 *Given a probability function w on SL the corresponding conditional probability function $w(.|.)$ is defined by*

$$w(\theta|\phi) = \frac{w(\theta \wedge \phi)}{w(\phi)}$$

whenever $\theta, \phi \in SL$ and $w(\phi) > 0$ and is undefined otherwise.

As is well known $w(\theta|\phi)$ for fixed ϕ with $w(\phi) > 0$ is, as a function of θ, a probability function on SL.

The condition (P3) can be considered a closed world assumption. It corresponds to the intention that every element in the domain/universe is referred to by some constant symbol. Gaifman (see [7]) provides a result showing that if w is defined only over $QFSL$ and w satisfies (P1) and (P2) then w has a unique extension to a probability function on SL (so satisfying not just (P1) and (P2) but also (P3)).

In what follows w will always denote a probability/belief function (on the sentences of some language, which should be clear from the context). Similarly $w(.|.)$ will always denote the corresponding conditional probability/belief function

Definition 5 *For the finite sublanguage L^n of L define the formulae*

$$\bigwedge_{j=1}^{n} P_j^{\epsilon_{ij}}(x)$$

to be the atoms *of SL^n, where $\epsilon_{ij} \in \{0,1\}$ and $P_j^1 = P_j$, $P_j^0 = \neg P_j$ for $j = 1, 2, \ldots, n$. Let $\alpha_1(x), \alpha_2(x), \ldots, \alpha_{2^n}(x)$ be some enumeration of the atoms of SL^n.*

Note that any sentence $\theta(a_i) \in QFSL$ is logically equivalent to a unique instantiation (by a_i) of a disjunction of atoms, $\bigvee_{j=1}^{s} \alpha_j(a_i)$. Hence, by (P1) and (P2), we have

$$w(\theta(a_i)) \;=\; w(\bigvee_{j=1}^{s} \alpha_j(a_i)) \;=\; \sum_{j=1}^{s} w(\alpha_j(a_i)) \qquad (1)$$

More generally, any $\theta(a_{i_1}, a_{i_2}, ..., a_{i_m}) \in QFSL$ is logically equivalent to an instantiation of a disjunction of conjunctions of atoms,

$$\bigvee_{j=1}^{s} \bigwedge_{k=1}^{m} \alpha_{jk}(a_{i_k}),$$

and

$$w(\theta(a_{i_1}, a_{i_2}, ..., a_{i_m})) = \sum_{j=1}^{s} w(\bigwedge_{k=1}^{m} \alpha_{jk}(a_{i_k})). \tag{2}$$

It follows from (2) and Gaifman's result in [7] that a belief function w on SL^n is determined by its values on the (instantiations) of conjunctions of atoms, $\bigwedge_{k=1}^{m} \alpha_{jk}(a_{i_k})$.

Principles of Inductive Reasoning

Here we present the criteria by which we judge a belief function to be 'rational'. After digesting these the reader is invited to jump ahead for a preliminary reading of the conclusion in which their context is discussed more thoroughly.

One principle of inductive reasoning we assume to hold of all the belief functions w we shall consider is that of *Regularity.*

The Principle of Regularity (R)
For $\theta \in QFSL$, $w(\theta) = 0$ if and only if $\vDash \neg\theta$.

Regularity is justified on the grounds that would be 'irrational' to assign zero belief to a sentence which was not impossible. Notice that this principle ensures that the conditional belief $w(\theta|\phi)$ is defined whenever $\phi \in QFSL$ is not self-contradictory.

Two more principles of inductive reasoning that we take to hold of all belief functions we shall consider are those of *Constant Exchangeability* and *Predicate Exchangeability*

The Constant Exchangeability Principle (Ex)
For $\theta, \theta' \in QFSL$, if θ' is obtained from θ by replacing those constant symbols $a_{i_1}, a_{i_2}, ..., a_{i_m}$ occurring in θ by $a_{k_1}, a_{k_2}, ..., a_{k_m}$ respectively then $w(\theta) = w(\theta')$.

Thus if $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ are the constant symbols occurring in a sentence θ then we may simultaneously replace each by $a_1, a_2, \ldots, a_m$ respectively to obtain a sentence with equal belief with respect to a belief function w satisfying Ex. For this reason we will be useful to adopt the convention that $\vec{a}$ stands for $a_1, a_2, \ldots, a_m$ where m will be clear from the context if not stated explicitly.

The Predicate Exchangeability Principle (Px)
For $\theta, \theta' \in QFSL$, if θ' is obtained from θ by replacing the predicate symbols $P_{j_1}, P_{j_2}, ..., P_{j_m}$ occurring in θ by $P_{s_1}, P_{s_2}, ..., P_{s_m}$ respectively then $w(\theta) = w(\theta')$.

Notice that by Ex and Px, the behavior of w on any sublanguage of L with n predicates and m constants is determined from the behavior of w on the sublanguage of L with predicates $P_1, P_2, ..., P_n$ and constants $a_1, a_2, ..., a_m$. We will use this fact frequently and without mention in what follows.

Together the principles Px and Ex capture the intuition that belief functions should be symmetric about renaming of predicate or constant symbols. In other words the belief in a sentence should be a function of the logical structure of the sentence and not contingent on the particular constants or predicates therein.

One consequence of Ex is the famous theorem of de Finetti (see [5]).

Definition 6 *For finite $n \in \mathbb{N}$ define $\mathbb{D}_n$ to be the simplex*

$$\{\langle x_1, x_2, \ldots, x_{2^n}\rangle \in \mathbb{R}^{2^n} \ : \ \sum_{i=1}^{2^n} x_i = 1 \ , \ x_i \geq 0 \ \text{ for } \ i = 1, 2, \ldots, 2^n\}$$

de Finetti's Representation Theorem 1 *Let w be a belief function over SL^n. If w satisfies Ex then there is a unique countably additive probability measure μ on $\mathbb{D}_n$ such that*

$$w(\bigwedge_{k=1}^{m} \alpha_{h_k}(a_k)) \ = \ \int_{\mathbb{D}_n} \prod_{r=1}^{2^n} x_r^{n_r} d\mu(\vec{x}) \tag{3}$$

where $n_r = |\{k : h_k = r\}|$ for $1 \leq r \leq 2^n$. We will call such a measure μ a de Finetti measure (for w).

Notice that, conversely, if μ is a countably additive measure on $\mathbb{D}_n$ and we define w via (3) then w is a belief function on SL^n satisfying Ex.

A few words of explanation here might not be amiss. Each point within the closed convex region $\mathbb{D}_n$ uniquely determines a belief function over $QFSL^n$ (and hence SL^n by [7]) for which the indivuals are assumed to be independently distributed. That is to say, for $\vec{x} \in \mathbb{D}_n$, if we take $w(\alpha_r(a_1)) = x_r$, for $r = 1, 2, \ldots, 2^n$, then by the independence assumption we have

$$w(\bigwedge_{i=1}^{m} \alpha_{r_i}(a_i) \ = \ \prod_{i=1}^{m} x_{r_i}$$

where $r_i \in \{1, 2, \ldots, 2^n\}$ for $i = 1, 2, \ldots, m$.

de Finetti's Representation Theorem tells us that if a belief function on SL^n satisfies constant exchangeability then that belief function looks like a weighted mixture of such 'Bernoulli belief functions'. It is precisely this mixture we will seek to define for 'rational' belief functions.

One consequence of Theorem 1 and Ex is the following *Principle of Instantial Relevance*, see for example [6], [9].

The Principle of Instantial Relevance (PIR)
For consistent $\psi(\vec{a}) \in QFSL$ and an atom[2] α

$$w(\alpha(a_{m+2})) \mid \alpha(a_{m+1}) \wedge \psi(\vec{a})) \;\; \geq \;\; w(\alpha(a_{m+1}) \mid \psi(\vec{a}))$$

Further to our considerations of symmetry between predicate symbols and constant symbols, we shall also require as a standing assumption on w symmetry between predicates and their negations.

The Strong[3] Negation Principle (SN)
For $\theta, \theta' \in SL$, if P is any predicate symbol of L and θ' is obtained from θ by replacing each occurrence of $\pm P$ in θ by $\mp P$ then $w(\theta) = w(\theta')$.

Henceforth we shall assume that all our belief functions satisfy R, Ex, Px, SN. By Ex we also have PIR.

We conclude this section with a final, useful, definition.

Definition 7 *If for each finite sublanguage L' of L, $w_{L'}$ is a belief function on SL' then this family of functions is called* language invariant *if whenever L'' is a sublanguage of L'*

$$w_{L'} \restriction SL'' \;=\; w_{L''}$$

Given a belief function w on SL, the restrictions of w to the SL', i.e. $w \restriction SL'$, for finite sublanguages L' of L are clearly a language invariant family of belief functions. Conversely any language invariant family of belief functions $w_{L'}$ determines a belief function w on SL (with $w \restriction SL' = w_{L'}$) by taking w to be the union of the $w_{L'}$. Furthermore Ex, Px, etc. hold for this w just if they hold for all the $w \restriction SL'$, equivalently all the $w \restriction SL^n$. For this reason we shall in what follows occasionally be rather lax about whether the domain of a belief function we are considering is SL or SL^n for some n.

The Generalized Principle of Instantial Relevance

The Principle of Instantial Relevance, PIR, fits nicely with one of our basic intuitions about inductive reasoning, that the observation of an object satisfying a

[2]Actually this result also holds with an arbitrary quantifier free $\theta(x) \in SL$ in place of $\alpha(x)$.
[3]An alternative, Weak Negation, Principle is where we require *all* the predicates in θ to change sign simultaneously.

set of properties should not decrease the belief in further objects, yet to be seen, also satisfying that same set of properties. Indeed, as already mentioned, the proof of PIR from Ex generalizes straightforwardly to give that for any quantifier free formula $\theta(x)$, consistent $\theta(a_{m+1}) \wedge \psi(\vec{a}) \in QFSL$ and w satisfying Ex and R,

$$w(\theta(a_{m+2}) \mid \theta(a_{m+1}) \wedge \psi(\vec{a})) \;\geq\; w(\theta(a_{m+1}) \mid \psi(\vec{a})).$$

For more on this, in particular the conditions under which equality can hold here, see [6], [9], [14].

