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Abstract

In this paper we determine detailed information on the conjugacy classes and centralizers of semisimple elements in the exceptional Lie-type group $E_8(2)$.

Keywords: exceptional group of Lie-type; semisimple element; centralizer.

1 Introduction

Down the years, considerable effort has been expended examining various properties of families of finite simple groups, or indeed of just one specific finite simple group. These properties range from those encountered in the vast area of representation theory (in the modular and the non-modular case) to various aspects of the substructure of these groups, such as subgroups and conjugacy classes. Often such investigations are a response to a particular program of work or to aid in the solution of certain questions. For example, the work of Deligne and Lusztig [8] on the degrees of semisimple complex irreducible representations of finite groups of Lie type motivated the comprehensive work of Carter [5], [6], and many subsequent papers. While the classification of the finite simple groups was another major impetus to amass an extraordinary amount of information about the structure and properties of simple groups (see [15]). In particular, the sporadic groups (and other small simple groups) have received extensive attention, the resulting data covering information about conjugacy classes, power maps, maximal subgroups and complex character tables being arrayed in the famed Atlas [4].

Families of groups of Lie type have increasingly been scrutinized. Here we are interested in semisimple elements of one such group, namely $E_8(2)$. The early study of semisimple elements of finite groups of Lie type can be traced in the papers of Steinberg [21], Mizuno [19], Carter [5], [6] and Deriziotis [9], [10]. In [19] and [21] their conjugacy classes are the focus of attention, while [5], [6], [9] and [10] are concerned with determining the structure of the centralizers of semisimple elements. This latter endeavour results in a generic description of the centralizers for all groups of Lie type, the twisted variants being dealt with in Deriziotis and Liebeck [12]. This description relies upon viewing the finite group G of Lie type of characteristic p as a subgroup of a certain algebraic group $\widetilde{G}$ defined over the algebraic closure of $GF(q)$, q an appropriate power of p. Let σ be a surjective endomorphism of $\widetilde{G}$ for which $G = \widetilde{G}_\sigma$, the set of σ-stable elements of $\widetilde{G}$. Then for x a semisimple element of G, the connected centralizer $C_{\widetilde{G}}(x)^\circ$ is a σ-stable reductive subgroup of $\widetilde{G}$ of maximal rank [2] and, morover, its connected centralizer $C_G(x)^\circ$ in G is the subgroup of σ-stable elements in $C_{\widetilde{G}}(x)^\circ$. Further $C_{\widetilde{G}}(x)^\circ$ factorizes as $\widetilde{M}\widetilde{S}$ where $\widetilde{M}$ is semisimple, $\widetilde{S}$ is a torus and $\widetilde{M} \cap \widetilde{S}$ is a finite group. Also $|C_{\widetilde{G}}(x)^\circ_\sigma| = |\widetilde{M}_\sigma||\widetilde{S}_\sigma|$. More recent work on semisimple elements and related topics is to be found in Lübeck [18], Liebeck and Seitz [17], Deriziotis and Holt [11], Fleischmann and Janiszczak [13] and Fleischmann, Janiszczak and Knörr [14].

As mentioned earlier, a substantial theory underpins the study of conjugacy classes and centralizers of semisimple elements of Lie type – our aim here, for $E_8(2)$, is to sharpen and give very explicit descriptions of these features. More specifically, we itemize the conjugacy classes, and use the Atlas conventions to name them, along with power maps and explicit structures of their centralizers. However we do not attempt to detail the slave classes. Additionally, in an accompanying electronic file we list representatives for each of the semisimple conjugacy classes. We do this employing the 248-dimensional $GF(2)$–module V for $E_8(2)$ – thus these representatives are 248×248 matrices over $GF(2)$. This data, apart from being of intrinsic interest, is an important part of a current project to determine all the maximal subgroups of $E_8(2)$ where this level of detail is vital. For example, the generic description of the centralizer of x where $x \in 3D$ gives centralizer possibilities $SU_9(2)$ and $3 \times U_9(2)$ (the latter being the actual centralizer). While for $x \in 19A$, the description $19 \times 3.PGU_3(2)$ covers five possible structures. As well as the obvious candidates $19 \times GU_3(2)$ and $19 \times 3 \times PGU_3(2)$, we have $19 \times (3 \times U_3(2))^{\cdot}3$ and two of shape $19 \times SU_3(2)^{\cdot}3$. In fact the centralizer is one of the latter two possibilities. Pinning down this possibility required knowing the number of conjugacy classes of elements of order 57 and 171. We remark that analogous information featured in Ballantyne, Bates and Rowley [1] where the maximal subgroups of $E_7(2)$ were classified. Three electronic files are available for this paper. The first, called `E8Setup.txt` contains the standard Magma commands used to construct the copy of $E_8(2)$ used for the calculations in this paper, along with generators for the Sylow 3-subgroup referred to in Lemma 2.2. The second file, `AllReps.txt` contains, as mentioned above, representatives for each conjugacy class of semisimple elements, while the third file, `Procedures.txt`, gives details of the procedures used throughout the paper.

In this paper we establish the following result.

Theorem 1.1 *The conjugacy classes of semisimple elements of $E_8(2)$, together with the structure of their centralizers, dimension of their fixed spaces on V, power maps and Lübeck numbers, are given in Table 1.*

The Lübeck numbers are to be found in Lübeck [18]. This paper is arranged as follows. In Section 2 we begin by determining lower bounds for the number of $E_8(2)$-conjugacy classes of elements of certain orders. This is achieved using an algorithm in the computer algebra system Magma (see [3]). This is followed in Sections 3 and 4 by the determination of the number and centralizers of $E_8(2)$-classes of elements of prime and composite order respectively. Throughout these sections, the lower bounds obtained in Section 2, together with the information in Lübeck [18], allow for easy determination of the number of classes of most orders. The paper concludes in Section 5 with an exploration into the remaining details given in Table 1, namely the dimensions of the fixed-point spaces of elements of each class acting on the associated Lie algebra, and the power maps.

