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THE MAXIMAL 2-LOCAL GEOMETRY FOR J_4, III

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INTRODUCTION

This is the final part of the saga, begun in [RW1] and [RW2], to understand certain aspects of Γ, the maximal 2-local geometry for the sporadic simple group J_4. We continue the section numbering of the earlier papers. The main object of our attentions is $\mathcal{G}$, the point-line collinearity graph of Γ. Here we shall be concentrating almost exclusively upon $\Delta_3(a)$, the third disc of a – a is a fixed point of $\mathcal{G}$. For further details, background and notation, not to mention statements of the main theorems we are endeavouring to prove, see Sections 1 and 2.

We start Section 9 with a preliminary result on non-sparse triangles followed by two results which will be used for orbit matching. Orbit matching comes to the fore in our later arguments when we have assembled certain information about $\mathcal{G}$. The remainder of Section 9 investigates lines incident with a point in $\Delta_2^3(a)$. In Section 10 we plunge into $\Delta_3(a)$ and look at the set $\Delta_3^1(a)$, defined earlier in Definition 6.1. This lengthy campaign begins with an analysis of $\{c,d\}^{\perp}$ and G_{ad} for $d \in \Delta_3^1(a)$ and $c \in \Delta_2^1(a) \cap \Delta_1(a)$, the main results being stated in Theorem 10.3. After examining the G_{ad}-orbits upon $\Gamma_1(d)$ for $d \in \Delta_3^1(a)$ we work through these line orbits determining their point distributions. Section 11, the final section, deals with the G_a-orbit $\Delta_3^2(a)$. Here we prove that

$$\Gamma_0 = \{a\} \cup \Delta_2^1(a) \cup \Delta_2^2(a) \cup \Delta_2^3(a) \cup \Delta_3^1(a) \cup \Delta_3^2(a),$$

which completes our study of $\mathcal{G}$ as well as enabling us to deduce Theorems A and B.

In view of the length of this work and the many results proved along the way, we include an appendix detailing where point distributions for particular lines are determined.

9. SOME MISCELLANEOUS RESULTS

Lemma 9.1 Let X,Y $\in \Gamma_2(x)$ for some x $\in \Gamma_0$. Suppose $l_1,l_2,l_3,l_4,l_5 \in \Gamma_1(x,X)$ form a non-sparse triangle.

(i) If Y $\in \gamma_0(X)$, then $l_i \in \beta_3(x,Y)$ for one $i \in \{1,...,5\}$ and $l_i \in \beta_1(x,Y)$ for four $i \in \{1,...,5\}$.

(ii) If Y $\in \gamma_1(X)$, then $l_i \in \beta_2(x,Y)$ for one $i \in \{1,...,5\}$ and $l_i \in \beta_1(x,Y)$ for four $i \in \{1,...,5\}$.

(iii) If Y $\in \gamma_3(X)$, then either $l_i \in \beta_0(x,Y)$ for one i $\in \{1,...,5\}$ and $l_i \in \beta_3(x,Y)$ for four $i \in \{1,...,5\}$,or $l_i \in \beta_2(x,Y)$ for three i $\in \{1,...,5\}$ and $l_i \in \beta_3(x,Y)$ for two $i \in \{1,...,5\}$.

Proof This follows from the intersection matrices given in (2.6) and the fact that every octad formed from a union of tetrads in X lies in exactly one of the trios $l_1,...,l_5$.

Lemma 9.2 Let Λ and Θ be distinct G_a-orbits of $\mathcal{G}$ and let $x \in \Lambda$, $y \in \Theta$ be such that $d(x,y) = 1$. If $\Delta_1(x) \cap \Theta$ is a G_{ax}-orbit, then $\Delta_1(y) \cap \Lambda$ is a G_{ay}-orbit and

$$|\Delta_1(y) \cap \Lambda| = [G_{ay} : G_{ayx}] = \frac{|\Lambda|}{|\Theta|}\, |\Delta_1(x) \cap \Theta| \ .$$

Proof So we have $|\Lambda| = [G_a : G_{ax}]$ and $|\Theta| = [G_a : G_{ay}]$. Also, since $\Delta_1(x) \cap \Theta$ is a G_{ax}-orbit, $|\Delta_1(x) \cap \Theta| = [G_{ax} : G_{axy}]$. By counting edges of $\mathcal{G}$ between Λ and Θ we deduce that

$$\begin{aligned}[G_a : G_{ay}]|\Delta_1(y) \cap \Lambda| &= |\Theta||\Delta_1(y) \cap \Lambda| = |\Lambda||\Delta_1(x) \cap \Theta| \\ &= [G_a : G_{ax}][G_{ax} : G_{axy}] = [G_a : G_{axy}] \\ &= [G_a : G_{ay}][G_{ay} : G_{ayx}] \ .\end{aligned}$$

Hence $|\Delta_1(y) \cap \Lambda| = [G_{ay} : G_{ayx}]$, which yields the lemma.

Lemma 9.3 Suppose that Λ and Θ are distinct G_a-orbits of $\mathcal{G}$ and let $x \in \Lambda$, $y \in \Theta$ be such that $d(x,y) = 1$. Assume that $|\Gamma_0(x+y) \cap \Lambda| = 1$ or 3 and $|\Gamma_0(x+y) \cap \Theta| = 2$ and that there exists $t \in G_a$ such that t interchanges the two points in $\Gamma_0(x+y) \cap \Theta$. Let O_x and O_y be, respectively, the G_{ax} (respectively G_{ay})-orbit of $\Gamma_1(x)$ (respectively $\Gamma_1(y)$) containing $x+y$ (respectively $y+x$). Set $L = \bigcup\{O_x^g \mid g \in G_a\}$ and $M = \bigcup\{O_y^g \mid g \in G_a\}$. Then $\Gamma_1(y) \cap L = O_y$, $\Gamma_1(x) \cap M = O_x$ and therefore $L = M$. Moreover,

$$2|O_x||\Lambda| = |O_y||\Theta||\Gamma_0(x+y) \cap \Lambda|.$$

Proof First, we verify that $\Gamma_1(y) \cap L = O_y$. Clearly we have $O_y \subseteq \Gamma_1(y) \cap L$. Let $k \in \Gamma_1(y) \cap L$. Then $k = (y+x)^g$ for some $g \in G_a$ (since L is a G_a-orbit of Γ_1), with y, $y^g \in \Gamma_0(k) \cap \Theta$. If $y = y^g$, then $g \in G_{ay}$. whence $k = (y+x)^g \in O_y$. If $y \neq y^g$, then since k is in the same G_a-orbit as $x+y$, there exists $s \in G_a$ which interchanges y and y^g. So $y = y^{gs}$ and $k^s = k$. Consequently $gs \in G_{ay}$ and hence we also have $k = k^s = (y+x)^{gs} \in O_y$. Therefore $\Gamma_1(y) \cap L = O_y$. Now arguing as in Lemma 9.2 we may complete the proof of the lemma.

Lemma 9.4 Let $d \in \Delta_2^3(a)$ and $l \in \Gamma_1(d)$.

(i) If $l \in (\beta_0, *)$, then $|\Gamma_0(l) \cap \Delta_1(a)| = 1$ and $\left|\Gamma_0(l) \cap \Delta_2^3(a)\right| = 4$.

(ii) If $l \in (\beta_0, **)$, then $\left|\Gamma_0(l) \cap \Delta_2^2(a)\right| = 3$ and $\left|\Gamma_0(l) \cap \Delta_2^3(a)\right| = 2$ and if $c \in \Gamma_0(l) \cap \Delta_2^2(a)$, then $l = c + d \in (\beta_2\beta_2\beta_3, \alpha_1)$.

(iii) If $l \in (\beta_3, 2^2 1^4; 2^2 1^4; 1^8)$, then $\left|\Gamma_0(l) \cap \Delta_2^2(a)\right| = 1$ and $\left|\Gamma_0(l) \cap \Delta_2^3(a)\right| = 4$.

Proof (i) If $l \in (\beta_0, *)$, then l is one of the five lines incident with a point in $\{a,d\}^\perp$. For any b $\in \Gamma_0(l) \cap \{a,d\}^\perp$ we have b+a $\in \alpha_0(b, b+d)$ because $\Delta_2^3(a) \neq \Delta_2^i(a)$ for i = 1,2. Hence $b + a \in \alpha_0$(b,b+x) for all $x \in \Gamma_0(l) \backslash \{b\}$ and $x \in \Delta_2^3(a)$ by definition.

(ii) Let $l \in (\beta_0, **)$. So $l \in \Gamma_1(X_d)$ where X_d is the unique plane in $\Gamma_2(d)$ fixed by G_{ad}^{*d}.

Then there exists k $\in (\beta_0, *)$ with k $\in \alpha_3(d,l)$. Let b $\in \Gamma_0(k) \cap \{a,d\}^\perp$ and m $\in \Gamma_1(b)$ be such that $\Gamma_0(m) \subseteq \{a,d\}^\perp$. Without loss of generality we may take b+a to be the standard sextet, m = $\begin{array}{|cc|cc|cc|} \hline + & + & + & + & o & o \\ + & + & + & + & o & o \\ - & - & o & o & - & - \\ - & - & o & o & - & - \\ \hline \end{array}$ and k = $\begin{array}{|cc|cc|cc|} \hline + & o & + & o & o & - \\ o & + & + & o & - & o \\ - & - & o & o & + & - \\ + & + & - & - & + & - \\ \hline \end{array}$. Since $l \in \alpha_3(d,k)$, the five lines in $\{b+x \,|\, x \in \Gamma_0(l)\}$ form a full triangle. None of these lines is m and so they must be k, $\begin{array}{|cc|cc|cc|}\hline + & + & + & + & o & - \\ + & + & + & + & - & o \\ o & o & o & o & - & o \\ - & - & - & - & - & o \\ \hline \end{array}$, $\begin{array}{|cc|cc|cc|}\hline + & o & + & o & + & o \\ o & + & + & o & o & + \\ - & - & + & + & - & - \\ - & - & o & o & - & - \\ \hline \end{array}$, $\begin{array}{|cc|cc|cc|}\hline o & + & o & + & - & - \\ + & o & o & + & - & - \\ o & o & - & - & + & o \\ + & + & - & - & + & o \\ \hline \end{array}$ and $\begin{array}{|cc|cc|cc|}\hline + & o & + & o & - & + \\ o & + & + & o & + & - \\ o & o & - & - & - & o \\ - & - & + & + & - & o \\ \hline \end{array}$. Three of these lines lie in $\alpha_1(b, b+a)$ and two lie in $\alpha_0(b, b+a)$. So $\left|\Gamma_0(l) \cap \Delta_2^2(a)\right| = 3$ and $\left|\Gamma_0(l) \cap \Delta_2^3(a)\right| = 2$ as required.

For part (iii) let b $\in \{$a,d$\}^\perp$ and k = b+d. We may suppose that b+a = $\begin{array}{|cc|cc|cc|}\hline + & o & + & o & o & - \\ o & + & + & o & - & o \\ - & - & o & o & + & - \\ + & + & - & - & + & - \\ \hline \end{array}$ and k is the standard trio. Let Y be the standard sextet. Then Y $\in \gamma_3(X_d, b) \cap \Gamma_3(k)$. Let $l_1 = \begin{array}{|cc|cc|cc|}\hline + & o & + & - & - & o \\ + & o & + & - & - & o \\ + & o & + & - & - & o \\ + & o & + & - & - & o \\ \hline \end{array}$, $l_2 = \begin{array}{|cc|cc|cc|}\hline + & o & - & + & o & - \\ + & o & - & + & o & - \\ + & o & - & + & o & - \\ + & o & - & + & o & - \\ \hline \end{array}$, $l_3 = \begin{array}{|cc|cc|cc|}\hline + & o & o & - & + & - \\ + & o & o & - & + & - \\ + & o & o & - & + & - \\ + & o & o & - & + & - \\ \hline \end{array}$ and $l_4 = \begin{array}{|cc|cc|cc|}\hline + & o & - & o & - & + \\ + & o & - & o & - & + \\ + & o & - & o & - & + \\ + & o & - & o & - & + \\ \hline \end{array}$. So k, l_1, l_2, l_3, l_4 form a full triangle in $\Gamma_1(b, Y)$. Let $l \in \Gamma_1(d) \backslash \{k\}$ with $\Gamma_0(l) \cap \Gamma_0(l_i) = \emptyset$ for all i = 1,...,4. Then $l \in \beta_3$(d,X$_d$) by Lemma 9.1. By considering the orbits of G$_{ad}$ on Γ_1(d) we see that $l \in (\beta_3; 2^2 1^4; 2^2 1^4; 1^8)$ because lines in $(\beta_3; 1^8; 1^8; 1^8)$ $\cup(\beta_3; 2^2 1^4; 2^2 1^4; 2^2 1^4)$ cannot lie in $\alpha_3(d, k)$. Since $l_3 \in \alpha_1(b, b+a)$ and $k, l_1, l_2, l_4 \in \alpha_0(b, b+a)$we have $\left|\Gamma_0(l) \cap \Delta_2^2(a)\right| = 1$ and $\left|\Gamma_0(l) \cap \Delta_2^3(a)\right| = 4$ as required.

Lemma 9.5 Let d $\in \Delta_2^3(a)$ and G_{ad}^{*d} be the centralizer of the involution τ. Then

(i) if $l \in (\beta_2; 2^4; 1^8; 1^8)$, then $l^\tau = l$; and
(ii) if $l \in (\beta_2; 2^4; 2^2 1^4; 2^2 1^4)$, then $l^\tau \neq l$.

Proof This is a consequence of Lemma 7.10.

Lemma 9.6 Let d $\in \Delta_2^3(a)$ and $l \in \Gamma_1(d)$.
(i) If $l \in (\beta_2; 2^4; 2^2 1^4; 2^2 1^4)$, then $\left|\Gamma_0(l) \cap \Delta_2^3(a)\right| = 3$ and $\left|\Gamma_0(l) \cap \Delta_3^1(a)\right| = 2$.
(ii) If $l \in (\beta_2; 2^4; 1^8; 1^8)$, then $\left|\Gamma_0(l) \cap \Delta_2^2(a)\right| = 1$, $\left|\Gamma_0(l) \cap \Delta_2^3(a)\right| = 2$ and $\left|\Gamma_0(l) \cap \Delta_3^1(a)\right| = 2$.