The *Generalized Principle of Instantial Relevance*, GPIR, proposed in [8], [14] takes the intuition behind PIR a step further. In words, GPIR says that the observation of an object satisfying a set of properties should not decrease the belief in further objects satisfying a *superset* of those properties.

Generalised Principle of Instantial Relevance (GPIR)

For $\theta(a_1), \phi(a_1), \psi(\vec{a}) \in QFSL$ and $\phi(a_{m+1}) \wedge \psi(\vec{a})$ consistent, if $\theta \models \phi$ then

$$w(\theta(a_{m+2}) \mid \phi(a_{m+1}) \wedge \psi(\vec{a})) \;\geq\; w(\theta(a_{m+1}) \mid \psi(\vec{a})). \tag{4}$$

It is the purpose of this paper to give necessary and sufficient conditions for a belief function w to satisfy GPIR.

It is easily shown (see [14]) that GPIR is equivalent to the principle: For $\theta(a_1), \phi(a_1)$ and $\psi(\vec{a})$ in $QFSL$ and $\phi(a_{m+1}) \wedge \psi(\vec{a})$ consistent, if $\phi \models \theta$ then

$$w(\theta(a_{m+2}) \mid \phi(a_{m+1}) \wedge \psi(\vec{a})) \;\geq\; w(\theta(a_{m+1}) \mid \psi(\vec{a})).$$

Thus GPIR says also that the observation of an object satisfying a set of properties should not decrease the belief in further objects satisfying a *subset* of those properties.

The belief functions w^δ

In this section we present a class of belief functions parameterized by a single value $\delta \in [0, 1]$. We shall then go on to show that these belief functions satisfy GPIR.

Definition 8 *For a point $\vec{b} \in \mathbb{D}_n$ and $\epsilon > 0$, define the ϵ-neighbourhood, $N_\epsilon(\vec{b})$, as*

$$N_\epsilon(\vec{b}) = \{\vec{x} \in \mathbb{D}_n \,:\, |\vec{x} - \vec{b}| < \epsilon\}$$

Definition 9 *For a measure function μ over (the Borel subsets of) $\mathbb{D}_n$, define point $\vec{b} = \langle b_1, b_2, \ldots, b_{2^n} \rangle$ in $\mathbb{D}_n$ to be* non-null *with respect to μ, if for all $\epsilon > 0$*

$$\mu(N_\epsilon(\vec{b})) > 0.$$

Definition 10 *Define a point $\vec{x}$ in $\mathbb{D}_n$ to be* semi-radial *if it is of the form*

$$\langle \gamma, \ldots, \gamma, \gamma + \delta, \gamma, \ldots, \gamma, \gamma \rangle$$

for some $\gamma \in [0,1]$, $\delta \in \mathbb{R}$. Define $\vec{x}$ to be radial *if $\delta \geq 0$. Clearly since $\vec{x} \in \mathbb{D}_n$ we have $2^n \gamma = 1 - \delta$ and $\delta \in [-(2^n - 1)^{-1}, 1]$.*

An illustration may be of use here. The simplex $\mathbb{D}_2$ is bounded by a tetrahedron in $\mathbb{R}^4$, the extreme points of which lie at $\langle 1,0,0,0 \rangle$, $\langle 0,1,0,0 \rangle$, $\langle 0,0,1,0 \rangle$ and $\langle 0,0,0,1 \rangle$. Figure 1 shows line segments lying within this tetrahedral region. These line segments lie between the centroid of $\mathbb{D}_2$, at $\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \rangle$, and the extreme points. These line segments constitute the radial points. Semi-radial points lie along the line segments obtained by extending the line segments shown to the opposing faces of the tetrahedron.

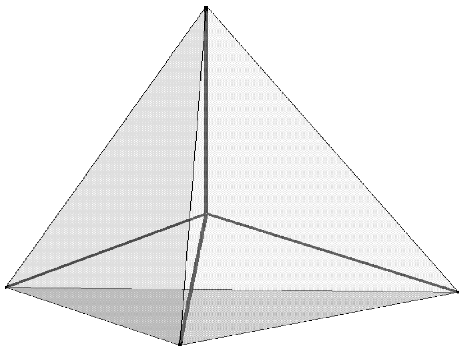

Figure 1: Radial points in $\mathbb{D}_2$

Definition 11 *Define a measure function μ over the (Borel subsets of) $\mathbb{D}_n$ to be* semi-radial *if all non-null points in $\mathbb{D}_n$ with respect to μ are semi-radial. Similarly, define a measure function μ over $\mathbb{D}_n$ to be* radial *if all non-null points in $\mathbb{D}_n$ with respect to μ are radial.*

Definition 12 *Amongst the semi-radial points we distinguish points of the form*

$$\langle \gamma, \ldots, \gamma, \gamma + \delta, \gamma, \ldots, \gamma \rangle$$

for a fixed *δ by the name* δ-semi-radial. *If $\delta \geq 0$ then we may call such points* δ-radial.

Clearly, if $\delta \neq 0$ then there are 2^n δ-semi-radial points. On the other hand if $\delta = 0$ then there is just a single δ-semi-radial point, namely $\langle 2^{-n}, 2^{-n}, ..., 2^{-n} \rangle$.

Definition 13 *For each integer $n \geq 1$ and $\delta \in [-(2^n - 1)^{-1}, 1]$ define the measure ν_n^δ over $\mathbb{D}_n$ to be that measure satisfying*

$$\nu_n^\delta(\{\vec{x}\}) = \begin{cases} 2^{-n} & \text{if } \delta \neq 0, \\ 1 & \text{if } \delta = 0, \end{cases}$$

for each δ-semi-radial point $\vec{x} \in \mathbb{D}_n$. We will refer to such a measure as being δ-semi-radial, *or possibly* δ-radial *if $\delta \geq 0$.*

For $\delta \in [-(2^n - 1)^{-1}, 1]$ let w_n^δ be the belief function defined from ν_n^δ according to (3).

Notice that all points that are not δ-semi-radial must be null-points with respect to ν_n^δ.

We call ν_n^0 and w_n^0 the *independent measure* and *independent belief function* respectively, and ν_n^1 and w_n^1 the *trivial measure* and *trivial belief function* respectively.

Theorem 14 *For any integer $n \geq 1$, and any δ-semi-radial measure ν_n^δ*

$$w_n^\delta(\bigwedge_{k=1}^m \alpha_{h_k}(a_k)) \quad = \quad \int_{\mathbb{D}_n} \prod_{r=1}^{2^n} x_r^{k_r} d\nu_n^\delta \quad = \quad \frac{\gamma^k}{2^n} \sum_{r=1}^{2^n} \left(1 + \frac{\delta}{\gamma}\right)^{k_r} \tag{5}$$

where $2^n\gamma = 1 - \delta$, $k_r = |\{i : h_i = r\}|$ and $k = \sum_{r=1}^{2^n} k_r$.
[In case $\gamma = 0$ we take this right hand expression to stand for $\frac{1}{2^n}\sum_{r=1}^{2^n} \gamma^{k-k_r}(\gamma + \delta)^{k_r}$.]

Proof. Suppose $\delta \neq 0$ then the above integral must be equivalent to a sum of 2^n terms since there are 2^n non-null points in $\mathbb{D}_n$ with respect to ν_n^δ of the form

$$\langle \gamma, \dots, \gamma, \gamma + \delta, \gamma, \dots, \gamma \rangle$$

each of which gets measure $\frac{1}{2^n}$, so

$$\begin{aligned}
\int_{\mathbb{D}_n} \prod_{r=1}^{2^n} x_r^{k_r} d\nu_n^\delta \quad &= \quad \frac{(\gamma+\delta)^{k_1}.\gamma^{k_2}.\gamma^{k_3}. \ \dots \ .\gamma^{k_{2^n}}}{2^n} \\
&+ \quad \frac{\gamma^{k_1}.(\gamma+\delta)^{k_2}.\gamma^{k_3}. \ \dots \ .\gamma^{k_{2^n}}}{2^n} \\
&\vdots \\
&+ \quad \frac{\gamma^{k_1}.\gamma^{k_2}.\gamma^{k_3}. \ \dots \ .(\gamma+\delta)^{k_{2^n}}}{2^n} \\
&= \quad \frac{\gamma^k}{2^n} \sum_{r=1}^{2^n} \left(\frac{\gamma+\delta}{\gamma}\right)^{k_r}
\end{aligned}$$

from which equation (5) follows. If $\delta = 0$ the only non-null point is

$$\langle 2^{-n}, 2^{-n}, \dots, 2^{-n} \rangle$$

and thus this point gets measure 1, hence our integral equals $(\frac{1}{2^n})^k$, which is just the value given by the right hand side of (5). ■

Theorem 15 *For $\delta \in [0, 1)$ the w_n^δ form a language invariant family of belief functions satisfying the standing assumptions R, Ex, Px and SN.*

Proof. That the w_n^δ satisfies the standing assumptions follows directly from (5). Indeed with the exception of R they all hold as well when $\delta = 1$.

