THE MAXIMAL 2-LOCAL GEOMETRY FOR J4, I

Walker, Louise and Rowley, Peter

2006

MIMS EPrint: **2006.81**

ISSN 1749-9097

THE MAXIMAL 2-LOCAL GEOMETRY FOR J_4, I

Peter Rowley Louise Walker

School of Mathematics
University of Manchester
Oxford Road
Manchester M13 9PL
U.K.

1. INTRODUCTION

In [Ja] Janko presented detailed evidence for the existence of a new finite simple group of order $2^{21}.3^3.5.7.11^3.23.29.31.37.43$. Subsequently this group, now called J_4, was constructed (with the extensive use of machine) by Conway, Norton, Parker and Thackray (see [No]), and recently a computer-free construction has been given by Ivanov and Meierfrankenfeld [IvMe].

A more recent development arising out of Buekenhout [Bu], Ronan and Smith [RoSm] and Ronan and Stroth [RoSt] is the study of geometries related to various of the sporadic simple groups. Of course, the motivation for such a programme is to obtain a better understanding of this assorted menagerie of groups and (hopefully) unify their study. In [RoSm] one such geometry, the so-called maximal 2-local geometry for J_4 is described. This paper, together with [RW1] and [RW2], is devoted to an exhaustive and detailed study of the structure of the maximal 2-local geometry for J_4 from a geometric slant. A major focus is the point-line collinearity graph associated with this geometry which has 173,067,389 vertices. We remark that the action of J_4 on this graph yields the smallest faithful permutation representation for J_4 . Also we note that the work in this paper, [RW1] and [RW2] does not rely on any machine calculations. In fact, for our investigations it is not necessary to assume that our group is J_4 - we only need assume certain local geometric data as listed in Hypothesis 1.1 below. Moreover, as will be seen, we shall be studying rank 3 geometries (the maximal 2-local geometry for J_4 has rank 4 plus a "ghost node").

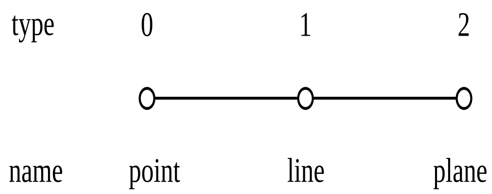
Our group theoretic notation is standard and it, together with additional geometric notation, will be reviewed towards the end of this section. We now introduce our main hypothesis.

Hypothesis 1.1 Γ is a residually connected string geometry with type set $\{0,1,2\}$ and G is a flag transitive subgroup of $Aut\Gamma$ which satisfy the following:-

(i) for $a \in \Gamma_0$, Γ_a is the rank 2 geometry of trios and sextets (defined on the Steiner system $S(5,8,24)$), $G_a/Q(a) \cong M_{24}$ and $Q(a)$ is the 11-dimensional M_{24}-Todd module; and,

(ii) for $X \in \Gamma_2$, Γ_X is the rank 2 geometry of duads and hexads (defined on the Steiner system $S(3,6,22)$), $G_X/Q(X) \cong M_{22} : 2$ and $Q(X) \cong 2^{1+12}3$ with $O_2(G_X) = O_2(Q(X))$ an extraspecial group of order 2^{1+12}.

Assume Hypothesis 1.1 holds. We shall use the following names for the elements of Γ.



Let $a \in \Gamma_0$, $X \in \Gamma_2$ and put $H = G_a/Q(a)(\cong M_{24})$ and $K = G_X/Q(X)(\cong M_{22} : 2)$. We recall that the geometry of trios and sextets consists of 3795 trios and 1771 sextets with, by definition, a trio being incident with a sextet whenever the three octads forming the trio may be obtained from a pairing of the tetrads of the sextet. While the geometry of duads and hexads has 231 duads (that is, 2-element subsets of the 22-element set) and 77 hexads and incidence here is just (set-wise) containment. Now the stabilizer in H of a trio (respectively, a sextet) is isomorphic to $2^6 : (L_3(2) \times S_3)$ (respectively, $2^6 : (3^{\cdot}S_6)$), and the stabilizer in K of a duad (respectively, a hexad) is isomorphic to $2^5 : S_5$ (respectively, $2^4 : S_6$). As a consequence of these observations and the flag transitivity of G we see that, in Γ_a, lines correspond to trios and planes correspond to sextets and, in Γ_X, points correspond to hexads and lines correspond to duads. Further, since $Q(a)$ is isomorphic to the 11-dimensional M_{24}-Todd module, when a, l, X is a maximal flag, $Q(a)|_{G_{aX}} \sim 1\backslash\overline{6}\backslash 4$ and $Q(a)|_{G_{al}} \sim \overline{3}.1\backslash 3.2\backslash 1.2$ (see [MeSt]). Thus ,

$$C_{G_X}(O_2(Q(X))/Z(O_2(Q(X))) \leq Q(X).$$

To prevent proliferation of notation, we adhere to the following convention: when working in Γ_a ($a \in \Gamma_0$) we identify the lines and planes with the trios and sextets of the Steiner system $S(5,8,24)$; and in Γ_X ($X \in \Gamma_2$) the points and lines will be identified with the hexads and duads of the Steiner system $S(3,6,22)$. In analysing the many configurations in Γ_a ($a \in \Gamma_0$) that confront us, extensive use is made of Curtis's MOG [Cu2]. We recommend the reader to have this miraculous calculating device to hand.

Next we discuss the point-line collinearity graph $\mathcal{G}$ of Γ. The vertex set $\mathcal{G}$ is just Γ_0 and two distinct vertices a and b are adjacent in $\mathcal{G}$ if and only if a and b are collinear points in Γ_0 (that is, there exists a line in Γ incident with both a and b). We recall, since Γ is a residually connected string geometry, that $\mathcal{G}$ is a connected graph. For a, a vertex of $\mathcal{G}$, and $i \in \mathbb{N}$, $\Delta_i(a)$ is the set of vertices (points of Γ) distance i in $\mathcal{G}$ from a. Our first theorem gives a numerical summary of our results.

Theorem A Suppose Hypothesis 1.1 holds and let a be a fixed point of $\mathcal{G}$. Then
(i) $|\Gamma| = 173,067,389$;
(ii) $\{a\}, \Delta_1(a), \Delta_2^1(a), \Delta_2^2(a), \Delta_2^3(a), \Delta_3^1(a)$ and $\Delta_3^2(a)$ are the orbits of G_a on Γ_0;
(iii) $\Delta_2(a) = \Delta_2^1(a) \cup \Delta_2^2(a) \cup \Delta_2^3(a)$, $\Delta_3(a) = \Delta_3^1(a) \cup \Delta_3^2(a)$; and
(iv) $|\Delta_1(a)| = 2^2.3.5.11.23$, $|\Delta_2^1(a)| = 2^4.7.11.23$, $|\Delta_2^2(a)| = 2^7.3.5.7.11.23$, $|\Delta_2^3(a)| = 2^{11}.3^2.7.11.23$, $|\Delta_3^1(a)| = 2^{11}.3.5.7.11.23$ and $|\Delta_3^2(a)| = 2^{18}.3^2.5.7$.

Further, the number of edges between the G_a-orbits is given in Figure 1.2 below.

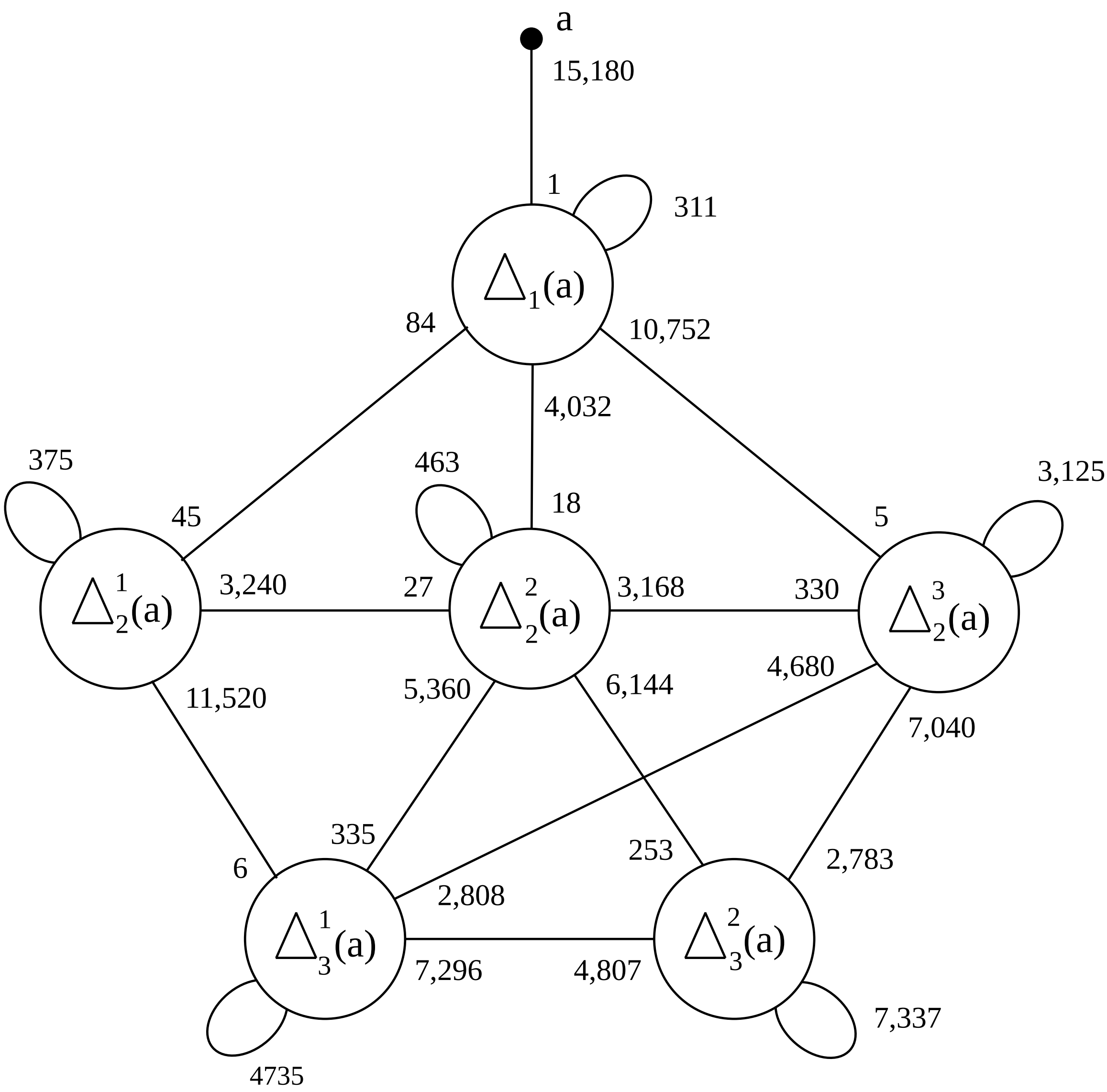

a
15,180
1
311
$\triangle_1(a)$
84
10,752
4,032
375
463
3,125
45
18
5
$\triangle_2^1(a)$
3,240
27
$\triangle_2^2(a)$
3,168
330
$\triangle_2^3(a)$
11,520
5,360
6,144
4,680
7,040
335
253
6
2,783
2,808
$\triangle_3^1(a)$
7,296
4,807
$\triangle_3^2(a)$
7,337
4735

Figure 1.2

A very explicit and detailed description of the edges of $\mathcal{G}$, in terms of the geometry Γ, is given below in Theorems C-G. A geometric definition of the G_a-orbits given in Theorem A appears later in this section.

In [Iv1] Ivanov gives partial results on the number of edges between points of certain G_a-orbits of $\mathcal{G}$ (G and Γ as in Theorem A). See also a more recent monograph [Iv2] which elaborates upon the material in [Iv1]. The graph $\mathcal{G}$ has also been studied by Meierfrankenfeld [Me].

From Theorem A we may readily deduce Theorem B.

Theorem B If Hypothesis 1.1 holds, then $G \cong J_4$ and Γ is isomorphic to a rank 3 subgeometry of the J_4 maximal 2-local geometry.

We note that the main result of [Iv1] follows from Theorem B.

Before presenting Theorems C-G we introduce some notation.

Geometric Notation

Suppose $\Gamma = (\Gamma, \tau, *)$ is a geometry over $\{0, 1, ..., n-1\}$; so Γ is a rank $n-1$ geometry with type map τ and symmetric incidence relation $*$. Let G be a subgroup of $Aut\Gamma$, the automorphism group of Γ. For $i \in \{0, 1, ..., n-1\}$, $x \in \Gamma$ and $\Sigma \subseteq \Gamma$,

$$\Gamma_i = \{y \in \Gamma | \tau(y) = i\} \text{ (the objects of } \Gamma \text{ of type } i\text{) and}$$

$$\Gamma_x = \{y \in \Gamma | x * y\} \text{ (the residue geometry of } x\text{).}$$

We use $\Gamma(\Sigma)$ to denote the set of objects in Γ incident with all objects in Σ and $\Gamma_i(\Sigma)$ denotes the set $\Gamma(\Sigma) \cap \Gamma_i$. If $\Sigma = \{x_1, ..., x_k\}$, then we sometimes write $\Gamma(x_1, ..., x_k)$ and $\Gamma_i(x_1, ..., x_k)$ instead of $\Gamma(\{x_1, ..., x_k\})$ and $\Gamma_i(\{x_1, ..., x_k\})$. By G_Σ or $G_{x_1 \dots x_k}$ we mean the subgroup of G fixing every object in $\Sigma = \{x_1, ..., x_k\}$. For $x \in \Gamma$ we define,

$$Q(x) = \{g \in G_x | g \text{ fixes every object in } \Gamma_x\}.$$

Clearly $Q(x)$ is a normal subgroup of G_x; if $H \leqslant G_x$ we denote $HQ(x)/Q(x)$, the image of H in $G/Q(x)$, by H^{*x}

Now assume that Γ_0 and Γ_1 are the "points" and "lines" of Γ. The point-line collinearity graph $\mathcal{G}(\Gamma)$ of Γ is the graph whose vertex set is Γ_0 with two vertices being adjacent whenever they are collinear points in Γ. We use $d(\ ,\ )$ to denote the usual distance in $\mathcal{G}(\Gamma)$ and so for $x \in \Gamma_0$,

$$\Delta_i(x) = \{y \in \Gamma_0 | d(y,x) = i\}.$$

When Hypothesis 1.1 holds we write $\mathcal{G}$ instead of $\mathcal{G}(\Gamma)$.

For $x,y \in \Gamma_0$ we put $\{x,y\}^\perp = \Delta_1(x) \cap \Delta_1(y)$ and define $Z_1(x) = \{g \in G_x | g$ fixes every point in $\Delta_1(x)\}$.

Suppose two distinct collinear points of Γ are always collinear with a unique line. (This will be the case when Hypothesis 1.1 pertains - see Lemma 3.5.) Let x and y be two collinear points of Γ; the unique line collinear with x and y will often be denoted by $x+y$ (respectively, $y+x$) if we are viewing the line as a line in Γ_x (respectively, Γ_y).