Conjugacy Class	Lübeck Number	$C_G(x)$	$\lvert C_G(x)\rvert$	$\dim(C_V(x))$	Powers
$1A$	1	$E_8(2)$	$\lvert E_8(2)\rvert$	248	-
$3A$	294	$3 \times E_7(2)$	$2^{63} \cdot 3^{12} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43 \cdot 73 \cdot 127$	134	-
$3B$	376	$3 \times \Omega_{14}^-(2)$	$2^{42} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 43$	92	-
$3C$	147	$3.({}^2E_6(2) \times U_3(2)).3$	$2^{39} \cdot 3^{13} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	86	-
$3D$	258	$3 \times U_9(2)$	$2^{36} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 43$	80	-
$5A$	480	$5 \times \Omega_{12}^-(2)$	$2^{30} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31$	68	-
$5B$	247	$SU_5(4)$	$2^{20} \cdot 3^2 \cdot 5^5 \cdot 13 \cdot 17 \cdot 41$	48	-
$7A$	441	$7 \times E_6(2)$	$2^{36} \cdot 3^6 \cdot 5^2 \cdot 7^4 \cdot 13 \cdot 17 \cdot 31 \cdot 73$	80	-
$7B$	516	$7 \times L_3(2) \times {}^3D_4(2)$	$2^{15} \cdot 3^5 \cdot 7^4 \cdot 13$	38	-
$9A$	560	$9 \times \Omega_{10}^-(2)$	$2^{20} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	48	3C
$9B$	656	$9 \times \mathrm{Sym}(3) \times {}^3D_4(2)$	$2^{13} \cdot 3^7 \cdot 7^2 \cdot 13$	34	3C
$9C$	580	$9 \times \mathrm{Sym}(3) \times U_5(2)$	$2^{11} \cdot 3^8 \cdot 5 \cdot 11$	30	3C
$9D$	366	$9 \times \mathrm{Sym}(3) \times U_3(8)$	$2^{10} \cdot 3^7 \cdot 7 \cdot 19$	28	3C
$11A$	679	$11 \times U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11^2$	28	-
$13A$	712	$13 \times {}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13^2$	32	-
$13B$	709	$13 \times U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13^2$	20	-
$15A$	540	$15 \times \Omega_{10}^+(2)$	$2^{20} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 17 \cdot 31$	48	3B,5A
$15B$	636	$5 \times 3^2 : 2 \times \Omega_8^-(2)$	$2^{13} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 17$	34	3A,5A
$15C$	686	$15 \times U_5(2)$	$2^{10} \cdot 3^6 \cdot 5^2 \cdot 11$	28	3D,5A
$15D$	621	$5 \times GU_3(2) \times L_4(2)$	$2^9 \cdot 3^6 \cdot 5^2 \cdot 7$	26	3C,5A
$15E$	600	$15 \times L_2(4) \times U_4(2)$	$2^8 \cdot 3^6 \cdot 5^3$	24	3B,5A
$15F$	706	$15 \times U_3(4)$	$2^6 \cdot 3^2 \cdot 5^3 \cdot 13$	20	3B,5B
$15G$	695	$15 \times L_2(16)$	$2^4 \cdot 3^2 \cdot 5^2 \cdot 17$	16	3D,5B
$17AB$	738	$17 \times \Omega_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17^2$	32	-
$17CD$	693	$17 \times L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17^2$	16	-
$19A$	823	$19 \times 3^{\cdot}PGU_3(2)$	$2^3 \cdot 3^4 \cdot 19$	14	-

Table 1: Conjugacy classes of semisimple elements of $E_8(2)$.

Conjugacy Class	Lübeck Number	$C_G(x)$	$\lvert C_G(x)\rvert$	$\dim(C_V(x))$	Powers
$21A$	610	$21 \times L_6(2)$	$2^{15} \cdot 3^5 \cdot 5 \cdot 7^3 \cdot 31$	38	3A,7A
$21B$	720	$21 \times {}^3D_4(2)$	$2^{12} \cdot 3^5 \cdot 7^3 \cdot 13$	32	3A,7B
$21C$	728	$21 \times \Omega_8^-(2)$	$2^{12} \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 17$	32	3B,7A
$21D$	469	$7 \times 3.(3^2 : Q_8 \times L_3(4)) : 3$	$2^9 \cdot 3^6 \cdot 5 \cdot 7^2$	26	3C,7A
$21E$	594	$21 \times L_3(2) \times L_2(8)$	$2^6 \cdot 3^4 \cdot 7^3$	20	3A,7B
$21F$	697	$7 \times L_3(2) \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^6 \cdot 3^5 \cdot 7^2$	20	3C,7B
$21G$	760	$21 \times 3 \times L_2(8)$	$2^3 \cdot 3^4 \cdot 7^2$	14	3B,7B
$21H$	826	$21 \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^3 \cdot 3^5 \cdot 7$	14	3D,7B
$31ABC$	672	$31 \times L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31^2$	28	-
$31D$	857	31^2	31^2	8	-
$33AB$	768	$33 \times U_4(2)$	$2^6 \cdot 3^5 \cdot 5 \cdot 11$	20	3D,11A
$33CD$	748	$11 \times \mathrm{Sym}(3) \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^4 \cdot 3^5 \cdot 11$	16	3C,11A
$33E$	811	$33 \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^3 \cdot 3^5 \cdot 11$	14	3A,11A
$33F$	790	$33 \times 3 \times \mathrm{Sym}(3)^2$	$2^2 \cdot 3^4 \cdot 11$	12	3B,11A
$35A$	778	$35 \times U_4(2)$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7$	20	5A,7A
$39A$	762	$13 \times \mathrm{Sym}(3) \times L_2(8)$	$2^4 \cdot 3^4 \cdot 7 \cdot 13$	14	3A,13A
$39B$	820	$13 \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^3 \cdot 3^4 \cdot 13$	14	3C,13A
$39C$	872	195	$3 \cdot 5 \cdot 13$	8	3B,13B
$41AB$	864	205	$5 \cdot 41$	8	-
$43ABC$	837	$129 \times \mathrm{Sym}(3)$	$2 \cdot 3^2 \cdot 43$	10	-
$45A$	773	$45 \times L_4(2)$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7$	20	3C,9A,5A,15D
$45B$	798	$45 \times 3 \times \mathrm{Alt}(5)$	$2^2 \cdot 3^4 \cdot 5^2$	12	3C,9A,5A,15D
$45C$	853	$45 \times 3 \times \mathrm{Sym}(3)$	$2 \cdot 3^4 \cdot 5$	10	3C,9C,5A,15D
$51AB$	783	$51 \times L_4(2)$	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 17$	20	3B,17AB
$51CD$	764	$51 \times \mathrm{Sym}(3) \times \mathrm{Alt}(5)$	$2^3 \cdot 3^3 \cdot 5 \cdot 17$	14	3A,17AB
$51EF$	832	$17 \times GU_3(2)$	$2^3 \cdot 3^4 \cdot 17$	14	3C,17AB