To show the language invariance it is enough, by (2), to show that for atoms $\alpha_1, \alpha_2, ..., \alpha_{2^n}$ of SL^n

$$w_n^\delta(\bigwedge_{i=1}^m \alpha_{h_i}(\alpha_i)) \;=\; w_{n+1}^\delta(\bigwedge_{i=1}^m \alpha_{h_i}(\alpha_i)).$$

Let the atoms of SL^{n+1} be $\alpha_1^+, \alpha_1^-, \alpha_2^+, \alpha_2^-, ..., \alpha_{2^n}^+, \alpha_{2^n}^-$ where $\alpha_i^+ = \alpha_i \wedge P_{n+1}(x)$, and $\alpha_i^- = \alpha_i \wedge \neg P_{n+1}(x)$. Then

$$\begin{aligned}
w_{n+1}^\delta(\bigwedge_{i=1}^m \alpha_{h_i}(a_i)) \;&=\; w_{n+1}^\delta(\bigvee_{\epsilon_1,...,\epsilon_m=\pm} \bigwedge_{i=1}^m \alpha_{h_i}^{\epsilon_i}(a_i))\\
&= \sum_{\epsilon_1,...,\epsilon_m=\pm} w_{n+1}^\delta(\bigwedge_{i=1}^m \alpha_{h_i}^{\epsilon_i}(a_i))\\
&= \sum_{\epsilon_1,...,\epsilon_m=\pm} 2^{-(n+1)}(\gamma/2)^m \sum_{r=1}^{2^n}\sum_{\epsilon=\pm}(1+2\delta/\gamma)^{k_r^\epsilon},\\
&\quad \text{where } k_r^\epsilon = |\{i : \alpha_{h_i}^{\epsilon_i} = \alpha_r^\epsilon\}|,\\
&= 2^{-(n+1)}(\gamma/2)^m \sum_{r=1}^{2^n}\sum_{s=0}^{k_r} 2^{m-k_r+1}\binom{k_r}{s}(1+2\delta/\gamma)^s,\\
&\quad \text{where } k_r = |\{i : \alpha_{h_i} = \alpha_r\}|, \text{ since } 2^{m-k_r}\binom{k_r}{s} \text{ is the number of}\\
&\quad \text{ways of choosing the } \epsilon_i \text{ to give the term } (1+2\delta/\gamma)^s \text{ for each of}\\
&\quad \text{the } \epsilon = \pm,\\
&= 2^{-(n+1)}(\gamma/2)^m 2^{m-k_r+1}\sum_{r=1}^{2^n}(2+2\delta/\gamma)^{k_r},\\
&= 2^{-n}\gamma^m \sum_{r=1}^{2^n}(1+\delta/\gamma)^{k_r},
\end{aligned}$$

as required. ■

Let w^δ be the belief function on SL such that $w^\delta \restriction SL^n = w_n^\delta$ for each n. Notice that for $\delta \in [0,1)$ w^δ satisfies R, Ex, Px, SN.

Theorem 16 *For $\delta \in [0,1)$, w^δ satisfies GPIR.*

For $\delta = 0$, w^δ is the independent belief function and equality holds in (4), see [13]. So from now on we can assume that $\delta > 0$. Before embarking on the proof however we need some useful notation.

Definition 17 *For w, a belief function satisfying R and $\theta(a_1), \phi(a_1)$, $\psi(\vec{a})$ sentences in SL define the function s^w by*

$$\begin{aligned} & s^w\left(\theta(a_{m+2}),\, \phi(a_{m+1}),\, \psi(\vec{a})\right) \\ &= \; w(\theta(a_{m+2}) \,|\, \phi(a_{m+1}) \,\wedge\, \psi(\vec{a})) \;-\; w(\theta(a_{m+2}) \,|\, \psi(\vec{a})) \end{aligned}$$

whenever the right hand side is defined and zero otherwise.

Thus, s^w singles out the contribution of $\phi(a_{m+1})$ to the degree of confirmation of the hypothesis, $\theta(a_{m+2})$, in the presence of background knowledge, $\psi(\vec{a})$. This formalises what we mean by support.

Henceforth we may abbreviate $s^w\left(\theta(a_{m+2}),\, \phi(a_{m+1}),\, \psi(\vec{a})\right)$ by $s^w_{\phi,\psi}(\theta)$. With this abbreviation notice that

$$s^w_{\phi,\psi}(\theta) \;=\; \sum_{\alpha_i \vDash \theta} s^w_{\phi,\psi}(\alpha_i). \tag{6}$$

Clearly w satisfies GPIR if and only if $s^w_{\phi,\psi}(\theta) \geq 0$ for $\phi \vDash \theta$. We now consider this function of support for the w^δ, $\delta \in [0,1)$.

Definition 18 *For a sequence,*

$$\vec{k} = \langle\, k_1,\, k_2,\, \ldots,\, k_{2^n} \,\rangle$$

of non-negative integers define for $i \in \{1, 2, \ldots, 2^n\}$ the sequence

$$\vec{k}^i = \langle\, k^i_1,\, k^i_2,\, \ldots,\, k^i_{2^n} \,\rangle$$

to be that sequence identical to $\vec{k}$ at all elements except k_i, for which $k^i_i = k_i + 1$. That is, for all $1 \leq r \leq 2^n$

$$k^i_r = \begin{cases} k_r + 1 & \textit{if } r = i, \\ k_r & \textit{otherwise.} \end{cases}$$

We extend the above definition to a sequence of integers $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, 2^n\}$ and define $\vec{k}^{i_1, i_2, \ldots, i_t}$ for all $1 \leq r \leq 2^n$ as

$$k^{i_1, i_2, \ldots, i_t}_r = \begin{cases} k^{i_2, \ldots, i_t}_r + 1 & \textit{if } r = i_1, \\ k^{i_2, \ldots, i_t}_r & \textit{otherwise.} \end{cases}$$

Lemma 19 *Let $w = w^\delta$, $\delta \in [0,1)$, let $\phi(a_{m+1})$ be a consistent sentence in $QFSL^n$, let*

$$\psi(\vec{a}) = \bigvee_{h=1}^{p} \bigwedge_{r=1}^{m} \alpha_{hr}(a_r),$$

and let

$$k_{hj} = |\{r \,:\, \alpha_{hr} = \alpha_j\}|$$

for $h = 1, 2, \ldots, p$ *and* $j = 1, 2, \ldots, 2^n$.

Then for some $\lambda \geq 0$ *and independent of* i,

$$s^w_{\phi,\psi}(\alpha_i) \;=\; \begin{cases} \lambda \displaystyle\sum_{h=1}^{p} \sum_{l=1}^{p} \sum_{\alpha_j \vDash \neg\phi} \left(1 + \frac{\delta}{\gamma}\right)^{k_{li}+k_{hj}} & \text{for } \alpha_i \vDash \phi, \\ -\lambda \displaystyle\sum_{h=1}^{p} \sum_{l=1}^{p} \sum_{\alpha_j \vDash \phi} \left(1 + \frac{\delta}{\gamma}\right)^{k_{li}+k_{hj}} & \text{for } \alpha_i \vDash \neg\phi, \end{cases}$$

and $\lambda = 0$ *if and only if* $\delta = 0$, *i.e.* w *is the independent belief function.*

Proof. By Theorem 1 and Theorem 14 it suffices to show that the difference between

$$\left[\sum_{\alpha_j \vDash \phi} \sum_{h=1}^{p} \gamma^{m+2} \sum_{r=1}^{2^n} \left(1 + \frac{\delta}{\gamma}\right)^{k^{i,j}_{hr}}\right] \cdot \left[\sum_{l=1}^{p} \gamma^{m} \sum_{s=1}^{2^n} \left(1 + \frac{\delta}{\gamma}\right)^{k_{ls}}\right] \tag{7}$$

and

$$\left[\sum_{\alpha_j \vDash \phi} \sum_{h=1}^{p} \gamma^{m+1} \sum_{r=1}^{2^n} \left(1 + \frac{\delta}{\gamma}\right)^{k^{i}_{hr}}\right] \cdot \left[\sum_{l=1}^{p} \gamma^{m+1} \sum_{s=1}^{2^n} \left(1 + \frac{\delta}{\gamma}\right)^{k^{j}_{ls}}\right] \tag{8}$$

is equal to

$$\tau \sum_{h=1}^{p} \sum_{l=1}^{p} \sum_{\alpha_j \vDash \neg\phi} \left(1 + \frac{\delta}{\gamma}\right)^{k_{hi}+k_{lj}} \qquad \text{or} \qquad -\,\tau \sum_{h=1}^{p} \sum_{l=1}^{p} \sum_{\alpha_j \vDash \phi} \left(1 + \frac{\delta}{\gamma}\right)^{k_{hi}+k_{lj}}$$

for $\alpha_i \vDash \phi$ and $\alpha_i \vDash \neg\phi$ respectively, for some $\tau > 0$, independent of i. For then we may set

$$\lambda \;=\; \frac{\tau}{\left[\sum_{l=1}^{p} \gamma^{m} \sum_{s=1}^{2^n} \left(1 + \frac{\delta}{\gamma}\right)^{k_{ls}}\right] \cdot \left[\sum_{l=1}^{p} \gamma^{m+1} \sum_{s=1}^{2^n} \left(1 + \frac{\delta}{\gamma}\right)^{k^{j}_{ls}}\right]}$$

and the result follows with the provision that $\tau = 0$ if and only if w is the independent belief function. For $h = 1, 2, \ldots, p$ and $i = 1, 2, \ldots, 2^n$ let

$$d^l(i) \;=\; \gamma^m \left(1 + \frac{\delta}{\gamma}\right)^{k_{li}}$$

If $\alpha_i \vDash \neg\phi$ then expression (7) is equal to

$$\sum_{h=1}^{p} \sum_{l=1}^{p} \sum_{\alpha_j \vDash \phi} \left[\gamma^2 \sum_{r=1}^{2^n} d^h(r) \;+\; \gamma\delta\, d^h(i) \;+\; \gamma\delta\, d^h(j)\right] \cdot \left[\sum_{s=1}^{2^n} d^l(s)\right]$$

Expression (8) is equal to

$$\sum_{h=1}^{p}\sum_{l=1}^{p}\sum_{\alpha_j\models\phi}\left[\gamma\sum_{r=1}^{2^n}d^h(r)\,+\,\delta\,d^h(i)\right]\,.\,\left[\gamma\sum_{s=1}^{2^n}d^l(s)\,+\,\delta\,d^l(j)\right]$$