Let a be a fixed point of $\mathcal{G}$. Suppose d is some other point of $\mathcal{G}$ with $d \notin \Delta_1(a)$, and let $l \in \Gamma_1(d)$. The point distribution of the line l is a sequence,

$$i_1\Delta_1 i_2\Delta_2^1 i_3\Delta_2^2 i_4\Delta_2^3 i_5\Delta_3^1 i_6\Delta_3^2$$

which means that $\Gamma_0(l)$ (the points of Γ incident with l) consists of i_1 $\Delta_1(a)$ points, i_2 $\Delta_2^1(a)$ points, i_3 $\Delta_2^2(a)$ points, i_4 $\Delta_2^3(a)$ points, i_5 $\Delta_3^1(a)$points and i_6 $\Delta_3^2(a)$ points. Whenever $i_n = 0$ we omit that particular term from the sequence.

Theorem C For $d \in \Delta_2^1(a)$, G_{ad} has 4 orbits on $\Gamma_1(d)$ with the following point distribution:-

ORBIT	ORBIT SIZE	POINT DISTRIBUTION
β_0	15	$3\Delta_1 2\Delta_2^1$
β_1	2880	$1\Delta_2^1 4\Delta_3^1$
β_2	180	$3\Delta_2^1 2\Delta_2^2$
β_3	720	$1\Delta_2^1 4\Delta_2^2$

Theorem D Let $d \in \Delta_2^2(a)$. Then G_{ad} has 10 orbits on $\Gamma_1(d)$ with the following point distribution:-

ORBIT	ORBIT SIZE	POINT DISTRIBUTION
$(\beta_0\beta_0\beta_0,\{l\})$	1	$3\Delta_2^1 2\Delta_2^2$
$(\beta_0\beta_2\beta_2,\alpha_2)$	18	$1\Delta_1 4\Delta_2^2$
$(\beta_0\beta_3\beta_3,\alpha_3)$	24	$1\Delta_2^1 4\Delta_2^2$
$(\beta_1\beta_1\beta_1,\alpha_0)$	1536	$1\Delta_2^2 4\Delta_3^2$
$(\beta_1\beta_1\beta_2,\alpha_1)$	288	$1\Delta_2^2 2\Delta_2^3 2\Delta_3^1$
$(\beta_1\beta_1\beta_3,\alpha_0)$	1152	$1\Delta_2^2 4\Delta_3^1$
$(\beta_1\beta_1\beta_3,\alpha_1)$	576	$1\Delta_2^2 4\Delta_2^3$
$(\beta_2\beta_2\beta_2,\alpha_2)$	24	$3\Delta_2^2 2\Delta_3^1$
$(\beta_2\beta_3\beta_3,\alpha_1)$	144	$3\Delta_2^2 2\Delta_2^3$
$(\beta_3\beta_3\beta_3,\alpha_3)$	32	$1\Delta_2^2 4\Delta_3^1$

Theorem E For $d \in \Delta_2^3(a)$, G_{ad} has 9 orbits on $\Gamma_1(d)$ with the following point distribution:-

ORBIT	ORBIT SIZE	POINT DISTRIBUTION
$(\beta_0,*)$	5	$1\Delta_1 4\Delta_2^3$
$(\beta_0,**)$	10	$3\Delta_2^2 2\Delta_2^3$
$(\beta_1;21^6;21^6;1^8)$	960	$3\Delta_2^3 2\Delta_3^2$
$(\beta_1;21^6;21^6;2^21^4)$	1920	$1\Delta_2^3 2\Delta_3^1 2\Delta_3^2$
$(\beta_2;2^4;1^8;1^8)$	60	$1\Delta_2^2 2\Delta_2^3 2\Delta_3^1$
$(\beta_2;2^4;2^21^4;2^21^4)$	120	$3\Delta_2^3 2\Delta_3^1$
$(\beta_3;1^8;1^8;1^8)$	160	$2\Delta_2^3 3\Delta_3^1$
$(\beta_3;2^21^4;2^21^4;1^8)$	240	$1\Delta_2^2 4\Delta_2^3$
$(\beta_3;2^21^4;2^21^4;2^21^4)$	320	$1\Delta_2^3 4\Delta_3^2$

Theorem F For $d \in \Delta_3^1(a)$, G_{ad} has 10 orbits on $\Gamma_1(d)$ with the following point distribution:-

ORBIT	ORBIT SIZE	POINT DISTRIBUTION
$\{l\}$	1	$3\Delta_2^2 2\Delta_3^1$
(β_0,α_2)	6	$1\Delta_2^1 4\Delta_3^1$
(β_0,α_3)	8	$1\Delta_2^2 4\Delta_3^1$
(β_1,α_0)	2304	$1\Delta_2^3 2\Delta_3^1 2\Delta_3^2$
(β_1,α_1)	576	$3\Delta_3^1 2\Delta_3^2$
(β_2,α_0)	144	$2\Delta_2^3 3\Delta_3^1$
(β_2,α_2)	36	$1\Delta_2^2 2\Delta_2^3 2\Delta_3^1$
(β_3,α_0)	384	$1\Delta_3^1 4\Delta_3^2$
(β_3,α_1)	288	$1\Delta_2^2 4\Delta_3^1$
(β_3,α_3)	48	$2\Delta_3^1 3\Delta_2^3$

Theorem G For $d \in \Delta_3^2(a)$, G_{ad} has 6 orbits on $\Gamma_1(d)$ with the following point distribution:-

ORBIT	ORBIT SIZE	POINT DISTRIBUTION
S_4	253	$1\Delta_2^2 4\Delta_3^2$
$Dih(24)$	253	$3\Delta_2^3 2\Delta_3^2$
S_4	253	$3\Delta_3^1 2\Delta_3^2$
$Dih(12)$	506	$1\Delta_2^3 4\Delta_3^2$
S_3	1012	$1\Delta_3^1 4\Delta_3^2$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	1518	$1\Delta_2^3 2\Delta_3^1 2\Delta_3^2$

The situations covered in Theorems C-G will be examined later in greater detail, at the appropriate moment. However, we discuss briefly some of the notation appearing in these theorems.

In Theorem C, G_{ad}^{*d} is the stabilizer of a plane (sextet) in $G_d^{*d} \cong M_{24}$ and the orbits of G_{ad} on the lines in $\Gamma_1(d)$ are just the orbits of the stabilizer of a sextet upon the trios in $S(5,8,24)$ (see (2.4) for the definition of $\beta_i(\ ,\ )$). While in Theorem D, G_{ad}^{*d} is the stabilizer of a sextet line in $G_d^{*d} \cong M_{24}$ - a sextet line consists of a certain triple of sextets $\{X_1, X_2, X_3\}$ which together determine a unique trio, l. The orbits of G_{ad} on $\Gamma_1(d)$ are parameterized by $(\beta_i\beta_j\beta_k, \alpha_m)$ meaning that, up to a possible reordering of X_1, X_2, X_3, a given trio is in $\beta_i(d,X_1), \beta_j(d,X_2), \beta_k(d,X_3)$ and $\alpha_m(d,l)$ (see (2.2) for the definition of $\alpha_m(\ ,\ )$). Turning to Theorem E, there we have that G_{ad}^{*d} is the centralizer in $G_d^{*d} (\cong M_{24})$ of a certain involution τ^*. Now τ^* (as a permutation in M_{24}) has cycle type 2^{12} and $C_{G_d^{*d}}(\tau^*)$ is also a subgroup of the stabilizer of a sextet X. So in this case the orbits of G_{ad} on $\Gamma_1(d)$ are parameterized by $(\beta_i;2^{\cdot}1^{\cdot};2^{\cdot}1^{\cdot};2^{\cdot}1^{\cdot})$ to indicate that the trio is in $\beta_i(d,X)$ and that the three octads of the trio cut the partition of 24 points into 12 pairs as given (in some order). The first two orbits listed are not easily described in this scheme, hence the ad hoc $*,**$. In Theorem F, G_{ad}^{*d} turns out to be the stabilizer of a trio l and sextet X where l and X are incident. Thus we have one G_{ad}-orbit on $\Gamma_1(d)$, $\{l\}$, while the remainder are described by (β_i,α_j) meaning a trio is in $\beta_i(d,X)$ and $\alpha_j(d,l)$.

Finally we come to Theorem G. Here we have $G_{ad}^{*d} \cong L_2(23)$. From the point of view of the Steiner system $S(5,8,24)$ the maximal subgroup $L_2(23)$ of M_{24} is "invisible" in the sense that it does not leave invariant any combinatorial configuration related to the Steiner system. Accordingly we are forced to label the orbits for G_{ad} on $\Gamma_1(d)$ in terms of their stabilizer structure (in M_{24}).

Next we define the G_a-orbits mentioned in Theorem A.

Definition 1.4 Let a be a fixed point of $\mathcal{G}$.

(i) $\Delta_2^1(a) = \{c \in \Delta_2(a) | \Gamma_2(a,c) \neq \emptyset\}$.

(ii) $\Delta_2^2(a) = \{c \in \Gamma_0 |$ there exists $b \in \{a,c\}^{\perp}$ such that $b+c \in \alpha_1(b,b+a)\}$.

(iii) $\Delta_2^3(a) = \{c \in \Gamma_0 |$ there exists $b \in \{a,c\}^{\perp}$ such that $b+c \in \alpha_0(b,b+a)\}$.

(iv) $\Delta_3^1(a) = \{d \in \Gamma_0 |$ there exists $c \in \Delta_2^1(a)$ such that $c+d \in \beta_1(c,X(c,a))\}$.

(v) $\Delta_3^2(a) = \{d \in \Gamma_0 |$ there exists $c \in \Delta_2^2(a)$ such that $c+d \in (\beta_1\beta_1\beta_1,\alpha_0)\}$.

We now review the contents of this paper. Section 2 establishes certain basic notation and assembles relevant properties of the residue geometries. The various subsets (of the set of trios) $\alpha_i(a,l)$ and $\beta_i(a,X)$ are continually being scrutinized throughout this work. This section also lays the groundwork, when studying the residue of a plane, for our later study of triangles in $\mathcal{G}$. In Section 3 we start to array several important results that we shall heavily rely upon. First and foremost we mention the involutions $\tau(X)$ and Lemma 3.3. The fact that, from the point of view of Γ, $\mathcal{G}$ has two different types of triangles (sparse and non-sparse triangles) is revealed in Lemmas 3.8 and 3.9. We emphasise the value of Lemma 3.9 which asserts that if a triangle a,b,c in $\mathcal{G}$ when viewed from a is a sparse (non-sparse) triangle, then the same is also true when viewed from b and c. This allows us to move our information around $\mathcal{G}$ in a "crab-like" fashion. Two other results in Section 3 worth mentioning are that a line in Γ is incident with exactly 5 points (Lemma 3.1) and two collinear points in Γ determine a unique line (Lemma 3.5). After analysing $\Delta_1(a)$ (a, a fixed point of $\mathcal{G}$) in Theorem 3.6, we begin disembowelling $\Delta_2(a)$. First, we investigate $\Delta_2^1(a)$ obtaining, in Theorem 4.3, all we need to know about this G_a-orbit without too much effort. On the other hand the analysis of $\Delta_2^2(a)$ is much more demanding but the resulting configurations are very pretty (see Theorem 4.7). We remark that the geometric structures associated with $\Delta_2^2(a)$ are pivotal in our study of $\Delta_3(a)$. We close Section 4 with Theorem 4.8 which establishes some easy properties of $\Delta_2^3(a)$ - we will return later to study this orbit again in [RW2].

In Section 5 we consider the edges between $\Delta_2^1(a)$ and $\Delta_2^2(a) \cup \Delta_2^3(a)$. We particularly mention Theorem 5.2 in which it is shown that there are no edges in $\mathcal{G}$ between $\Delta_2^1(a)$ points and $\Delta_2^3(a)$ points.

We follow the ATLAS [$\mathbb{A}$] conventions for describing groups; our other group theoretic notation is standard as given in, for example, [Go], [Su]. We end this section by showing that Theorems C-G, together with other results, yield the data displayed in Figure 1.2.

By Theorem 3.6(i) $|\Delta_1(a)| = 2^2.3.5.11.23 = 15,180$.

(Using Theorem C),
$\Delta_2^1(a) \longrightarrow \Delta_1(a) : 15 \times 3 = 45$
$\Delta_2^1(a) \longrightarrow \Delta_2^1(a) : 15 + 180 \times 2 = 375$
$\Delta_2^1(a) \longrightarrow \Delta_2^2(a) : 180 \times 2 + 720 \times 4 = 3240$
$\Delta_2^1(a) \longrightarrow \Delta_3^1(a) : 2880 \times 4 = 11,520$

(Using Theorem D)
$\Delta_2^2(a) \longrightarrow \Delta_1(a) : 18$
$\Delta_2^2(a) \longrightarrow \Delta_2^1(a) : 24 + 3 = 27$
$\Delta_2^2(a) \longrightarrow \Delta_2^2(a) : 144 \times 2 + 18 \times 3 + 24 \times 2 + 24 \times 3 + 1 = 463$
$\Delta_2^2(a) \longrightarrow \Delta_2^3(a) : 288 \times 2 + 576 \times 4 + 144 \times 2 = 3,168$
$\Delta_2^2(a) \longrightarrow \Delta_3^1(a) : 288 \times 2 + 24 \times 2 + 32 \times 4 + 1152 \times 4 = 5,360$
$\Delta_2^2(a) \longrightarrow \Delta_3^2(a) : 1536 \times 4 = 6,144$

(Using Theorem E)
$\Delta_2^3(a) \longrightarrow \Delta_1(a) : 5$
$\Delta_2^3(a) \longrightarrow \Delta_2^2(a) : 10 \times 3 + 60 + 240 = 330$
$\Delta_2^3(a) \longrightarrow \Delta_2^3(a) : 5 \times 3 + 10 + 960 \times 2 + 60 + 120 \times 2 + 240 \times 3 + 160 = 3,125$

$\Delta_2^3(a) \longrightarrow \Delta_3^1(a) : 1920 \times 2 + 60 \times 2 + 120 \times 2 + 160 \times 3 = 4,680$
$\Delta_2^3(a) \longrightarrow \Delta_3^2(a) : 960 \times 2 + 1920 \times 2 + 320 \times 4 = 7,040$

(Using Theorem F)
$\Delta_3^1(a) \longrightarrow \Delta_2^1(a) : 6$
$\Delta_3^1(a) \longrightarrow \Delta_2^2(a) : 3 + 8 \times 1 + 36 + 288 = 335$
$\Delta_3^1(a) \longrightarrow \Delta_2^3(a) : 36 \times 2 + 48 \times 3 + 144 \times 2 + 2304 = 2,808$
$\Delta_3^1(a) \longrightarrow \Delta_3^1(a) : 1 + 6 \times 3 + 8 \times 3 + 36 + 48144 \times 2 + 288 \times 3 + 576 \times 2 + 2304 = 4,735$
$\Delta_3^1(a) \longrightarrow \Delta_3^2(a) : 384 \times 4 + 576 \times 2 + 2304 \times 2 = 7,296$

(Using Theorem G)
$\Delta_3^2(a) \longrightarrow \Delta_2^2(a) : 253$
$\Delta_3^2(a) \longrightarrow \Delta_2^3(a) : 253 \times 3 + 506 + 1518 = 2,783$
$\Delta_3^2(a) \longrightarrow \Delta_3^1(a) : 253 \times 3 + 506 + 1518 \times 2 = 4,807$
$\Delta_3^2(a) \longrightarrow \Delta_3^2(a) : 253 \times 3 + 253 + 253 + 506 \times 3 + 1012 \times 3 + 1518 = 7,337$

The edges emanating from $\Delta_1(a)$ can now be calculated using the sizes of $\Delta_2^1(a)$, $\Delta_2^2(a)$ and $\Delta_2^3(a)$ (see Theorems 4.3(iv), 4.7(vi) and 4.8(ii)).