Conjugacy Class	Lübeck Number	$C_G(x)$	$\|C_G(x)\|$	$\dim(C_V(x))$	Powers
$51GH$	870	255	$3\cdot5\cdot17$	8	3D,17CD
$55A$	877	165	$3\cdot5\cdot11$	8	5A,11A
$57AB$	823	$19\times3^{\cdot}PGU_3(2)$	$2^3\cdot3^4\cdot19$	14	3C,19A
$57C$	861	$3\times19\times9$	$3^3\cdot19$	8	3A,19A
$57DE$	863	57×3	$3^2\cdot19$	8	3D,19A
$63ABC$	754	$63\times\mathrm{Sym}(3)\times L_3(2)$	$2^4\cdot3^4\cdot7^2$	16	3C,9B,7B,21F
$63D$	802	$63\times\mathrm{Alt}(5)$	$2^2\cdot3^3\cdot5\cdot7$	12	3C,9A,7A,21D
$63E$	843	$63\times7\times\mathrm{Sym}(3)$	$2\cdot3^3\cdot7^2$	10	3C,9B,7A,21D
$63FGH$	849	$63\times3\times\mathrm{Sym}(3)$	$2\cdot3^4\cdot7$	10	3C,9D,7B,21F
$65ABCD$	800	$65\times\mathrm{Alt}(5)$	$2^2\cdot3\cdot5^2\cdot13$	12	5B,13B
$65EF$	858	13×5^2	$5^2\cdot13$	8	5A,13B
$73ABCD$	814	$73\times L_3(2)$	$2^3\cdot3\cdot7\cdot73$	14	-
$85AB$	804	$85\times\mathrm{Sym}(3)^2$	$2^2\cdot3^2\cdot5\cdot17$	12	5A,17AB
$85CDEF$	870	255	$3\cdot5\cdot17$	8	5B,17CD
$91ABC$	817	$91\times L_3(2)$	$2^3\cdot3\cdot7^2\cdot13$	14	7B,13A
$91D$	865	91×7	$7^2\cdot13$	8	7A,13A
$93ABC$	808	$93\times L_3(2)$	$2^3\cdot3^2\cdot7\cdot31$	14	3A,31ABC
$93DEF$	788	$93\times\mathrm{Alt}(5)$	$2^2\cdot3^2\cdot5\cdot31$	12	3B,31ABC
$99AB$	841	$99\times\mathrm{Sym}(3)$	$2\cdot3^3\cdot11$	10	3C,9C,11A,33CD
$99CD$	867	99×3	$3^3\cdot11$	8	3C,9A,11A,33CD
$105AB$	829	$35\times GU_3(2)$	$2^3\cdot3^4\cdot5\cdot7$	14	3C,5A,7A,15D,21D,35A
$105C$	794	$105\times\mathrm{Sym}(3)^2$	$2^2\cdot3^3\cdot5\cdot7$	12	3B,5A,7A,15A,21C,35A
$105D$	851	$105\times3\times\mathrm{Sym}(3)$	$2\cdot3^3\cdot5\cdot7$	10	3A,5A,7A,15B,21A,35A
$117ABC$	845	$117\times\mathrm{Sym}(3)$	$2\cdot3^3\cdot13$	10	3C,9B,13A,39B
$119AB$	878	357	$3\cdot7\cdot17$	8	7A,17AB
$127ABCDEFGHI$	835	$127\times\mathrm{Sym}(3)$	$2\cdot3\cdot127$	10	-

Conjugacy Class	Lübeck Number	$C_G(x)$	$\lvert C_G(x)\rvert$	$\dim(C_V(x))$	Powers
$129ABCDEF$	837	$129 \times \mathrm{Sym}(3)$	$2 \cdot 3^2 \cdot 43$	10	3D,43ABC
$129GHI$	859	129×3	$3^2 \cdot 43$	8	3A,43ABC
$129JKLMNO$	859	129×3	$3^2 \cdot 43$	8	3B, 43ABC
$151ABCDE$	868	151	151	8	-
$153AB$	879	153	$3^2 \cdot 17$	8	3C,9A,17AB,51EF
$155ABC$	876	465	$3 \cdot 5 \cdot 31$	8	5A,31ABC
$165AB$	877	165	$3 \cdot 5 \cdot 11$	8	3D,5A,11A,15C,33AB,55A
$171ABCDEF$	847	$171 \times \mathrm{Sym}(3)$	$2 \cdot 3^3 \cdot 19$	10	3C,9D,19A,57AB
$195ABCD$	872	195	$3 \cdot 5 \cdot 13$	8	3B,5B,13B,15F,39C,65ABCD
$205ABCDEFGH$	864	205	$5 \cdot 41$	8	5B,41AB
$217ABCDEF$	839	$217 \times \mathrm{Sym}(3)$	$2 \cdot 3 \cdot 7 \cdot 31$	10	7A,31ABC
$219ABCD$	875	219	$3 \cdot 73$	8	3A,73ABCD
$241ABCDEFGHIJ$	866	241	241	8	-
$255ABCD$	855	$255 \times \mathrm{Sym}(3)$	$2 \cdot 3^2 \cdot 5 \cdot 17$	10	3A,5A,15B,17AB,51CD,85AB
$255EF$	860	255×3	$3^2 \cdot 5 \cdot 17$	8	3B,5A,15A,17AB,51AB,85AB
$255GHIJKLMN$	870	255	$3 \cdot 5 \cdot 17$	8	3D,5B,15G,17CD,51GH,85CDEF
$273ABC$	873	273	$3 \cdot 7 \cdot 13$	8	3A,7B,13A,21B,39A,91ABC
$315AB$	871	315	$3^2 \cdot 5 \cdot 7$	8	3C,5A,7A,9A,15D,21D,35A,45A,63D,105AB
$331ABCDEFGHIJK$	869	331	331	8	-
$357ABCD$	878	357	$3 \cdot 7 \cdot 17$	8	3B,7A,17AB,21C,51AB,119AB
$381ABCDEFGHI$	874	381	$3 \cdot 127$	8	3A,127ABCDEFGHI
$465ABCDEF$	876	465	$3 \cdot 5 \cdot 31$	8	3B,5A,15A,31ABC,93DEF,155ABC
$511ABCDEFGH$	862	511	$7 \cdot 73$	8	7A,73ABCD
$651ABCDEF$	880	651	$3 \cdot 7 \cdot 31$	8	3A,7A,21A,31ABC,93ABC,217ABCDEF

2 Preliminary Results

For the remainder of the paper we set $G = E_8(2)$, and we recall that

$$|G| = 2^{120} \cdot 3^{13} \cdot 5^5 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 31^2 \cdot 41 \cdot 43 \cdot 73 \cdot 127 \cdot 151 \cdot 241 \cdot 331.$$

We shall have recourse to some of the subgroup structure of G, the relevant details being contained in the next two results.

Theorem 2.1 *The following groups are isomorphic to a subgroup of G:*

(i) $\Omega^+_{16}(2)$;

(ii) $\mathrm{Sym}(3) \times E_7(2)$;

(iii) $L_3(2) \times E_6(2)$;

(iv) $SU_5(4)$;

(v) $PGU_5(4)$;

(vi) ${}^3D_4(2) \times {}^3D_4(2)$;

(vii) $U_5(2) \times U_5(2)$;

(viii) $L_5(2) \times L_5(2)$;

(ix) $U_3(4) \times U_3(4)$;

(x) $U_3(16)$.

Proof Either consult the ATLAS [4] or Liebeck and Seitz [17]. □

Lemma 2.2 *Let $P \in Syl_3(G)$. Then P has exponent 9.*

Proof An explicit copy of P of order 3^{13} is located in the accompanying file `E8Setup.txt`. A straightforward MAGMA call yields the statement. □

Our quest to uncover the properties of the semisimple elements begins with the determination of the number of conjugacy classes of G for each given element order. By Steinberg [22] (see also Theorem 3.7.6 of [7]), we know there are $2^8 = 256$ conjugacy classes of semisimple elements.