Therefore subtracting expression (8) from (7) gives

$$-\sum_{h=1}^{p}\sum_{l=1}^{p}\sum_{\alpha_j\models\phi}\delta^2\,d^h(i)\,d^l(j)\;\;=\;\;-\gamma^m\,\delta^2\sum_{h=1}^{p}\sum_{l=1}^{p}\sum_{\alpha_j\models\phi}\left(1+\frac{\delta}{\gamma}\right)^{k_{li}+k_{hj}}$$

Notice that since $\delta < 1$, $\gamma > 0$, so $\tau = \gamma^m\delta^2 = 0$ just if $\delta = 0$. So the result is proven for $\alpha_i \models \neg\phi$. For $\alpha_i \models \phi$ we have expression (7) equal to

$$\sum_{h=1}^{p}\sum_{l=1}^{p}\sum_{\substack{\alpha_j\models\phi\\ j\neq i}}\left[\gamma^2\sum_{r=1}^{2^n}d^h(r)\,+\,\gamma\delta\,d^h(i)\,+\,\gamma\delta\,d^h(j)\right]\,.\,\left[\sum_{s=1}^{2^n}d^l(s)\right]$$
$$+\;\;\sum_{h=1}^{p}\sum_{l=1}^{p}\left[\gamma^2\sum_{r=1}^{2^n}d^h(r)\,+\,2\gamma\delta d^h(i)\,+\,\delta^2\,d^h(i)\right]\,.\,\left[\sum_{s=1}^{2^n}d^l(s)\right]$$

and expression (8) is similarly equal to

$$\sum_{h=1}^{p}\sum_{l=1}^{p}\sum_{\substack{\alpha_j\models\phi\\ j\neq i}}\left[\gamma\sum_{r=1}^{2^n}d^h(r)\,+\,\delta\,d^h(i)\right]\,.\,\left[\gamma\sum_{s=1}^{2^n}d^l(s)\,+\,\delta\,d^l(j)\right]$$
$$+\;\;\sum_{h=1}^{p}\sum_{l=1}^{p}\left[\gamma\sum_{r=1}^{2^n}d^h(r)\,+\,\delta\,d^h(i)\right]\,.\,\left[\gamma\sum_{s=1}^{2^n}d^l(s)\,+\,\delta\,d^l(i)\right].$$

Hence in this case, employing the previous result concerning the difference between the first two terms, subtracting expression (8) from (7) gives

$$-\sum_{h=1}^{p}\sum_{l=1}^{p}\sum_{\substack{\alpha_j\models\phi\\ j\neq i}}\delta^2 d^h(i)d^l(j)+\sum_{h=1}^{p}\sum_{l=1}^{p}\sum_{\substack{j=1\\ j\neq i}}^{2^n}\delta^2 d^h(i)d^l(j)$$

$$=\;\;-\,\gamma^m\,\delta^2\sum_{h=1}^{p}\sum_{l=1}^{p}\sum_{\substack{\alpha_j\models\phi\\ j\neq i}}\left(1+\frac{\delta}{\gamma}\right)^{k_{li}+k_{hj}}\;+\,\gamma^m\,\delta^2\sum_{h=1}^{p}\sum_{l=1}^{p}\sum_{\substack{j=1\\ j\neq i}}^{2^n}\left(1+\frac{\delta}{\gamma}\right)^{k_{li}+k_{hj}}$$

which is just

$$\gamma^m\,\delta^2\sum_{h=1}^{p}\sum_{l=1}^{p}\sum_{\alpha_j\models\neg\phi}\left(1+\frac{\delta}{\gamma}\right)^{k_{li}+k_{hj}}.$$

Again notice that since $\delta < 1$, $\gamma > 0$, and $\tau = \gamma^m \delta^2 = 0$ just if $\delta = 0$. ■

Proof of Theorem 16. Let $w = w^\delta$ with $\delta \in [0, 1)$ and let $\theta(a_{m+2}), \phi(a_{m+1})$ and $\psi(\vec{a})$ be in $QFSL^n$ with $\phi(a_{m+1}) \wedge \psi(\vec{a})$ consistent. Since w gives the same value to logically equivalent sentences we may assume that $\psi(\vec{a}) = \bigvee_{h=1}^{p} \bigwedge_{r=1}^{m} \alpha_{h_r}(a_r)$. Let

$$k_{hj} = |\{ r \, : \, h_r = j \}|$$

for $h = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, 2^n$. By Lemma 19

$$s^w_{\phi,\psi}(\alpha_i) \;=\; \begin{cases} \lambda \displaystyle\sum_{h=1}^{p} \sum_{l=1}^{p} \sum_{\alpha_j \vDash \neg\phi} c^{k_{li}+k_{hj}} & \text{for } \; \alpha_i \vDash \phi, \\ -\lambda \displaystyle\sum_{h=1}^{p} \sum_{l=1}^{p} \sum_{\alpha_j \vDash \phi} c^{k_{li}+k_{hj}} & \text{for } \; \alpha_i \vDash \neg\phi, \end{cases}$$

where

$$c \;=\; \left(1 + \frac{\delta}{\gamma}\right).$$

Thus

$$\begin{aligned} & w(\theta(a_{m+2}) \,|\, \phi(a_{m+1}) \wedge \psi(\vec{a})) \quad \geq \quad w(\theta(a_{m+1}) \,|\, \psi(\vec{a})) \qquad (9)\\ \Leftrightarrow \;\; & s^w_{\phi,\psi}(\theta) \;\geq\; 0 \\ \Leftrightarrow \;\; & \sum_{\alpha_i \vDash \theta} s^w_{\phi,\psi}(\alpha_i) \;\geq\; 0 \\ \Leftrightarrow \;\; & \sum_{h=1}^{p} \sum_{l=1}^{p} \sum_{\alpha_i \vDash \theta\wedge\phi} \sum_{\alpha_j \vDash \neg\phi} c^{k_{li}+k_{hj}} \;\geq\; \sum_{h=1}^{p} \sum_{l=1}^{p} \sum_{\alpha_i \vDash \neg\phi\wedge\theta} \sum_{\alpha_j \vDash \phi} c^{k_{li}+k_{hj}} \\ \Leftrightarrow \;\; & \sum_{h=1}^{p} \sum_{l=1}^{p} \sum_{\alpha_i \vDash \phi\wedge\theta} \sum_{\alpha_j \vDash \neg\phi\wedge\neg\theta} c^{k_{li}+k_{hj}} \;\geq\; \sum_{h=1}^{p} \sum_{l=1}^{p} \sum_{\alpha_i \vDash \neg\phi\wedge\theta} \sum_{\alpha_j \vDash \phi\wedge\neg\theta} c^{k_{li}+k_{hj}}. \qquad (10) \end{aligned}$$

But if $\phi \vDash \theta$ then

$$\{ \alpha_j \, : \, \alpha_j \vDash \phi \wedge \neg\theta \} \quad = \quad \emptyset$$

hence the right most sum on the right hand side of equation (10) is equal to zero, so the whole right hand side is equal to zero. The left hand side is clearly non-negative so we have our result. ■

It follows from the preceding proof that Theorem 16 can be improved marginally in that if $\phi(a_{m+1})$ and $\psi(\vec{a})$ are consistent and $\phi \models \theta$ then

$$w^\delta \left(\theta(a_{m+2}) \,|\, \phi(a_{m+1}) \wedge \psi(\vec{a})\right) > w^\delta \left(\theta(a_{m+1}) \,|\, \psi(\vec{a})\right)$$

unless θ is a tautology, in which case both sides will be equal. The reason for this is that with the given condition on θ, ϕ there will be atoms α_i, α_j such that $\alpha_i \models \phi \wedge \theta$ and $\alpha_j \models \neg\phi \wedge \neg\theta$ just if θ is not a tautology.

GPIR and the w^δ

We now turn our attention to proving the converse of Theorem 16. Namely we shall show that the belief functions satisfying GPIR (and R, Px, Ex and SN as always) are precisely the w^δ for $\delta \in [0,1)$.

As a first step it will be useful to provide some notation for the set of permutations of indices of atoms such that the equality

$$w(\bigwedge_{j=1}^{m} \alpha_{i_j}(a_j)) \;=\; w(\bigwedge_{j=1}^{m} \alpha_{\sigma(i_j)}(a_j))$$

is guaranteed by SN and Px.

For the language L^n let

$$\alpha_i(x) \;=\; \bigwedge_{j=1}^{n} P_j^{\epsilon_{ij}}(x)$$

where as usual $\epsilon_{ij} \in \{0,1\}$ and $P_j^1 = P_j$, $P_j^0 = \neg P_j$ for $j = 1,2,\ldots,n$. For a permutation ρ of $\{1,2,...,n\}$ let σ_ρ be the permutation of $\{1,2,...,2^n\}$ such that for $i \in \{1,2,...,n\}$,

$$\alpha_{\sigma_\rho(i)}(x) \;\equiv\; \bigwedge_{j=1}^{n} P_{\rho(j)}^{\epsilon_{ij}}(x).$$

Notice then that

$$\alpha_{\sigma_\rho(i)}(x) \;=\; \bigwedge_{j=1}^{n} P_j^{\epsilon_{\sigma_\rho(i)j}}(x) \;\equiv\; \bigwedge_{j=1}^{n} P_{\rho(j)}^{\epsilon_{ij}}(x) \;\equiv\; \bigwedge_{j=1}^{n} P_j^{\epsilon_{i\rho^{-1}(j)}}(x),$$

so σ_ρ could equally well have been defined by the identities

$$\epsilon_{\sigma_\rho(i)j} \;=\; \epsilon_{i\rho^{-1}(j)}.$$

Let

$$\Sigma_n^{\mathrm{Px}} \;=\; \{\sigma_\rho \;:\; \rho \text{ is a permutation of } \{1,2,...,n\}\}.$$

It is clear that each permutation in Σ_n^{Px} is defined by a unique permutation of the indices of predicate symbols. Conversely, each permutation of the indices of predicate symbols defines a single permutation in Σ_n^{Px}. It is also hopefully obvious that Σ_n^{Px} is the set of permutations σ such that the equality

$$w(\bigwedge_{j=1}^{m} \alpha_{i_j}(a_j)) \;=\; w(\bigwedge_{j=1}^{m} \alpha_{\sigma(i_j)}(a_j))$$

is guaranteed by Px. Clearly Σ_n^{Px} is closed under formation of inverses and compositions, indeed, $(\sigma_\rho)^{-1} = \sigma_{\rho^{-1}}$ and $\sigma_\rho\sigma_\tau = \sigma_{\rho\tau}$.