2. PROPERTIES OF THE RESIDUE GEOMETRIES

In our M_{24} related calculations we shall employ Curtis's MOG as described in [Cu2]. The term "standard trio" and "standard sextet" refer, respectively, to the following trio and sextet.

+	+	-	-	○	○
+	+	-	-	○	○
+	+	-	-	○	○
+	+	-	-	○	○

⋆	×	·	□	+	○
⋆	×	·	□	+	○
⋆	×	·	□	+	○
⋆	×	·	□	+	○

For our M_{22} related calculations we will use the 22 element set formed by removing the top row of the left-most heavy brick of the MOG.

Let $a \in \Gamma_0$ be fixed and set $\Lambda = \Gamma_a$. By Hypothesis 1.1 Λ is isomorphic to the geometry of trios and sextets of the Steiner system S(5,8,24) where we regard the trios as being objects of type 1 and the sextets objects of type 2. As is well-known, $M_{24} \cong G_a/Q(a)$ acts flag transitively upon Λ.

(2.1) (i) $|\Lambda_1| = 3795$ and $|\Lambda_2| = 1771$.

(ii) For $l \in \Lambda_1$ and $X \in \Lambda_2$, $|\Lambda_2(l)| = 7$ and $|\Lambda_1(X)| = 15$.

Proof Consult [Cu2].

For distinct trios l and m there are four possible ways in which their octads intersect. They are as below, where the $(i,j)^{th}$ entry of the matrix is the number of elements in the intersection of the i^{th} octad of l with the j^{th} octad of m (assuming an appropriate, fixed, labelling of the octads of l and m).

$$T_0 = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \; T_1 = \begin{pmatrix} 2 & 2 & 4 \\ 2 & 2 & 4 \\ 4 & 4 & 0 \end{pmatrix}$$
$$T_2 = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{pmatrix} \; T_3 = \begin{pmatrix} 4 & 4 & 0 \\ 4 & 0 & 4 \\ 0 & 4 & 4 \end{pmatrix}$$

Definition 2.2 Let l be a fixed trio in Λ_1.

$\alpha_0(a,l) = \{m \in \Lambda_1 | l$ and m have T_0-intersection matrix $\}$
$\alpha_1(a,l) = \{m \in \Lambda_1 | l$ and m have T_1-intersection matrix $\}$
$\alpha_2(a,l) = \{m \in \Lambda_1 | l$ and m have T_2-intersection matrix $\}$
$\alpha_3(a,l) = \{m \in \Lambda_1 | l$ and m have T_3-intersection matrix $\}$

Our notation here is chosen so as the subscript of $\alpha_i(a,l)$ is the number of octads of m intersecting two octads of l each in exactly 4 elements.

(2.3) Let l be a fixed trio in Λ_1.

(i) The G_{al} orbits of Λ_1 are $\{l\}$, $\alpha_0(a,l)$, $\alpha_1(a,l)$,$\alpha_2(a,l)$ and $\alpha_3(a,l)$.

(ii)$|\alpha_0(a,l)| = 2688$, $|\alpha_1(a,l)| = 1008$, $|\alpha_2(a,l)| = 42$ and $|\alpha_3(a,l)| = 56$.

Proof See Section 2, Chapter 1 of [Cu1].

Let l be a trio and X be a sextet. Then l cuts the tetrads of X in one of the following four possible ways.

$$\begin{array}{c} 44|44|44 \\ 2222|311111|311111 \\ 44|2222|2222 \\ 2222|2222|2222. \end{array}$$

Each of the three partitions gives the size of the non-empty intersection of an octad of l with each of the six tetrads of X. So, for example, 2222|311111|311111 means that one of the octads of l cuts four of the tetrads of X in 2 elements and the other two each cut one tetrad of X in 3 elements and the remaining 5 tetrads in exactly one element.

Next we describe certain sets of trios which are of great importance in our subsequent arguments.

Definition 2.4 Let X be a fixed sextet in Λ_2

$\beta_0(a,X) = \{l \in \Lambda_1 | l$ cuts X in 44|44|44$\}$
$\beta_1(a,X) = \{l \in \Lambda_1 | l$ cuts X in 2222|311111|311111$\}$
$\beta_2(a,X) = \{l \in \Lambda_1 | l$ cuts X in 44|2222|2222$\}$
$\beta_3(a,X) = \{l \in \Lambda_1 | l$ cuts X in 2222|2222|2222$\}$.

As an aide de memoir, the subscript of $\beta_i(a,X)$ is the number of octads of l that cut X in 2222.

(2.5) Let X be a fixed sextet in Λ_2

(i) The G_{aX} orbits of Λ_1 are $\beta_0(a,X)$, $\beta_1(a,X)$, $\beta_2(a,X)$ and $\beta_3(a,X)$.

(ii)$|\beta_0(a,X)| = 15$, $|\beta_1(a,X)| = 2880$, $|\beta_2(a,X)| = 180$ and $|\beta_3(a,X)| = 1720$.

Proof See Section 2, Chapter 1 of [Cu1]

Let X and Y be distinct sextets. There are three possibilities for the intersection matrix of X and Y; the $(i,j)^{th}$ entry being the number of elements in the intersection of the i^{th} tetrad of X with the j^{th} tetrad of Y, again assuming an appropriate labelling of the tetrads of X and Y.

$$S_0 = \begin{pmatrix} 2 & & & & 1 & 1 \\ & 2 & & & 1 & 1 \\ & & 2 & & 1 & 1 \\ & & & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & & \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 3 & 1 & & & & \\ 1 & 3 & & & & \\ & & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 2 & 2 & & & & \\ 2 & 2 & & & & \\ & & 2 & 2 & & \\ & & 2 & 2 & & \\ & & & & 2 & 2 \\ & & & & 2 & 2 \end{pmatrix}$$

Definition 2.6 For X a fixed sextet of Λ_2,

$\gamma_0(a,X) = \{Y \in \Lambda_2 | X$ and Y have S_0-intersection matrix$\}$

$\gamma_1(a,X) = \{Y \in \Lambda_2 | X$ and Y have S_1-intersection matrix$\}$

$\gamma_3(a,X) = \{Y \in \Lambda_2 | X$ and Y have S_3-intersection matrix$\}$

Again we draw the reader's attention to the following mnemonic used above: the subscript of $\gamma_i(a,X)$ denotes the number of octads X and Y have in common.

(2.7) Let X be a fixed sextet in Λ_2.
(i) The G_{aX} orbits of Λ_2 are $\{X\}$, $\gamma_0(a,X)$, $\gamma_1(a,X)$ and $\gamma_3(a,X)$.
(ii)$|\gamma_0(a,X)| = 1440$, $|\gamma_1(a,X)| = 240$ and $|\gamma_3(a,X)| = 90$.

Proof See Section 2, Chapter 1 of [Cu1].

(2.8) Let X and Y be sextets.
(i) If $Y \in \gamma_0(a,X)$, then of the 15 trios incident with Y three are in $\beta_3(a,X)$ and twelve in $\beta_1(a,X)$.
(ii) If $Y \in \gamma_1(a,X)$, then of the 15 trios incident with Y three are in $\beta_2(a,X)$ and twelve in $\beta_1(a,X)$.
(iii) If $Y \in \gamma_3(a,X)$, then of the 15 trios incident with Y one is in $\beta_0(a,X)$, six in $\beta_2(a,X)$ and eight in $\beta_3(a,X)$.

Proof A straightforward case-by-case check yields the result.

(2.9) Let l and m be trios with $m \in \alpha_j(a,l)$ ($j \in \{0,1,2,3\}$).
(i) If $j \in \{0,1\}$, then $\Lambda_2(l,m) = \emptyset$.
(ii) If $j \in \{2,3\}$, then $\Lambda_2(l,m)$ contains a unique sextet.

Proof Observe that $\Lambda_2(l,m) \neq \emptyset$ implies that an octad of l intersects the three octads of m in either 8|0|0 or 4|4|0. Hence (i) follows from the definition of $\alpha_0(a,l)$ and $\alpha_1(a,l)$.

Since G_a is transitive on Λ_1 and using (2.3)(i) there is no loss in generality in assuming

$$l = \begin{array}{|cc|cc|cc|}\hline + & + & - & - & \circ & \circ \\ + & + & - & - & \circ & \circ \\ + & + & - & - & \circ & \circ \\ + & + & - & - & \circ & \circ \\ \hline\end{array}$$

$$m = \begin{array}{|cc|cc|cc|}\hline + & + & - & - & \circ & \circ \\ + & + & - & - & \circ & \circ \\ + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ \hline\end{array} \text{ if } m \in \alpha_2(a,l); \text{ and}$$

$$m = \begin{array}{|cc|cc|cc|}\hline + & - & + & \circ & \circ & - \\ + & - & + & \circ & \circ & - \\ - & + & \circ & + & - & \circ \\ - & + & \circ & + & - & \circ \\ \hline\end{array} \text{ if } m \in \alpha_3(a,l).$$

If $m \in \alpha_2(a,l)$ let

$$X = \begin{array}{|cc|cc|cc|}\hline \star & \star & \cdot & \cdot & + & + \\ \star & \star & \cdot & \cdot & + & + \\ \times & \times & \square & \square & \circ & \circ \\ \times & \times & \square & \square & \circ & \circ \\ \hline\end{array}$$

and if $m \in \alpha_3(a,l)$ let

$$X = \begin{array}{|cc|cc|cc|}\hline \star & \times & \cdot & \square & + & \circ \\ \star & \times & \cdot & \square & + & \circ \\ \times & \star & \square & \cdot & \circ & + \\ \times & \star & \square & \cdot & \circ & + \\ \hline\end{array}$$

Then, since a tetrad is contained in a unique sextet, we see that $\Lambda_2(l,m) = \{X\}$.

NOTATION When we have $m \in \alpha_j(a,l)$ with $j \in \{2,3\}$ we shall denote the unique sextet in $\Lambda_2(l,m)$ by $X(a,l,m)$.

(2.10) Let X be a sextet and fix $l \in \Lambda_1(X)$, the set of trios incident with X. Then the orbits of G_{alX} on $\Lambda_1(X)$ are $\{l\}$, $\Lambda_1(X) \cap \alpha_2(a,l)$ and $\Lambda_1(X) \cap \alpha_3(a,l)$ with sizes 1, 6 and 8 respectively.

Proof Without loss of generality we take l to be the standard trio and X the standard sextet. It is an easy matter to verify the sizes of $\Lambda_1(X) \cap \alpha_i(a,l)$ for $i = 2,3$. Recall that G_{alX} contains a subgroup which induces S_3 upon the three octads of l and also a subgroup fixing one octad of l pointwise and acting as a Klein fours group on the remaining 4 columns of X. This readily yields (2.10).

(2.11) Let X be a fixed sextet in Λ_2 and let l be a trio in Λ_1.

(i) If $l \in \beta_0(a,X)$, then $|\alpha_3(a,l) \cap \Lambda_1(X)| = 6$ and $|\alpha_2(a,l) \cap \Lambda_1(X)| = 8$.

(ii) If $l \in \beta_1(a,X)$, then $|\alpha_0(a,l) \cap \Lambda_1(X)| = 12$ and $|\alpha_1(a,l) \cap \Lambda_1(X)| = 3$.

(iii) If $l \in \beta_2(a,X)$, then $|\alpha_1(a,l) \cap \Lambda_1(X)| = 12$ and $|\alpha_2(a,l) \cap \Lambda_1(X)| = 3$.

(iv) If $l \in \beta_3(a,X)$, then $|\alpha_0(a,l) \cap \Lambda_1(X)| = 8$, $|\alpha_1(a,l) \cap \Lambda_1(X)| = 6$ and $|\alpha_3(a,l) \cap \Lambda_1(X)| = 1$.

Proof This follows from the definitions using straightforward counting arguments.

On a few occasions it is convenient to have a labelling of the MOG elements and, as in [Cu2], we use the following:

∞	14	17	11	22	19
0	8	4	13	1	9
3	20	16	7	12	5
15	18	10	2	21	6

For tetrads s and t in the MOG, $s \oplus t$ denotes the symmetric difference of s and t. Sextet lines will play a prominent role in many of our later arguments. We recall that a sextet line is a triple $\{X_1, X_2, X_3\}$ of sextets with the property that if s_i and t_j are tetrads of X_i and X_j such that $s_i \cap t_j \neq \emptyset$ then $s_i \oplus t_j$ is a tetrad of X_k (for $\{i,j,k\} = \{1,2,3\}$).

Next we itemize properties of the plane residue. So let X be a fixed plane of Γ and set $\Lambda = \Gamma_X$. From Hypothesis 1.1, Λ is the geometry of hexads (objects of type 0) and duads (objects of type 1) of the Steiner system S(3,6,22). Here $a \in \Lambda_0$ and $l \in \Lambda_1$ are incident by definition if $l \subseteq a$. Also, by Hypothesis 1.1, we have $G_X/Q(X) \cong M_{22}:2$ and $M_{22}:2$ acts flag transitively on Λ. Note that Λ_0 plays the role of points and Λ_1 that of lines in $\mathcal{G}(\Lambda)$.

Remark Λ is isomorphic to the geometry of hexads and quintets for $M_{22}:2$ as described in [RoSm].

(2.12)(i) $|\Lambda_0| = 77$ and $|\Lambda_1| = 231$.

(ii) For $a \in \Lambda_0$ and $l \in \Lambda_1$, $|\Lambda_1(a)| = 15$ and $|\Lambda_0(l)| = 5$.

(iii) $\mathcal{G}(\Lambda)$ has diameter two and if $a, b \in \Lambda_0$ with $d(a,b) = 2$, then $|\Delta_1(a)| = 60$, $|\Delta_2(a)| = 16$ and $|\{a,b\}^\perp| = 45$ with G_{aX} transitive on $\Delta_1(a)$ and $\Delta_2(a)$.

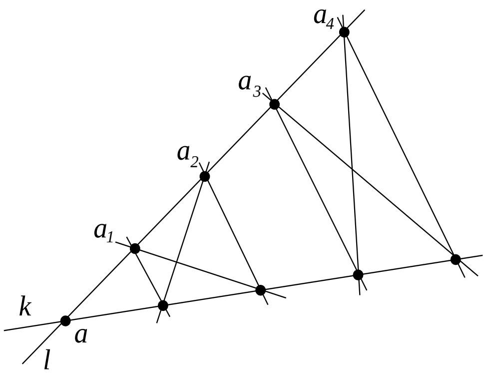
For $a \in \Lambda_0$ and $l \in \Lambda_1(a)$ we define,

$$\delta_0(X,a,l) = \{k \in \Lambda_1(a) \,||k \cap l| = 0\} \text{ and}$$
$$\delta_1(X,a,l) = \{k \in \Lambda_1(a) \,||k \cap l| = 1\}\ .$$

(2.13) Let $a \in \Lambda_0$ and $l \in \Lambda_1(a)$.