Lemma 2.3 *Lower bounds for the number of conjugacy classes of G of elements of each possible odd order are listed in Table 2.*

Proof By using the procedure `LowerBoundPower`, as given in the accompanying file `Procedures.txt` we obtain the given lower bounds. □

We note that the conjugacy classes itemized in Lemma 2.3 account for 197 of the 256 G-conjugacy classes of semisimple elements.

Element Order, o	Lower Bound, ℓ	Powering Parameter, p
1	1	1
3	4	1
5	2	1
7	2	1
9	4	1
11	1	1
13	2	1
15	7	1
17	4	1
21	8	1
31	4	1
33	6	1
35	1	1
39	3	1
43	3	1
45	3	1
51	8	1
55	1	1
63	8	1
65	6	1
73	4	1
85	6	1
91	4	1
93	6	1
99	4	3
105	4	1
117	3	1
119	2	7
127	9	15
129	15	1
151	5	1
153	2	9
155	3	5
165	2	5
195	4	1
217	6	1
219	4	3
255	14	1
273	3	1
315	2	1
381	9	1
511	8	1

Table 2: Lower bounds on the number of conjugacy classes of semisimple elements of G.

3 Prime Order Elements

In this section we determine the information relating to the semisimple elements of prime order, beginning with the elements of order 3. The centralizers of elements of order 3 are to be found in Table 3 of [20]. These are easily matched with their respective entries in Lübeck's list [18].

G-class	Lübeck Number	Centralizer
3A	294	$3 \times E_7(2)$
3B	376	$3 \times \Omega^-_{14}(2)$
3C	147	$3.({}^2E_6(2) \times U_3(2)).3$
3D	258	$3 \times U_9(2)$

Table 3: The G-classes of elements of order 3.

From Lemma 2.3 we know there are at least two G-conjugacy classes of elements of order 5. By Theorem 2.1 (iv) we have $H_1 \leq G$ with $H_1 \cong SU_5(4)$, and so for $P \in Syl_5(H_1)$, $P \in Syl_5(G)$. Checking the centralizers in H_1 of non-trivial elements of P we see they are all divisible by 5^4. There are only two G-conjugacy classes satisfying this on Lübeck's list, namely numbers 247 and 480. Now, for $x \in Z(H_1)$ we have $SU_5(4) \cong H_1 \leq C_G(x)$. By Theorem 2.1 (i), there is $H_2 \leq G$ with $H_2 \cong \Omega^+_{16}(2)$, and there exist $y \in H_2$ for which $C_{H_2}(y) \cong 5 \times \Omega^-_{12}(2)$. Hence there are two G-conjugacy classes of elements of order 5, and by orders their centralizers are as given in Table 4.

G-class	Lübeck Number	Centralizer
5A	480	$5 \times \Omega^-_{12}(2)$
5B	247	$SU_5(4)$

Table 4: The G-classes of elements of order 5.

By Theorem 2.1 (vi) we have $H_1 \leq G$ with $H_1 \cong {}^3D_4(2) \times {}^3D_4(2)$, whence H_1 contains a Sylow 7-subgroup of G which we see must be elementary abelian of order 7^4. So centralizers of elements of order 7 must be divisible by 7^4, the only possibilities on Lübeck's list being numbers 441 and 576. Therefore, by Lemma 2.3, there are exactly two G-conjugacy classes of elements of order 7. Now we have $H_2 \leq G$ with $H_2 \cong L_3(2) \times E_6(2)$. So inside H_1 and H_2 we see order 7 elements x and y with $7 \times L_3(2) \times {}^3D_4(2) \leq C_G(x)$ and $7 \times E_6(2) \leq C_G(y)$. Hence, by orders and the fact that $7 \times L_3(2) \times {}^3D_4(2)$ is not a subgroup of $7 \times E_6(2)$ (see [16]), the centralizer structure of order 7 elements are given in Table 5.

G-class	Lübeck Number	Centralizer
7A	441	$7 \times E_6(2)$
7B	516	$7 \times L_3(2) \times {}^3D_4(2)$

Table 5: The G-classes of elements of order 7.

Moving onto elements of order 11, we observe using Theorem 2.1 (vii) that $U_5(2) \times U_5(2) \cong H_1 \leq G$ contains a Sylow 11-subgroup of G which is elementary abelian of order 11^2. Since number 679 on Lübeck's list is the only centralizer divisible by 11^2, we conclude that there is only one G-conjugacy class of elements of order 11. Moreover, there is an element x of order 11 in H_1 with $C_{H_1}(x) \cong 11 \times U_5(2)$ whence by orders $C_G(x) \cong 11 \times U_5(2)$.

G-class	Lübeck Number	Centralizer
11A	679	$11 \times U_5(2)$

Table 6: The G-class of elements of order 11.

By Theorem 2.1 (vi) and Lemma 2.3 we have a Sylow 13-subgroup P of G contained in $H_1 \cong {}^3D_4(2) \times {}^3D_4(2)$ and at least two G-conjugacy classes of elements of order 13. Since P is elementary abelian and only number 709 and 712 on Lübeck's list have centralizer order divisible by 13^2, there must be exactly two classes. Additionally we have $x \in H_1$ of order 13 with $C_{H_1}(x) \cong 13 \times {}^3D_4(2)$ and, using Theorem 2.1 (ix), $y \in H_2 \cong U_3(4) \times U_3(4)$ with $C_{H_2}(x) \cong 13 \times U_3(4)$. Consideration of orders yields the centralizers in Table 7.

G-class	Lübeck Number	Centralizer
13A	712	$13 \times {}^3D_4(2)$
13B	709	$13 \times U_3(4)$

Table 7: The G-classes of elements of order 13.

There are at least four G-conjugacy classes of order 17 elements by Lemma 2.3. We have $U_3(16) \cong H_1 \leq G$ by Theorem 2.1 (x) which must contain a Sylow 17-subgroup of order 17^2. A total of four centralizers on Lübeck's list are divisible by 17^2, namely numbers 693 and 738 (each with multiplicity two). We observe there are elements x and y of order 17 with $x \in H_1$ and $y \in H_2 \cong \Omega_{16}^+(2)$ for which $C_{H_1}(x) \cong 17 \times L_2(16)$ and $C_{H_2}(y) \cong 17 \times \Omega_8^-(2)$ whence, by orders, we deduce the information in Table 8.

G-class	Lübeck Number	Centralizer
17AB	738	$17 \times \Omega_8^-(2)$
17CD	693	$17 \times L_2(16)$

Table 8: The G-classes of elements of order 17.

Since 19 divides $|E_7(2)|$ and $E_7(2)$ has just one conjugacy class of elements of order 19 by [1, Table 2], there is a single G-conjugacy class of elements of order 19. The exact centralizer structure will be determined in Section 4.