Given a function $f : \{1, 2, ..., n\} \to \{0, 1\}$ let η_f be the permutation of $\{1, 2, ..., 2^n\}$ such that for $i \in \{1, 2, ..., n\}$,

$$\alpha_{\eta_f(i)}(x) \;\; = \;\; \bigwedge_{j=1}^{n} P_j^{|\epsilon_{ij} - f(j)|}(x).$$

In other words $\alpha_{\eta_f(i)}$ is that atom obtained from α_i if the sign of each $P_j(x)$ in $\alpha_i(x)$ is altered if $f(j) = 1$ and kept the same if $f(j) = 0$. Define

$$\Sigma_n^{\mathrm{SN}} \;\; = \;\; \{\eta_f \; : \; f : \{1, 2, ..., n\} \to \{0, 1\}\}.$$

Hopefully it is obvious that Σ_n^{SN} is the set of permutations σ such that the equality

$$w(\bigwedge_{j=1}^{m} \alpha_{i_j}(a_j)) \;\; = \;\; w(\bigwedge_{j=1}^{m} \alpha_{\sigma(i_j)}(a_j))$$

is guaranteed by SN. Again, as with Σ_n^{Px}, it is easily shown that Σ_n^{SN} is closed under formation of inverses and compositions, indeed, $(\eta_f)^{-1} = \eta_f$ and $\eta_f \eta_g = \eta_{|f-g|}$.

Let Σ_n be the closure of $\Sigma_n^{\mathrm{SN}} \cup \Sigma_n^{\mathrm{Px}}$ under composition. Since $\sigma_\rho \eta_f = \eta_{f\rho^{-1}} \sigma_\rho$ it is clear that every element of Σ_n is of the form $\sigma_\rho \eta_f$ $(\eta_f \sigma_\rho)$ for some ρ and f. Combining our earlier observations then, for any w satisfying R, Ex, Px and SN,

$$w(\bigwedge_{i=1}^{m} \alpha_{h_i}(a_i)) \;\; = \;\; w(\bigwedge_{i=1}^{m} \alpha_{\sigma(h_i)}(a_i))$$

whenever $\sigma \in \Sigma_n$.

Lemma 20 *For $n > 0$, let w be a belief function over sentences of language L^n satisfying GPIR with e Finetti measure μ and let $\sigma \in \Sigma_n$. Then for any non-null point $\vec{b} \in \mathbb{D}_n$ with respect μ*

$$(b_i - b_{\sigma(i)})(b_j - b_{\sigma(j)}) \leq 0,$$

where $1 \leq i < j \leq 2^n$.

Proof. The proof of this lemma is a generalization of that given for Theorem 6 in [14].

Suppose $\vec{b} \in \mathbb{D}_n$ is non-null with respect to μ. Let $\sigma \in \Sigma_n$. Now, consider the point $\langle \frac{k_1}{k}, \frac{k_2}{k}, \ldots, \frac{k_{2^n}}{k} \rangle \in \mathbb{D}_n$ with $k, k_1, k_2, \ldots, k_{2^n} \in \mathbb{N}$. Clearly, for k suitably large, we can pick such a point lying within $\frac{2^n}{k}$ of $\vec{b}$.

To simplify the notation we shall write

$$\bigwedge_{r=1}^{2^n} \alpha_r^{k_r}$$

for

$$\bigwedge_{r=1}^{2^n} \bigwedge_{i=1}^{k_r} \alpha_r(a_{p_r+i}),$$

where $p_r = \sum_{j=1}^{r-1} k_j$, and write α_j, α_i for $\alpha_j(a_{k+1}), \alpha_i(a_{k+2})$, respectively.

Now for $1 \leq i, j \leq 2^n$, $i \neq j$, GPIR gives that

$$w\,(\,\alpha_i \,|\, \bigwedge_{r=1}^{2^n} \alpha_r^{k_r} \,\vee\, \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,) \;\geq\; w\,(\,\alpha_i \,|\, \alpha_j \wedge (\,\bigwedge_{r=1}^{2^n} \alpha_r^{k_r} \,\vee\, \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,)) \tag{11}$$

since $\alpha_i(x) \vDash \neg\alpha_j(x)$. We show that as $\langle \frac{k_1}{k}, \frac{k_2}{k}, \ldots, \frac{k_{2^n}}{k} \rangle \to \vec{b}$ the LHS and RHS of (11) converge to

$$\frac{b_i + b_{\sigma(i)}}{2} \quad \text{and} \quad \frac{b_i b_j + b_{\sigma(i)} b_{\sigma(j)}}{b_j + b_{\sigma(j)}}$$

respectively, and hence

$$\frac{b_i + b_{\sigma(i)}}{2} \;\geq\; \frac{b_i b_j + b_{\sigma(i)} b_{\sigma(j)}}{b_j + b_{\sigma(j)}},$$

from which we obtain our result,

$$(b_i - b_{\sigma(i)})(b_j - b_{\sigma(j)}) \leq 0.$$

Firstly consider the LHS of (11). Assuming that σ is not the identity (if it is the result is immediate) the sentences $\bigwedge_{r=1}^{2^n} \alpha_r^{k_r}$ and $\bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}$ are disjoint, so

$$\begin{aligned} & w\,(\,\alpha_i \,|\, \bigwedge_{r=1}^{2^n} \alpha_r^{k_r} \,\vee\, \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,) \\ &= \frac{w\,(\,\alpha_i \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,) \,+\, w\,(\,\alpha_i \wedge \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,)}{w\,(\,\bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,) \,+\, w\,(\,\bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,)} \end{aligned} \tag{12}$$

Note that by SN and Px,

$$w\,(\,\bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,) \;=\; w\,(\,\bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,) \tag{13}$$

$$w\,(\,\alpha_i \wedge \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,) \;=\; w\,(\,\alpha_{\sigma(i)} \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,) \tag{14}$$

hence by (12), (13) and (14) and Theorem 1 we have

$$w\,(\,\alpha_i \,|\, \bigwedge_{r=1}^{2^n} \alpha_r^{k_r} \,\vee\, \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,)$$

$$
\begin{aligned}
&= \frac{w\,(\,\alpha_i \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,)}{2\,.\,w\,(\,\bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,)} + \frac{w\,(\,\alpha_{\sigma(i)} \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,)}{2\,.\,w\,(\,\bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,)} \\
&= \frac{\int_{\mathbb{D}_n} x_i \prod_{r=1}^{2^n} x_r^{k_r}\,d\mu}{2\,.\int_{\mathbb{D}_n} \prod_{r=1}^{2^n} x_r^{k_r}\,d\mu} + \frac{\int_{\mathbb{D}_n} x_{\sigma(i)} \prod_{r=1}^{2^n} x_r^{k_r}\,d\mu}{2\,.\int_{\mathbb{D}_n} \prod_{r=1}^{2^n} x_r^{k_r}\,d\mu} \qquad (15)
\end{aligned}
$$

Finally, by the asymptotic consistency of Bayes posteriors (see for example Theorem 7.78 of [16] or page 25 of [14]) we see that the RHS of (15) converges to

$$\frac{b_i + b_{\sigma(i)}}{2}$$

as $k \to \infty$, as required. So we have proved convergence for the LHS of (11), now for the RHS. Assuming again that σ is not the identity,

$$
\begin{aligned}
&w\,(\,\alpha_i \,|\, \alpha_j \wedge (\bigwedge_{r=1}^{2^n} \alpha_r^{k_r} \;\vee\; \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,)) \\
&\quad = \frac{w\,(\,\alpha_i \wedge \alpha_j \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,) + w\,(\,\alpha_i \wedge \alpha_j \wedge \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,)}{w\,(\,\alpha_j \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,) + w\,(\,\alpha_j \wedge \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,)}. \qquad (16)
\end{aligned}
$$

Note that by SN and Px,

$$
\begin{aligned}
w\,(\,\alpha_j \wedge \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,) &= w\,(\,\alpha_{\sigma(j)} \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,) \qquad (17) \\
w\,(\,\alpha_i \wedge \alpha_j \wedge \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,) &= w\,(\,\alpha_{\sigma(i)} \wedge \alpha_{\sigma(j)} \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,), \qquad (18)
\end{aligned}
$$

hence by (16),(17) and (18) we have

$$
\begin{aligned}
&w\,(\,\alpha_i \,|\, \alpha_j \wedge (\bigwedge_{r=1}^{2^n} \alpha_r^{k_r} \;\vee\; \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,)) \\
&\quad = \frac{w\,(\,\alpha_i \wedge \alpha_j \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,) + w\,(\,\alpha_{\sigma(i)} \wedge \alpha_{\sigma(j)} \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,)}{w\,(\,\alpha_j \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,) + w\,(\,\alpha_{\sigma(j)} \wedge \bigwedge_{r=1}^{2^n} \alpha_r^{k_r}\,)} \\
&\quad = \frac{\int_{\mathbb{D}_n} x_i x_j \prod_{r=1}^{2^n} x_r^{k_r}\,d\mu + \int_{\mathbb{D}_n} x_{\sigma(i)} x_{\sigma(j)} \prod_{r=1}^{2^n} x_r^{k_r}\,d\mu}{\int_{\mathbb{D}_n} x_j \prod_{r=1}^{2^n} x_r^{k_r}\,d\mu + \int_{\mathbb{D}_n} x_{\sigma(j)} \prod_{r=1}^{2^n} x_r^{k_r}\,d\mu}. \qquad (19)
\end{aligned}
$$