(i) The G_{alX}-orbits on $\Lambda_1(a)$ are $\{l\}$, $\delta_0(X,a,l)$ and $\delta_1(X,a,l)$ with $|\delta_0(X,a,l)| = 6$ and $|\delta_1(X,a,l)| = 8$.

(ii) Suppose $k \in \delta_0(X,a,l)$ and let $\Lambda_0(l)\backslash\{a\} = \{a_1,a_2,a_3,a_4\}$. Each a_i is collinear with precisely two points of $\Lambda_0(k)\backslash\{a\}$; moreover, up to relabelling, a_1 and a_2 are collinear with the same pair of points in $\Lambda_0(k)\backslash\{a\}$ and a_3 and a_4 are collinear with the remaining pair of points in $\Lambda_0(k)\backslash\{a\}$. That is

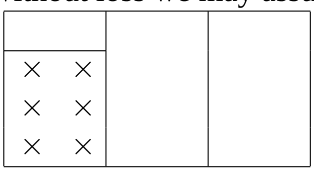
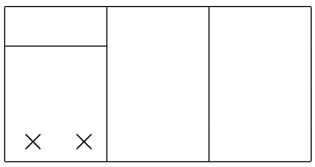


and so k determines a pairing of the points of $\Lambda_0(l)\backslash\{a\}$.

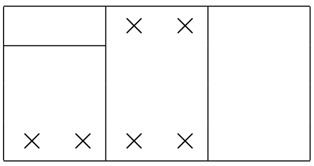
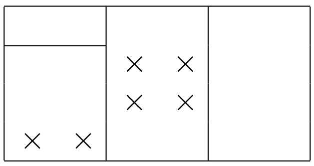
(iii) If $k \in \delta_1(X,a,l)$, then every point in $\Lambda_0(l)\backslash\{a\}$ is collinear with each of the points in $\Lambda_0(k)\backslash\{a\}$.

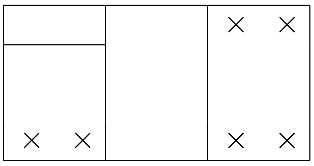
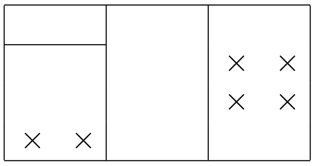
(iv) Let k_i ($1 \leqslant i \leqslant 6$) be the lines in $\delta_0(X,a,l)$. Then k_i and k_j ($i \neq j$) determine the same pairing of $\Lambda_0(l)\backslash\{a\}$ if and only if $k_i \in \delta_0(X,a,k_j)$.

Proof Without loss we may assume,

$a =$  and $l =$ 

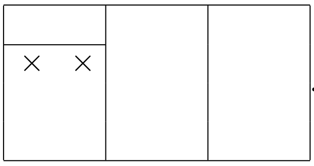
Then $\Lambda_0(l)\backslash\{a\} = \{a_1,a_2,a_3,a_4\}$ where

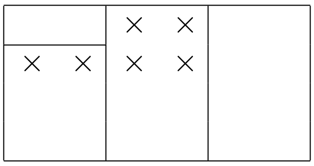
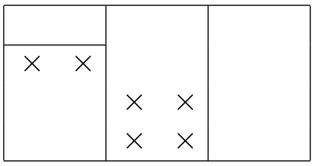
$a_1 =$  , $a_2 =$  ,

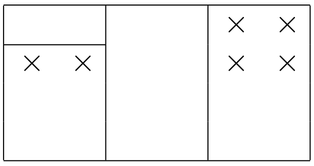
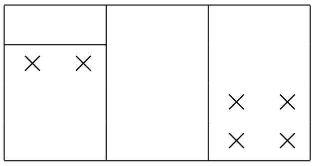
$a_3 =$  and $a_4 =$  .

Since G_{alX} contains a subgroup inducing S_4 upon the set $a\backslash l$ and also contains an element interchanging the two (MOG) points of l, we see that (i) holds.

(ii) In view of part (i) we may suppose,

$k =$  .

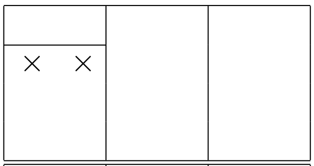
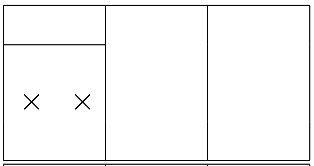
Let $b_1 =$  , $b_2 =$  ,

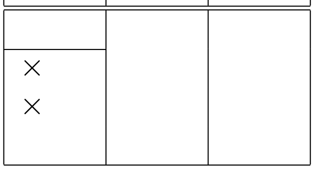
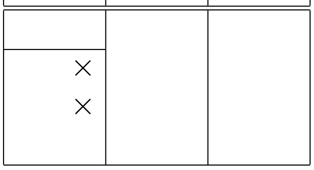
$b_3 =$  and $b_4 =$  .

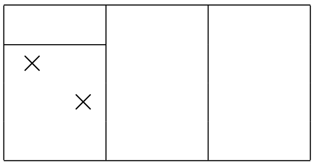
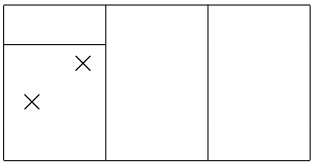
So $\Lambda_0(k)\backslash\{a\} = \{b_1, b_2, b_3, b_4\}$. It is now easy to check that a_1 and a_2 are only collinear with b_1 and b_2 as are a_3 and a_4 with b_3 and b_4.

(iii) Since $k \cap l \neq \emptyset$, a hexad containing l must intersect each hexad containing k in a duad, whence part (iii) follows.

(iv) Here we have ,

$k_1 =$  , $k_2 =$  ,

$k_3 =$  , $k_4 =$  ,

$k_5 =$  and $k_6 =$  .

It may be verified that k_1 and k_2 determine the pairing $\{a_1,a_2\}\{a_3,a_4\}$; k_3 and k_4 determine the pairing $\{a_1,a_4\}\{a_2,a_3\}$; and k_5 and k_6 determine the pairing $\{a_1,a_3\}\{a_2,a_4\}$. Part (iv) now follows immediately.

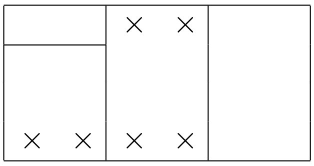
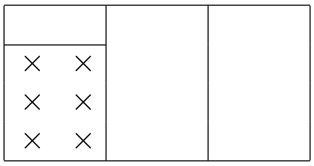
(2.14) Let a and b be collinear points in Λ with $\Lambda_1(a,b)=\{l\}$. Suppose $l_1,l_2,m\in\Lambda_1(b)$ satisfy

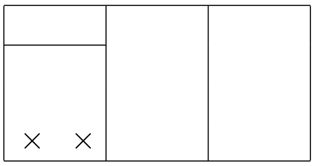
(i) $l_1,l_2\in\delta_1(X,b,l)$, $m\in\delta_0(X,b,l)$; and

(ii) $l_2\in\delta_0(X,b,l_1)$, $m\in\delta_0(X,b,l_i)$ for $i=1,2$.

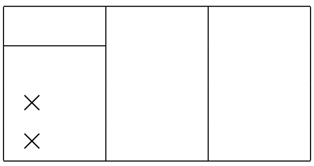
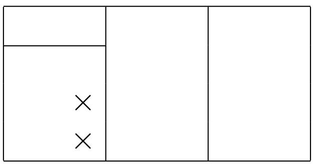
Let $d_1,d_2\in\Lambda_0(l_1)$ and $e_1,e_2\in\Lambda_0(l_2)$ be such that $\{d_1,d_2,e_1,e_2\}$ are pairwise collinear (see (2.13)(ii)). Then there is a unique point x in $\Lambda_0(m)$ which is collinear with a and (up to a relabelling of e_1 and e_2) $\Lambda_1(d_1,e_1,x)\neq\emptyset\neq\Lambda_1(d_2,e_2,x)$.

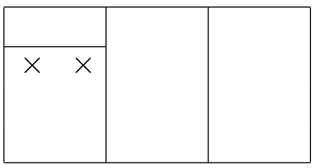
Proof Since G_{Xal} is transitive on $\Lambda_0(l)\backslash\{a\}$, we may suppose that

$a =$  , $b =$ 

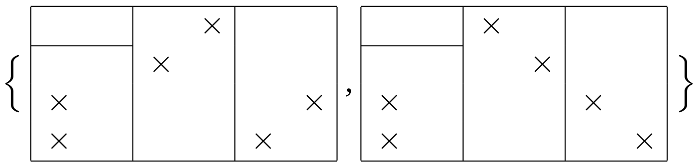
and $l =$  .

Given the conditions on l_1,l_2 and m we may further assume that

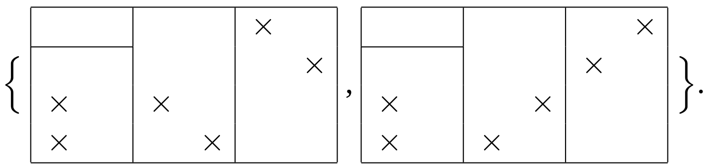
$l_1 =$  , $l_2 =$ 

and $m =$  .

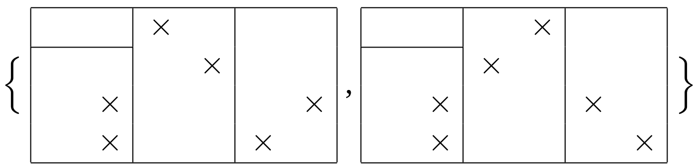
By (2.13)(ii) we must have $\{d_1, d_2\}$ equal to

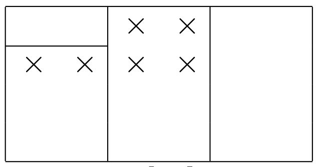
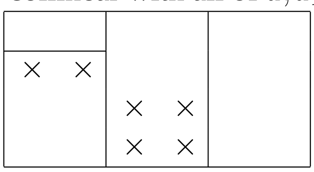


or



If the former holds, then $\{e_1, e_2\}$ must be



in which case it can be checked that  is the unique point of $\Lambda_0(m)$ collinear with all of a, d_1, d_2, e_1, e_2. In the latter case the unique point of $\Lambda_0(m)$ is  .

(2.15) Let $\{a,b,c\}$ be a triangle in $\mathcal{G}(\Lambda)$ with $k \in \Lambda_1(a,b)$, $l \in \Lambda_1(b,c)$ and $m \in \Lambda_1(a,c)$, and let $i \in \{0,1\}$. If $k \in \delta_i(X,a,m)$, then $l \in \delta_i(X,b,k)$.

Proof Suppose $k \in \delta_0(X,a,m)$. Then by (2.13)(ii) c is collinear with exactly 3 points of k, two of which are a and b. Hence $l \in \delta_0(X,b,k)$ by (2.13). A similar argument shows that $k \in \delta_1(X,a,m)$ implies that $l \in \delta_1(X,b,k)$.

Remark From (2.15) it follows that for a triangle a,b,c in $\mathcal{G}(\Lambda)$ (and using the notation in (2.15)) the size of the intersections $k \cap l$, $k \cap m$ and $l \cap m$ are the same.

(2.16) Let $a,c \in \Lambda_0$ with $d(a,c) = 2$. If $l \in \Lambda_1(a)$, then $|\Lambda_0(l) \cap \{a,c\}^{\perp}| = 3$.

Proof As hexads in Λ, a and c are disjoint. Since l is a duad incident with a, l is disjoint from c. Therefore there are exactly three hexads incident with l which have a non-empty intersection with c. These three hexads correspond to the three points of $\Lambda_0(l) \cap \{a,c\}^{\perp}$.

3. $\tau(X)$ AND TRIANGLES IN $\mathcal{G}$

From Γ being a string geometry and (2.12)(ii) we have

Lemma 3.1 For $l \in \Gamma_1$, $|\Gamma_0(l)| = 5$.

For $X \in \Gamma_2$, we have that $|Z(O_2(Q(X)))| = 2$ by Hypothesis 1.1(ii). We put $Z(O_2(Q(X))) = <\tau(X)>$. So $\tau(X) \in Z(G_X)$. Also note that for $a \in \Gamma_0(X)$, $\tau(X) \in Q(a)$. This follows from $Q(a), Q(X) \leqslant G_{aX}$ and $C_{G_a}(Q(a)) = Q(a)$.

Lemma 3.2 If $a \in \Gamma_0$ and $Y \in \Gamma_2(a)$, then the following hold.

(i) $G_{aY}^{*Y} \cong 2^4 : S_6$.

(ii) $Q(a)^{*Y} = O_2(G_{aY}^{*Y}) \cong 2^4$.

(iii) Let $1 \neq g \in O_2(G_{aY}^{*Y})$. Then for three of the lines l in $\Gamma_1(a,Y)$, g fixes each point in $\Gamma_0(l)$ and for the other twelve lines l in $\Gamma_1(a,Y)$ g acts regularly on $\Gamma_0(l) \backslash \{a\}$.

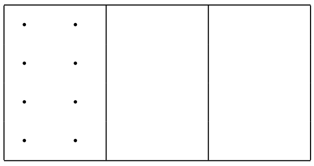
(iv) $Z_1(a) = 1$.

Proof Because G_{aY}^{*Y} is the stabilizer in $M_{22} : 2$ of a hexad, we have part (i).

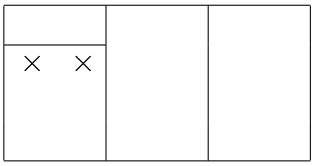
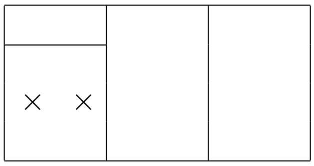
Since $Q(a)$ is elementary abelian and $O_2(Q(Y))$ is an extraspecial 2-group of order 2^{1+12}, $|Q(a) \cap O_2(Q(Y))| \leqslant 2^7$. Now $Q(a) \cong 2^{11}$, $Q(a) \leqslant O_2(G_{aY})$ and part (i) yield (ii).

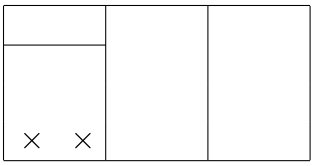
In Γ_Y we may, without loss of generality, choose $a = \begin{array}{|cc|cc|cc|} \hline & & & & & \\ \cline{1-2} \times & \times & & & & \\ \times & \times & & & & \\ \times & \times & & & & \\ \hline \end{array}$.

Since the involutions in $O_2(G_{aY}^{*Y})$ are all conjugate in G_{aY}^{*Y}, there is no loss in further supposing that,

$g =$  .

By inspecting each of the 15 lines in Γ_Y incident with a we see that g fixes all the points in $\Gamma_0(l)$ for

$l =$  , 

and  .

While for the remaining twelve lines l, g acts regularly on $\Gamma_0(l)\backslash\{a\}$, and this proves (iii).