G-class	Lübeck Number	Centralizer
19A	823	$19 \times 3^{\cdot}PGU_3(2)$

Table 9: The G-class of elements of order 19.

From Lemma 2.3, G has at least four conjugacy classes of elements of order 31. Using Theorem 2.1 ($viii$) gives $H \leq G$ with $H \cong L_5(2) \times L_5(2)$ and so a Sylow 31-subgroup of G is elementary abelian. Also 31^2 and $31 \times L_5(2)$ are centralizers of elements of order 31 in H. Since there are only four centralizers in [18] divisible by 31^2, namely numbers 672 and 857, we conclude there are four conjugacy classes of elements of order 31 with centralizers as given in Table 10.

G-class	Lübeck Number	Centralizer
31ABC	672	$31 \times L_5(2)$
31D	857	31^2

Table 10: The G-classes of elements of order 31.

We next consider elements of order 41, and need to look at centralizers of order 205, of which there are 10 classes by [18]. Now these correspond to centralizers of elements of order 41 or 205. We have $H \leq G$ where $H \cong PGU_5(4)$ by Theorem 2.1 (v) and note that H has 8 conjugacy classes of elements of order 41. Since all G-conjugacy classes of such elements will be of equal size, we see that there are 8, 4, 2 or 1 G-conjugacy classes of elements of order 41. Further, each G-conjugacy class of elements of order 41 is the union of fifth powers of G-conjugacy classes of elements of order 205. Consequently the 10 G-conjugacy classes of elements having centralizer of order 205 consist of two classes of elements of order 41 and eight of elements of order 205. Having accounted for number 247 of [18] as being the centralizer of a $5B$-element, there remain 10 centralizer orders divisible by 41 (and all of order 205). So the centralizers of the two G-classes of element of order 41 are cyclic of order 205.

G-class	Lübeck Number	Centralizer
41AB	864	205

Table 11: The G-classes of elements of order 41.

For order 43 elements in G, we have at least three G-classes by Lemma 2.3. Since G contains $H \cong E_7(2)$ with H having three H-conjugacy classes of elements of order 43, G has precisely three classes. Looking inside $\mathrm{Sym}(3) \times E_7(2)$, a subgroup of G by Theorem 2.1 (ii), and using [18], we see the centralizers have structure given in Table 12.

G-class	Lübeck Number	Centralizer
43ABC	837	$3 \times \mathrm{Sym}(3) \times 43$

Table 12: The G-classes of elements of order 43.

A Sylow 73-subgroup of G may be viewed in $H \cong E_7(2)$ by Theorem 2.1 (ii) whence, by [1] and Lemma 2.3, we deduce there are four G-conjugacy classes. Their centralizers are, by Theorem 2.1 (iii) and [18], to be seen in the subgroup isomorphic to $L_3(2) \times E_6(2)$.

G-class	Lübeck Number	Centralizer
73ABCD	814	$73 \times L_3(2)$

Table 13: The G-classes of elements of order 73.

There are at least nine G-conjugacy classes of elements of order 127 by Lemma 2.3. Since a Sylow 127-subgroup of G is contained in a subgroup $H \cong E_7(2)$, [1] shows there are precisely nine such classes. Using Theorem 2.1 (*ii*), [1] and [18] reveals their centralizers.

G-class	Lübeck Number	Centralizer
127ABCDEFGHI	835	$\mathrm{Sym}(3) \times 127$

Table 14: The G-classes of elements of order 127.

For $p \in \{151, 241, 331\}$ the only centralizers in Lübeck's list divisible by p are the cyclic subgroups of order p. We conclude that the remaining centralizers are those listed in Table 15.

G-class	Lübeck Number	Centralizer
151ABCDE	868	151
241ABCDEFGHIJ	866	241
331ABCDEFGHIJK	869	331

Table 15: The G-classes of elements of orders 151, 241 and 331.

4 Composite Order Elements

We now proceed to realise the bounds given in Table 2 for the number of classes of elements of a given composite order o. Indeed, assume that $o = pm$ for some prime number p such that $(p, m) = 1$. Except for the G-class $5B$, elements of order p have centralizers of the form $p \times H$ for some subgroup H as determined in Section 3. By considering all such elements of order p in G and elements of order m in the respective groups H, we may obtain an upper bound on the number of G-classes of elements of order o. We illustrate this strategy with an example. For the sake of brevity, we do not include the relevant numbers from Lübeck's list in the tables of this section. However, these numbers are given in Table 1.

In Section 3 we saw that there were two classes of elements of order 5 in G, namely $5A$ and $5B$, whose representatives had respective centralizers $5 \times \Omega_{12}^-(2)$ and $SU_5(4)$. By conjugating by a suitable element of G, we may assume that every element of G of order 15 may be written as an element of order 5 multiplied by an element of order 3 in either $\Omega_{12}^-(2)$ or $SU_5(4)$. Since there are five $\Omega_{12}^-(2)$-classes of elements of order 3 and two $SU_5(4)$-classes, we deduce that G contains at most seven classes of elements of order 15. Since this is the lower bound found in Table 2, we conclude that G contains precisely seven classes of elements of order 15. Moreover, since the centralizer of an element of order 15 must be contained within the respective centralizer of an element of order 5, we may fully determine the centralizers of elements of order 15 in G.

A similar approach using centralizers of elements of order 3, 7, 11, 13, 17, 31, 43, 73 and 127, yields the number of G-classes and centralizers of all elements except

Element Order	H-class $h*$	$C_H(h*)$	$C_G(5A \times h*)$	G-class of $5A \times h*$
15	$3A$	$3 \times \Omega_{10}^{+}(2)$	$15 \times \Omega_{10}^{+}(2)$	$15A$
	$3B$	$3 \times \mathrm{Sym}(3) \times \Omega_8^{-}(2)$	$15 \times \mathrm{Sym}(3) \times \Omega_8^{-}(2)$	$15B$
	$3C$	$3 \times U_5(2)$	$15 \times U_5(2)$	$15C$
	$3D$	$GU_3(2) \times L_4(2)$	$5 \times GU_3(2) \times L_4(2)$	$15D$
	$3E$	$3 \times L_2(4) \times U_4(2)$	$15 \times L_2(4) \times U_4(2)$	$15E$
35	$7A$	$7 \times U_4(2)$	$35 \times U_4(2)$	$35A$
45	$9A$	$9 \times L_4(2)$	$45 \times L_4(2)$	$45A$
	$9B$	$9 \times 3 \times \mathrm{Alt}(5)$	$45 \times 3 \times \mathrm{Alt}(5)$	$45B$
	$9C$	$9 \times 3 \times \mathrm{Sym}(3)$	$45 \times 3 \times \mathrm{Sym}(3)$	$45C$
55	$11A$	33	165	$55A$
65	$13AB$	65	5×65	$65EF$
85	$17AB$	$17 \times \mathrm{Sym}(3)^2$	$85 \times \mathrm{Sym}(3)^2$	$85AB$
105	$21AB$	$7 \times GU_3(2)$	$35 \times GU_3(2)$	$105AB$
	$21C$	$21 \times \mathrm{Sym}(3)^2$	$105 \times \mathrm{Sym}(3)^2$	$105C$
	$21D$	$21 \times 3 \times \mathrm{Sym}(3)$	$105 \times 3 \times \mathrm{Sym}(3)$	$105D$
155	$31ABC$	93	465	$155ABC$
165	$33AB$	33	165	$165AB$
255	$51ABCD$	$51 \times \mathrm{Sym}(3)$	$255 \times \mathrm{Sym}(3)$	$255ABCD$
	$51EF$	51×3	255×3	$255EF$
315	$63AB$	63	315	$315AB$