Proof Fix $b \in \{a,d\}^{\perp}$. In Γ_b we may suppose that $b+a$ is the standard sextet and $b+d = \begin{array}{|cc|cc|cc|}\hline + & o & + & + & + & + \\ o & + & - & - & o & o \\ - & + & o & - & o & - \\ - & + & - & o & o & - \\ \hline\end{array}$ because $b+d \in \alpha_0(b,b+a)$. Then $b+b' = \begin{array}{|cc|cc|cc|}\hline o & o & + & + & o & o \\ o & o & + & + & o & o \\ + & + & - & - & - & - \\ + & + & - & - & - & - \\ \hline\end{array}$, X = $\begin{array}{|cc|cc|cc|}\hline + & o & \Box & \Box & + & + \\ o & + & - & - & o & o \\ - & \Box & \cdot & * & \cdot & * \\ - & \Box & * & \cdot & \cdot & * \\ \hline\end{array}$ and Y = $\begin{array}{|cc|cc|cc|}\hline + & + & \Box & \Box & o & o \\ + & + & \Box & \Box & o & o \\ - & - & \cdot & \cdot & * & * \\ - & - & \cdot & \cdot & * & * \\ \hline\end{array}$, where b$'$ $\in \{a,d\}^{\perp}\backslash\{b\}$, X $\in \Gamma_2(d,b,b')$ and Y $\in \Gamma_2(a,b,b')$. By Theorem 7.2(iv), $\tau = \tau(Y)^{*d}$ is the involution fixed by G$^{*d}_{ad}$. Let $y_1 \in \Delta_1(b)$ with $b+y_1 = \begin{array}{|cc|cc|cc|}\hline + & o & + & + & + & + \\ - & + & - & o & o & - \\ o & + & - & - & o & o \\ - & + & o & - & o & - \\ \hline\end{array}$. So $b+y_1 \in \alpha_0(b,b+a)\cup\alpha_2(b,b+a)$. By Lemma 3.8 we may suppose $y_1 \in \Delta_1(d)$ with $\{b,d,y_1\}$ a sparse triangle. We have $y_1 \in \Delta_2^3(a)$ by definition. If $y_2 \in \Delta_1(b)\cap\Gamma_0(d+y_1)$ with $y_2 \neq d,y_1$, then $b+y_2 = \begin{array}{|cc|cc|cc|}\hline + & o & + & + & + & + \\ - & + & o & - & o & - \\ - & + & - & o & o & - \\ o & + & - & - & o & o \\ \hline\end{array}$ and so $y_2 \in \Delta_2^3(a)$ because $b+y_2 \in \alpha_0(b,b+a)$. The unique element in $\Gamma_2(b,d,y_1,y_2)$ is $X_1 = \begin{array}{|cc|cc|cc|}\hline \Box & o & \Box & \Box & + & \Box \\ \cdot & + & * & - & o & \cdot \\ - & + & \cdot & * & o & - \\ * & + & - & \cdot & o & * \\ \hline\end{array}$ as a sextet in Γ_b. Since $d+y_1 \in \alpha_2(d,d+b)\backslash\beta_0(d,X)$, Lemma 9.4 implies that $d+y_1 \in \beta_2(d,X)$. Furthermore $y_1^{\tau} \neq y_1$ because $b+y_1 \in \beta_1(b,Y)$. Hence $(d+y_1)^{\tau} \neq x+y_1$ and we must have $d+y_1 \in (\beta_2;2^4;2^21^4;2^21^4)$ by Lemma 9.5. If $z \in \Gamma_0(x+y_1)\backslash\{x,y_1,y_2\}$, then $z \in \Delta_2^1(b)$ by definition, and since $b+a \in \beta_1(b,X_1)$ we have a $\in \Delta_3^1(z)$. Now Lemma 6.5 implies that z $\in \Delta_3^1(a)$ and so part (i) is proved because $(\beta_2;2^4;2^21^4;2^21^4)$ is a G_{ad}-orbit.

For part (ii) let $x_1 \in \Delta_1(b)$ with $b+x_1 = \begin{array}{|cc|cc|cc|}\hline + & o & + & + & + & + \\ o & + & o & o & o & o \\ o & + & - & - & - & - \\ o & + & - & - & - & - \\ \hline\end{array}$. Then $b+x_1 \in \alpha_1(b,b+a)\cap\alpha_2(b,b+d)$ and we may suppose that $x_1 \in \Delta_1(d)$. We have $x_1 \in \Delta_2^2(a)$ and if $x_2 \in (\Gamma_0(d+x_1)\backslash\{d,x_1\})\backslash\Delta_1(b)$, then $b+x_2 = \begin{array}{|cc|cc|cc|}\hline + & - & + & + & + & + \\ - & + & o & o & - & - \\ o & + & o & - & o & - \\ o & + & - & o & o & - \\ \hline\end{array}$ with $x_2 \in \Delta_2^3(a)$ by definition. As in part (i), Lemma 9.4 implies that $d+x_1 \in \beta_2(d,X)$. However $x_1^{\tau} = x_1$ because $b+x_1 \in \beta_2(b,Y)$ and so $d+x_1 \in (\beta_2;2^4;1^8;1^8)$ by Lemma 9.5. Again, as in part (i), we can show that if $z \in \Gamma_0(d+y_1)\backslash\{d,x_1,x_2\}$, then $z \in \Delta_3^1(a)$ because $b+a \in \beta_1(b,X_2)$ for X_2 the unique element in $\Gamma_2(b,d,x_1,x_2)$. This completes the proof.

10. $\Delta_3^1(a)$ REVISITED

Lemma 10.1 Let $d \in \Delta_3^1(a)$ and $c \in \Delta_2^1(a) \cap \Delta_1(d)$. Set $S := \Delta_2^1(a) \cap \Delta_1(c) \cap \Delta_1(d)$. Then $|S| = 5$, $|\Gamma_1(c, X(c,a)) \cap \alpha_1(c, c+d)| = 3$ and there exists a unique set of three points b_1, b_2, b_3 in $\{a,c\}^{\perp} \cap \Delta_2^2(d)$ such that

(i) $a + b_i \neq a + b_j$ for $i, j \in \{1,2,3\}$, $i \neq j$;
(ii) each point in S is collinear with b_i for $i = 1,2,3$;
(iii) for each $i = 1,2,3$, $S = \Gamma_0(Z) \cap \{b_i, d\}^{\perp}$ for some $Z \in \mathcal{S}(d, b_i)$;and
(iv) for each $i = 1,2,3$, $b_i + a \in (\beta_2\beta_2\beta_2, \alpha_2)$.

Proof Let $X = X(c,a)$. Since $d \in \Delta_3(a)$, we have $c + d \in \beta_1(c, X)$ by Theorem C and so, without loss of generality, we may suppose that X is the standard sextet and $c + d = \begin{array}{|cc|cc|cc|}\hline + & o & o & o & o & o \\ o & + & + & + & + & + \\ o & + & - & - & - & - \\ o & + & - & - & - & - \\ \hline \end{array}$ in Γ_c. By inspection there exist exactly three trios in $\Gamma_1(X) \cap \alpha_1(c, c+d)$, namely

$l_1 = \begin{array}{|cc|cc|cc|}\hline + & + & o & o & - & - \\ + & + & o & o & - & - \\ + & + & o & o & - & - \\ + & + & o & o & - & - \\ \hline \end{array}$, $l_2 = \begin{array}{|cc|cc|cc|}\hline + & + & o & - & o & - \\ + & + & o & - & o & - \\ + & + & o & - & o & - \\ + & + & o & - & o & - \\ \hline \end{array}$ and $l_3 = \begin{array}{|cc|cc|cc|}\hline + & + & o & - & - & o \\ + & + & o & - & - & o \\ + & + & o & - & - & o \\ + & + & o & - & - & o \\ \hline \end{array}$.

We now view the situation in the residue geometry Γ_X. Without loss we may assume that $c = \begin{array}{|cc|cc|cc|}\hline & & & & & \\ \cline{1-2} \times & \times & & & & \\ \times & \times & & & & \\ \times & \times & & & & \\ \hline \end{array}$ and $l_1 = \begin{array}{|cc|cc|cc|}\hline & & & & & \\ \cline{1-2} \times & \times & & & & \\ & & & & & \\ & & & & & \\ \hline \end{array}$, $l_2 = \begin{array}{|cc|cc|cc|}\hline & & & & & \\ \cline{1-2} & & & & & \\ \times & \times & & & & \\ & & & & & \\ \hline \end{array}$ and

$l_3 = \begin{array}{|cc|cc|cc|}\hline & & & & & \\ \cline{1-2} & & & & & \\ & & & & & \\ \times & \times & & & & \\ \hline \end{array}$ using Lemma 3.7. Since $d(a,c) = 2$, as a hexad in Γ_X, a is disjoint from c. By (2.12) we may suppose that $a = \begin{array}{|cc|cc|cc|}\hline & & \times & \times & \times & \\ \cline{1-2} & & & & & \times \\ & & & & & \times \\ & & & & & \times \\ \hline \end{array}$. Let

$b_1 = \begin{array}{|cc|cc|cc|}\hline & & \times & \times & & \\ \cline{1-2} \times & \times & \times & \times & & \\ & & & & & \\ & & & & & \\ \hline \end{array}$, $b_2 = \begin{array}{|cc|cc|cc|}\hline & & \times & \times & & \\ \cline{1-2} & & & & & \\ \times & \times & \times & \times & & \\ & & & & & \\ \hline \end{array}$ and $b_3 = \begin{array}{|cc|cc|cc|}\hline & & \times & \times & & \\ \cline{1-2} & & & & & \\ & & & & & \\ \times & \times & \times & \times & & \\ \hline \end{array}$.

Then $b_i \in \{a,c\}^{\perp} \cap \Gamma_0(l_i)$ and $a + b_i = \begin{array}{|cc|cc|cc|}\hline & & \times & \times & & \\ \cline{1-2} & & & & & \\ & & & & & \\ & & & & & \\ \hline \end{array}$ for each $i \in \{1,2,3\}$, and there are no other points which satisfy these conditions. This gives (i).

By Theorem 5.8(ii) if $x \in \Delta_2^1(a) \cap \Delta_1(c)$, then $c+x \in \beta_2(c,X)$. The only lines in $\beta_2(c,X)$ which lie in $\alpha_2(c,c+d) \cup \alpha_3(c,c+d)$ are $l = \begin{array}{|cc|cc|cc|} \hline + & + & o & o & o & o \\ + & + & - & - & - & - \\ + & + & - & - & - & - \\ + & + & o & o & o & o \\ \hline \end{array}$ $k = \begin{array}{|cc|cc|cc|} \hline + & + & o & o & o & o \\ + & + & - & - & - & - \\ + & + & o & o & o & o \\ + & + & - & - & - & - \\ \hline \end{array}$ and $m = \begin{array}{|cc|cc|cc|} \hline + & + & o & o & o & o \\ + & + & o & o & o & o \\ + & + & - & - & - & - \\ + & + & - & - & - & - \\ \hline \end{array}$. By inspection, $l,k,m \in \alpha_2(c,l_i)$ for each $i = 1,2,3$. Let $\Delta_2^1(a) \cap \Gamma_0(l) = \{c,y_1,z_1\}$ and $\Delta_2^1(a) \cap \Gamma_0(k) = \{c,y_2,z_2\}$. Since $l \in \alpha_2(c,l)$, there exists $b \in \Gamma_0(l_1) \cap \{a,c\}^\perp$ with $b \in \Delta_1(y_1)$. In Γ_b, $b+y_1 \in \alpha_2(b,b+a) \cap \alpha_2(b,b+c)$ and since $y_1 \notin \Gamma_0(X)$, this forces the trios $b+a, b+y_1$ and $b+c$ to contain the same octad. If $x \in \Delta_1(b) \cap \Gamma_0(l)$ with $x \neq c,y_1$, then $b+x$ contains this octad and so $x \in \Delta_2^1(a)$. Hence $x = z$. Similarly there exist $b' \in \Gamma_0(l_2) \cap \{a,c\}^\perp$ and $b'' \in \Gamma_0(l_3) \cap \{a,c\}^\perp$ with $z_1,y_1 \in \Delta_1(b') \cap \Delta_1(b'')$. We know that $X(a,y_1), X(a,y_2) \in \gamma_3(a,X)$ by Lemma 5.1, whence $|\Gamma_1(X(a,y_1),X)| = |\Gamma_1(X(a,z_1),X)| = 1$. This forces b, b' and b'' to be incident with the same line in $\Gamma_1(a,X)$ and by uniqueness we have $\{b,b',b''\} = \{b_1,b_2,b_3\}$. Using a similar argument it can be shown that y_2,z_2 and the two points in $(\Delta_2^1(a) \cap \Gamma_0(m)) \backslash \{c\}$ are collinear with each of b_1,b_2 and b_3. Since $l,k \in \alpha_3(c,c+d)$, $y_1,y_2,z_1,z_2 \in \Delta_1(d)$. By Theorem 4.7 d is collinear with a unique point in $(\Delta_1(b) \cap \Gamma_0(m)) \backslash \{c\}$ and this point lies in $\Delta_2^1(a)$. Let $Z = \begin{array}{|cc|cc|cc|} \hline * & \times & o & o & o & o \\ \times & * & - & - & - & - \\ \times & * & \square & \square & \square & \square \\ \times & * & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$. Then $l,k,m \in \Gamma_1(Z)$. Since $l,k \in \alpha_2(c,c+b_i) \cap \alpha_3(c,c+d)$ and $l,k \in \alpha_2(c,c+b_i) \cap \alpha_2(c,c+d)$ for each $i = 1,2,3$, Theorem 4.7 implies that $Z \in \mathcal{S}(d,b_i)$. Thus we have shown (ii) and (iii).