If (iv) is false, then $Q(a)$ being a G_a-chief factor gives $Z_1(a) = Q(a)$ (note that $Z_1(a) \leqslant Q(a)$ by Hypothesis 1.1(i)). Therefore, by (ii), $O_2(G_{aY}^{*Y}) = Z_1(a)^{*Y}$ fixes all the points in $\Gamma_0(l)$ for all $l \in \Gamma_1(a,Y)$, contradicting part (iii).

The next result describes the action of $\tau(X)$ in a point residue and is a vital ingredient of many of our later arguments.

Lemma 3.3 Let $a \in \Gamma_0$, $l \in \Gamma_1(a)$ and $X \in \Gamma_2(a)$. Then $x^{\tau(X)} \neq x$ for each $x \in \Gamma_0(l)\backslash\{a\}$ if and only if $l \in \beta_1(a,X)$.

Proof From Lemma 3.2(iv) there exists $l \in \Gamma_1(a)$ such that $x^{\tau(X)} \neq x$ for some $x \in \Gamma_0(l)\backslash\{a\}$. Let $Y \in \Gamma_2(a,l)$ and put $\tau_Y = \tau(X)^{*Y}$. Then $\tau_Y \in Q(a)^{*Y} = O_2(G_{aY}^{*Y})$ by Lemma 3.2(ii). Noting that $\tau_Y \neq 1$ (as $x^{\tau_Y} \neq x$ and $x \in \Gamma_Y$) using Lemma 3.2(iii) we infer that $\tau(X)$ acts regularly on $\Gamma_0(l)\backslash\{a\}$. Now suppose that $l \in \beta_i(a,X)$ with $i \neq 1$. Since $\tau(X) \in Z(G_{aX})$, (2.5)(i) implies that $\tau(X)$ acts regularly on $\Gamma_0(k)\backslash\{a\}$ for all $k \in \beta_i(a,X)$. Let $Z \in \Gamma_2(a)$ be such that $Z \in \gamma_3(a,X)$. Again by Lemma 3.2(ii) $\tau_Z = \tau(X)^{*Z} \in Q(a)^{*Z} = O_2(G_{aZ}^{*Z})$. Consulting (2.8) we see that of the 15 lines in $\Gamma_1(a,Z)$, 1 is in $\beta_0(a,X)$, 6 are in $\beta_2(a,X)$ and 8 in $\beta_3(a,X)$. So we have $\tau_Z \neq 1$. Moreover τ_Z acts regularly on $\Gamma_0(k)\backslash\{a\}$ for either 1,7,9,14 or 15 of the lines k in $\Gamma_1(a,Z)$. This contradicts Lemma 3.2(iii) and thus we conclude that $\tau(X)$ acts regularly on $\Gamma_0(l)\backslash\{a\}$ if and only if $l\in\beta_1(a,X)$.

One consequence of Lemma 3.3 is that for $X_1,X_2\in\Gamma_2(a)$ with $a\in\Gamma_0$, $\tau(X_1)=\tau(X_2)$ if and only if $X_1=X_2$. A further consequence of Lemma 3.3 is given in

Lemma 3.4. Let $a\in\Gamma_0$ and $X,Z\in\Gamma_2(a)$ with $X\neq Z$. Then $\tau(Z)\in Q(X)$ if and only if $Z\in\gamma_3(a,X)$.

Proof By (2.8)(i),(ii) if $Z\notin\gamma_3(a,X)$, then there exists $l\in\Gamma_1(a,X)$ such that $l\in\beta_1(a,Z)$. Therefore $\tau(Z)$ acts regularly on $\Gamma_0(l)\backslash\{a\}$ by Lemma 3.3 and $\tau(Z)\notin Q(X)$. Conversely, if $Z\in\gamma_3(a,X)$, then $l\notin\beta_1(a,Z)$ for every $l\in\Gamma_1(a,X)$ by (2.8)(iii). Hence, using Lemma 3.3, $\tau(Z)$ fixes $\Gamma_0(l)$ point-wise for each $l\in\Gamma_1(a,X)$. Then Lemma 3.2(iii) forces $\tau(Z)\in Q(X)$, so proving the lemma.

Lemma 3.5 Let $l,k\in\Gamma_1$. If $|\Gamma_0(l)\cap\Gamma_0(k)|\geqslant 2$, then $l=k$.

Proof Suppose $l\neq k$, and let $\{a,b\}\subseteq\Gamma_0(l)\cap\Gamma_0(k)$ with $a\neq b$. If there exists $Y\in\Gamma_0(l,k)$, then we also get $a,b\in\Gamma_Y$ since Γ is a string geometry. But two points in Γ_Y are incident with a unique line. Therefore $\Gamma_2(l,k)=\emptyset$. Hence, by (2.9), $k\in\alpha_0(a,l)\cup\alpha_1(a,l)$. Taking l to be the standard trio and, because of (2.3)(i), we may also take $k=\begin{array}{|cc|cc|cc|}\hline - & + & + & + & + & + \\ + & - & \circ & \circ & - & - \\ + & \circ & \circ & - & - & \circ \\ + & \circ & - & \circ & - & \circ \\ \hline\end{array}$ (if $k\in\alpha_0(a,l)$) and $k=$ $\begin{array}{|cc|cc|cc|}\hline - & + & + & + & + & + \\ + & - & - & - & - & - \\ + & - & \circ & \circ & \circ & \circ \\ + & - & \circ & \circ & \circ & \circ \\ \hline\end{array}$ (if $k\in\alpha_1(a,l)$). Letting $X\in\Gamma_2(a)$ be the standard sextet we have that (in either case) $k\in\beta_1(a,X)$ and $l\in\beta_0(a,X)$. Now applying Lemma 3.3 to both l and k gives the impossible $b^{\tau(X)}\neq b=b^{\tau(X)}$. Thus we infer that $l=k$.

Let a and b be distinct collinear points in Γ_0. Then by Lemma 3.5 $\Gamma_1(a,b)=\{l\}$ for some $l\in\Gamma_1$. We shall frequently denote l by $a+b$ or $b+a$; $a+b$ indicates that we are viewing l as being in the residue Γ_a while $b+a$ that we are viewing l in Γ_b. We next introduce the following set of involutions in G

$$T(a+b)=\{\tau(X)|X\in\Gamma_2(a+b)\}.$$

Observe that $T(a+b) = T(b+a)$ and that, by (2.1)(ii), $|T(a+b)| = 7$.

For the remainder of this paper a is a fixed point of $\mathcal{G}$.

Theorem 3.6

(i) $|\Delta_1(a)| = 2^2.3.5.11.23 = 15,180$ and $\Delta_1(a)$ is a G_a-orbit.

(ii) For $b \in \Delta_1(a)$, $G_{ab}^{*a} \cong 2^6 : (S_3 \times L_2(7))$ and $Q(a)_b \cong 2^9$.

(iii) For $b \in \Delta_1(a)$, $< T(a+b) >\cong 2^3$; in particular $T(a+b)$ consists of the non-trivial elements of $< T(a+b) >$.

Proof Together (2.1)(i) and Lemma 3.1 give $|\Delta_1(a)|$. From Lemma 3.5 $G_{ab} \leqslant G_{al}$. Selecting an $X \in \Gamma_2(a)$ such that $a+b \in \beta_1(a,X)$ and employing Lemma 3.3 we see that $Q(a)_b \neq Q(a)$ and, by Lemma 3.1, $[G_{al} : G_{ab}] = 2$ or 2^2. Now [Lemma 3.5(b); MeSt] forces $Q(a)_b \cong 2^9$. Hence $G_{ab}^{*a} \cong 2^6 : (S_3 \times L_2(7))$ and $\Delta_1(a)$ is a G_a-orbit, and so we have (i) and (ii). Let $b \in \Delta_1(a)$, and let $X \in \Gamma_2(a+b)$. By [Lemma 3.5(b); MeSt] G_{aa+b} has a minimal normal subgroup N of order 2^3 contained in $Q(a)$. Since $\tau(X) \in Z(G_{aX})$, $< \tau(X) >= Z(T)$ for some $T \in Syl_2(G_a)$ and so $\tau(X) \in N$. Now N is a 3-dimensional $GF(2)L_2(7)$-module and so $N^\#$ is a G_{aa+b}-conjugacy class. Thus $N^\# \subseteq T(a+b)$ which yields (iii).

We now consider triangles in $\mathcal{G}$.

Lemma 3.7 Let $\{a,l,X\}$ be a maximal flag in Γ and let $k \in \Gamma_1(a,X)\backslash\{l\}$. Then

(i) $k \in \delta_0(X,a,l)$, when viewed in Γ_X, if and only if $k \in \Gamma_1(X) \cap \alpha_2(a,l)$ when viewed in Γ_a; and

(ii) $k \in \delta_1(X,a,l)$, when viewed in Γ_X, if and only if $k \in \Gamma_1(X) \cap \alpha_3(a,l)$ when viewed in Γ_a;

Proof Combining (2.10) and (2.13) (i) yields the result.

Lemma 3.8 Let $X \in \Gamma_2(a)$ and $l,k \in \Gamma_1(a,X)$. Put $\Gamma_0(l)\backslash\{a\} = \{a_1,a_2,a_3,a_4\}$.

(i) If $k \in \alpha_2(a,l)$, then each a_i is collinear with precisely two points of $\Gamma_0(k)\backslash\{a\}$ and, up to relabelling, a_1 and a_2 are collinear with the same pair of points in $\Gamma_0(k)\backslash\{a\}$ and a_3 and a_4 are collinear with the remaining pair of points in $\Gamma_0(k)\backslash\{a\}$.

(ii) If $k \in \alpha_3(a,l)$, then each a_i is collinear with every point in $\Gamma_0(k)\backslash\{a\}$.

Proof Combining (2.13) and Lemma 3.7 gives the lemma.

Lemma 3.9 Let $\{a,b,c\}$ be a triangle in $\mathcal{G}$ and let $i \in \{2,3\}$. If $a+b \in \alpha_i(a,a+c)$, then $b+c \in \alpha_i(b,b+a)$.

Proof This is a consequence of (2.15) and Lemma 3.7.

Again we remark that for a triangle $\{a,b,c\}$ in $\mathcal{G}$, Lemma 3.9 means that the relationship between $a+c$ and $a+b$ (at a) is the same as that between $b+a$ and $b+c$ (at b) and that between $c+b$ and $c+a$ (at c). This fact enables us to readily translate information between point residues.

We shall say that a triangle $\{a,b,c\}$ in $\mathcal{G}$ is **sparse** if $a+b \in \alpha_2(a,a+c)$ (that is Lemma 3.8(i) holds for $a+b$ and $a+c$) and that it is **non-sparse** if $a+b \in \alpha_3(a,a+c)$ (that is Lemma 3.8(ii) holds for $a+b$ and $a+c$). By Lemma 3.9 sparse triangles and non-sparse triangles are well-defined.

Lemma 3.10 Let $X \in \Gamma_2$ and $b,c \in \Gamma_0(X)$. If $l \in \Gamma_1(b,c)$, then $l \in \Gamma_1(X)$.

Proof Suppose $l \notin \Gamma_1(X)$. Since G_{bX} has 2 orbits on $\Gamma_0(X)\setminus\{b\}$, using Lemma 3.5 it follows that every point of $\Gamma_0(X)\setminus\{b\}$ is collinear with b in Γ. If S is the set of pairs (x,Y) such that $Y \in \Gamma_2(b)$ and $x \in \Delta_1(b) \cap \Gamma_0(Y)$, then $|S| = |\Gamma_2(b)||\Gamma_0(X) - 1| = 1771.76$. Also, since $\Delta_1(a)$ is a G_a-orbit,

$$|S| = |\Delta_1(b)||\Gamma_2(b,c)| = 2^2.5.11.23.|\Gamma_2(b,c)|.$$

However this implies that $|\Gamma_2(b,c)|$ is not an integer, a contradiction.

Lemma 3.11 Let $\{a_1,a_2,a_3\}$ form a triangle in $\mathcal{G}$, and set $l_1 = a_1+a_2$, $l_2 = a_2+a_3$ and $l_3 = a_3+a_1$. Then $\Gamma_2(a_1,a_2,a_3) \neq \emptyset$ and for each $X \in \Gamma_2(a_1,a_2,a_3)$, $\Gamma_0(l_i) \subseteq \Gamma_0(X)$ ($i = 1,2,3$). If, furthermore, the l_i are distinct lines, then $|\Gamma_2(a_1,a_2,a_3)| = 1$.

Proof First we show that $|\Gamma_2(a_1,a_2,a_3)| = 1$ when the l_i are all distinct. Assume that $a_1+a_3 \in \alpha_i(a_1,a_1+a_2)$ where $i = 0$ or 1. Let a_1+a_2 be the standard trio in Γ_{a_1} and, without loss of generality, we may choose a_1+a_3 to be as in Lemma 3.5. Let X be the standard sextet in Γ_a. Then $\tau(X) \in T(a_1+a_2)$ and $a_1+a_3 \in \beta_1(a_1,X)$. Hence, by Lemma 3.3, $a_3^{\tau(X)} \neq a_3$. Since $\tau(X) \in Q(a_1) \cap Q(a_2)$, Lemma 3.5 forces

$$a_2+a_3 = a_3+a_3^{\tau(X)} = a_2+a_1,$$

a contradiction. Thus $a_1+a_3 \in \alpha_i(a_1,a_1+a_2)$ with $i = 2,3$ whence (2.9)(ii) yields that $|\Gamma_2(a_1,a_2,a_3)| = 1$.

If two of l_1, l_2 and l_3 are equal, then $\Gamma_1(a_1, a_2, a_3) \neq \emptyset$ and then we also have that $\Gamma_2(a_1, a_2, a_3) \neq \emptyset$. That $\Gamma_0(l_i) \subseteq \Gamma_0(X)$ (for $i = 1, 2, 3$) follows from Γ being a string geometry and Lemma 3.10.

4. THE SECOND DISC

Recall that a is a fixed point of $\mathcal{G}$. In this section we tear $\Delta_2(a)$ limb from limb. Much of our attention is focused on the intermediate points between a and $\Delta_2(a)$. In fact we dwell at great length on $\{a,c\}^\perp$ for $c \in \Delta_2^2(a)$; our labours being rewarded by the picture displayed after Theorem 4.7.

Here we exhibit the subsets of $\Delta_2(a)$ with which we will become intimate.

Definition 4.1
$\Delta_2^1(a) := \{c \in \Delta_2(a) | \Gamma_2(a,c) \neq \emptyset\}$
$\Delta_2^2(a) := \{c \in \Gamma_0(a) |$ there exists $b \in \{a,c\}^\perp$ such that $b+c \in \alpha_1(b, b+a)\}$
$\Delta_2^3(a) := \{c \in \Gamma_0(a) |$ there exists $b \in \{a,c\}^\perp$ such that $b+c \in \alpha_0(b, b+a)\}$.

Lemma 4.2 $\Delta_2(a) = \Delta_2^1(a) \cup \Delta_2^2(a) \cup \Delta_2^3(a)$.

Proof First we show that $\Delta_2^1(a) \cup \Delta_2^2(a) \cup \Delta_2^3(a) \subseteq \Delta_2(a)$. Let $c \in \Delta_2^2(a)$ with $b \in \{a,c\}^\perp$ such that $b+c \in \alpha_1(b, b+a)$. If $c \in \Delta_1(a)$, then by Lemma 3.11 $\Gamma_2(a,b,c) = \{X\}$ with $b+a, b+c \in \Gamma_1(X)$. This is impossible by (2.9)(i) and thus $\Delta_2^2(a) \subseteq \Delta_2(a)$; a similar argument gives $\Delta_2^3(a) \subseteq \Delta_2(a)$. Now using (2.3)(i) and(2.9)(ii) we obtain the result.