Table 16: The G-classes of elements of order $5m$ for some $(5, m) = 1$ and class $5A$ with $C_G(5A) \cong 5 \times H \cong 5 \times \Omega_{12}^{-}(2)$.

Element Order	G-class $h*$	$C_G(5B \times h*)$	G-class of $5B \times h*$
15	$3A$	$15 \times U_3(4)$	$15F$
	$3B$	$15 \times L_2(16)$	$15G$
65	$13ABCD$	$65 \times \mathrm{Alt}(5)$	$65ABCD$
85	$17ABCD$	255	$85CDEF$
195	$39ABCD$	195	$195ABCD$
205	$41ABCDEFGH$	205	$205ABCDEFGH$
255	$51ABCDEFGH$	255	$255GHIJKLMN$

Table 17: The G-classes of elements of order $5m$ for some $(5, m) = 1$ and class $5B$ with $C_G(5B) \cong SU_5(4)$.

for elements of order 9, 19, 57, 63, 171, 357, 465, 651. Details of the G-classes and centralizers of elements of these orders are given in Tables 16–22.

Element Order	G-class $7*$	$C_G(7*)$, $7*\times H$	H-class $h*$	$C_H(h*)$	$C_G(7*\times h*)$	G-class of $7*\times h*$
21	$7A$	$7\times E_6(2)$	$3A$	$3\times L_6(2)$	$21\times L_6(2)$	$21A$
			$3B$	$3\times \Omega_8^-(2)$	$21\times \Omega_8^-(2)$	$21C$
			$3C$	$3.(3^2:Q_8\times L_3(4)):3$	$7\times 3.(3^2:Q_8\times L_3(4)):3$	$21D$
	$7B$	$7\times L_3(2)\times {}^3D_4(2)$	$3A$	$3\times {}^3D_4(2)$	$21\times {}^3D_4(2)$	$21B$
			$3B$	$3\times L_3(2)\times L_2(8)$	$21\times L_3(2)\times L_2(8)$	$21E$
			$3C$	$L_3(2)\times 3_+^{1+2}:2\mathrm{Alt}4$	$7\times L_3(2)\times 3_+^{1+2}:2\mathrm{Alt}4$	$21F$
			$3D$	$3^2\times L_2(8)$	$21\times 3\times L_2(8)$	$21G$
			$3E$	$3\times 3_+^{1+2}:2\mathrm{Alt}4$	$21\times 3_+^{1+2}:2\mathrm{Alt}4$	$21H$
91	$7A$	$7\times E_6(2)$	$13A$	91	7×91	$91D$
	$7B$	$7\times L_3(2)\times {}^3D_4(2)$	$13ABC$	$13\times L_3(2)$	$91\times L_3(2)$	$91ABC$
119	$7A$	$7\times E_6(2)$	$17AB$	51	357	$119AB$
217	$7A$	$7\times E_6(2)$	$31ABCDEF$	$31\times \mathrm{Sym}(3)$	$217\times \mathrm{Sym}(3)$	$217ABCDEF$
511	$7A$	$7\times E_6(2)$	$73ABCDEFGH$	73	511	$511ABCDEFGH$

Table 18: The G-classes of elements of order $7m$ for some $(7,m)=1$.

Element Order	G-class $11*$	$C_G(11*)$, $11*\times H$	H-class $h*$	$C_H(h*)$	$C_G(11*\times h*)$	G-class of $11*\times h*$
33	$11A$	$11\times U_5(2)$	$3AB$	$3\times U_4(2)$	$33\times U_4(2)$	$33AB$
			$3CD$	$\mathrm{Sym}(3)\times 3_+^{1+2}:2\mathrm{Alt}4$	$11\times \mathrm{Sym}(3)\times 3_+^{1+2}:2\mathrm{Alt}4$	$33CD$
			$3E$	$3\times 3_+^{1+2}:2\mathrm{Alt}4$	$33\times 3_+^{1+2}:2\mathrm{Alt}4$	$33E$
			$3F$	$3^2\times \mathrm{Sym}(3)^2$	$33\times 3\times \mathrm{Sym}(3)^2$	$33F$
99	$11A$	$11\times U_5(2)$	$9AB$	$9\times \mathrm{Sym}(3)$	$99\times \mathrm{Sym}(3)$	$99AB$
			$9CD$	9×3	99×3	$99CD$

Table 19: The G-classes of elements of order $11m$ for some $(11,m)=1$.

Element Order	G-class $13*$	$C_G(13*)$, $13*\times H$	H-class $h*$	$C_H(h*)$	$C_G(13*\times h*)$	G-class of $13*\times h*$
39	$13A$	$13\times {}^3D_4(2)$	$3A$ $3B$	$3\times L_2(8)$ $3_+^{1+2}:2\mathrm{Alt}4$	$39\times L_2(8)$ $13\times 3_+^{1+2}:2\mathrm{Alt}4$	$39A$ $39B$
	$13B$	$13\times U_3(4)$	$3A$	15	195	$39C$
117	$13A$	$13\times {}^3D_4(2)$	$9ABC$	$9\times \mathrm{Sym}(3)$	$117\times \mathrm{Sym}(3)$	$117ABC$
273	$13A$	$13\times {}^3D_4(2)$	$21ABC$	21	273	$273ABC$

Table 20: The G-classes of elements of order $13m$ for some $(13,m)=1$.

Element Order	G-class $17*$	$C_G(17*)$, $17*\times H$	H-class $h*$	$C_H(h*)$	$C_G(17*\times h*)$	G-class of $17*\times h*$
51	$17AB$	$17\times \Omega_8^-(2)$	$3A$ $3B$ $3C$	$3\times L_4(4)$ $3\times \mathrm{Sym}(3)\times \mathrm{Alt}5$ $GU_3(2)$	$51\times L_4(2)$ $51\times \mathrm{Sym}(3)\times \mathrm{Alt}5$ $17\times GU_3(2)$	$51AB$ $51CD$ $51EF$
	$17CD$	$17\times L_2(16)$	$3A$	15	255	$51GH$
153	$17AB$	$17\times \Omega_8^-(2)$	$9A$	9	153	$153AB$

Table 21: The G-classes of elements of order $17m$ for some $(17,m)=1$.