By the asymptotic consistency of Bayes posteriors, we have for $k \to \infty$,

$$\frac{\int_{\mathbb{D}_n} x_j \prod_{r=1}^{2^n} x_r^{k_r}\,d\mu}{\int_{D_n} \prod_{r=1}^{2^n} x_r^{k_r}\,d\mu} \longrightarrow b_j, \qquad (20)$$

$$\frac{\int_{\mathbb{D}_n} x_{\sigma(j)} \prod_{r=1}^{2^n} x_r^{k_r}\, d\mu}{\int_{\mathbb{D}_n} \prod_{r=1}^{2^n} x_r^{k_r}\, d\mu} \longrightarrow b_{\sigma(j)}, \tag{21}$$

$$\frac{\int_{\mathbb{D}_n} x_i x_j \prod_{r=1}^{2^n} x_r^{k_r}\, d\mu}{\int_{\mathbb{D}_n} \prod_{r=1}^{2^n} x_r^{k_r}\, d\mu} \longrightarrow b_i \,.\, b_j, \tag{22}$$

$$\frac{\int_{\mathbb{D}_n} x_{\sigma(i)} x_{\sigma(j)} \prod_{r=1}^{2^n} x_r^{k_r}\, d\mu}{\int_{\mathbb{D}_n} \prod_{r=1}^{2^n} x_r^{k_r}\, d\mu} \longrightarrow b_{\sigma(i)} \,.\, b_{\sigma(j)}. \tag{23}$$

Hence by (19),(20),(21),(22), and (23), as $\langle \frac{k_1}{k}, \frac{k_2}{k}, \ldots, \frac{k_{2^n}}{k} \rangle \to \vec{b}$

$$w\,(\,\alpha_i \,|\, \alpha_j \wedge (\bigwedge_{r=1}^{2^n} \alpha_r^{k_r} \;\vee\; \bigwedge_{r=1}^{2^n} \alpha_{\sigma^{-1}(r)}^{k_r}\,)) \;\longrightarrow\; \frac{b_i b_j + b_{\sigma(i)} b_{\sigma(j)}}{b_j + b_{\sigma(j)}}.$$

■

Given Lemma 20 above we may go on show that μ is semi-radial.

Lemma 21 *For $n > 0$, let w be a belief function over sentences of language L^n satisfying GPIR whose prior w has De Finetti measure μ. Then μ is semi-radial.*

Proof. Suppose that $\langle b_1, b_2, ..., b_{2^n} \rangle$ is a non-null point of μ. By Lemma 20 we know that for any $\sigma \in \Sigma_n$,

$$(b_i - b_{\sigma(i)})(b_j - b_{\sigma(j)}) \;\leq\; 0. \tag{24}$$

Suppose that $\langle b_1, b_2, ..., b_{2^n} \rangle$ was not semi-radial. The first possibility is that three of the b_i are different, say $b_1 > b_2 > b_3$. Let $\eta \in \Sigma_n^{\mathrm{SN}}$ be such that $\eta(1) = 3$, so $\eta(3) = 1$. If $b_2 \neq b_{\eta(2)}$ then (24) gives a contradiction (with $\sigma = \eta$, $i = 1$, $j = 2$ if $b_2 > b_{\eta(2)}$ and $i = 1$, $j = \eta(2)$ otherwise). So $b_{\eta(2)} = b_2$. Let $\kappa \in \Sigma_n^{\mathrm{SN}}$ be such that $\kappa(3) = \eta(2)$. Then

$$b_{\kappa\eta(1)} \;=\; b_{\kappa(3)} \;=\; b_{\eta(2)} \;=\; b_2 \;<\; b_1$$

and

$$b_{\kappa\eta(2)} \;=\; b_{\kappa\kappa(3)} \;=\; b_3 \;<\; b_2$$

so again we have a contradiction to (24) with $i = 1$, $j = 2$, $\sigma = \kappa\eta$.

The second possibility is that the b_i take just two values but each of them is taken at least twice, say $b_1 = b_3 > b_2 = b_4$. Let $\sigma \in \Sigma_n^{\mathrm{SN}}$ be such that $\sigma(1) = 2$. Then the cycles of σ are all of size 2 and we would contradict (24) if for some $j \neq 1, 2$, $b_j \neq b_{\sigma(j)}$. So it must be that $b_j = b_{\sigma(j)}$ for all $j \neq 1, 2$. Similarly if $\tau \in \Sigma_n^{\mathrm{SN}}$ is such that $\tau(3) = 4$ it must be that $b_j = b_{\tau(j)}$ for all $j \neq 3, 4$. But then

$$b_{\tau\sigma(1)} \;=\; b_{\tau(2)} \;=\; b_2 \;<\; b_1,$$

and

$$b_{\tau\sigma\sigma(3)} = b_{\tau(3)} = b_4 < b_3 = b_{\sigma(3)}$$

so again (24) gives a contradiction. ■

At this point we know that if the belief function w on SL is to satisfy GPIR then the non-null points of the de Finetti measures μ_n for the $w_n = w \restriction SL^n$ must all be semi-radial[4]. We now show that in fact the non-null points must all be radial.

Theorem 22 *Let w be a belief function SL satisfying GPIR and let μ_n be the de Finetti measure for $w_n = w \restriction SL^n$. Then all the non-null points of μ_n are radial.*

Proof. First note that for A a Borel subset of $\mathbb{D}_n$

$$\mu_n(A) = \mu_{n+1}\{\langle x_1, x_2, ..., x_{2^{n+1}-1}, x_{2^{n+1}}\rangle \,:\, \langle x_1 + x_2, ..., x_{2^{n+1}-1} + x_{2^{n+1}}\rangle \in A\}. \tag{25}$$

To see this notice that if we define the measure μ' by

$$\mu'(A) = \mu_{n+1}\{\langle x_1, x_2, ..., x_{2^{n+1}-1}, x_{2^{n+1}}\rangle \,:\, \langle x_1 + x_2, ..., x_{2^{n+1}-1} + x_{2^{n+1}}\rangle \in A\}$$

then straightforward calculation shows that the belief function w' on SL^n determined by μ' agrees with w_n. Hence by the uniqueness of the de Finetti measure, $\mu' = \mu_n$.

Now suppose that $\langle b, c, c, ..., c\rangle$ was a non-null point with respect to μ_n with $b < c$. Pick $0 < \epsilon < 1/2(c - b)$ and $m > n$ so large that

$$(b + \epsilon) - (1 - 2^{-m})(c - \epsilon) < 0. \tag{26}$$

Let

$$A = \{\langle x_1, x_2, x_3, ..., x_{2^n}\rangle \in \mathbb{D}_n \,:\, |\langle x_1, x_2, x_3, ..., x_{2^n}\rangle - \langle b, c, c, ..., c\rangle| \leq \epsilon\}.$$

Then by repeated use of (25) $\mu_n(A) = \mu_{n+m}(A_{n+m})$ where

$$A_{n+m} = \{\langle x_{12}, x_{12}, ..., x_{12^m}, x_{22}, ..., x_{2^m 2^m}\rangle \,:\, \langle \sum_{j=1}^{2^m} x_{1j}, \sum_{j=1}^{2^m} x_{2j}, ..., \sum_{j=1}^{2^m} x_{2^m j}\rangle \in A\}.$$

But if $\langle x_{12}, x_{12}, ..., x_{12^m}, x_{22}, ..., x_{2^m 2^m}\rangle \in A_{n+m}$ then it is not possible than this point is semi-radial, since suppose it was, say all the x_{ij} are equal except possibly

[4]It might be noted that such a w satisfies the Principle of Atom Exchangeability given in [14]. Atom Exchangeability is essentially a stronger alternative to the combination of SN and Px.

x_{st}. Then all the $\sum_{j=1}^{2^m} x_{ij}$ for $i \neq s$ are equal and within ϵ of c whilst $\sum_{j=1}^{2^m} x_{sj}$ is within ϵ of b. But then $|x_{ij} - c/2^m| \leq \epsilon/2^m$ for $i \neq s$ so

$$\sum_{j=1}^{2^m} x_{sj} \;\geq\; (2^m - 1)(c - \epsilon)/2^m \;>\; b + \epsilon$$

and this point is not in A_{n+m} after all!

It follows that all points in A_{n+m} must be null (with respect to μ_{n+m}) and hence than $0 = \mu_{n+m}(A_{n+m}) = \mu_n(A)$, contradicting the non-nullness of the point $\langle b, c, c, ..., c\rangle$, as required. ■

We are now ready to prove the converse of Theorem 16, namely:

Theorem 23 *Let w be a belief function satisfying GPIR (and R, Ex, Px, SN). Then $w = w^\delta$ for some $\delta \in [0, 1)$.*

Proof. Let $n > 1$ and $w_n = w \restriction SL^n$. By Lemmas 21 and 22 we know that the de Finetti measure μ_n, for w_n is radial. It will be enough to show that there is just one δ_n such that the non-null points of μ_n are δ_n-radial since then by the language invariance of the w_n^δ these δ_n must be the same for all n, giving $w = w^\delta$ for this common value.

So suppose on the contrary that there is more than one such $\delta_n (\in [0, 1])$. Define a measure η (extending uniquely to the Borel subsets of $[2^{-n}, 1]$) by setting

$$\begin{aligned} \eta[x, y] &= \mu_n\{\langle \gamma + \delta, \gamma, \gamma, ..., \gamma\rangle \,:\, x \leq \gamma + \delta \leq y\}, \\ \eta\{1/2^n\} &= 2^{-n}\mu_n\{\langle 1/2^n, 1/2^n, ..., 1/2^n\rangle\}, \end{aligned}$$

for $1/2^n < x \leq y \leq 1$.