Theorem 4.3 Let $c \in \Delta_2^1(a)$.
(i) $|\Gamma_2(a,c)| = 1$.
Set $\Gamma_2(a,c) = \{X(a,c)\}$.
(ii) $\{a,c\}^\perp \subseteq \Gamma_0(X(a,c))$.
(iii) $|\{a,c\}^\perp| = 45$.
(iv) $|\Delta_2^1(a)| = 2^4.7.11.23$.
(v) $\Delta_2^1(a)$ is a G_a-orbit and $G^{*a}_{aX(a,c)} = G^{*a}_{ac} \cong 2^6 : 3.S_6$.

Proof Since $c \in \Delta_2^1(a)$, there exists $X \in \Gamma_2(a,c)$. Set $\tau = \tau(X)$. We begin by establishing

(4.3.1) $\{a,c\}^\perp \subseteq \Gamma_0(X)$

Let $d \in \{a,c\}^\perp$. We show that $d \notin \Gamma_0(X)$ leads to a contradiction. From $d \notin \Gamma_0(X)$, Hypothesis 1.1 implies that $a+d \notin \Gamma_1(X)$. Therefore $a+d \in \beta_i(a,X)$ with $i \in \{1,2,3\}$. If $a+d \in \beta_1(a,X)$, then $d^\tau \neq d$ by Lemma 3.3. Since $\tau \in Q(a) \cap Q(c)$, Lemma 3.5 then forces $d+c = d+d^\tau = d+a$, contrary to $d(a,c) = 2$. Thus $a+d \in \beta_2(a,X) \cup \beta_3(a,X)$. We claim that there exists $Y \in \Gamma_2(a,c,d)$ such that $Y \in \gamma_3(a,X)$. Without loss of generality we may assume X is the standard sextet. By (2.5), without loss, we may take

$$a+d = \begin{array}{|cc|cc|cc|} \hline + & + & + & + & \circ & \circ \\ + & + & + & + & \circ & \circ \\ - & - & - & - & \circ & \circ \\ - & - & - & - & \circ & \circ \\ \hline \end{array} \text{ if } a+d \in \beta_2(a,X)\text{; and}$$

$$a+d = \begin{array}{|cc|cc|cc|} \hline + & + & + & + & - & - \\ + & + & + & + & - & - \\ \circ & \circ & - & - & \circ & \circ \\ \circ & \circ & - & - & \circ & \circ \\ \hline \end{array} \text{ if } a+d \in \beta_3(a,X).$$

Let Z be the following sextet

$$\begin{array}{|cc|cc|cc|} \hline \star & \star & \cdot & + & + & \cdot \\ \star & \times & \square & \circ & \circ & \square \\ \times & \star & \square & \circ & \circ & \square \\ \times & \times & \cdot & + & + & \cdot \\ \hline \end{array}$$

Evidently we have $Z \in \gamma_3(a,X)$ and $a+d \in \beta_1(a,Z)$. Consequently $d^{\tau(Z)} \neq d$ by Lemma 3.3. From $Z \in \gamma_3(a,X)$, $\tau(Z) \in Q(X)$ by Lemma 3.4, and so $\tau(Z)$ fixes c. Thus $\{c,d,d^{\tau(Z)}\}$ is a triangle and hence $\Gamma_2(c,d,d^{\tau(Z)})$ contains a unique plane by Lemma 3.11. Let Y denote this unique plane; we now verify that Y has the required properties. Because $\tau(Z) \in Q(a)$ and $d,d^{\tau(Z)} \in \Gamma_0(Y)$ we get $a \in \Gamma_0(Y)$ by Lemma 3.10. Thus $Y \in \Gamma_2(a,c,d)$. It only remains to show that $Y \in \gamma_3(a,X)$. From (2.16) for each $l \in \Gamma_1(a,X)$ there exists $b \in \Gamma_0(l)$ such that $b \in \{a,c\}^{\perp}$. Now assume that $Y \notin \gamma_3(a,X)$. Then there exists $l \in \Gamma_1(a,X)$ such that $l \in \beta_1(a,Y)$ by (2.8). Thus we may find $b \in \{a,c\}^{\perp}$ such that $a+b \in \beta_1(a,Y)$. Since $\tau(Y) \in Q(a) \cap Q(c)$, Lemma 3.3 gives the contradiction $b+a = b+b^{\tau} = b+c$. So $Y \in \gamma_3(a,X)$, which establishes our claim. From $Y \in \gamma_3(a,X)$, we see there exists $l \in \Gamma_1(X) \cap \Gamma_1(Y)$ with $l \in \Gamma_1(a)$. Employing (2.16) yields $e, f \in \Gamma_0(l)$ such that $\{c,e,f\}$ forms a triangle. Now Lemma 3.11 yields $X = Y$, contrary to $Y \in \gamma_3(a,X)$. This completes the proof of (4.3.1).

Let $X_1 \in \Gamma_2(a,c)$. By (4.3.1) $\{a,c\}^{\perp} \subseteq \Gamma_0(X) \cap \Gamma_0(X_1)$. Combining (2.9)(ii) and (2.12)(iii) yields $X = X_1$, so proving part (i). Part (ii) follows from part (i) and (4.3.1), while part (ii) and (2.12) (iii) imply (iii).

From part (i), (2.12)(iii) and (2.1)(i),

$$|\Delta_2^1(a)| = 16|\Gamma_2(a)| = 16.1771 = 2^4.7.11.23,$$

as required.

Finally we consider (v). That $\Delta_2^1(a)$ is a G_a-orbit follows from G_a being transitive on $\Gamma_2(a)$ (by the flag-transitivity of G on Γ) and (2.12)(iii). Thus, by (iv), $|G_{ac}| = 2^{17}.3^3.5$. Also, from part (i), $G_{ac} \leqslant G_{aX(a,c)}$ ($\sim 2^{10}2^6 3S_6$). By order considerations, $G^{*a}_{ac} \cap O_2(G^{*a}_{aX(a,c)}) \neq 1$. Because the only subgroups of $3^{\cdot}S_6$ of 2 power index are $3^{\cdot}A_6$ and $3^{\cdot}S_6$, we see that $G^{*a}_{ac}O_2(G^{*a}_{aX(a,c)})/O_2(G^{*a}_{aX(a,c)}) \cong 3^{\cdot}A_6$ or $3^{\cdot}S_6$. Now $O_2(G^{*a}_{aX(a,c)})$ is a chief factor of $2^{10}2^6 3A_6$ and therefore $G^{*a}_{ac} \cong 2^6 3S_6$ or $2^6 3A_6$. The latter possibility implies that $[Q(a) : Q(a) \cap G_{ac}] = 2^5$ with $Q(a) \cap G_{ac}$ normalized by the $2^{10}2^6 3A_6$ which contradicts the module structure of the M_{24}-Todd module (see [Lemma 3.5(b); MeSt]). Consequently $G^{*a}_{aX(a,c)} = G^{*a}_{ac} \cong 2^6 : 3.S_6$.

Our next result will be called upon in Lemma 4.6 and Theorem 4.8.

Lemma 4.4 Suppose $c \in \Delta_2^3(a)$ and let $b \in \{a,c\}^{\perp}$ be such that $b+c \in \alpha_0(b, b+a)$. Then there exists a unique $l \in \Gamma_1(b)$ such that $l \in \alpha_3(b, b+a) \cap \alpha_3(b, b+c)$ and $\Delta_1(b) \cap \{a,c\}^{\perp} = \Gamma_0(l) \backslash \{b\}$.

Proof In view of (2.3) we may, without loss, assume

$$b+a = \begin{array}{|cc|cc|cc|} \hline + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ \hline \end{array} \text{ and } b+c = \begin{array}{|cc|cc|cc|} \hline - & + & + & + & + & + \\ + & - & \circ & \circ & - & - \\ + & \circ & \circ & - & - & \circ \\ + & \circ & - & \circ & - & \circ \\ \hline \end{array}.$$

For $d \in \Delta_1(b) \cap \{a,c\}^{\perp}$ we now determine $b+d$. We assert that $b+d$ and $b+a$ do not have an octad in common. For if they did then that octad of $b+d$ would have to cut the octads of $b+c$ in $4, 2^2$. This is impossible by (2.9) and Lemma 3.11, since $\{b,c,d\}$ forms a triangle. Similarly, $b+d$ and $b+c$ do not share a common octad. Using (2.9) and Lemma 3.11 again we obtain $b+d \in \alpha_3(b, b+a) \cap \alpha_3(b, b+c)$. Then by inspection we see that there is only one possibility for $b+d$, namely $b+d = \begin{array}{|cc|cc|cc|} \hline + & + & - & - & + & + \\ + & + & - & - & + & + \\ - & - & \circ & \circ & \circ & \circ \\ - & - & \circ & \circ & \circ & \circ \\ \hline \end{array}$. Applying Lemma 3.8 now gives the lemma.

Lemma 4.5 Let $c \in \Delta_2^i(a)$ where $i = 1,2$ or 3 and let $b \in \{a,c\}^{\perp}$. If $i = 2$ (respectively $i = 3$) we further assume that $b+c \in \alpha_1(b,b+a)$ (respectively $b+c \in \alpha_0(b,b+a)$).

(i) If $i = 1$, then $\tau(X)$ fixes c for all $\tau(X) \in T(a+b)$.

(ii) If $i = 2$, then precisely three elements of $T(a+b)$ fix c.

(iii) If $i = 3$, then only one element of $T(a+b)$ fixes c.

Proof We may suppose that $b+a$ is the standard trio and that

$$b+c = \begin{array}{|cc|cc|cc|} \hline - & + & + & + & + & + \\ + & - & - & - & - & - \\ + & - & \circ & \circ & \circ & \circ \\ + & - & \circ & \circ & \circ & \circ \\ \hline \end{array} \text{ if } b+c \in \alpha_1(b,b+a)\text{; and}$$

$$b+c = \begin{array}{|cc|cc|cc|} \hline - & + & + & + & + & + \\ + & - & \circ & \circ & - & - \\ + & \circ & \circ & - & - & \circ \\ + & \circ & - & \circ & - & \circ \\ \hline \end{array} \text{ if } b+c \in \alpha_0(b,b+a).$$

In Γ_b, the seven sextets incident with $b+a$ are as listed.

$$X_1 = \begin{array}{|cc|cc|cc|} \hline \star & \times & \cdot & \square & \circ & + \\ \star & \times & \cdot & \square & \circ & + \\ \star & \times & \cdot & \square & \circ & + \\ \star & \times & \cdot & \square & \circ & + \\ \hline \end{array} \qquad X_2 = \begin{array}{|cc|cc|cc|} \hline \star & \times & \cdot & \square & \circ & + \\ \star & \times & \cdot & \square & \circ & + \\ \times & \star & \square & \cdot & + & \circ \\ \times & \star & \square & \cdot & + & \circ \\ \hline \end{array}$$

$$X_3 = \begin{array}{|cc|cc|cc|} \hline \star & \times & \cdot & \square & \circ & + \\ \times & \star & \square & \cdot & + & \circ \\ \star & \times & \cdot & \square & \circ & + \\ \times & \star & \square & \cdot & + & \circ \\ \hline \end{array} \qquad X_4 = \begin{array}{|cc|cc|cc|} \hline \star & \times & \cdot & \square & + & \circ \\ \times & \star & \square & \cdot & \circ & + \\ \times & \star & \square & \cdot & \circ & + \\ \star & \times & \cdot & \square & + & \circ \\ \hline \end{array}$$

$$X_5 = \begin{array}{|cc|cc|cc|} \hline \star & \star & \cdot & \cdot & + & + \\ \star & \star & \cdot & \cdot & + & + \\ \times & \times & \square & \square & \circ & \circ \\ \times & \times & \square & \square & \circ & \circ \\ \hline \end{array} \qquad X_6 = \begin{array}{|cc|cc|cc|} \hline \star & \star & \cdot & \cdot & + & + \\ \times & \times & \square & \square & \circ & \circ \\ \star & \star & \cdot & \cdot & + & + \\ \times & \times & \square & \square & \circ & \circ \\ \hline \end{array}$$

$$X_7 = \begin{array}{|cc|cc|cc|} \hline \star & \star & \cdot & \cdot & + & + \\ \times & \times & \square & \square & \circ & \circ \\ \times & \times & \square & \square & \circ & \circ \\ \star & \star & \cdot & \cdot & + & + \\ \hline \end{array}$$

If $i = 1$, then $b \in \Gamma_0(X(a,c))$ by Theorem 4.3(ii). In the $i = 1$ case we may suppose that, in Γ_b, $X(a,c)$ is the sextet X_1. Then $b+c \notin \beta_1(b,X_j)$ for all $j \in \{1,2,3,4,5,6,7\}$. For $i = 2$ we see that $b+c \in \beta_1(b,X_j)$ for $j = 1,2,3,4$ and $b+c \notin \beta_1(b,X_j)$ for $j = 5,6,7$. And for $i = 3$ we observe that $b+c \in \beta_1(b,X_j)$ for $j \neq 5$ and $b+c \notin \beta_1(b,X_5)$. Employing Lemma 3.3 now gives the lemma.

Lemma 4.6 Let $i \in \{1,2,3\}$. Suppose that $c \in \Delta_2^i(a)$ and that $b,d \in \{a,c\}^{\perp}$.

(i) Let $j \in \{0,1,2\}$. Then $b+c \in \beta_j(b,b+a)$ if and only if $d+c \in \alpha_j(d,d+a)$.

(ii) If $d(b,d) = 2$, then $d \in \Delta_2^1(b)$.

Proof (i) First we consider $j = 2$. So $c \in \Delta_2^1(a)$. By Theorem 4.3 we have $\{X(a,c)\} = \Gamma_2(a,c)$ and $d \in \Gamma_0(X(a,c))$. Since $d(a,c) = 2$, Lemma 3.8(ii) (applied to $d, d+a$ and $d+c$) implies that $d+c \in \alpha_2(d,d+a)$, and so (i) holds when j = 2.

Now we suppose (i) is false and seek a contradiction. Thus we may assume that $b+c \in \alpha_0(b,b+a)$ and $d+c \in \alpha_1(d,d+a)$. As a consequence $c \in \Delta_2^2(a) \cap \Delta_2^3(a)$ and, by Theorem 4.3(ii), $c \notin \Delta_2^1(a)$.

(4.6.1) $d(b,d) = 2$.