Element Order	G-class $p*$	$C_G(p*)$, $p*\times H$	H-class $h*$	$C_H(h*)$	$C_G(p*\times h*)$	G-class of $p*\times h*$
93	$31ABC$	$31\times L_5(2)$	$3A$ $3B$	$3\times L_3(2)$ $3\times \mathrm{Alt}5$	$93\times L_3(2)$ $93\times \mathrm{Alt}5$	$93ABC$ $93DEF$
129	$43ABC$	$43\times 3\times \mathrm{Sym}(3)$	$3AB$ $3CDE$	$3\times \mathrm{Sym}(3)$ 3^2	$129\times \mathrm{Sym}(3)$ 129×3	$129ABCDEF$ $129GHIJKLMNO$
219	$73ABCD$	$73\times L_3(2)$	$3A$	3	219	$219ABCD$
381	$127ABCDEFGHI$	$127\times \mathrm{Sym}(3)$	$3A$	3	381	$381ABCDEFGHI$

Table 22: The G-classes of elements of order $3p$ for some $p\in\{31,43,73,127\}$.

G-class, $9*$	Lübeck Number	$C_G(9*)$	Maximal subgroup $M \leq G$ satisfying $C_G(9*) \leq M$
$9A$	560	$9 \times \Omega_{10}^-(2)$	$\Omega_{16}^+(2)$
$9B$	656	$9 \times \mathrm{Sym}(3) \times {}^3D_4(2)$	$({}^3D_4(2))^2.6$
$9C$	580	$9 \times \mathrm{Sym}(3) \times U_5(2)$	$(U_5(2))^2.4$
$9D$	366	$9 \times \mathrm{Sym}(3) \times U_3(8)$	$\mathrm{Sym}(3) \times E_7(2)$

Table 23: The G-classes of elements of order 9.

G-class $9*$	$C_G(9*)$, $9* \times H$	H-class $h*$	$C_H(h*)$	$C_G(9* \times h*)$	G-class of $9* \times h*$
$9A$	$9 \times \Omega_{10}^-(2)$	$7A$	$7 \times \mathrm{Alt5}$	$63 \times \mathrm{Alt5}$	$63D$
$9B$	$9 \times \mathrm{Sym}(3) \times {}^3D_4(2)$	$7ABC$	$7 \times \mathrm{Sym}(3) \times L_3(2)$	$63 \times \mathrm{Sym}(3) \times L_3(2)$	$63ABC$
		$7D$	$7^2 \times \mathrm{Sym}(3)$	$63 \times 7 \times \mathrm{Sym}(3)$	$63E$
$9D$	$9 \times \mathrm{Sym}(3) \times U_3(8)$	$7ABC$	$7 \times 3 \times \mathrm{Sym}(3)$	$63 \times 3 \times \mathrm{Sym}(3)$	$63FGH$

Table 24: The G-classes of elements of order 63.

Consider the G-classes having numbers 366, 560, 580 and 656 in [18]. By the work of Section 3, these elements correspond to elements of composite order. Moreover, considering the order of the centralizers of these elements together with the orders of centralizers of elements of prime order, we see that these elements must have order 3^i for some $i > 1$. Hence as the exponent of a Sylow 3-subgroup, S, of G is 9 by Lemma 2.2, we deduce that these classes all correspond to elements of order 9. Moreover, as the centralizer in S of each element of order 9 has order at least 3^4. Their centralizers can be identified by looking within known maximal subgroups of G and are summarized in Table 23.

To determine the number of classes and centralizers of elements of order 63 in G, we consider an element of order 63 in G to be the product of an element of order 9 and an element of order 7 from one of the subgroups $\Omega_{10}^-(2)$, $\mathrm{Sym}(3) \times {}^3D_4(2)$, $\mathrm{Sym}(3) \times U_5(2)$ or $\mathrm{Sym}(3) \times U_3(8)$. Following the approach used at the beginning of this section and the lower bound obtained in Table 2, we conclude that there are eight classes of elements of order 63 in G, with centralizers as given in Table 24.

Since there are no remaining centralizers in Lübeck's list with order divisible by 3^4, we conclude that there are no further classes of elements of order 9, and hence G contains precisely four classes of elements of order 9, with details as given in Table 23.

Next we consider elements of order $19m$ for some $(19, m) = 1$. We know there is a unique G-conjugacy class of elements of order 19. Thus there exists $x \in 19A$ such that

$$x \in C_G(3C) \sim 3.({}^2E_6(2) \times U_3(2)).3,$$

with x lying in the subgroup $3^{\cdot 2}E_6(2)$. Since the centralizer of x in $3^{\cdot 2}E_6(2)$ has structure 3×19 (see [4]), we have that $C_G(x)$ must contain a subgroup of shape $19 \times 3.U_3(2).3$. Comparing orders with [18], we deduce that this is the whole centralizer $C_G(x)$.

Element Order	H-class $h*$	$C_H(h*)$	$C_G(19 * \times h*)$	G-class of $19 * \times h*$
57	$3AB$	$3{\cdot}PGU_3(2)$	$19 \times 3{\cdot}PGU_3(2)$	$57AB$
	$3C$	3×9	57×9	$57C$
	$3DE$	3^2	57×3	$57DE$
171	$9ABCDEF$	$9 \times \mathrm{Sym}(3)$	$171 \times \mathrm{Sym}(3)$	$171ABCDEF$

Table 25: The G-classes of elements of order $19m$ for some $(19, m) = 1$.

By [20], the structure of the group $3.U_3(2).3$ may be refined to $3.PGU_3(2)$. There are five possible extensions of this shape. We know that there are eleven unaccounted centralizers in [18] with orders divisible by 19. This condition eliminates two of the possible extensions.

Of the remaining three possibilities for $3.PGU_3(2)$, one contains exactly two classes of elements of order 9, each with centralizer of order 3^2. This would imply there are exactly two classes of elements of order 171 in G, each with centralizer $3^2.19$. However, since $E_7(2)$ contains elements of order 171, and $C_G(3A) \sim \mathrm{Sym}(3) \times E_7(2)$, we see that there exist elements y of order 171 in G with $|C_G(y)| \geq 2.3^3.19$. Thus this extension of $3.PGU_3(2)$ is ruled out.

Another of the possibilities for $3.PGU_3(2)$ contains exactly three classes of elements of order 3, with centralizer orders $2^3.3^4$ (twice) and 3^3. This would imply that there are exactly three classes of elements of order 57 in G, with centralizer orders $2^3.3^4.19$ (twice) and $3^3.19$. However, since for $g \in 3D$ we have $C_G(g) = \langle g \rangle \times U_9(2)$, which contains elements h of order 57 such that $h^{19} = g$, and the centralizer in $U_9(2)$ of an element of order 19 has order 3.19, we see that there exists z of order 57 in G with $|C_G(z)| = 3^2.19$. Therefore this extension of $3.PGU_3(2)$ is also ruled out.