For (iv), notice that $b_i + a \in \alpha_2(b_i, b_i+c)$ $(i = 1,2,3)$ and so if $Y \in \mathcal{S}(d,b_i) \cap \Gamma_2(c)$, then $b_i + a \in \beta_2(b_i,Y)$ by (2.11). Theorem D implies that $b_i + a \in (\beta_2\beta_2\beta_2,\alpha_2) \cup (\beta_1\beta_1\beta_2,\alpha_1)$ because $a \in \Delta_3^1(d)$. Appealing to Theorem 4.7 again, $y_1 \in \Gamma_0(Y')$ for some $Y' \in \mathcal{S}(d,b_i) \backslash \{Y\}$. Therefore $b_i + a \notin \beta_1(b_i,Y)$ by (2.11)(ii) because $b_i + y_1 \in \alpha_2(b_i, b_i+a)$ by part (ii). Hence $b_i + a \in (\beta_2\beta_2\beta_2,\alpha_2)$ and we have part (iv).

Lemma 10.2 Let $d \in \Delta_3^1(a)$. Then

(i) $\Delta_2^1(a) \cap \Delta_1(d)$ consists of six pairwise collinear points incident with the same sextet in $\Gamma_2(d)$; and

(ii) there exists a unique line $l \in \Gamma_1(a)$ such that $\{X(a,x) \,|\, x \in \Delta_2^1(a) \cap \Delta_1(d)\} \subseteq \Gamma_2(l)$ with $|\Gamma_0(l) \cap \Delta_2^2(d)| = 3$ and $|\Gamma_0(l) \cap \Delta_3^1(d)| = 2$.

Proof By Lemma 10.1, for part (i) it is enough to prove that if $x,y \in \Delta_2^1(a) \cap \Delta_1(d)$, then $y \in \Delta_1(x) \cup \{x\}$. We suppose there exist $x,y \in \Delta_2^1(a) \cap \Delta_1(d)$ with $y \in \Delta_2(x)$ and argue for a contradiction. Let $X = X(a,x)$ and $Y = Y(a,y)$. Notice that $x+d \in \beta_1(x,X)$ by Theorem 5.8.

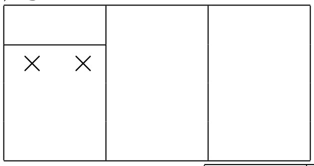
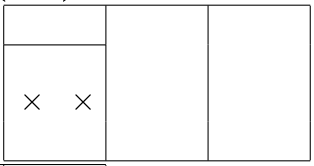
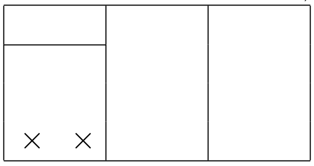
(10.2.1) (i) There exists one $l \in \Gamma_1(a,X)$ such that $|\Gamma_0(l) \cap \Delta_2^2(d)| = 3$ and $|\Gamma_0(l) \cap \Delta_3^1(d)| = 2$.

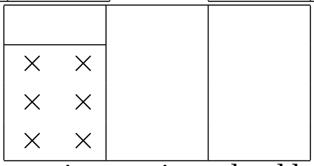
(ii) There exist six $k \in \Gamma_1(a,X)$ such that $|\Gamma_0(k) \cap \Delta_2^2(d)| = 1$, $|\Gamma_0(k) \cap \Delta_2^3(d)| = 2$ and

$\left|\Gamma_0(k)\cap\Delta_3^1(d)\right|=2$.

(iii) There exist eight $m\in\Gamma_1(a,X)$ such that $\left|\Gamma_0(m)\cap\Delta_2^3(d)\right|=3$ and $\left|\Gamma_0(m)\cap\Delta_3^1(d)\right|=2$.

Since $x+d\in\beta_1(x,X)$, of the 15 lines in $\Gamma_1(x,X)$, 3 lie in $\alpha_1(x,x+d)$ and 12 lie in $\alpha_0(x,x+d)$. Let $\alpha_1(x,x+d)\cap\Gamma_1(x,X)=\{k_1,k_2,k_3\}$. So $\Gamma_0(k_i)\backslash\{x\}\subseteq\Delta_2^2(d)$ for $i=1,2,3$. As trios in Γ_x, k_1,k_2,k_3 contain the same octad. By (2.13) and Lemma 3.7 we may assume that, as duads in Γ_X,

$k_1=$  , $k_2=$  and $k_3=$ 

where x is the hexad  . There are 16 hexads in $\Gamma_0(X)$ which are disjoint from x and each one contains a unique duad lying in a hexad with k_i for each $i=1,2,3$. Since $a\in\Delta_2^1(x)$, a is a hexad in Γ_X disjoint from x. So there exists a unique $l\in\Gamma_1(a,X)$ with $\Gamma_0(l)\cap\Gamma_0(k_i)\neq\emptyset$ for $i=1,2,3$. Hence $\left|\Gamma_0(l)\cap\Delta_2^2(d)\right|=3$ and since $a\in\Delta_3^1(d)$, Theorem D implies that $\left|\Gamma_0(l)\cap\Delta_3^1(d)\right|=2$. In Γ_X there are exactly 6 duads contained in a which lie in a hexad with precisely one of the duads k_i ($i=1,2,3$), because a and x are disjoint hexads. Moreover the remaining eight duads in a do not lie in any hexad in $\Gamma_0(k_i)$ for every $i=1,2,3$. Therefore there are 6 lines $k\in\Gamma_1(a,X)$ with $\left|\Gamma_0(k)\cap\Delta_2^2(d)\right|=1$ and 8 lines $m\in\Gamma_1(a,X)$ with $\Gamma_0(m)\cap\Delta_2^2(d)=\emptyset$. Let $k\in\Gamma_1(a,X)$ with $\left|\Gamma_0(k)\cap\Delta_2^2(d)\right|=1$. Since x is collinear with precisely three points in $\Gamma_0(k)$ and $\Delta_2^1(d)\cap\{a,x\}^\perp=\emptyset$ we conclude that $\left|\Gamma_0(k)\cap\Delta_2^3(d)\right|\geq 2$. However $a\in\Delta_3^1(d)$, whence $\left|\Gamma_0(k)\cap\Delta_2^3(d)\right|=2$ and $\left|\Gamma_0(k)\cap\Delta_3^1(d)\right|=2$ by Theorem D. Suppose $m\in\Gamma_1(a,X)$ with $\Gamma_0(k)\cap\Delta_2^2(d)=\emptyset$. Then the three points in $\Gamma_0(m)$ which are collinear with x must lie in $\Delta_2^3(d)$ with the other two points of $\Gamma_0(m)$ lying in $\Delta_3^1(d)$ by Theorem D again. This proves (10.2.1).

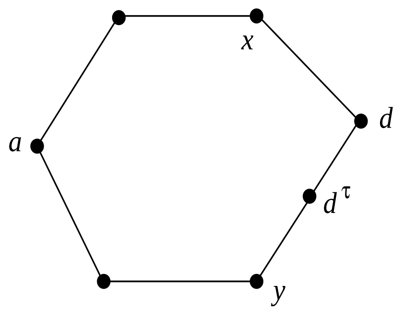
Combining (2.8) and Lemma 3.3 we deduce that $\tau(Y)$ fixes $\Gamma_0(l)$ pointwise for at least three $l\in\Gamma_1(a,X)$. Set $\tau=\tau(Y)$. By (10.2.1) we may find $e,e'\in\Delta_2^3(d)\cap\{a,x\}^\perp$ such that $e^\tau=e$, $e'^\tau=e'$ and $a+e\neq a+e'$. Since $\left|\Gamma_0(a+e')\cap\Delta_2^3(d)\right|\geq 2$ by (10.2.1) and two collinear points determine a unique line, we may assume further that $x+e\neq x+e'$. Let $z\in\{e,d\}^\perp\backslash\{x\}$. Then $e+z\in\alpha_3(e,e+x)$ by Theorem 4.8(i), whence $e+z\in\beta_0(e,X)\cup\beta_3(e,X)$ using (2.11). If $e+z\in\beta_0(e,X)$, then $z\in\Gamma_0(X)$ which forces $z\in\Delta_2^1(a)$ because $d(a,d)=3$. This contradicts Lemma 10.1 because $z\in\Delta_1(x)$and so $e+z\in\beta_3(e,X)$.

(10.2.2) $x^\tau=x$.

Assume $x^\tau\neq x$ and argue for a contradiction. Since $y+d\in\beta_1(y,Y)$, $d^\tau\neq d$. However $e^\tau=e$, whence $d^\tau\in\Gamma_0(d+y)\cap\Delta_2^3(e)$ and $a+e\notin\beta_1(y,Y)$ using (2.8). Therefore $e\in\Delta_2(y)$ by Theorem 5.8 because $a\in\Delta_2^1(y)$. By Theorem 5.2, $e\notin\Delta_2^1(y)$ because $d\in\Delta_2^3(e)\cap\Delta_1(y)$ and $e\notin\Delta_2^3(y)$ because $a\in\Delta_2^1(y)\cap\Delta_1(e)$. Hence $e\in\Delta_2^2(y)$ and there exists $z\in\{e,y\}^\perp\cap\{e,d\}^\perp$ by Lemma 8.4. Furthermore $z\in\Delta_1(d^\tau)$, whence $z^\tau\in\{e,d\}^\perp$. Thus if $z^\tau\neq z$, then $\{e,d\}^\perp=\{e,d^\tau\}^\perp$ by Theorem 4.8(i) and so $x^\tau\in\{e,d\}^\perp$. However $x^\tau\in\Delta_2^1(a)\cap\Delta_1(x)\cap\Gamma_0(X)$ because $\tau\in Q(a)$ which contradicts Lemma 10.1. Therefore $z^\tau=z$ and so $e+x,e+x^\tau\in\alpha_3(e,e+z)\cap\Gamma_1(e,X)$. Since

$e+z \in \beta_3(e,X)$, $e+z \in \alpha_3(e,l)$ for a unique $l \in \Gamma_1(e,X)$ by (2.11). This forces $e+x = e+x^\tau$. Using exactly the same argument with e replaced by e' we have $e'+x = e'+x^\tau$. However this implies that $e+x = x+x^\tau = e'+x$ contrary to our choice of e and e'. Therefore we have (10.2.2).

Since $d^\tau \neq d$ we have



Thus $x \in \Delta_2^1(y)$ because $x \notin \Delta_1(y)$

(10.2.3) $X \in \gamma_3(a,Y)$.

If $X \notin \gamma_3(a,Y)$, then there exists $b \in \{a,x\}^\perp$ with $a+b \in \beta_1(a,Y)$ by (2.8). However x is collinear with exactly three points in $\Gamma_0(a+b)$ and τ interchanges two pairs of points in $\Gamma_0(a+b)\backslash\{a\}$, which is impossible because $x^\tau = x$ by (10.2.2).

By (10.2.3) we can find $b \in \{a,x\}^\perp \cap \{a,y\}^\perp$. Since $b \in \Delta_2^2(d) \cup \Delta_2^3(d)$ and $x \in \Delta_2^1(y)$, Lemma 4.6(ii) yields the required contradiction. This proves part (i).

Part (ii) follows from (10.2.1)(i).

Theorem 10.3 Let $d \in \Delta_3^1(a)$.

(i) $\left|\Delta_3^1(a)\right| = 2^{11}.3.5.7.11.23$.

(ii) $G_{ad}^{*d} \cong 2^6(S_3 \times S_4)$ with $Q(a)=1$; G_{ad}^{*d} is the stabilizer in G_d^{*d} of a unique line l_d and a unique plane X_d incident with l_d in Γ_d.

Proof By Lemma 10.2 $\left|\Delta_2^1(a) \cap \Delta_1(d)\right| = 6$. Therefore since $\Delta_2^1(a)$ and $\Delta_3^1(a)$ are G_a-orbits by Theorem 4.3(v) and Lemma 6.3, we have

$$|\Delta_3^1(a)| = 2^{11}.\frac{|\Delta_2^1(a)|.|\beta_1(c,X(c,a))|.4}{6} = \frac{2^4.7.11.23.2^6.3^2.5.2^2}{6} = 2^{11}.3.5.7.11.23.$$

for $c \in \Delta_2^1(a) \cap \Delta_1(d)$ with $c+d \in \beta_1(c,X(c,a))$.

For part (ii) let l_d be the unique line described in Lemma 10.2(ii) and X_d be the unique plane in Γ_d incident with every point in $\Delta_2^1(a) \cap \Delta_1(d)$. Then $X_d \in \Gamma_2(l_d)$ by Lemma 10.2(ii) and G_{ad}^{*d} is a subgroup of the stabilizer in G_d^{*d} of l_d and X_d.

By definition there exists $c \in \Delta_2^1(a) \cap \Delta_1(d)$ with $c+d \in \beta_1(c,X(c,a))$. From Theorem 4.3 $G_{ac}^{*c} \cong 2^6 : 3{\cdot}S_6$ and $Q(c)_a \cong 2^7$. Also $Q(c)_a / < \tau(X(d,a)) >$ is isomorphic to a 6-dimensional irreducible $GF(2)3{\cdot}S_6$-module. Further $G_{acd}^{*c} \cong S_4 \times 2$ and $Q(c)_{ad} \cong 2^5$ with $Q(c)_{ad} \cap < \tau(X(d,a)) >= 1$. We show that

(10.3.1) $T(c+d) \cap G_a = \emptyset$.

In Γ_c we may suppose that $c+d$ is the standard trio and $X(c,a) = \begin{array}{|cc|cc|cc|} \hline \times & \times & \times & o & - & \cdot \\ \times & o & o & o & * & \square \\ * & \cdot & - & \square & - & * \\ \square & - & \cdot & * & \square & \cdot \\ \hline \end{array}$.