If (4.6.1) is false, then $d(b,d) = 1$. Hence, by Lemma 4.4, $b+d \in \alpha_3(b,b+a) \cap \alpha_3(b,b+c)$. Therefore $d+b \in \alpha_3(d,d+a) \cap \alpha_3(d,d+c)$ by Lemma 3.9. Without loss of generality we may take $d+a$ to be the standard trio and

$$d+c = \begin{array}{|cc|cc|cc|} \hline - & + & + & + & + & + \\ + & - & - & - & - & - \\ + & - & \circ & \circ & \circ & \circ \\ + & - & \circ & \circ & \circ & \circ \\ \hline \end{array}\,.$$

Then, by inspection, the possibilities for $d+b$ are as follows.

$$\begin{array}{|cc|cc|cc|} \hline + & + & - & - & - & - \\ + & + & - & - & - & - \\ + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ \hline \end{array} \qquad \begin{array}{|cc|cc|cc|} \hline - & - & + & + & - & - \\ - & - & + & + & - & - \\ + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ \hline \end{array}$$

$$\begin{array}{|cc|cc|cc|} \hline + & + & + & + & - & - \\ + & + & + & + & - & - \\ - & - & \circ & \circ & \circ & \circ \\ - & - & \circ & \circ & \circ & \circ \\ \hline \end{array} \qquad \begin{array}{|cc|cc|cc|} \hline + & + & - & - & - & - \\ + & + & \circ & \circ & \circ & \circ \\ + & + & - & - & - & - \\ + & + & \circ & \circ & \circ & \circ \\ \hline \end{array}$$

+	+	−	−	−	−
+	+	○	○	○	○
+	+	○	○	○	○
+	+	−	−	−	−

However, none of these lie in both $\alpha_3(d, d+a)$ and $\alpha_3(d, d+c)$, so (4.6.1) must hold.

If $d \in \Delta_2^1(b)$, then Theorem 4.3(ii) forces $c \in \Delta_2^1(a)$ whereas $c \notin \Delta_2^1(a)$. So $d \notin \Delta_2^1(a)$. Hence, by (4.6.1), (2.3)(i) and the definition of $\Delta_2^1(b)$,

$$a+b \in \alpha_0(a, a+d) \cup \alpha_1(a, a+d).$$

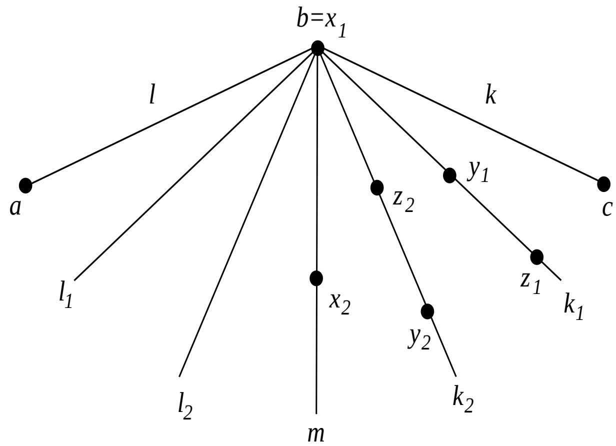
Suppose $a+b \in \alpha_0(a, a+d)$. Then $d \in \Delta_2^3(b)$. By Lemma 4.5(ii) there are three elements of $T(a+d)$ which fix c and from Lemma 4.5(iii) there is a unique element of $T(a+b)$ fixing b. So we may select $\tau \in T(a+d)$ such that $c^\tau = c$ and $b^\tau \neq b$. So $\{c, b, b^\tau\}$ forms a triangle and hence $\Gamma_2(c, b, b^\tau) \neq \emptyset$. Because $\tau \in Q(a)$, Lemma 3.5 implies that $b + b^\tau = b + a$ and hence Lemma 3.10 gives the untenable $\Gamma_2(a, c) \neq \emptyset$.

Turning to the possibility $a+b \in \alpha_1(a, a+d)$, using Lemma 4.5 again we may find a $\tau \in T(a+b)$ such that $d = d^\tau$ and $c^\tau \neq c$. Then arguing as above we deduce that $\Gamma_2(b, d) \neq \emptyset$, against $d \notin \Delta_2^1(b)$. From these contradictions we infer that (i) holds.

(ii) If $c \in \Delta_2^1(a)$, then Theorem 4.3(ii) easily yields $d \in \Delta_2^1(b)$. So we only need show that assuming $c \in \Delta_2^2(a)$, then $d \in \Delta_2^3(b)$ is impossible. By Lemma 4.5 there exists $\tau \in T(a+b)$ such that $c^\tau = c$ and $d^\tau \neq d$. Arguing as in part (i) then yields $c \in \Delta_2^1(a)$ which is impossible since $c \in \Delta_2^2(a)$. So (ii) is proven.

Theorem 4.7 Let $c \in \Delta_2^2(a)$, and let $b \in \{a, c\}^\perp$. Set $b + a = l$ and $b + c = k$.

(i) We have $|\Delta_1(b) \cap \{a, c\}^\perp| = 9$ with the points in $\Delta_1(b) \cap \{a, c\}^\perp$ incident with the lines $l_1, l_2, m, k_1, k_2 \in \Gamma_1(b)$, as shown.



Further l, l_1, l_2 and m are incident with a (unique) plane $X_a \in \Gamma_2(a)$. Likewise k, k_1, k_2 and m are also incident with a (unique) plane $X_c \in \Gamma_2(c)$. Also m is the unique line of $\Gamma_1(b)$ in $\alpha_2(b, b+a) \cap \alpha_2(b, b+c)$.

(ii) Using the notation of (i), there exists (unique) Y_a, Z_a in $\Gamma_2(a)$ with $y_1, y_2 \in \Gamma_0(Y_a)$ and $z_1, z_2 \in \Gamma_0(Z_a)$. Moreover $\mathcal{S}_b(a,c) := \{X_a, Y_a, Z_a\}$ forms a sextet line in Γ_a and the six points in $\Gamma_0(X_a) \cap \{a,c\}^{\perp}$ are pairwise collinear, with a similar statement for Y_a and Z_a.

Set $I = (\Gamma_0(X_a) \cup \Gamma_0(Y_a) \cup \Gamma_0(Z)) \cap \{a,c\}^{\perp}$.

(iii) If $b' \in I$, then $\mathcal{S}_{b'}(a,c) = \mathcal{S}_b(a,c)$.

(iv) For each $S \in \mathcal{S}_b(a,c)$, $\tau(S)$ fixes c.

(v) Every point of $\{a,c\}^{\perp}$ is incident with exactly one of X_a, Y_a and Z_a . Hence $|\{a,c\}^{\perp}| = 18$.

(vi) $|\Delta_2^2(a)| = 2^7.3.5.7.11.23$.

Proof First we establish part (i). By Lemma 4.6(i) we have that $b+c \in \alpha_1(b, b+a)$ and so, because of (2.3)(i), we may take $l = b+a$ to be the standard trio and

$$k = b + c = \begin{array}{|cc|cc|cc|} \hline - & + & + & + & + & + \\ + & - & - & - & - & - \\ + & - & \circ & \circ & \circ & \circ \\ + & - & \circ & \circ & \circ & \circ \\ \hline \end{array}\,.$$

Let $x \in \Delta_1(b) \cap \{a,c\}^{\perp}$. Lemma 3.11 and (2.9) imply that $b+x \in \alpha_2(b,b+a) \cup \alpha_3(b,b+c)$. By inspection, the only possibilities for $b+x$ are as follows.

$$m = \begin{array}{|cc|cc|cc|} \hline + & + & - & - & - & - \\ + & + & - & - & - & - \\ + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ \hline \end{array};$$

$$l_1 = \begin{array}{|cc|cc|cc|} \hline - & - & + & + & - & - \\ - & - & + & + & - & - \\ + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ \hline \end{array}; \qquad l_2 = \begin{array}{|cc|cc|cc|} \hline + & + & + & + & - & - \\ + & + & + & + & - & - \\ - & - & \circ & \circ & \circ & \circ \\ - & - & \circ & \circ & \circ & \circ \\ \hline \end{array};$$

$$k_1 = \begin{array}{|cc|cc|cc|} \hline + & + & - & - & - & - \\ + & + & \circ & \circ & \circ & \circ \\ + & + & - & - & - & - \\ + & + & \circ & \circ & \circ & \circ \\ \hline \end{array}; \text{ and } \qquad k_2 = \begin{array}{|cc|cc|cc|} \hline + & + & - & - & - & - \\ + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ + & + & - & - & - & - \\ \hline \end{array}.$$

Then we observe that

(4.7.1) (i)$m \in \alpha_2(b,b+a) \cap \alpha_2(b,b+c)$
(ii) $l_i \in \alpha_3(b,b+a) \cap \alpha_2(b,b+c)$ for $i = 1,2$
(iii) $k_i \in \alpha_3(b,b+c) \cap \alpha_2(b,b+a)$ for $i = 1,2$.

Appealing to Lemma 3.8 we conclude that $|\Gamma_0(l_i) \cap \{a,c\}^{\perp}| = 3 = |\Gamma_0(k_i) \cap \{a,c\}^{\perp}|$ for $i = 1,2$. We further observe that, in Γ_b, m, l_1, l_2 and l are incident with the sextet $S_l = \begin{array}{|cc|cc|cc|} \hline \star & \star & \cdot & \cdot & + & + \\ \star & \star & \cdot & \cdot & + & + \\ \times & \times & \square & \square & \circ & \circ \\ \times & \times & \square & \square & \circ & \circ \\ \hline \end{array}$ while, also in Γ_b, m, k_1, k_2 and k are incident with the sextet $S_k = \begin{array}{|cc|cc|cc|} \hline \star & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \star & + & + & + & + \\ \times & \star & \square & \square & \square & \square \\ \times & \star & \circ & \circ & \circ & \circ \\ \hline \end{array}$.

Let X_a , respectively X_c, be the plane in Γ corresponding to S_l, respectively S_k. Note that $k_i \notin \Gamma_1(X_a)$ and $l_i \notin \Gamma_1(X_c)$ for $i = 1,2$.

Inspecting l_1, l_2, m yet again yields that

(4.7.2) $m \in \alpha_2(b, l_i)$ for $i = 1, 2$ and $l_2 \in \alpha_2(b, l_1)$.

(4.7.3) The points of $\Gamma_0(X_a) \cap \{a, c\}^{\perp}$ are pairwise collinear.

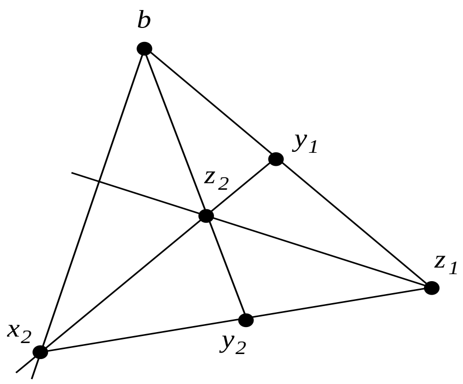
If (4.7.3) were false, then Theorem 4.3(ii) forces $c \in \Gamma_0(X_a)$, contrary to $c \in \Delta_2^2(a)$.

Using Lemma 3.7 we interpret (4.7.1) (i), (ii) and (4.7.2) in Γ_{X_a} as follows.

(4.7.4) (i) $l_1, l_2 \in \delta_1(X_a, b, l)$, $m \in \delta_0(X_a, b, l)$; and
(ii) $l_2 \in \delta_0(X_a, b, l_1)$, $m \in \delta_0(X_a, b, l_i)$ for $i = 1, 2$.

Combining (4.7.3), (4.7.4) and (2.14) we see that there is a unique point x_2, say, in $\Gamma_0(m)$ which is collinear with a and such that $\Gamma_1(d_1, e_1, x_2) \neq \emptyset$ where $d_1 \in \Gamma_0(l_1) \cap \{a, c\}^{\perp}$ and $e_1 \in \Gamma_0(l_2) \cap \{a, c\}^{\perp}$. Because of (4.7.3) we now deduce that $\Gamma_0(m) \cap \{a, c\}^{\perp} \subseteq \{b, x_2\}$. We now show that $x_2 \in \{a, c\}^{\perp}$. Supposing $x_2 \notin \{a, c\}^{\perp}$ we argue for a contradiction. Hence the points of $\Gamma_0(X_a) \cap \{a, c\}^{\perp}$, when viewed from b, lie on three lines, two of which are each incident with three points of $\{a, c\}^{\perp}$ and the third incident with just one point of $\{a, c\}^{\perp}$. This latter statement also applies to any other point in $\Gamma_0(X_a) \cap \{a, c\}^{\perp}$ by Lemma 4.6(i). However we have seen that the line incident with d_1, e_1, x_2 has only two points in $\{a, c\}^{\perp}$ (namely d_1, e_1) which is the desired contradiction. Therefore $\Gamma_0(m) \cap \{a, c\}^{\perp} = \{b, x_2\}$ and so we have established part (i).

We now consider part (ii); let y_1, y_2, z_1, z_2 be as in the diagram in part (i). Note that all the previous assertions for X_a have appropriate analogues for X_c. In particular, (analogue of (4.7.4)) $k_1, k_2 \in \delta_1(X_c, b, k)$, $m \in \delta_0(X_c, b, k)$, $k_2 \in \delta_0(X_c, b, k_1)$ and $m \in \delta_0(X_c, b, k_i) (i = 1, 2)$. So, by (2.14), we may assume y_1, y_2, z_1, z_2 are labelled so as



.

(That is $\Gamma_1(x_2,y_2,z_1) \neq \emptyset \neq \Gamma_1(x_2,z_2,y_1)$ and $\Gamma_1(y_1,y_2) \neq \emptyset \neq \Gamma_1(z_1,z_2)$.) In X_c, y_2+b and y_2+x_2 are each incident with three points of $\{a,c\}^\perp$ while y_2+y_1 is incident with just two points of $\{a,c\}^\perp$. Letting y_2 play the role of b and X_c the role of X_a in part (ii) we deduce that there is a plane $Y_a \in \Gamma_2(a)$ incident with the line y_1+y_2 (this line plays the part of m) and two more lines in $\Gamma_1(y_2,Y_a)$ incident with a further four points of $\{a,c\}^\perp$. None of these four points of $\{a,c\}^\perp$ lie in $\Gamma_0(l_1) \cup \Gamma_0(l_2)$. This is because of (2.9)(i) and Lemma 3.11 and the fact that $k \in \alpha_1(b,l_i)$ for $i=1,2$. Repeating this argument with z_2 in place of y_2 produces a plane $Z_a \in \Gamma_2(a)$ and a further four points of $\{a,c\}^\perp$ which are not incident with X_a nor Y_a. Our next goal is to show that $\{X_a,Y_a,Z_a\}$ forms a sextet line in Γ_a; this will be done by identifying the lines $a+x_1(=a+b)$, $a+x_2$, $a+y_1$, $a+z_1$ and $a+z_2$ in Γ_a. Without loss of generality we may take X_a to be the standard sextet and $l=a+b$ to be $\begin{array}{|cc|cc|cc|}\hline + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ \hline\end{array}$. Since G_{aX_al} is transitive on the six sextets in $\Gamma_2(a,l)\backslash\{X_a\}$, we may also assume that

$$X(a,l,a+y_1) = \begin{array}{|cc|cc|cc|}\hline \star & \star & + & \cdot & \cdot & + \\ \times & \times & \cdot & + & \cdot & + \\ \times & \star & \square & \circ & \square & \circ \\ \star & \times & \square & \circ & \square & \circ \\ \hline\end{array}.$$

Note that $a+z_1$ is also incident with $X(a,l,a+y_1)$. Because $k_1 \in \alpha_2(b,b+a)$ by (4.7.1)(iii), Lemma 3.9 implies that $a+b(=l)$, $a+y_1$, and $a+z_1$, are trios which pairwise share an octad. Since these three trios are distinct this can only happen if all three share the same octad. Now the stabilizer in G_a of X_a, l and $X(a,l,a+y_1)$ is transitive on the three octads of l and so we may suppose the common octad is the leftmost block of the MOG. This forces $a+y_1$ and $a+z_1$ to be as follows (with a possible interchanging of y_1 and z_1):-

$$a+y_1 = \begin{array}{|cc|cc|cc|}\hline + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ + & + & - & - & - & - \\ + & + & - & - & - & - \\ \hline\end{array} \qquad a+z_1 = \begin{array}{|cc|cc|cc|}\hline + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ + & + & - & \circ & - & \circ \\ + & + & - & \circ & - & \circ \\ \hline\end{array}.$$

Applying part (i) with x_2 playing the role of b gives that $x_2+y_1 \in \alpha_2(x_2,x_2+a)$ and $x_2+z_1 \in \alpha_2(x_2,x_2+a)$. From (4.7.1)(i) we recall that $m \in \alpha_2(b,b+a)$.