We consider two cases. Firstly suppose that there are $1/2^n < a < b < c < d < 1$ such that $\eta[a, b], \eta[c, d] > 0$. We shall derive a contradiction by showing that

$$w(\alpha_1(a_{m+2}) \,|\, \alpha_2(a_{m+1}) \wedge \bigwedge_{j=1}^m \alpha_3(a_j)) \;>\; w(\alpha_1(a_{m+1}) \,|\, \bigwedge_{j=1}^m \alpha_3(a_j)),$$

equivalently that

$$Y_1 Y_3 > Y_2^2, \tag{27}$$

where

$$Y_1 \;=\; \int_{\mathbb{D}_n} x_3^m d\mu_n = \int_{[2^{-n},1]} x^m + (2^n - 1) y^m d\eta,$$

$$\begin{aligned}
Y_2 &= \int_{\mathbb{D}_n} x_1 x_3^m d\mu_n = \int_{[2^{-n},1]} yx^m + xy^m + (2^n-2)y^{m+1} d\eta, \\
Y_3 &= \int_{\mathbb{D}_n} x_1 x_2 x_3^m d\mu_n = \int_{[2^{-n},1]} y^2x^m + 2xy^{m+1} + (2^n-3)y^{m+2} d\eta, \\
y &= (1-x)/(2^n-1).
\end{aligned}$$

Expanding (27) and using the fact that by Schwartz Inequality

$$\begin{aligned}
&\left(\int_{[2^{-n},1]} x^m + (2^n-1)y^m d\eta\right) \cdot \left(\int_{[2^{-n},1]} y^2(x^m + (2^n-1)y^m) d\eta\right) \\
&\qquad \geq \left(\int_{[2^{-n},1]} y(x^m + (2^n-1)y^m) d\eta\right)^2
\end{aligned}$$

we see that to show (27) it is enough to show that

$$\begin{aligned}
&\frac{\int_{[2^{-n},1]} y^{m+1}(x-y) d\eta}{\int_{[2^{-n},1]} y^m(x-y) d\eta} \\
&\quad > \frac{\int_{[2^{-n},1]} y(x^m + (2^n-1)y^m) d\eta}{\int_{[2^{-n},1]} x^m + (2^n-1)y^m d\eta} + \frac{\int_{[2^{-n},1]} y^m(x-y) d\eta}{2\int_{[2^{-n},1]} x^m + (2^n-1)y^m d\eta}. \qquad (28)
\end{aligned}$$

Concerning the rightmost term of (28) we have

$$\begin{aligned}
0 \leq \int_{[2^{-n},1]} y^m(x-y) d\eta &\leq \left(\int_{[2^{-n},a]} + \int_{[a,1]}\right) y^m(x-y) d\eta \\
&\leq (1/(2^n-1))^m \eta[0,a] + ((1-a)/(2^n-1))^m \eta[a,1],
\end{aligned}$$

whilst

$$\int_{[2^{-n},1]} x^m + (2^n-1)y^m d\eta \geq \int_{[c,d]} x^m d\eta \geq c^m \eta[c,d] > 0.$$

Hence since $(1-a)/(2^n-1) < a < c$ the rightmost term of (28) tends to 0 as $m \to \infty$.

A similar argument shows that each of

$$\frac{\int_{[2^{-n},1]} y(x^m + (2^n-1)y^m) d\eta}{\int_{[c,1]} yx^m d\eta}, \quad \frac{\int_{[2^{-n},1]} x^m + (2^n-1)y^m d\eta}{\int_{[c,1]} x^m d\eta},$$

$$\frac{\int_{[2^{-n},1]} y^{m+1}(x-y) d\eta}{\int_{[2^{-n},b]} y^{m+1}(x-y) d\eta}, \quad \frac{\int_{[2^{-n},1]} y^m(x-y) d\eta}{\int_{[2^{-n},b]} y^m(x-y) d\eta},$$

tends to 1 as $m \to \infty$ so it is enough to show that

$$\frac{\int_{[2^{-n},b]} y^{m+1}(x-y) d\eta}{\int_{[2^{-n},b]} y^m(x-y) d\eta} - \frac{\int_{[c,1]} yx^m d\eta}{\int_{[c,1]} x^m d\eta} > e$$

some some fixed $e > 0$ and all m eventually. But this is clear since by the mean value theorem

$$\frac{1-b}{2^n-1} \leq \frac{\int_{[2^{-n},b]} y^{m+1}(x-y)d\eta}{\int_{[2^{-n},b]} y^m(x-y)d\eta}$$

whilst

$$\frac{\int_{[c,1]} yx^m d\eta}{\int_{[c,1]} x^m d\eta} \leq \frac{1-c}{2^n-1}.$$

This gives the required contradiction in the case where there are such $2^{-n} < a < b < c < d < 1$. On the other hand if these do not exist then μ_n must be discrete giving measure just to δ-radial points for one or both of $\delta = 0$ or $\delta = 1$ together with $\delta = a$ for some $0 < a < 1$. That is, μ_n looks like a weighted combination of ν_n^0, ν_n^a, and ν_n^1 with at least two of the weights non-zero (by our assumption that μ has more than one non null-point). Let $\delta_1 = 0$, $\delta_2 = a$ and $\delta_3 = 1$ and let $2^n\gamma_t = 1 - \delta_t$ for $t = 1, 2, 3$, let the respective weights be c_1, c_2, c_3 where $c_1 + c_2 + c_3 = 1$. For any Borel subset, A, of $\mathbb{D}_n$

$$\mu_n(A) = \sum_{t=1,2,3} c_t \nu_n^{\delta_t}(A).$$

First suppose that $c_3 > 0$. Then as $k \to \infty$,

$$\begin{aligned} & w(\alpha_3(a_{k+1}) \mid \bigwedge_{i=1}^{k} \alpha_1(a_i)) \\ & = \frac{(c_1\gamma_1(\gamma_1+\delta_1)^k + c_2\gamma_2(\gamma_2+\delta_2)^k)(1+o(1))}{c_3(1+o(1))}, \end{aligned} \tag{29}$$

$$\begin{aligned} & w(\alpha_3(a_{k+2}) \mid \alpha_2(a_{k+1}) \wedge \bigwedge_{i=1}^{k} \alpha_1(a_i)) \\ & = \frac{(c_1\gamma_1^2(\gamma_1+\delta_1)^k + c_2\gamma^2(\gamma_2+\delta)^k)(1+o(1))}{(c_1\gamma_1(\gamma_1+\delta_1)^k + c_2\gamma_2(\gamma_2+\delta_2)^k)(1+o(1))}, \end{aligned} \tag{30}$$

so for large k

$$w(\alpha_3(a_{k+1}) \mid \bigwedge_{i=1}^{k} \alpha_1(a_i)) < w(\alpha_3(a_{k+2}) \mid \alpha_2(a_{k+1}) \wedge \bigwedge_{i=1}^{k} \alpha_1(a_i)),$$

contradicting GPIR. Hence $c_3 = 0$ and both c_1 and c_2 must be non-zero.

Again then, as $k \to \infty$,

$$w(\alpha_3(a_{k+2}) \mid \bigwedge_{i=1}^{k} \alpha_1(a_i)) = \frac{(c_1\gamma_1(\gamma_1+\delta_1)^k + c_2\gamma^2(\gamma_2+\delta)^k)(1+o(1))}{(c_1(\gamma_1+\delta_1)^k + c_2(\gamma_2+\delta_2)^k)(1+o(1))},$$

so combining this with (29), (30) to give a refutation of GPIR that for large k

$$w(\alpha_3(a_{k+2}) \,|\, \bigwedge_{i=1}^{k} \alpha_1(a_i)) \;\; < \;\; w(\alpha_3(a_{k+2}) \,|\, \alpha_2(a_{k+1}) \wedge \bigwedge_{i=1}^{k} \alpha_1(a_i))$$

it is enough to show that for large k

$$\begin{aligned} &(c_1\gamma_1(\gamma_1+\delta_1)^k + c_2\gamma^2(\gamma_2+\delta)^k)^2 \\ &\quad < \;\; (c_1\gamma_1^2(\gamma_1+\delta_1)^k + c_2\gamma_2^2(\gamma_2+\delta_2)^k)(c_1(\gamma_1+\delta_1)^k + c_2(\gamma_2+\delta_2)^k), \end{aligned}$$

equivalently,

$$2\gamma_1\gamma_2(\gamma_1+\delta_1)^k(\gamma_2+\delta_2)^k \;\; < \;\; \gamma_1^2(\gamma_1+\delta_1)^k(\gamma_2+\delta_2)^k + \gamma_2^2(\gamma_1+\delta_1)^k(\gamma_2+\delta_2)^k,$$

which clearly holds since $\gamma_1 \neq \gamma_2$.

This concludes the proof of the theorem.

■

Combining Theorems 16 and 23 we can finally state the main result of this paper.

Theorem 24 *A belief function w on SL satisfies R, Ex, Px, SN and GPIR if and only if $w = w^\delta$ for some $\delta \in [0,1)$.*

Conclusion

In this paper we have introduced a 'continuum of inductive methods', w^δ, and shown that they are characterized by the principles of Regularity, Constant and Predicate Exchangeability, Strong Negation and Generalized Instantial Relevance.

The situation here then is analogous to that pertaining to Carnap's celebrated continuum of inductive methods (see [2]), c_λ, defined, on SL^n, by

$$c_\lambda\,(\bigwedge_{k=1}^{m} \alpha_{h_k}(a_k)) = \frac{\prod_{r=1}^{2^n}\prod_{j=0}^{k_r-1}(j+2^{-n}\lambda)}{\prod_{j=0}^{k-1}(j+\lambda)}.$$

where $k_r = |\{i : h_i = r\}|$ and $k = \sum_{r=1}^{2^n} k_r$, $0 \leq \lambda \leq \infty$.