It is easy to check that for each $Y \in \Gamma_2(c+d)$, there exists $\Gamma_1(X(c,a))$ with $k \in \beta_1(c,Y)$. So $\tau(Y)$ acts regularly on $\Gamma_0(k)\backslash\{c\}$ which consists of three points in $\Delta_1(a)$ and one point in $\Delta_2^1(a)$. Therefore $\tau(Y) \notin G_a$ and we have (10.3.1).

Since $< T(c+\dot{d}) >= Q(c) \cap Q(d)$, (10.3.1) implies that

(10.3.2) $Q(c) \cap Q(d) \cap G_a = 1$.

Suppose $g \in G_{acd}\backslash Q(c)$ and that $g \in Q(d)$. Then, using (10.3.2),

$$[g, Q(c)_{ad}] \leqslant Q(c) \cap Q(d) \cap G_a = 1.$$

Hence g centralizes a subgroup of $Q(c)_a / < \tau(X(d,a)) >$ of order 2^5, which is impossible for a 6-dimensional irreducible $GF(2)3{}^{\cdot}S_6$-module. So, together with (10.3.2), we must have $G_{acd} \cap Q(d) = 1$. Since $\Gamma_0(d+c) \cap \Delta_2^1(a) = \{c\}, G_{ad} \cap Q(d) \leqslant G_{acd}$ whence $G_{ad} \cap Q(d) = 1$ and therefore $G_{ad} \cong G_{ad}^{*d} \cong 2^6(S_3 \times S_4)$.

Lemma 10.4 Let $d \in \Delta_3^1(a)$. Then the G_{ad}^{*d}-orbits on $\Gamma_1(d)$ are as described in Theorem F.

Proof Let X_d and l_d be, respectively, the sextet and trio fixed by G_{ad}^{*d}. Then we may take X_d to be the standard sextet and l_d to be the standard trio. Clearly each of the sets listed in Theorem F is a union of G_{ad}^{*d}-orbits on $\Gamma_1(d)$. Let $H = G_{ad}^{*d} (\cong 2^6 3(S_4 \times 2))$ and $Q = O_{2,3}(H) (\cong 2^6 3)$ with $\overline{H} = H / Q$. We look at several cases.

Case 1 Clearly $\{l_d\}$ is a $G_{ad}^{*d}-$orbit. Consider the set of lines in $\beta_0(X_d)\backslash\{l_d\}$. Since H is transitive on the columns of X_d and the three octads of l_d, H has two orbits on this set, namely (β_0, α_2) of length 6 and (β_0, α_3) of length 8.

Case 2 Let $l = \begin{array}{|cc|cc|cc|} \hline + & - & + & + & + & + \\ o & + & - & o & - & o \\ o & + & o & - & - & o \\ - & + & o & o & - & - \\ \hline \end{array}$. So $l \in (\beta_1, \alpha_0)$. Every element of Q fixes the columns of the standard sextet. Each 3-element cycles three entries in each column and each 2-element acts like an element of a fours group on each column and fixes the entries in 2 or 0 of the columns. Therefore $Q_l = 1$. As a subroup of S_6 acting on the columns of the standard sextet $\overline{H}_l$ is generated by (2,5)(1,6) and (3,4). So $|H_l|\,|4$ and $2304|\left|l^H\right|$. Since $|\Gamma_1(d)| = 3795$ we must have $\left|l^H\right| = 2304$ and so (β_1, α_0) is a G_{ad}-orbit of length 2304 as required.

Case 3 Next suppose $l = \begin{array}{|cc|cc|cc|} \hline + & o & + & + & + & + \\ o & + & o & o & o & o \\ o & + & - & - & - & - \\ o & + & - & - & - & - \\ \hline \end{array}$, so $l \in (\beta_1, \alpha_1)$. As in the previous case $Q_l = 1$. Since any element of H_l must fix O_1, H_l contains no 3-elements. Hence $|H_l| \leq 2^4$ and

$|l^H| \geq 576$. However we already have 2304 lines in $\beta_1(X_d)$ and so by (2.5) we must have $|l^H| \leq 576$. Hence $|l^H| = 576$ and (β_1, α_0) is a G_{ad}-orbit.

Case 4 Let $l = \begin{array}{|cc|cc|cc|} + & + & + & - & - & + \\ + & - & o & o & - & - \\ o & + & o & - & o & - \\ o & - & + & o & o & + \end{array}$, so $l \in (\beta_3, \alpha_0)$. By observation (see Appendix 2 in [RW2]) and the action of 3-elements on the column entries, $|Q_l| = 2^2$. If $S \in Syl_2(\overline{H}_l)$, then every non-trivial element of S acts on the columns of the standard sextet and fixes one of the octads of l. Each element of S fixes the same octad of l, otherwise the product of a certain pair of elements in S would be a 3-element. Without loss of generality we may suppose S fixes the octad "+" of l. Then there is only one possibility for a non-trivial element of S, namely (1,2)(3,6)(4,5) as an element of the group S_6 acting on the columns. Hence $|H_l| | 2^3.3$ and $384 | |l^H|$. However we already have 2304 lines in $\alpha_0(d, l_d)$ and so $|l^H| \leq 528$ by (2.3). So (β_3, α_0) is a G_{ad}-orbit of length 384.

Case 5 Now suppose $l = \begin{array}{|cc|cc|cc|} + & - & + & - & - & - \\ + & - & + & - & o & o \\ + & o & + & o & o & - \\ + & o & + & o & - & o \end{array}$, whence $l \in (\beta_2, \alpha_0)$. Then $|Q_l| = 2^4$ using the Appendix and the action of 3-elements in Q. Also, as a subroup of the S_6 acting on the columns, $\overline{H}_l$ is generated by (1,2)(3,4) and (5,6). So $|H_l| \leq 2^6$ and $|l^H| \geq 144$. Using Cases 2 and 4 and the size of $\alpha_0(l(d,a))$ given in (2.3) we have that (β_2, α_0) is a G^{*d}_{ad}-orbit of length 144.

Case 6 Let $l = \begin{array}{|cc|cc|cc|} + & + & + & o & o & + \\ + & o & - & - & - & - \\ o & + & - & - & - & - \\ o & o & + & o & o & + \end{array}$, and so $l \in (\beta_3, \alpha_1)$. By inspection $|Q_l| = 2^2$. Also $\overline{H}_l$ has no 3-elements because $\overline{H}_l$ fixes O_1. Hence $|H_l| \leq 2^5$ and $|l^H| \geq 288$. Since we already have 576 lines in $\alpha_1(l_d)$, (2.3) implies that $|l^H| = 288$. Therefore (β_3, α_1) is a G^{*d}_{ad}-orbit as required.

Case 7 Let $l = \begin{array}{|cc|cc|cc|} + & + & o & o & o & o \\ + & + & o & o & o & o \\ + & + & - & - & - & - \\ + & + & - & - & - & - \end{array} \in (\beta_2, \alpha_2)$. As in Cases 3 and 5 H_l has no 3-elements. Using the Appendix we see that $|Q_l| = 2^4$. So $|H_l| \leq 2^8$ and $|l^H| \geq 36$. Together with Case 5 and (2.5) this yields that (β_2, α_2) is a G^{*d}_{ad}-orbit of length 36.

Case 8 Lastly let $l = \begin{array}{|cc|cc|cc|} + & + & o & o & o & o \\ + & + & o & o & o & o \\ - & - & + & + & - & - \\ - & - & + & + & - & - \end{array}$. So $l \in (\beta_3, \alpha_3)$. By inspection $|Q_l| = 2^2$ and therefore $|H_l| \leq 2^6.3$ and $|l^H| \geq 48$. Since we already have considered 672 lines in $\beta_3(d, X_d)$ we must have $l^H = (\beta_3, \alpha_3)$, a G^{*d}_{ad} -orbit of length 48. This completes the proof.

Lemma 10.5 Let $d \in \Delta^1_3(a)$ and Let l_d be the unique line in $\Gamma_1(d)$ fixed by G^{*d}_{ad}. Suppose

$l \in \Gamma_1(d)$.

(i) $\left|\Gamma_0(l_d)\cap\Delta_2^2(a)\right| = 3$ and $\left|\Gamma_0(l_d)\cap\Delta_3^1(a)\right| = 2$ and for any $c \in \Gamma_0(l_d)\cap\Delta_2^2(a)$, $c+d \in (\beta_2\beta_2\beta_2,\alpha_2)$.

(ii) If $l \in (\beta_0,\alpha_2)$, then $\left|\Gamma_0(l)\cap\Delta_2^1(a)\right| = 1$ and $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right| = 4$ and for any $c \in \Delta_1(d)\cap\Delta_2^1(a)$, $c+d \in \beta_1(c,X(c,a))$.

(iii) If $l \in (\beta_0,\alpha_3)$, then $\left|\Gamma_0(l)\cap\Delta_2^2(a)\right| = 1$ and $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right| = 4$ and for any $c \in \Gamma_0(l)\cap\Delta_2^2(a)$, $c+d \in (\beta_3\beta_3\beta_3,\alpha_3)$.

(iv) If $l \in (\beta_2,\alpha_2)$, then $\left|\Gamma_0(l)\cap\Delta_2^2(a)\right| = 1$, $\left|\Gamma_0(l)\cap\Delta_2^3(a)\right| = 2$ and $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right| = 2$. Also for any $c \in \Gamma_0(l)\cap\Delta_2^2(a)$, and any $c' \in \Gamma_0(l)\cap\Delta_2^3(a)$, $c+d \in (\beta_1\beta_1\beta_2,\alpha_1)$ and $c'+d \in (\beta_2;2^4;1^8;1^8)$.

(v) If $l \in (\beta_3,\alpha_3)$, then $\left|\Gamma_0(l)\cap\Delta_2^3(a)\right| = 3$ and $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right| = 2$ and for any $c \in \Gamma_0(l)\cap\Delta_2^3(a)$, $c+d \in (\beta_2;2^4;2^21^4;2^21^4)$.

(vi) If $l \in (\beta_3,\alpha_1)$, then $\left|\Gamma_0(l)\cap\Delta_2^2(a)\right| = 1$ and $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right| = 4$ and for any $c \in \Gamma_0(l)\cap\Delta_2^2(a)$, $c+d \in (\beta_1\beta_1\beta_3,\alpha_0)$.

(vii) If $\Gamma_0(l)\cap\Delta_2^i(a) \neq \emptyset$ for $i = 1,2$, then $l \in \{l_d\}\cup(\beta_j,\alpha_k)$, for $(j,k) \in \{(0,2),(0,3),(2,2),(3,1)\}$.

Proof Part (i) is a consequence of Theorem D and Lemmas 6.5, 10.1 and 10.2 together with the definition of l_d given in Theorem 10.3(ii). For any $c \in \Delta_1(d)\cap\Delta_2^1(a)$ we have $c+d \in \beta_1(c,X(c,a))$ by Theorem C and so

$$|\Delta_1(d)\cap\Delta_2^1(a)| = \frac{|\Delta_2^1(a)|.2880.4}{|\Delta_3^1(a)|} = 6$$

by Theorems 4.7(vi) and 4.8(ii) and Lemma 9.2.

Since $\{d+x \,|\, x \in \Delta_1(d)\cap\Delta_2^1(a)\}$ is a G_{ad}^{*d}-orbit of lines we must have $d+c \in (\beta_0,\alpha_2)$ and so part (ii) follows from Theorem C.

Set l_d and X_d to be the standard trio and standard sextet respectively. Let $c,x \in \Delta_1(d)$ with $d+c = \begin{array}{|cc|cc|cc|}\hline + & + & o & - & o & - \\ + & + & o & - & o & - \\ + & + & o & - & o & - \\ + & + & o & - & o & - \\ \hline\end{array}$ and $d+x = \begin{array}{|cc|cc|cc|}\hline + & - & + & o & o & - \\ + & - & + & o & o & - \\ + & - & + & o & o & - \\ + & - & + & o & o & - \\ \hline\end{array}$. So $d+c \in (\beta_0,\alpha_2)$ and $d+x \in (\beta_0,\alpha_3)$. By part (ii) we may suppose that $c \in \Delta_2^1(a)$. Also there exists $c' \in \Delta_1(d)\cap\Delta_2^1(a)$ with $d+c' = \begin{array}{|cc|cc|cc|}\hline + & o & o & + & - & - \\ + & o & o & + & - & - \\ + & o & o & + & - & - \\ + & o & o & + & - & - \\ \hline\end{array}$. By Lemma 3.8(ii) we may suppose that c,x,c' are incident with the same line because $d+c, d+x, d+c'$ are three lines of a non-sparse triangle at

d. Since $d(a,d)=3$, $x \notin \Delta_1(a)$ and so Theorem C implies that $x \in \Delta_2^2(a)$. By Theorem D $x+c$ is the unique line in $\Gamma_1(x)$ fixed by G_{ax}^{*x}. Also $x+d \in \alpha_3(x,x+c)$, whence $x+d \in (\beta_3\beta_3\beta_3,\alpha_3)$ and $\left|\Gamma_0(d+x)\cap\Delta_2^2(a)\right| = 1$ and $\left|\Gamma_0(d+x)\cap\Delta_3^1(a)\right| = 4$ by Theorem D. This proves part (iii) because (β_0,α_3) is a G_{ad}^{*d} -orbit.