The correct extension $3.PGU_3(2)$ is therefore determined, and is the unique group of shape $SU_3(2){\cdot}3$ which contains elements of order 3 which lie outside the subgroup $SU_3(2)$. This gives rise to five classes of elements of order 57 in G, and six classes of elements of order 171 in G. The details of the corresponding centralizers are summarized in Table 25.

There are 16 classes remaining in [18]. These correspond to numbers 876 (six classes having centralizers of order 465), 878 (four classes having centralizers of order 357) and 880 (six classes having centralizers of order 651). Since all proper factors of 357, 465 and 651 have been accounted for, we deduce that classes 876, 878 and 880 must correspond to centralizers of elements of order 465, 357 and 651 respectively.

This completes the identification of the orders and centralizers of the G-classes of semisimple elements.

5 Fixed-Point Spaces and Power Maps

We conclude this paper by considering the fixed-point spaces on V and power maps of semisimple elements of G. A description of the methods used to obtain the dimension of fixed-point spaces of elements of each semisimple conjugacy class is given in Subsection 5.1, whilst an exploration of the power maps is detailed in Subsection 5.2.

Combining these details with the structure of centralizers in G, representatives of each semisimple conjugacy class may be obtained. Such representatives are available in the accompanying electronic file `AllReps.txt`.

5.1 Fixed-Point Spaces

Given its status as a conjugacy class invariant, the fixed-point space dimension is a useful asset in our arsenal when investigating maximal subgroups of G. For an element x in G, the dimension of its fixed-point space on V may be determined in MAGMA using the command `Dimension(Eigenspace(x,1))`. In most cases, determining the dimensions of fixed point spaces was a straight forward exercise. Let x and y be elements of orders 155 and 205 respectively in G. By considering the order of centralizers for elements of order 5, we see quickly that x must power to an element of $5A$ whilst y powers into $5B$. By taking appropriate powers we obtain representatives for $5A$ and $5B$ and their respective fixed-point space dimensions. Repeating this exercise for appropriate elements x and y is sufficient to determine the majority of the fixed-point space dimensions.

Some cases prove slightly more problematic. Consider the conjugacy classes for elements of order 33. Utilising the `LowerBoundPower` procedure detailed in the accompanying file `Procedures.txt`, we find representatives for the 6 classes. We find four different fixed-point space dimensions with two repeated twice. These repeated dimensions must belong to the classes $33AB$ and $33CD$ whilst the unique fixed-point space dimensions 12 and 14 are attributed to classes $33E$ and $33F$ in some order. Let x be the representative with dimension 12 and y the representative with dimension 14. By considering their eleventh powers, we find x powers into $3B$ and y into $3A$. The centralizer of a $3A$ element is of the form $3 \times E_7(2)$, and the centralizer of an element of order 11 in $E_7(2)$ has the form $11 \times 3_+^{1+2} : \mathrm{Alt}(4)$, [1]. Hence we deduce that y is contained in $33E$ and thus elements of $33E$ have fixed-point space dimension 14, whilst x is contained in $33F$ and such elements have fixed-point space dimension 12.

5.2 Power Maps

With the fixed-point space dimensions determined, information relating to the power maps for each conjugacy class follows smoothly. The representatives gained in the previous subsection were used to fill in much of the information and the conjugacy classes of powers were in many cases determined by fixed-point space dimension and Lagrange's theorem on the order of the centralizers. Where this was not quite enough to distinguish between classes, the `BrauerCharacter` procedure given in `Procedures.txt` proved a useful tool. However, there are some cases where a more detailed approach is still required.

In the case that $o \in \mathbb{N}$ and there exist distinct classes of elements of order o which power into each other, the same methodology is used to determine these interactions. For the sake of brevity, we relegate these details to the electronic file `AllReps.txt`. However, we briefly mention the cases where a more specific approach is needed.

Consider the primes $p = 241, 331$. In each case, a Sylow p-subgroup, P, of G is cyclic of order p. Moreover, the automorphism group of a cyclic group of order 241 (respectively 331) is cyclic of order 240 (respectively 330) and intersects G in a cyclic subgroup of order 24 (respectively 30). It follows that the fusion of non-trivial elements of P is determined uniquely by these cyclic subgroups, and is given by

$$\begin{array}{lll} 241A^7 = 241B, & 241B^7 = 241C, & 241C^7 = 241D, \\ 241D^7 = 241E, & 241E^7 = 241F, & 241F^7 = 241G, \\ 241G^7 = 241H, & 241H^7 = 241I, & 241I^7 = 241J, \end{array}$$

and

$$\begin{array}{llll} 331A^3 = 331B, & 331B^3 = 331C, & 331C^3 = 331D, & 331D^3 = 331E, \\ 331E^3 = 331F, & 331F^3 = 331G, & 331G^3 = 331H, & 331H^3 = 331I, \\ & 331I^3 = 331J & \text{and} & 331J^3 = 331K. \end{array}$$

The power maps for elements of order 57 and 171 may be explicitly calculated within the centralizer of an element of order 19. We see that $57A^{20} = 57B$, $57D^{20} = 57E$, $171A^{20} = 171B$, $171B^{20} = 171C$, $171C^{20} = 171D$, $171D^{20} = 171E$ and $171E^{20} = 171F$. A similar approach can be used for elements of order 205, 357, 465 and 651. We illustrate this in the case of elements of order 205. Indeed, we know that an element of order 205 powers up into $C_G(x) \cong SU_5(4)$, where $x \in 5B$. Moreover, there are eight G-classes of elements of order 205 and eight classes of $SU_5(4)$-elements of order 41. Thus the fusion of elements of order 205 in G is fully determined by the fusion of elements of order 41 within $SU_5(4)$. Consider an element $w = xy$ where $y \in C_G(x)$ has order 41. Taking successive powers w^{6k} for $k \in \mathbb{N}$ and using the fusion of conjugacy classes within $U_5(4)$, we deduce that

$$\begin{array}{llll} 205A^{11} = 205B, & 205B^{11} = 205C, & 205C^{11} = 205D, & 205D^{11} = 205E, \\ 205E^{11} = 205F, & 205F^{11} = 205G & \text{and} & 205G^{11} = 205H. \end{array}$$

Repeating the above method for elements of order 357, 465 and 651 inside the respective centralizers of $21C$, $15A$ and $21A$ we obtain the power maps $357A^{22} = 357B$, $357B^{22} = 357C$, $357^{22} = 357D$, $465A^{106} = 465B$, $465B^{106} = 465C$, $465C^{106} = 465D$, $465D^{106} = 465E$, $465E^{106} = 465F$, $651A^{22} = 651B$, $651B^{22} = 651C$, $651C^{22} = 651D$, $651D^{22} = 651E$ and $651E^{22} = 651F$.

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