The resulting conditional belief functions are more resistent to the influence of new information for larger values of λ. The two extreme cases $\lambda = 0$ and $\lambda = \infty$ behave, in the former case, as though the world is entirely uniform and thus a single observation suffices to predict all others (being undefined when variety is observed), and in the latter case, as though observations have no bearing on belief and thus the posterior belief after conditioning is equal to the prior.

This intuitive explanation of the behaviour of c_λ may also be offered for the belief functions w^δ, where high λ corresponds to low δ and vice versa. When $\delta = 0$, w^δ agrees with c_∞ and is not affected by conditioning and when $\delta = 1$ $w^1 = c_0$ and we have again the function expecting homogeneous observations. Between these two extremes however the w^δ and c_λ are different (as we shall shortly see).

Interestingly, Carnap had reservations about what he called *complex evidence.* The following is an excerpt from [1][5].

> Let us assume again that we have an adequate definition for the concept of confirming case. Then it may happen that the evidence e available to [agent] X does not quite suffice to make the individual b a confirming case. For example, let h be the law '$(x)(Mx \supset M'x)$' ('all swans are white') and let e contain '$Mb.(M'b \vee Pb)$' ('b is a swan and is either white or small') and nothing else about b. Here, X does not know whether the swan b is white or not; but, still the information that b is either white or small is more than nothing. Should it not count for something in weighting the evidence for the law h? But how much? Perhaps as half a confirming case? Or should it be left aside as an irrelevant case? Suppose, furthermore, that another part of the evidence e says that, of 100 observed small things, 90 were white. Then the assumption the b is white becomes much more probable. Therefore it seems no longer justified to disregard b entirely in determining $c(h, e)$. Although it cannot be counted as a whole confirming case, it must be counted in some way.

However, Carnap's continuum seems to offer no intuitive answers to the question of complex evidence.

The c_λ are again characterized by 'principles', namely our standing assumptions of Regularity and Constant Exchangeability and in addition,

Johnson's Sufficientness Principle (JSP), see [10] For all $1 \leq h_1, h_2, \ldots, h_m \leq 2^n$,

$$w(\alpha_j(a_{m+1}) \mid \bigwedge_{i=1}^{m} \alpha_{h_i}(a_i))$$

depends only on m and $|\{\, l \;:\; h_l = j \,\}|$.

JSP is in fact strong enough to imply our remaining standing assumptions of Strong Negation and Predicate Exchangeability. So the choice between continua

[5]What we have called 'belief', Carnap calls 'confirmation'. Carnaps confirmation functions are here denoted by 'c'.

becomes a contest between JSP and GPIR. The observation that Carnap's λ-continuum satisfies PIR[6], and indeed Carnap's insistence upon PIR as a principle of inductive reasoning ([3], [4]), might suggest that adherents of the λ-continuum could feel sympathy towards GPIR[7].

To emphasize the incompatibility of these two principles note, as shown in [14], that, away from the extreme cases, JSP implies Reichenbach's Axiom, and that Reichenbach's Axiom contradicts GPIR.

Reichenbach's Axiom (RA)

Let $\alpha_{k_i}(x)$ for $i = 1, 2, 3, ...$ be an infinite sequence of atoms of SL^n. Then for $\alpha_s(x)$ an atom of SL^n,

$$\lim_{n\to\infty}\left(w(\alpha_s(a_{n+1})|\bigwedge_{i=1}^{n}\alpha_{k_i}(a_i)) - \frac{u(n)}{n}\right) = 0$$

where $u(n) = |\{i : 1 \le i \le n \text{ and } k_i = s\}|$,

This contradiction might well have been expected in advance since RA deals with complete (i.e. atomic) knowledge and ensures that belief converges on the observed proportions of objects in the real world. On the other hand, GPIR deals with uncertain knowledge (or complex evidence as Carnap puts it). So, for example, if a swan, b, is known to be white or small though not which, then neither the complete state of b nor the proportion of white (or small) objects amongst those observed is known.

Admittedly, the appeal of RA in itself seems no less for the above consideration. Functions assigning beliefs varying wildly from observed frequencies appear particularly undesirable. Given a straight choice RA might well seem preferable to GPIR.

That GPIR contradicts JSP may be shown more directly by considering a criticism oft levelled against JSP. Suppose our evidence consists of $\bigwedge_{i=1}^{m}\alpha_{h_i}(a_i)$ in which $\alpha_j(x)$ is not instantiated. It might be argued that $\alpha_j(a_{m+1})$ should be prescribed a larger magnitude of belief when the α_{h_i} vary greatly than when they are all the same, for the latter case suggests more uniformity in a world biased against the satisfaction of α_j. Indeed this is just how the functions w^δ behave for $0 < \delta < 1$, whereas JSP refutes this outright.

Another nice property that can be shown to hold for the w^δ is hinted at by inequality (10). Here we see that for any w^δ and non-disjoint formulae $\theta(x), \phi(x)$ such that $\nvDash \theta \vee \phi$, the sentences $\theta(a_{n+2}), \phi(a_{n+1})$ may be brought to support one another by conditioning on individuals satisfying both, or neither, formulae. That

[6]Solely by its satisfaction of Ex, a pleasing fact, apparently not appreciated at the time Carnap published 'The continuum of inductive methods'

[7]As already noted in [14] GPIR does hold for the c_λ if we require the background knowledge to be what is there called a *state description*, that is, knowledge of the form $\bigwedge_{i=1}^{m}\alpha_{h_i}(a_i)$.

is to say, if the evidence tells us that two formulae $\theta(x), \phi(x)$ look to be *empirically* equivalent, or look to be almost so, then the two sentences $\theta(a_{n+2}), \phi(a_{n+1})$ support one another. On the other hand, if $\vDash \neg(\theta \wedge \phi)$ or $\vDash \theta \vee \phi$, then no amount of conditioning can change the two sentences $\theta(a_{n+2}), \phi(a_{n+1})$ prior relation of support, which is necessarily not positive.

Conversly, if neither $\theta \vDash \phi$ nor $\phi \vDash \theta$, then it can be arranged for $\theta(a_{n+2})$ and $\phi(a_{n+1})$ to negatively support one another by conditioning on individuals satisfying one formulae but not the other. On the other hand, if $\theta \vDash \phi$ or $\phi \vDash \theta$ then, as before, no amount of conditioning can change the two sentences $\theta(a_{n+2})$ and $\phi(a_{n+1})$ prior relation of support, which is necessarily not negative.

The property described in the previous two paragraphs can also be shown to hold for c_λ in Carnap's continuum (see [12]), the difference in behaviour being only in the sentences $\theta(a_{n+2})$ and $\phi(a_{n+1})$ prior relation of support.

As a penultimate remark, we should note one weighty objection to which both formalisms, Carnap's and ours, fall foul. Belief functions of either variety fail to take account of the possible 'closeness' between the α_j and the $\alpha_{h_1}, ..., \alpha_{h_m}$. For example, giving the same belief to $\alpha_j(a_{m+1})$ if all the $\alpha_{h_i}(x)$ were different and differed from $\alpha_j(x)$ in just one predicate P_k as when the $\alpha_{h_i}(x)$ were all the same and disagreed with $\alpha_j(x)$ at an apparently 'random' set of predicates P_k. [For a more detailed discussion on this point see [14], and [11] for a contrary view.].

The final choice between w^δ and c_λ, seems to be only a matter of taste (or distaste). Nevertheless it seems to us interesting that simply demanding that the notion of supporting evidence 'extend' that of entailment, as embodied in GPIR, restricts the possible belief functions to such a narrow class as these w^δ.

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References

[1] Carnap, R., *Logical Foundations of Probability*, University of Chicago Press, Chicago, and Routledge & Kegan Paul Ltd., 1950.

[2] Carnap, R., *The Continuum of Inductive Methods*, University of Chicago Press, 1952.

[3] Carnap, R., On the Comparitive Concept of Confirmation, *British Journal for the Philosophy of Science*, 1953, **3**:311-318.

[4] Carnap, R., *Induktive Logik und Wahrscheinlichkeit. (Bearbeitet von Wolfgang Stegmuller)*, Springer, Wein, 1950. (Anhang B: Ein neues Axiomensystem).

[5] Feller, W., *An Introductions to Probability Theory and its Applications, volume II*, John Wiley & Sons, New York, 1966.

[6] Gaifman, H., Applications of de Finetti's Theorem to inductive logic, in *Studies in Inductive Logic and Probability (Volume I),* Eds. R.Carnap & R.C.Jeffrey, University of California Press, Berkeley and Los Angeles, 1971.

[7] Gaifman, H., Concerning Measures on First Order Calculi, *Isreal Journal of Mathematics*, 1964, **2**:1-18.

[8] Hill, M. J., *Aspects of Induction and the Principle of Duality*, Ph.D.Thesis, Manchester University, 1999.

[9] Humburg, J., The principle of instantial relevance, in *Studies in Inductive Logic and Probability (Volume I),* Eds. R.Carnap & R.C.Jeffrey, University of California Press, Berkeley and Los Angeles, 1971.

[10] Johnson, W. E., Probability: The Deductive and Inductive Problems, *Mind*, 1932, **41**(164):409-423.

[11] Miller, D., Popper's qualitative theory of versimilitude, *British Journal for the Philosophy of Science*, 1974, **25**:565-575.

[12] Nix, C. J., *Probabilistic Induction in the Predicate Calculus*, Ph.D. Thesis, Manchester University, 2005.

[13] Paris, J.B., *The Uncertain Reasoner's Companion,* Cambridge university Press, 1994.

[14] Hill, M. J., Paris, J. B. and Wilmers, G. M., Some Observations on Induction in Predicate Probabilistic Reasoning, *Journal of Philosophical Logic*, 2002, **31**(1):43-75.

[15] Quine, W. V., *Ontological Relativity and Other Essays*, Columbia University Press, 1977.

[16] Schervish, M., *Theory of Statistics*, Springer-Verlag, 1995.