For part (iv) let $x \in \Delta_1(d)\cap\Delta_2^2(a)$ with $x+d \in (\beta_1\beta_1\beta_2,\alpha_1)$. Without loss of generality we may take $x+d = \begin{array}{|cc|cc|cc|}\hline + & + & \mathrm{o} & + & \mathrm{o} & + \\ + & - & \mathrm{o} & - & \mathrm{o} & - \\ - & - & \mathrm{o} & + & \mathrm{o} & + \\ - & + & \mathrm{o} & - & \mathrm{o} & - \\ \hline \end{array}$ where $\mathcal{S}_d$ is the standard sextet line. Let $x+c = \begin{array}{|cc|cc|cc|}\hline + & + & \mathrm{o} & - & \mathrm{o} & - \\ + & + & \mathrm{o} & - & \mathrm{o} & - \\ + & + & \mathrm{o} & - & \mathrm{o} & - \\ + & + & \mathrm{o} & - & \mathrm{o} & - \\ \hline \end{array} \in (\beta_0\beta_2\beta_2,\alpha_2)\cap\alpha_2(x,x+d)$. Then we may suppose $c \in \Delta_2^1(a)$ by Theorem D. Since $\Delta_1(c)\cap\Delta_2^3(a)=\emptyset$ by Theorem 5.2 and

$$\left|\Gamma_0(x+d)\cap\Delta_2^3(a)\right| = \left|\Gamma_0(x+d)\cap\Delta_3^1(a)\right| = 2,$$

we must have $c \in \Delta_1(d)$. By part(ii), $d+c \in (\beta_0,\alpha_2)$ and so $d+x \in \beta_2(d,X_d)$ because $d+x \in \alpha_2(d,d+c)$ and $d+x \notin \beta_0(d,X_d)$ by (ii) and (iii). Furthermore $d+x \notin \alpha_0(d,l_d)$ because $d+c \in \alpha_2(d,d+x)\cap\alpha_2(d,l_d)$. Examining the possible G_{ad}^{*d} -orbits on $\Gamma_1(d)$ yields $d+x \in (\beta_2,\alpha_2)$ and (iv) follows from Lemma 9.2(iii) because (β_2,α_2) is a G_{ad}^{*d}-orbit.

Turning to part (v), let $x \in \Gamma_0(l_d)\cap\Delta_2^2(a)$. Then $l_d = x+d \in (\beta_2\beta_2\beta_2,\alpha_2)$ by part (i) and we may suppose that $x+d = \begin{array}{|cc|cc|cc|}\hline + & + & - & - & - & - \\ + & + & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} \\ + & + & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} \\ + & + & - & - & - & - \\ \hline \end{array}$ in Γ_x where $\mathcal{S}_x$ is the standard sextet line. If l_x is the unique line fixed by G_{ax}^{*x}, then d is collinear with three points in $\Gamma_0(l_x)$ one of which lies in $\Delta_2^1(a)$ by Theorem D. Taking this with part (ii) we get that $X_d \in \Gamma_2(x+d,l_x)$ and so $X_d = \begin{array}{|cc|cc|cc|}\hline + & + & \cdot & \cdot & - & - \\ \ast & \ast & \mathrm{o} & \mathrm{o} & \square & \square \\ \ast & \ast & \mathrm{o} & \mathrm{o} & \square & \square \\ + & + & \cdot & \cdot & - & - \\ \hline \end{array}$. Let $l_1 = \begin{array}{|cc|cc|cc|}\hline \mathrm{o} & + & + & + & + & + \\ + & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} \\ + & \mathrm{o} & - & - & - & - \\ + & \mathrm{o} & - & - & - & - \\ \hline \end{array}$, $l_2 = \begin{array}{|cc|cc|cc|}\hline \mathrm{o} & + & - & - & - & - \\ + & \mathrm{o} & - & - & - & - \\ + & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} \\ + & \mathrm{o} & + & + & + & + \\ \hline \end{array}$, $l_3 = \begin{array}{|cc|cc|cc|}\hline \mathrm{o} & + & - & - & - & - \\ + & \mathrm{o} & + & + & + & + \\ + & \mathrm{o} & - & - & - & - \\ + & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} \\ \hline \end{array}$ and $l_4 = \begin{array}{|cc|cc|cc|}\hline \mathrm{o} & + & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} \\ + & \mathrm{o} & - & - & - & - \\ + & \mathrm{o} & + & + & + & + \\ + & \mathrm{o} & - & - & - & - \\ \hline \end{array}$. Then $l_1,l_2,l_3,l_4,x+d$ are the lines of a full triangle. Also $l_1,l_2 \in (\beta_1\beta_1\beta_2,\alpha_1)$ and $l_3,l_4 \in (\beta_1\beta_1\beta_3,\alpha_1)$ and so $\left|\Gamma_0(l_i)\cap\Delta_2^3(a)\right| = \left|\Gamma_0(l_i)\cap\Delta_3^1(a)\right| = 2$ for $i=1,2$ and $\left|\Gamma_0(l_j)\cap\Delta_2^3(a)\right| = 4$ for $j=3,4$ by Theorem D. Let k_1,k_2,k_3,k_4 be the distinct lines in $\Gamma_1(d)\backslash\{d+x\}$ with $\Gamma_0(l_i)\cap\Gamma_0(k_j)\neq\emptyset$ for each $i,j\in\{1,2,3,4\}$. Then $l_d(=d+x),k_1,k_2,k_3,k_4$ are the lines of a non-sparse triangle and so $k_i \in \alpha_3(d,l_d)$ for each $i=1,2,3,4$. Moreover $k_i \notin \Gamma_1(X_d)$ because $l_i \notin \Gamma_1(X_d)$ for each i. Therefore $k_i \in (\beta_3,\alpha_3)$ for $i=1,2,3,4$ and then $\left|\Gamma_0(k_i)\cap\Delta_2^3(a)\right| = 3$ and $\left|\Gamma_0(k_i)\cap\Delta_3^1(a)\right| = 2$. So we have proved part (v).

Let $c \in \Delta_1(d) \cap \Delta_2^2(a)$. By Theorem D $\left|\Delta_1(c) \cap \Delta_3^1(a)\right| = 2.288 + 4.1152 + 2.24 + 4.32 = 5360$. So

$$\left|\Delta_1(d) \cap \Delta_2^2(a)\right| = \frac{\left|\Delta_2^2(a)\right|.5360}{\left|\Delta_3^1(a)\right|} = \frac{2^7.3.5.7.11.23.5360}{2^{11}.3.5.7.11.23} = 335$$

by Theorems 4.7(vi) and 10.3(i). Using parts (i),(iii) and (iv) we have 288 points $x \in \Delta_1(d) \cap \Delta_2^2(a)$ with $d+x \notin \{l_d\} \cup (\beta_0, \alpha_3) \cup (\beta_2, \alpha_2)$ and $x+d \in (\beta_1\beta_1\beta_3, \alpha_0)$. Also $\left|\Gamma_0(d+x) \cap \Delta_2^2(a)\right| = 1$ for each of these points x. Looking at the possible orbit sizes listed in Theorem F we must have $d+x \in (\beta_3, \alpha_1)$ for each x and so we have shown part (vi).

Finally part (vii) follows from parts (i),(ii),(iii),(iv) and (vi) together with Theorems C and D.

Lemma 10.6 Let $d \in \Delta_3^1(a)$ and $l \in (\beta_3, \alpha_0)$. Then $\left|\Gamma_0(l) \cap \Delta_3^2(a)\right| = 4$.

Proof Without loss of generality we may suppose that, in Γ_d, $l_d = \begin{array}{|cc|cc|cc|} \hline + & + & o & o & - & - \\ + & + & o & o & - & - \\ + & + & o & o & - & - \\ + & + & o & o & - & - \\ \hline \end{array}$

$X_d = \begin{array}{|cc|cc|cc|} \hline + & \cdot & o & \square & * & - \\ + & \cdot & o & \square & * & - \\ + & \cdot & o & \square & * & - \\ + & \cdot & o & \square & * & - \\ \hline \end{array}$ and $l = \begin{array}{|cc|cc|cc|} \hline + & + & + & - & - & + \\ + & - & o & o & - & - \\ o & + & o & - & o & - \\ o & - & + & o & o & + \\ \hline \end{array}$. Set

$d+x = \begin{array}{|cc|cc|cc|} \hline + & o & + & - & - & o \\ + & o & + & - & - & o \\ + & o & + & - & - & o \\ + & o & + & - & - & o \\ \hline \end{array}$. Then $d+x \in (\beta_0, \alpha_3) \cap \alpha_3(d,l) \cap \alpha_3(d,l_d)$. By Lemma 10.4 we may suppose that $x \in \Delta_2^2(a)$ with $x+d \in (\beta_3\beta_3\beta_3, \alpha_3)$. If $k_1 = \begin{array}{|cc|cc|cc|} \hline + & + & o & - & o & - \\ + & + & o & - & o & - \\ + & + & o & - & o & - \\ + & + & o & - & o & - \\ \hline \end{array}$, $k_2 =$

$\begin{array}{|cc|cc|cc|} \hline + & o & o & + & - & - \\ + & o & o & + & - & - \\ + & o & o & + & - & - \\ + & o & o & + & - & - \\ \hline \end{array}$, $k_3 = \begin{array}{|cc|cc|cc|} \hline + & - & o & o & - & + \\ + & - & o & o & - & + \\ + & - & o & o & - & + \\ + & - & o & o & - & + \\ \hline \end{array}$ and $k_4 = \begin{array}{|cc|cc|cc|} \hline + & o & - & o & + & - \\ + & o & - & o & + & - \\ + & o & - & o & + & - \\ + & o & - & o & + & - \\ \hline \end{array}$, then $k_1, k_2, k_3, k_4, d+x$ form a non-sparse triangle at d with $k_1, k_2, k_3 \in (\beta_0, \alpha_2)$ and $k_4 \in (\beta_0, \alpha_3)$. By Lemma 10.4 there exists $y \in \Gamma_0(k_1)$ with $y \in \Delta_2^1(a)$. By Theorem C, $\Gamma_0(x+y) \subseteq \Delta_2(a)$ and using the point distributions described in Lemma 10.4 we must have $x+y \in (\beta_0\beta_0\beta_0, \{l\})$. Since $l \in \alpha_0(d, d+x)$, for any $z \in \Gamma_0(l) \backslash \{d\}$, $z \in \Delta_1(x)$ and $x+z \in \alpha_3(x, x+d) \cap \alpha_0(x, x+y)$. So $x+z \in (\beta_1\beta_1\beta_1, \alpha_0) \cup (\beta_1\beta_1\beta_3, \alpha_0)$. By Lemma 10.4 $z \notin \Delta_2^2(a)$ and so $x+z \notin (\beta_1\beta_1\beta_3, \alpha_0)$ by Lemma 8.7. Therefore $x+z \in (\beta_1\beta_1\beta_1, \alpha_0)$ and $z \in \Delta_3^2(a)$ by definition.

We next uncover the point distribution of certain line orbits for points in $\Delta_2^3(a)$.

Lemma 10.7 Let $d \in \Delta_2^3(a)$ and $l \in \Gamma_1(d)$.

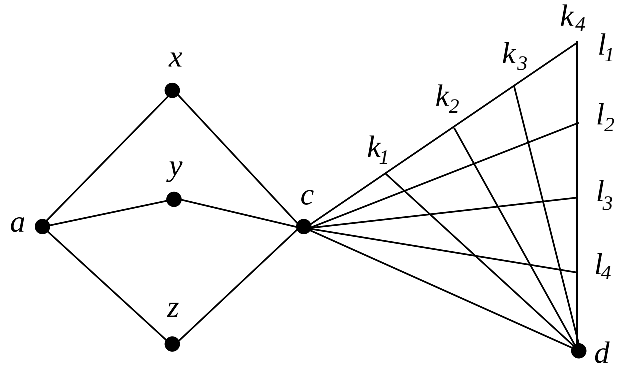
(i) If $l \in (\beta_3; 2^2 1^4; 2^2 1^4; 2^2 1^4)$, then $\left|\Gamma_0(l) \cap \Delta_3^2(a)\right| = 4$.

(ii) If $l \in (\beta_3; 1^8; 1^8; 1^8)$, then $\left|\Gamma_0(l) \cap \Delta_2^3(a)\right| = 2$, $\left|\Gamma_0(l) \cap \Delta_3^1(a)\right| = 3$ and for any $e \in \Gamma_0(l) \cap \Delta_3^1(a)$, $e + d \in (\beta_2, \alpha_0)$.

Proof Let $c \in \Delta_2^2(a) \cap \Delta_1(d)$ with $c + d = \begin{array}{|cc|cc|cc|}\hline + & \text{o} & + & \text{o} & \text{o} & \text{o} \\ + & \text{o} & + & \text{o} & - & - \\ + & - & + & - & - & \text{o} \\ + & - & + & - & \text{o} & - \\ \hline\end{array} \in (\beta_2\beta_3\beta_3, \alpha_1)$ (where $\mathcal{S}_c$ is the standard sextet line). If $x, y, z \in \{a, c\}^\perp$, with $c + x = \begin{array}{|cc|cc|cc|}\hline + & \text{o} & + & \text{o} & - & - \\ + & \text{o} & + & \text{o} & - & - \\ + & \text{o} & + & \text{o} & - & - \\ + & \text{o} & + & \text{o} & - & - \\ \hline\end{array}$, $c + y = \begin{array}{|cc|cc|cc|}\hline \text{o} & \text{o} & \text{o} & \text{o} & - & - \\ \text{o} & \text{o} & \text{o} & \text{o} & - & - \\ + & + & + & + & - & - \\ + & + & + & + & - & - \\ \hline\end{array}$ and $c + z = \begin{array}{|cc|cc|cc|}\hline + & \text{o} & + & \text{o} & - & - \\ + & \text{o} & + & \text{o} & - & - \\ \text{o} & + & \text{o} & + & - & - \\ \text{o} & + & \text{o} & + & - & - \\ \hline\end{array}$, then $c + x \in \alpha_2(c, c+d)$ and $c + y, c + z \in \alpha_3(c, c+d)$ and $x, y, z \in \{a, d\}^\perp$ by Lemma 5.6. We define lines $l_1, l_2, l_3, l_4 \in \Gamma_1(c)$ in two cases.

Case 1: Let $l_1 = \begin{array}{|cc|cc|cc|}\hline + & \text{o} & + & - & \text{o} & - \\ - & \text{o} & - & - & \text{o} & + \\ + & \text{o} & - & + & \text{o} & - \\ - & \text{o} & + & + & \text{o} & + \\ \hline\end{array}$, $l_2 = \begin{array}{|cc|cc|cc|}\hline \text{o} & + & \text{o} & - & + & - \\ + & + & + & - & \text{o} & - \\ \text{o} & \text{o} & + & - & \text{o} & - \\ + & \text{o} & \text{o} & - & + & - \\ \hline\end{array}$,

$l_3 = \begin{array}{|cc|cc|cc|}\hline + & + & + & - & + & - \\ \text{o} & + & \text{o} & - & - & \text{o} \\ + & - & \text{o} & \text{o} & - & - \\ \text{o} & - & + & \text{o} & + & \text{o} \\ \hline\end{array}$ and $l_4 = \begin{array}{|cc|cc|cc|}\hline + & - & + & + & - & + \\ \text{o} & - & \text{o} & + & \text{o} & - \\ + & \text{o} & \text{o} & - & \text{o} & + \\ \text{o} & \text{o} & + & - & - & - \\ \hline\end{array}$. Then $c + d, l_1, l_2, l_3, l_4$ are the lines of a non-sparse triangle at c with $l_1, l_2 \in (\beta_1\beta_1\beta_2, \alpha_1)$ and $l_3, l_4 \in (\beta_1\beta_1\beta_3, \alpha_0)$. We have



where k_1, k_2, k_3, k_4 are distinct lines in $\Gamma_1(d)$ with $k_i \in \alpha_3(d, d+c)$ and $\Gamma_0(k_i) \cap \Gamma_0(l_j) \neq \emptyset$ for each $i, j = 1, 2, 3, 4$. By inspection, $l_i \in \alpha_1(c, c+x) \cap \alpha_0(c, c+y) \cap \alpha_0(c, c+z)$. By Lemma 9.2(ii), $d + c \in (\beta_0, **)$. Since $d + c, k_1, k_2, k_3, k_4$ form a non-sparse triangle at d, Lemma 9.1(iii) implies that $k_i \in \beta_3(d, X_d)$. Also using the point distribution for $l_i, i = 1,2,3,4$ given in Theorem D we have that $\Gamma_0(k_i) \cap \Delta_3^1(a) \neq \emptyset$ for each i.

Case 2: Now let $l_1 = \begin{array}{|cc|cc|cc|} \hline o & + & + & + & + & + \\ + & - & o & - & - & o \\ + & - & o & - & o & - \\ + & o & o & o & - & - \\ \hline \end{array}$, $l_2 = \begin{array}{|cc|cc|cc|} \hline o & - & + & - & - & - \\ + & + & o & + & o & - \\ + & o & o & o & - & + \\ + & - & o & - & + & o \\ \hline \end{array}$,

$l_3 = \begin{array}{|cc|cc|cc|} \hline o & - & + & - & - & - \\ + & o & o & o & - & + \\ + & - & o & - & + & o \\ + & + & o & + & o & - \\ \hline \end{array}$ and $l_4 = \begin{array}{|cc|cc|cc|} \hline o & o & + & o & o & o \\ + & - & o & - & + & - \\ + & + & o & + & - & - \\ + & - & o & - & - & + \\ \hline \end{array}$. Then $c+d, l_1, l_2, l_3, l_4$ are the lines of a non-sparse triangle at c with $l_1, l_2, l_3, l_4 \in (\beta_1\beta_1\beta_1, \alpha_0)$. If we define k_1, k_2, k_3, k_4 as in Case 1, then Lemma 9.1(iii) again implies that $k_i \in \beta_3(d, X_d)$. Moreover $\left|\Gamma_0(k_i) \cap \Delta_3^2(a)\right| = 4$ using the point distribution of l_i given in Theorem D, for $i = 1,2,3,4$.

We now shift our attention to d. By Lemma 9.2(v) $k_i \notin (\beta_3; 2^2 1^4; 2^2 1^4; 1^8)$ and so $k_i \in (\beta_3; 1^8; 1^8; 1^8) \cup (\beta_3; 2^2 1^4; 2^2 1^4; 2^2 1^4)$. Suppose $k_1 \in (\beta_3; 1^8; 1^8; 1^8)$. Then there exists a unique line in $\alpha_3(d, k_1) \cap (\beta_0, **)$ and this must be $d+c$. Then k_2, k_3, k_4 are uniquely determined and lie in $(\beta_3; 1^8; 1^8; 1^8)$. Similarly if $k_1 \in (\beta_3; 2^2 1^4; 2^2 1^4; 2^2 1^4), d+c$ is the unique line in $\alpha_3(d, k_1) \cap (\beta_0, **)$ and k_2, k_3, k_4 are uniquely determined and lie in $(\beta_3; 2^2 1^4; 2^2 1^4; 2^2 1^4)$. Taking this with the information in Cases 1 and 2 and the point distributions of $l_i, i = 1,2,3,4$ we get

(10.7.1) (i)$\left|\Gamma_0(k_1) \cap \Delta_2^3(a)\right| = 2$ and $\left|\Gamma_0(k_1) \cap \Delta_3^1(a)\right| = 3$ for k_1 in exactly one of $(\beta_3; 2^2 1^4; 2^2 1^4; 2^2 1^4)$ or $(\beta_3; 1^8; 1^8; 1^8)$.

(ii) $\left|\Gamma_0(k_1) \cap \Delta_2^3(a)\right| = 1$ and $\left|\Gamma_0(k_1) \cap \Delta_3^2(a)\right| = 4$ for k_1 in exactly one of $(\beta_3; 2^2 1^4; 2^2 1^4; 2^2 1^4)$ or $(\beta_3; 1^8; 1^8; 1^8)$.

Suppose that $k_1 \in (\beta_3; 2^2 1^4; 2^2 1^4; 2^2 1^4) \cup (\beta_3; 1^8; 1^8; 1^8)$ with $\left|\Gamma_0(k_1) \cap \Delta_2^3(a)\right| = 2$ and $\left|\Gamma_0(k_1) \cap \Delta_3^1(a)\right| = 3$. So we are in Case 1. Let $f \in \Gamma_0(k_1) \cap \Delta_3^1(a)$. Then $f \in \Gamma_0(l_i)$ for some $i = 1, \ldots, 4$ and by Lemma 10.5 $f + c \in \beta_2(f, X_f) \cup \beta_3(f, X_f)$. Let $\Gamma_0(c+d) = \{c, c_1, c_2, d_1, d\}$. By Lemma 8.4(ii) we may suppose that $c_1, c_2 \in \Delta_2^2(a)$ and $d_1 \in \Delta_2^3(a)$. Using Lemma 10.5(vii) we have $f + c_1, f + c_2 \in \beta_i(f, X_f)$ for $i = 0,2$ or 3. Thus if $Y \in \Gamma_2(c, d, f)$, then $Y \in \gamma_3(X_f)$ by Lemma 9.1. Using Lemma 9.1 again and the fact that $f + c \in \beta_2(f, X_f) \cup \beta_3(f, X_f)$ we have $f + d \in \beta_2(f, X_f) \cup \beta_3(f, X_f)$ because $f + d \in \beta_0(f, X_f)$ by Lemma 10.5. Considering the possible G_{af}^{*f}-orbits and the known point distributions given in Lemmas 10.5 and 10.6 we conclude that $f + d \in (\beta_2, \alpha_0)$. So we have proved

(10.7.2). If $l \in \Gamma_1(f)$ with $l \in (\beta_2, \alpha_0)$ then $\left|\Gamma_0(l) \cap \Delta_2^3(a)\right| = 2$ and $\left|\Gamma_0(l) \cap \Delta_3^1(a)\right| = 3$.

For a contradiction suppose that $k_1 \in (\beta_3; 2^2 1^4; 2^2 1^4; 2^2 1^4)$. Using the orbit sizes in Theorem E and the point distributions described in Lemma 9.2 we have

$\left|\Delta_1(d) \cap \Delta_3^1(a)\right| = 960n + 1920m + 2.60 + 2.120 + 3.320 = 960q + 1320$
for some $n, m, q \in \mathbb{Z}$. So

$$\left|\Delta_1(f) \cap \Delta_2^3(a)\right| = \frac{(960q + 1320).2^{11}.3^2.7.11.23}{2^{11}.3.5.7.11.23} = 576q + 792$$

using the orbit sizes in Theorems 4.7(vi) and 10.3(i). By Lemma 10.5 and (10.7.2) we have already accounted for $2.36+3.48+2.144=504$ of these points. Therefore there are $576q+288$ points in $\Delta_1(f)\cap\Delta_2^3(a)$ which lie in $(\beta_1,\alpha_0)\cup(\beta_1,\alpha_1)$. We now have the required contradiction because $576\,||(\beta_1,\alpha_0)\cup(\beta_1,\alpha_1)|$. Hence $k_1=f+d\in(\beta_3;1^8;1^8;1^8)$. The lemma now follows from (10.7.1).

Lemma 10.8 Let $d\in\Delta_3^1(a)$ and $l\in\Gamma_1(d)$.

(i) If $l\in(\beta_1,\alpha_0)$, then $\left|\Gamma_0(l)\cap\Delta_2^3(a)\right|=1$ and $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right|=\left|\Gamma_0(l)\cap\Delta_3^2(a)\right|=2$ and for any $x\in\Gamma_0(l)\cap\Delta_2^3(a)$, $x+d=l\in(\beta_1;21^6;21^6;2^21^4)$.

(ii) If $l\in(\beta_1,\alpha_1)$, then $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right|=3$ and $\left|\Gamma_0(l)\cap\Delta_3^2(a)\right|=2$.

Proof We first show

(10.8.1) .(i) There exist lines $l\in\Gamma_1$ with $\left|\Gamma_0(l)\cap\Delta_2^3(a)\right|=1$ and $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right|=\left|\Gamma_0(l)\cap\Delta_3^2(a)\right|=2$.

(ii) There exist lines $l\in\Gamma_1$ with $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right|=3$ and $\left|\Gamma_0(l)\cap\Delta_3^2(a)\right|=2$.

Let $c\in\Delta_2^2(a)$. Then there exist lines $l_1,l_2,l_3,l_4,l_5\in\Gamma_1(c)$ forming the lines of a full triangle at c with $l_1,l_2\in(\beta_1\beta_1\beta_1,\alpha_0)$, $l_3,l_4\in(\beta_1\beta_1\beta_3,\alpha_0)$ and $l_5\in(\beta_1\beta_1\beta_2,\alpha_1)$. For example, with $\mathcal{S}_c$ the standard sextet line let $l_1=\begin{array}{|cc|cc|cc|}\hline + & - & + & + & + & + \\ o & + & - & o & - & o \\ o & + & o & - & - & o \\ - & + & o & o & - & - \\ \hline\end{array}$, $l_2=\begin{array}{|cc|cc|cc|}\hline + & - & + & + & o & + \\ - & o & + & o & - & - \\ o & o & - & + & - & o \\ + & o & o & - & - & + \\ \hline\end{array}$,

$l_3=\begin{array}{|cc|cc|cc|}\hline + & - & + & + & o & + \\ + & o & o & - & - & + \\ - & o & + & o & - & - \\ o & o & - & + & - & o \\ \hline\end{array}$, $l_4=\begin{array}{|cc|cc|cc|}\hline + & + & + & + & o & + \\ o & o & - & - & + & o \\ - & o & o & - & + & - \\ - & o & - & o & + & - \\ \hline\end{array}$ and

$l_5=\begin{array}{|cc|cc|cc|}\hline + & o & + & + & o & + \\ - & o & - & + & o & - \\ + & o & - & - & o & + \\ - & o & + & - & o & - \\ \hline\end{array}$. So $\left|\Gamma_0(l_i)\cap\Delta_3^2(a)\right|=4$ for $i=1,2$, $\left|\Gamma_0(l_i)\cap\Delta_3^1(a)\right|=4$, for $i=3,4$ and $\left|\Gamma_0(l_5)\cap\Delta_2^3(a)\right|=\left|\Gamma_0(l_5)\cap\Delta_3^1(a)\right|=2$. Since the lines l_i form a non-sparse triangle at c there exist lines $l\notin\Gamma_1(c)$ with $\Gamma_0(l_i)\cap\Gamma_0(l)\neq\emptyset$ for each $i=1,\ldots,5$ and $\Gamma_0(l_5)\cap\Gamma_0(l)\subseteq\Delta_2^3(a)$. For such lines l we have $\left|\Gamma_0(l)\cap\Delta_2^3(a)\right|=1$ and $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right|=\left|\Gamma_0(l)\cap\Delta_3^2(a)\right|=2$. Similarly there exist lines $l\notin\Gamma_1(c)$ with $\Gamma_0(l_i)\cap\Gamma_0(l)\neq\emptyset$ for each $i=1,\ldots,5$ and $\Gamma_0(l_5)\cap\Gamma_0(l)\subseteq\Delta_3^1(a)$. For such lines l we have $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right|=3$ and $\left|\Gamma_0(l)\cap\Delta_3^2(a)\right|=2$ and so we have proved (10.8.1).

Let $l\in\Gamma_1(d)$ with $\left|\Gamma_0(l)\cap\Delta_2^3(a)\right|=1$ and $\left|\Gamma_0(l)\cap\Delta_3^1(a)\right|=\left|\Gamma_0(l)\cap\Delta_3^2(a)\right|=2$ and suppose $x\in\Gamma_0(l)\cap\Delta_2^3(a)$. The only line orbits of G_{ad}^{*d} whose point distributions are still unknown are (β_1,α_0) and (β_1,α_1) by Lemmas 10.5,10.6 and 10.7. So

(10.8.2) l lies in one of the orbits (β_1,α_0) or (β_1,α_1) and the other orbit contains lines k with $\left|\Gamma_0(k)\cap\Delta_3^1(a)\right|=3$ and $\left|\Gamma_0(k)\cap\Delta_3^2(a)\right|=2$.

Suppose $l\in(\beta_1,\alpha_1)$. Then

$$\left|\Delta_1(d)\cap\Delta_2^3(a)\right| = 576+2.144+2.36+3.48 = 1080$$

by Lemmas 10.4,10.5 and 10.6. Therefore, using the orbit sizes in Theorems 4.8 and 10.3(i) we get

$$\left|\Delta_1(x)\cap\Delta_3^1(a)\right| = \frac{1080.2^{11}.3.5.7.11.23}{2^{11}.3^2.7.11.23} = 1800.$$

Using Lemmas 9.2 and 10.7 we must have $x+d \in (\beta_1;21^6;21^6;1^8)\cup(\beta_1;21^6;21^6;2^21^4)$. We

now have a contradiction to the orbit sizes given in Theorem E. Therefore $l \in (\beta_1,\alpha_0)$ and by a calculation similar to that above we get $\left|\Delta_1(x)\cap\Delta_3^1(a)\right| = 4680$. Using the point distributions given in Lemmas 9.2 and 10.7 yields that $x+d$ lies in a G_{ax}^{*x}-orbit of $\Gamma_1(x)$ of size $\frac{4680-2.60-2.120-3.160}{2} = 1920$. Therefore $x+d \in (\beta_1;21^6;21^6;2^21^4)$ and part (i) is proved.

Part (ii) now follows from (10.8.2).

We have now proved Theorem F.

We conclude this section by completing the proof of Theorem E.

Lemma 10.9 Let $d \in \Delta_2^3(a)$ and $l \in \Gamma_1(d)$ with $l \in (\beta_1;21^6;21^6;1^8)$. Then $\left|\Gamma_0(l)\cap\Delta_2^3(a)\right| = 3$ and $\left|\Gamma_0(l)\cap\Delta_3^2(a)\right| = 2$.

Proof Without loss of generality we may suppose that $l = \begin{array}{|cc|cc|cc|}\hline o & + & + & + & + & + \\ + & o & o & o & o & o \\ + & o & - & - & - & - \\ + & o & - & - & - & - \\ \hline\end{array}$. Let $k = \begin{array}{|cc|cc|cc|}\hline + & + & - & - & - & - \\ + & + & o & o & o & o \\ + & + & - & - & - & - \\ + & + & o & o & o & o \\ \hline\end{array}$, $l_1 = \begin{array}{|cc|cc|cc|}\hline o & + & - & - & - & - \\ + & o & + & + & + & + \\ + & o & o & o & o & o \\ + & o & - & - & - & - \\ \hline\end{array}$,

$l_2 = \begin{array}{|cc|cc|cc|}\hline o & + & - & - & - & - \\ + & o & - & - & - & - \\ + & o & + & + & + & + \\ + & o & o & o & o & o \\ \hline\end{array}$ and $l_3 = \begin{array}{|cc|cc|cc|}\hline o & + & o & o & o & o \\ + & o & - & - & - & - \\ + & o & - & - & - & - \\ + & o & + & + & + & + \\ \hline\end{array}$. Then

(10.9.1) $k \in (\beta_2;2^4;2^21^4;2^21^4)$ and $l_1,l_2,l_3 \in (\beta_1;21^6;21^6;2^21^4)$ with l,k,l_1,l_2,l_3 forming the lines of a non-sparse triangle at d.

Since $l \in \beta_1(x,X(d,a))$, $\tau(X(d,a))$ interchanges two pairs of points in $\Gamma_0(l)$ by Lemma 3.3. Also we know that $\Gamma_0(l) \subseteq \Delta_2^3(a)\cup\Delta_3^2(a)$ by Theorems C,D and F. Therefore either

(1) $\left|\Gamma_0(l)\cap\Delta_3^2(a)\right| = 4$; or

(2) $\left|\Gamma_0(l)\cap\Delta_2^3(a)\right| = 5$; or

(3) $\left|\Gamma_0(l)\cap\Delta_2^3(a)\right| = 3$ and $\left|\Gamma_0(l)\cap\Delta_3^2(a)\right| = 2$.

Let $y \in \Gamma_0(k) \cap \Delta_3^1(a)$. Suppose we have case (1). Then $\Gamma_0(y+z) \subseteq \Delta_3^1(a) \cup \Delta_3^2(a)$ for all $z \in \Gamma_0(l) \backslash \{d\}$ and so $y+z \in (\beta_1, \alpha_1) \cup (\beta_3, \alpha_0)$ by Theorem F. Also $y+d \in (\beta_3, \alpha_3)$ by Lemma 10.4(v). Appealing to Lemma 9.1 we must have $y+z \in (\beta_1, \alpha_1)$ for all $z \in \Gamma_0(l) \backslash \{d\}$ and so $\left|\Gamma_0(y+z) \cap \Delta_3^1(a)\right| = 3$ and $\left|\Gamma_0(y+z) \cap \Delta_3^2(a)\right| = 2$ for all $z \in \Gamma_0(l) \backslash \{d\}$. So the set $\{\Gamma_0(y+z) | z \in \Gamma_0(l)\}$ contains exactly 8 points in $\Delta_3^2(a)$, 10 points in $\Delta_3^1(a)$ and 3 points in $\Delta_2^3(a)$. However $\{\Gamma_0(y+z) | z \in \Gamma_0(l)\} = \Gamma_0(l) \cup \Gamma_0(k) \cup \Gamma_0(l_1) \cup \Gamma_0(l_2) \cup \Gamma_0(l_3)$ because $\{y+z) | z \in \Gamma_0(l)\}$ are the lines of a non-sparse triangle at y. However (10.9.1) together with Lemmas 9.2 and 10.8 implies that $\Gamma_0(l) \cup \Gamma_0(k) \cup \Gamma_0(l_1) \cup \Gamma_0(l_2) \cup \Gamma_0(l_3)$ contains exactly 10 points in $\Delta_3^2(a)$, 8 points in $\Delta_3^1(a)$ and 3 points in $\Delta_2^3(a)$. This contradiction excludes case (1).

Next assume case (2) holds. Then $\left|\Gamma_0(y+z) \cap \Delta_2^3(a)\right| = 1$ and $\Gamma_0(y+z) \subseteq \Delta_2^3(a) \cup \Delta_3^1(a) \cup \Delta_3^2(a)$ whence $y+z \in (\beta_2, \alpha_2)$ for all $z \in \Gamma_0(l) \backslash \{d\}$. However $y+x \in (\beta_3, \alpha_3)$ and so we have a contradiction to Lemma 9.1. So case (3) holds and the lemma is proved.

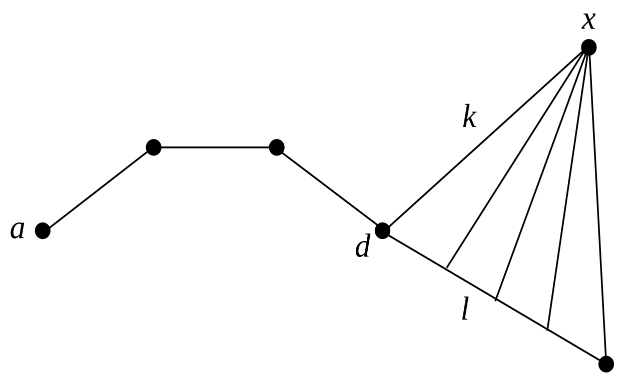
§11 THE LAST LAP

Lemma 11.1 Let $d \in \Delta_3^2(a)$. Then for all $l \in \Gamma_1(d)$, $\Gamma_0(l)$ contains a point in $\Delta_2^3(a) \cup \Delta_3^1(a) \cup \Delta_3^2(a)$.

Proof Let $L = \{l \in \Gamma_1(d) | \Gamma_0(l) \cap (\Delta_2(a) \cup \Delta_3^1(a)) \neq \emptyset\}$. We suppose the result is false and argue for a contradiction. Let $l \in \Gamma_1(d)$ with $\Gamma_0(l) \subseteq \Delta_3^2(a) \cup \Delta_4(a)$ (we already know $\Gamma_0(l) \cap \Delta_1(a) = \Gamma_0(l) \cap \Delta_2^1(a) = \emptyset$). Either there exists $k \in L$ with $k \in \alpha_3(l)$ or we can find $k = k_1, k_2, \ldots, k_r = l \in \Gamma_1(x)$ with $k_i \in \alpha_3(k_{i+1})$ $(i = 1, \ldots, r-1)$ for some $r \geq 2$. In the latter case we can find $j \in \{1, \ldots, r-1\}$ with $k_{j-1} \in L$ and $k_j \notin L$. So in either case, possibly with a new choice of l, we may suppose there exists $k \in L$ with $k \in \alpha_3(l)$.

Since $\Delta_1(y) \cap \Delta_4(a) = \emptyset$ for all $y \in \Delta_2(a) \cup \Delta_3^1(a)$ by Theorems C,D,E and F, we must have $\Gamma_0(l) \cap \Delta_4(a) = \emptyset$. Therefore we have reduced the problem to the case when $\Gamma_0(l) \subseteq \Delta_2^3(a)$.

Let $x \in \Gamma_0(k) \cap (\Delta_2^2(a) \cup \Delta_2^3(a) \cup \Delta_3^1(a))$. So $x \in \Delta_1(y)$ for all $y \in \Gamma_0(l)$ and we have



We consider three cases separately.

Case 1 $x \in \Delta_2^2(a)$.

By Theorem D, $x+y \in (\beta_1\beta_1\beta_1, \alpha_0)$ for each $y \in \Gamma_0(l)$. However $\{x+y | y \in \Gamma_0(l)\}$ form the lines of a non-sparse triangle at x and so we have a contradiction to Lemma 9.1.

Case 2 $x \in \Delta_3^1(a)$.

Let l_x be the unique line in $\Gamma_1(x)$ fixed by G_{ax}^{*x}. By Theorem F $x+y \in (\beta_1, \alpha_0) \cup (\beta_1, \alpha_1) \cup (\beta_3, \alpha_0)$ for each $y \in \Gamma_0(l)$. By definition, if X_x is unique plane in $\Gamma_2(x)$ fixed by G_{ax}^{*x}, then $x+y \in \beta_1(x, X_x) \cup \beta_3(x, X_x)$ for all $y \in \Gamma_0(l)$. Now Lemma 9.1 implies that $x+y \in \beta_1(x, X_x)$ for four $y \in \Gamma_0(l)$ and $x+y \in \beta_3(x, X_x)$ for one $y \in \Gamma_0(l)$. Let $Y \in \Gamma_2(d,x,k,l)$. Then $Y \in \gamma_0(x, X_x)$ by Lemma 9.1 again. Appealing to (2.8) there are three lines in $\beta_3(X_x)$ and one of these lines must lie in $\alpha_1(l_x)$. For instance if X_x is the standard sextet and $Y = \begin{array}{|cc|cc|cc|} \hline \times & \times & \times & \text{o} & \square & \cdot \\ \times & \text{o} & \text{o} & \text{o} & - & * \\ - & \cdot & \square & * & \square & - \\ * & \square & \cdot & - & * & \cdot \\ \hline \end{array}$, then the three lines in $\beta_3(X_x)$ are $m_1 = \begin{array}{|cc|cc|cc|} \hline + & + & + & + & - & - \\ + & + & + & + & \text{o} & \text{o} \\ \text{o} & - & - & \text{o} & - & \text{o} \\ \text{o} & - & - & \text{o} & \text{o} & - \\ \hline \end{array}$, $m_2 = \begin{array}{|cc|cc|cc|} \hline + & + & + & \text{o} & \text{o} & + \\ + & \text{o} & \text{o} & \text{o} & - & - \\ - & + & \text{o} & - & \text{o} & - \\ - & \text{o} & + & - & - & + \\ \hline \end{array}$ and $m_3 = \begin{array}{|cc|cc|cc|} \hline + & + & + & \text{o} & + & \text{o} \\ + & \text{o} & \text{o} & \text{o} & - & - \\ - & \text{o} & + & - & + & - \\ - & + & \text{o} & - & - & \text{o} \\ \hline \end{array}$. It is easy to check that all trios in $\Gamma_1(X_x)$ lie in $\alpha_1(m_i)$ for some $i = 1,2,3$. So there exists $m \in \Gamma_1(d,Y) \cap (\beta_3, \alpha_1)$ and a point $c \in \Gamma_0(m) \cap \Delta_2^2(a)$ (see Lemma 10.5(vi)). Since $m \in \Gamma_1(Y)$, we have $c \in \Delta_1(z)$ for three or five points $z \in \Gamma_0(l)$. By case (1), $c \in \Delta_1(z)$ for exactly three points $z \in \Gamma_0(l)$ and we have a sparse triangle at c. Since $\Gamma_0(l) \subseteq \Delta_3^2(a)$ we must have $c+z \in (\beta_1\beta_1\beta_1, \alpha_0)$ for each $z \in \Gamma_0(l) \cap \Delta_1(c)$. This contradicts the fact that three lines of a sparse triangle at c cannot each lie in $\alpha_0(c, l_c)$ and so case (2) cannot occur.

Case 3 $x \in \Delta_2^3(a)$.

By cases (1) and (2) for every $m \in \Gamma_1(x)$ with $\Gamma_0(m) \cap \Gamma_0(l) \neq \emptyset$, we have $\Gamma_0(m) \cap (\Delta_2^2(a) \cup \Delta_3^1(a)) = \emptyset$. So each of these lines lies in $(\beta_1; 21^6; 21^6; 1^8) \cup (\beta_3; 2^21^4; 2^21^4; 2^21^4)$ by Theorem E. Using Lemma 9.1 we have one line in $\beta_3(x, X(x,a))$ and four lines in $\beta_1(x, X(x,a))$ with $Y \in \gamma_0(x, X(x,a))$. Let t be the unique involution in centralized by G_{ax}^{*x}. Then t defines a partition of Ω_x into 12 pairs of elements. There are 15 octads which can be formed by taking unions of tetrads in Y. Of these 15 octads, 4 cut this partition in 1^8, 8 cut it in 21^6 and 3 hit it in 2^21^4. By (2.8) there are three trios in $\Gamma_1(x,Y) \cap \beta_3(x, X_x)$ and each one contains the same octad formed from tetrads in Y. Exactly one of these trios m has $\Gamma_0(m) \cap \Gamma_0(l) \neq \emptyset$ and the octads of m each cut the partition defined by t in 2^21^4 by the above. Putting all this together, if $m' \in \Gamma_1(x,Y) \cap \beta_3(x, X_x)$ with $m' \neq m$, then exactly one of the octads in m' cuts the partition in 2^21^4 (namely the octad which m and m' have in common). However looking at the possibilities for lines in $\beta_3(x, X_x)$, described in Theorem E we see that there exists no such trio m'.

Therefore we have obtained a contradiction in all three cases and so the lemma is proved.

Theorem 11.2 Let $d \in \Delta_3^2(a)$. Then

(i) $\left|\Delta_3^2(a)\right| = 2^{18}.3^2.5.7$;

(ii) $G_{ad}^{*d} \cong L_2(23)$ and $Q(d)_a = 1$; and
(iii) The G_{ad}-orbits on $\Gamma_1(d)$ and their sizes are as described in Theorem G.

Proof Let $n = \left|\Delta_3^2(a)\right|$. Combining Lemmas 10.6, 10.7, 10.8, 10.9, 11.1 and the definition of $\Delta_3^2(a)$, we have

$$\frac{1}{n}\left\{4.1536\left|\Delta_2^2(a)\right| + \left(\frac{2.960}{3} + 2.1920 + 4.320\right)\left|\Delta_2^3(a)\right| + \left(\frac{2.576}{3} + 4.384\right)\left|\Delta_3^1(a)\right|\right\} = 3795.$$

This gives part (i).

For part (ii) we first show that $Q(d)_a = 1$.

(11.2.1) For all $l \in \Gamma_1(d)$ and $g \in Q(d)$ either g fixes $\Gamma_0(l)$ pointwise or g interchanges two pairs of points in $\Gamma_0(l)$.

Since $\{\tau(X) | X \in \Gamma_2(d)\}$ has 1771 elements it clearly generates $Q(d)$. Now (11.2.1) follows from Lemma 3.3.

By Lemmas 10.5, 10.6, 10.7, 10.8, 11.1 and the definition of $\Delta_3^2(a)$, for every line $l \in \Gamma_1(d)$, $\Gamma_0(l)$ contains either one or three points exactly lying in the same G_a-orbit of points. Now (11.2.1) implies that every element in $Q(d)_a$ fixes $\Gamma_0(l)$ pointwise for all $l \in \Gamma_1(d)$. So $Q(d)_a = 1$ by Lemma 3.2(iv). Therefore by part (i) G_{ad}^{*d} is isomorphic to a subgroup of M_{24} and has order $2^3.3.11.23$. Perusing the maximal subgroups of M_{24} and M_{23} in [$\mathbb{A}$] reveals that the only possibility is $G_{ad}^{*d} \cong L_2(23)$. So we have part (ii).

We now prove part (iii). First we exhibit three G_{ad}-orbits of size 253 and one orbit of size 1518 of $\Gamma_1(d)$. Let $c \in \Delta_2^2(a) \cap \Delta_1(d)$; so by Theorem D $c + d \in (\beta_1\beta_1\beta_1, \alpha_0)$. Applying Lemma 9.2 with the orbit sizes given in Theorem 4.7(vi) and part (i) we have

$$\left|\Delta_2^2(a) \cap \Delta_1(d)\right| = \frac{\left|\Delta_2^2(a)\right| \left|(\beta_1\beta_1\beta_1, \alpha_0)\right|.4}{\left|\Delta_3^2(a)\right|} = \frac{2^7.3.5.7.11.23.1536.4}{2^{18}.3^2.5.7} = 253$$

and $\{l \in \Gamma_1(d) | \Gamma_0(l) \cap \Delta_2^2(a) \neq \emptyset\}$ is a G_{ad}-orbit. Now let $c' \in \Delta_2^3(a) \cap \Delta_1(d)$. First suppose that $c' + d \in (\beta_1; 21^6; 21^6; 1^8)$. Since $c' + d \in \beta_1(c', X(c', a))$, Lemma 3.3 implies that $\tau(X(c', a))$ interchanges the two points in $\Gamma_0(c' + d) \cap \Delta_3^2(a)$. Let O be the G_{ad}-orbit of $\Gamma_1(d)$ containing $d + c'$. Then we may appeal to Lemma 9.3 to show that

$$|O| = \frac{\left|\Delta_2^3(a)\right| \left|(\beta_1; 21^6; 21^6; 1^8)\right|.2}{\left|\Delta_3^2(a)\right|.3} = \frac{2^{11}.3^2.7.11.23.960.2}{2^{18}.3^2.5.7.3} = 253$$

by Theorem 4.8(ii) and part (i). Next suppose that $c' + d \in (\beta_1; 21^6; 21^6; 2^21^4)$. By a similar argument to the previous case, using Lemma 9.3 yields that $d + c'$ lies in a G_{ad}-orbit of $\Gamma_1(d)$ of size 1518. Finally let $e \in \Delta_3^1(a) \cap \Delta_1(d)$ with $e + d \in (\beta_1, \alpha_1)$. So $\tau(X_e)$ interchanges the two points in $\Gamma_0(e + d) \cap \Delta_3^2(a)$ by Lemma 3.3. Therefore we can again use Lemma 9.3, together with the orbit sizes given in Theorem 10.3(i) and part (i) to show that $d + e$ lies in a G_{ad}-orbit of $\Gamma_1(d)$ of size 253.

Consulting the M_{24} page of [$\mathbb{A}$] gives the permutation character of G_d^{*d} on $\Gamma_1(x)$ and classes under restriction to $G_{ax}^{*x} \cong L_2(23)$. A straightforward calculation shows that G_{ax}^{*x} has 6 orbits upon $\Gamma_1(d)$. Similar calculations reveal that

(11.2.2) if $g \in G_{ad}^{*d}$ has order, respectively 4, 6, 12, then g fixes, respectively 7, 3, 1 lines of $\Gamma_1(d)$.

In view of an element of order 12 fixing a unique line, one of the orbits (and only one) must be permutation isomorphic to $L_2(23)$ on a $Dih(24)$ subgroup. Since, in the representation of $L_2(23)$ on an S_4 subgroup, an element of order 4 fixes 3 lines, $L_2(23)$ must have two orbits like this. The only way to produce two more fixed points for an element of order 6 is to have an orbit of $L_2(23)$ on a $Dih(12)$ subgroup. Now the two remaining orbits must not have any fixed lines for elements of order 4 or 6. Hence the 1518 orbit must have stabilizer isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the last orbit of size 1012 must have stabilizer isomorphic to S_3.

Finally we uncover the point distribution for the 506 and 1012 orbits. Let $c \in \Delta_2^3(a) \cap \Delta_1(d)$ with $c + d \in (\beta_3; 2^21^4; 2^21^4; 2^21^4)$. We have $\left|\Delta_3^2(a) \cap \Delta_1(c)\right| = 960.2 + 1920.2 + 320.4$ by Theorem E and so

$$\left|\Delta_2^3(a) \cap \Delta_1(d)\right| = \frac{\left|\Delta_2^3(a)\right|.(960.2 + 1920.2 + 320.4)}{\left|\Delta_3^2(a)\right|} = 2783.$$

Since we have already accounted for $253.3 + 1518 = 2277$ of these points, we have a set of 506 lines in $\Gamma_1(d)$ containing $d + c$ which is therefore a G_{ad}-orbit. If $e \in \Delta_3^1(a) \cap \Delta_1(d)$ with $e + d \in (\beta_3, \alpha_0)$, then $d + e$ must lie in the last remaining orbit of $\Gamma_1(d)$ of size 1012. Hence we have part (iii) and so the theorem is proved.

Since $\mathcal{G}$ is a connected graph, Lemma 11.1 and Theorems F and G imply that

$$\Gamma_0 = \{a\} \cup \Delta_2^1(a) \cup \Delta_2^2(a) \cup \Delta_2^3(a) \cup \Delta_3^1(a) \cup \Delta_3^2(a).$$

This, together with Theorems 3.6(i), 4.3(iv), 4.7(vi), 4.8(ii), 10.3(i) and 11.2(i), then yields Theorem A.

The last item on the agenda is Theorem B.

Proof of Theorem B

By Theorem A(i) $|\Gamma_0| = 11^2.29.31.37.43$ and hence

$$|G| = |G_a|\,|\Gamma_0| = 2^{21}.3^3.5.7.11^3.23.29.31.37.43.$$

Combining Theorems 3.6(ii), 4.3(v), 7.1(v), 7.2(iii) 10.3(ii) and 11.2(ii) yields that for $x, y \in \Gamma_0, x \neq y$, we have $Q(x)_y < Q(x)$. This implies that $N_G(Q(a)) = G_a$, and so $C_G(Q(a)) = Q(a)$. Let $T \in Syl_2 G_a$. Since $Q(a)$ is not an $FF-$module for $G_a/Q(a) \cong M_{24}$ (see [Theorem 1;A]), $Q(a)$ is the unique elementary abelian subgroup of T of order 2^{11}. Thus $N_G(T) \leqslant N_G(Q(a)) = G_a$.

Let N be a non-trivial normal subgroup of G. Suppose that $N \cap G_a = 1$. Then N is soluble and therefore contains a characteristic subgroup N_0 of order a prime or 11^2. Since $Aut N_0$ is either abelian or $GL_2(11)$, this forces $G_a \leqslant C_G(N_0)$, contrary to $C_G(Q(a)) = Q(a)$. Therefore $N \cap G_a \neq 1$ and hence either $N \cap G_a = Q(a)$ or $G_a \leqslant N$. If the former holds, then $N \cap G_a = N_N(N \cap G_a) = C_N(N \cap G_a)$ with $N \cap G_a \in Syl_2 G_a$. By Burnsides' normal p-complement theorem, $N = (N \cap G_a)O_{2'}(N)$. Since $O_{2'}(N) \trianglelefteq G$ and $O_{2'}(N) \cap G_a = 1$, we must have $O_{2'}(N) = 1$ and so $N \leqslant G_a$ which is not possible as $G \leqslant Aut\Gamma$. Therefore $G_a \leqslant N$ which by the Frattini argument, as $N_G(T) \leqslant G_a$, implies that $G = NN_G(T) = NG_a = N$. Thus G must be a simple group.

Let $X \in \Gamma_2(a)$. Then $G_X \leqslant C_G(\tau(X))$ and $\tau(X)$ is central in a Sylow 2-subgroup of G. Assume $M = O_{2'}(C_G(\tau(X))) \neq 1$. By the structure of G_X, $M \cap G_X = 1$. Since M is soluble, it contains a characteristic subgroup M_0 of order a prime or 11^2. Consequently M_0 centralizes $G'_X \cong 2^{1+12}3M_{22}$. Now, by Lemma 3.2(i),(ii), $Q(a) \leqslant G'_X$ and this contradicts $C_G(Q(a)) = Q(a)$. Thus $O_{2'}(C_G(\tau(X))) = 1$ and we may now appeal to [Theorem B;R] to conclude that $G \cong J_4$. Knowing that $G \cong J_4$ it is straightforward to see that Γ is isomorphic to the relevant rank 3 subgeometry of the J_4 maximal 2-local geometry.

This completes the proof of Theorem B.

APPENDIX

$\Delta_2^1(a)$

ORBIT	LOCATION
β_0	Lemma 6.4
β_1	Lemma 6.4
β_2	Lemma 6.4
β_3	Lemma 6.4

$\Delta_2^2(a)$

ORBIT	LOCATION
$(\beta_0\beta_0\beta_0, \{l\})$	Definition 4.1
$(\beta_0\beta_2\beta_2, \alpha_2)$	Lemma 8.1
$(\beta_0\beta_3\beta_3, \alpha_3)$	Lemma 8.3
$(\beta_1\beta_1\beta_1, \alpha_0)$	Definition of $\Delta_3^2(a)$
$(\beta_1\beta_1\beta_2, \alpha_1)$	Lemma 8.4(i)
$(\beta_1\beta_1\beta_3, \alpha_0)$	Lemma 8.7
$(\beta_1\beta_1\beta_3, \alpha_1)$	Lemma 8.4(iii)
$(\beta_2\beta_2\beta_2, \alpha_2)$	Lemma 8.5
$(\beta_2\beta_3\beta_3, \alpha_1)$	Lemma 8.4(ii)
$(\beta_3\beta_3\beta_3, \alpha_3)$	Lemma 8.6

$\Delta_2^3(a)$

ORBIT	LOCATION
$(\beta_0, *)$	Lemma 9.4(i)
$(\beta_0, **)$	Lemma 9.4(ii)
$(\beta_1; 21^6; 21^6; 1^8)$	Lemma 10.9
$(\beta_1; 21^6; 21^6; 2^2 1^4)$	Lemma 10.8(i)
$(\beta_2; 2^4; 1^8; 1^8)$	Lemma 9.6(ii)
$(\beta_2; 2^4; 2^2 1^4; 2^2 1^4)$	Lemma 9.6(i)
$(\beta_3; 1^8; 1^8; 1^8)$	Lemma 10.7(ii)
$(\beta_3; 2^2 1^4; 2^2 1^4; 1^8)$	Lemma 9.4(iii)
$(\beta_3; 2^2 1^4; 2^2 1^4; 2^2 1^4)$	Lemma 10.7(i)

$\Delta_3^1(a)$

ORBIT	LOCATION
$\{l\}$	Lemma 10.5(i)
(β_0, α_2)	Lemma 10.5(ii)
(β_0, α_3)	Lemma 10.5(iii)
(β_1, α_0)	Lemma 10.8(i)
(β_1, α_1)	Lemma 10.8(ii)
(β_2, α_0)	Lemma 10.7(ii)
(β_2, α_2)	Lemma 10.5(iv)
(β_3, α_0)	Lemma 10.6
(β_3, α_1)	Lemma 10.5(vi)
(β_3, α_3)	Lemma 10.5(v)

$\Delta_3^2(a)$

ORBIT	LOCATION
S_4	Theorem 11.2(iii)
$Dih(24)$	Theorem 11.2(iii)
S_4	Theorem 11.2(iii)
$Dih(12)$	Theorem 11.2(iii)
S_3	Theorem 11.2(iii)
$\mathbb{Z}_2 \times \mathbb{Z}_2$	Theorem 11.2(iii)

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