Appealing to Lemma 3.9 we infer that $a+x_2$, $a+y_1$, $a+z_1$ and $a+b(=l)$, as trios in Γ_a, share an octad in common. This forces $a+x_2$ to be one of the following two trios.

$$\begin{array}{|cc|cc|cc|}\hline + & + & - & \circ & \circ & - \\ + & + & - & \circ & \circ & - \\ + & + & - & \circ & \circ & - \\ + & + & - & \circ & \circ & - \\ \hline \end{array} \qquad \begin{array}{|cc|cc|cc|}\hline + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ \hline \end{array}$$

We assume the first possibility holds; the argument being similar if the other possibility holds. The sextets $X(a,a+x_2,a+y_1)$ and $X(a,a+x_2,a+z_1)$ are now uniquely determined given the above identification of $a+x_2$, $a+y_1$ and $a+z_1$ and are

$$X(a,a+x_2,a+y_1) = \begin{array}{|cc|cc|cc|}\hline \star & \star & + & \cdot & \cdot & + \\ \times & \times & + & \cdot & \cdot & + \\ \star & \times & \square & \circ & \circ & \square \\ \times & \star & \square & \circ & \circ & \square \\ \hline \end{array} \text{ and}$$

$$X(a,a+x_2,a+z_1) = \begin{array}{|cc|cc|cc|}\hline \star & \times & + & \square & \circ & \cdot \\ \times & \star & + & \square & \circ & \cdot \\ \star & \star & \cdot & \circ & \square & + \\ \times & \times & \cdot & \circ & \square & + \\ \hline \end{array}.$$

Since $z_2 \in \Gamma_0(x_2+y_1)$ the trio $a+z_2$ is incident with $X(a,a+x_2,a+y_1)$. We already have that $x_2+y_1 \in \alpha_2(x_2,x_2+a)$ and so Lemma 3.9 forces $a+z_2$ to have a common octad with $a+x_2$ and $a+y_1$, whence we must have $a+z_2 =$ $\begin{array}{|cc|cc|cc|}\hline + & + & \circ & - & - & \circ \\ + & + & \circ & - & - & \circ \\ + & + & - & \circ & \circ & - \\ + & + & - & \circ & \circ & - \\ \hline \end{array}$. Similarly we find that $a+y_2 = \begin{array}{|cc|cc|cc|}\hline + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ + & + & - & - & \circ & \circ \\ + & + & - & - & \circ & \circ \\ \hline \end{array}$. Because Y_a is the unique sextet incident with both $a+y_1$ and $a+y_2$ we see that $Y_a = \begin{array}{|cc|cc|cc|}\hline \star & \star & \cdot & \cdot & + & + \\ \star & \star & \cdot & \cdot & + & + \\ \circ & \circ & \square & \square & \times & \times \\ \circ & \circ & \square & \square & \times & \times \\ \hline \end{array}$.

Likewise $a+z_1$ and $a+z_2$ uniquely specify $Z_a = \begin{array}{|cc|cc|cc|}\hline \star & \circ & \times & \cdot & + & \square \\ \star & \circ & \times & \cdot & + & \square \\ \circ & \star & \cdot & \times & \square & + \\ \circ & \star & \cdot & \times & \square & + \\ \hline \end{array}$.

Hence $\{X_a, Y_a, Z_a\}$ in a sextet line in Γ_a and so, since (4.7.3) also holds for Y_a and Z_a, we have proved part (ii).

Before tackling part (iii) we develop further our earlier concrete description of the sextet line $\{X_a, Y_a, Z_a\}$. Recall that $l = a+b$ was assumed to be the standard trio and, without loss of generality, $a+x_2 = \begin{array}{|cc|cc|cc|}\hline + & + & - & \circ & \circ & - \\ + & + & - & \circ & \circ & - \\ + & + & - & \circ & \circ & - \\ + & + & - & \circ & \circ & - \\ \hline \end{array}$.

We now aim to describe the 18 lines incident with a and each of the points in I (=$(\Gamma_0(X_a) \cup \Gamma_0(Y_a) \cup \Gamma_0(Z_a)) \cap \{a, c\}^{\perp}$). Now there are two lines (l_1 and l_2) in $\Gamma_1(X_a)$ incident with b, each containing two points of $(\{a, c\}^{\perp} \cap \Gamma_0(X_a)) \backslash \{b, x_2\}$, with a similar statement for x_2. Consequently, for $x \in (\{a, c\}^{\perp} \cap \Gamma_0(a)) \backslash \{b, x_2\}$, $b+a \in \alpha_3(b, b+x)$ and thus $a+x \in \alpha_3(a, a+b)$ by Lemma 3.9. Likewise $a+x \in \alpha_3(a, a+x_2)$. Therefore the trios $a+x$ and $a+b$ cannot have an octad in common and the trios $a+x$ and $a+x_2$ cannot have an octad in common. Surveying the trios in X_a (the standard sextet) we see that there are only four trios satisfying these conditions and so we have pinned down the lines $a+x$ for $x \in (\{a, c\}^{\perp} \cap \Gamma_0(X_a)) \backslash \{b, x_2\}$. They are:-

$$\underset{a+x_3}{\begin{array}{|cc|cc|cc|}\hline \circ & - & + & + & \circ & - \\ \circ & - & + & + & \circ & - \\ \circ & - & + & + & \circ & - \\ \circ & - & + & + & \circ & - \\ \hline \end{array}} \qquad \underset{a+x_4}{\begin{array}{|cc|cc|cc|}\hline \circ & - & + & + & - & \circ \\ \circ & - & + & + & - & \circ \\ \circ & - & + & + & - & \circ \\ \circ & - & + & + & - & \circ \\ \hline \end{array}}$$

$$\begin{array}{|cc|cc|cc|}\hline \circ & - & \circ & - & + & + \\ \circ & - & \circ & - & + & + \\ \circ & - & \circ & - & + & + \\ \circ & - & \circ & - & + & + \\ \hline \end{array} \qquad \begin{array}{|cc|cc|cc|}\hline - & \circ & \circ & - & + & + \\ - & \circ & \circ & - & + & + \\ - & \circ & \circ & - & + & + \\ - & \circ & \circ & - & + & + \\ \hline \end{array}.$$

Starting with $a+y_1$ and $a+y_2$ (as described earlier) we may repeat the above strategy and so obtain the lines $a+y$ for $y \in (\{a, c\}^{\perp} \cap \Gamma_0(Y_a)) \backslash \{y_1, y_2\}$ thus:

$$\underset{a+y_3}{\begin{array}{|cc|cc|cc|}\hline \circ & \circ & + & + & \circ & \circ \\ \circ & \circ & + & + & \circ & \circ \\ - & - & + & + & - & - \\ - & - & + & + & - & - \\ \hline\end{array}} \qquad \underset{a+y_4}{\begin{array}{|cc|cc|cc|}\hline \circ & \circ & + & + & - & - \\ \circ & \circ & + & + & - & - \\ - & - & + & + & \circ & \circ \\ - & - & + & + & \circ & \circ \\ \hline\end{array}}$$

$$\begin{array}{|cc|cc|cc|}\hline \circ & \circ & \circ & \circ & + & + \\ \circ & \circ & \circ & \circ & + & + \\ - & - & - & - & + & + \\ - & - & - & - & + & + \\ \hline\end{array} \qquad \begin{array}{|cc|cc|cc|}\hline \circ & \circ & - & - & + & + \\ \circ & \circ & - & - & + & + \\ - & - & \circ & \circ & + & + \\ - & - & \circ & \circ & + & + \\ \hline\end{array}.$$

And starting with $a+z_1$ and $a+z_2$ yields $a+z$ for $z \in (\{a,c\}^{\perp} \cap \Gamma_0(Z_a)) \setminus \{z_1, z_2\}$:

$$\underset{a+z_3}{\begin{array}{|cc|cc|cc|}\hline - & \circ & + & + & - & \circ \\ - & \circ & + & + & - & \circ \\ \circ & - & + & + & \circ & - \\ \circ & - & + & + & \circ & - \\ \hline\end{array}} \qquad \underset{a+z_4}{\begin{array}{|cc|cc|cc|}\hline - & \circ & + & + & \circ & - \\ - & \circ & + & + & \circ & - \\ \circ & - & + & + & - & \circ \\ \circ & - & + & + & - & \circ \\ \hline\end{array}}$$

$$\begin{array}{|cc|cc|cc|}\hline \circ & - & \circ & - & + & + \\ \circ & - & \circ & - & + & + \\ - & \circ & - & \circ & + & + \\ - & \circ & - & \circ & + & + \\ \hline\end{array} \qquad \begin{array}{|cc|cc|cc|}\hline \circ & - & - & \circ & + & + \\ \circ & - & - & \circ & + & + \\ - & \circ & \circ & - & + & + \\ - & \circ & \circ & - & + & + \\ \hline\end{array}.$$

We now begin the proof of part (iii). So as to avoid additional notation, we suppose $b' \in I$ is such that $a+b' = \begin{array}{|cc|cc|cc|}\hline - & \circ & + & + & - & \circ \\ - & \circ & + & + & - & \circ \\ \circ & - & + & + & \circ & - \\ \circ & - & + & + & \circ & - \\ \hline\end{array}$ (the argument being similar for any other choice of $a+b'$). So $b' \in \Gamma_0(Z_a)$. From (ii) we have five points in $\Gamma_0(Z_a) \cap \{a,c\}^{\perp}$ collinear with b' and so $Z_a \in \mathcal{S}_{b'}(a,c)$ (here Z_a is playing the role of X_a in part (i)).

Clearly we have $a+x_i$, $a+y_i \in \alpha_2(a, a+b')$ for $i = 3,4$. For $x \in I$ with $a+x$ equal to $a+x_i$ or $a+y_i$ (where $i \in \{3,4\}$) we must have $x \in \Delta_1(b')$ for otherwise we would have $d(b',x) = 2$, $x \in \Delta_2^1(b')$ and $c \in \Delta_2^2(a)$ which contradicts Lemma 4.6(ii). Thus we have located the four points in $\Delta_1(b) \cap \{a,c\}^{\perp}$ not contained in $\Gamma_0(Z_a)$; we may assume these points are x_3, x_4, y_3 and y_4. Note that $x_3, x_4 \in \Gamma_0(X_a)$ and $y_3, y_4 \in \Gamma_0(Y_a)$. Since $b' \notin \Gamma_0(X_a)$, b', x_3 and x_4 cannot be incident with a common line and, similarly, b', y_3 and y_4 cannot be incident with a common line.

Now (see the construction of the sextet line in (ii); note $\{x_3,x_4\},\{y_3,y_4\}$ play the role of $\{y_2,y_3\},\{z_1,z_2\}$) the other two sextets in $\mathcal{S}_{b'}(a,c)$ are uniquely determined by $\{a,x_3,x_4\}$ and $\{a,y_3,y_4\}$ and so must be X_a and Y_a. Thus $\mathcal{S}_{b'}(a,c) = \mathcal{S}_b(a,c)$ as asserted.

Set $\tau = \tau(X_a)$. Then $b,x_2 \in \Gamma_0(X_a)$ implies that $\tau \in Q(b) \cap Q(x_2)$ and so $c^\tau \in \Gamma_0(c+b) \cap \Gamma_0(c+x_2)$. Since $c+b$ and $c+x_2$ are distinct lines, this forces $c = c^\tau$. A similar argument works for $\tau(Y_a)$ and $\tau(Z_a)$, so proving (iv).

Next, we consider part (v). Supposing (v) is false we argue for a contradiction. Thus there exists $d \in \{a,c\}^\perp$ with $d \notin I$. By part (i) we have a sextet line $\mathcal{S}_d(a,c)$ in Γ_a. Set $\mathcal{S}_d(a,c) = \{X'_a, Y'_a, Z'_a\}$ and $I' = (\Gamma_0(X'_a) \cup \Gamma_0(Y'_a) \cup \Gamma_0(Z'_a)) \cap \{a,c\}^\perp$.

(4.7.5) $I \cap I' = \emptyset$.

If $I \cap I' \neq \emptyset$, then (iii) forces $I = I'$, contrary to $d \notin I$.

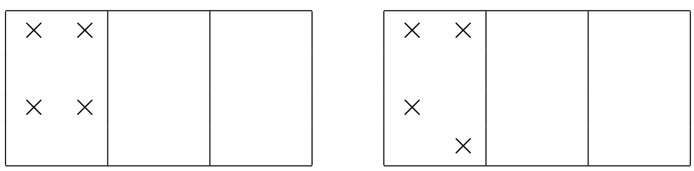
(4.7.6) For $S \in \mathcal{S}_b(a,c)$ and $S' \in \mathcal{S}_d(a,c)$ we have $S' \in \gamma_3(a,S)$.

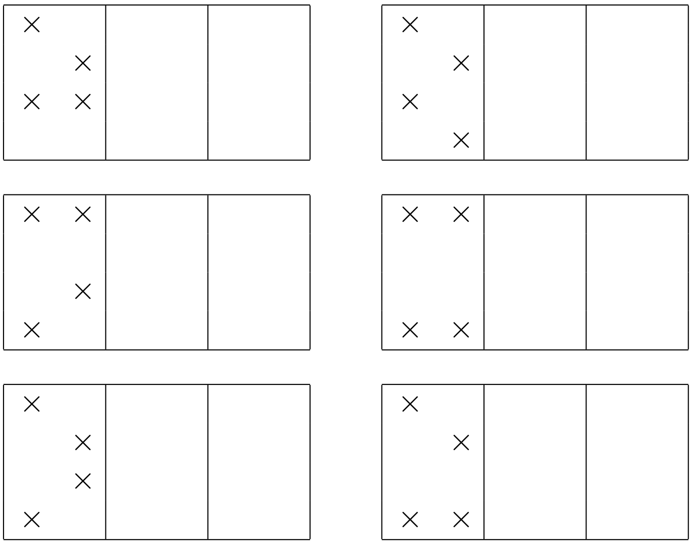
First observe that, for $x \in I$, $\Delta_1(x) \cap \{a,c\}^\perp \subseteq I$ by parts (i) (with $x = b$) and (iii). Hence, for $x \in I$ and $x' \in I'$, $d(x,x') = 2$ by (4.7.5). So, appealing to Lemma 4.6(ii), $x' \in \Delta_2^2(x)$. Set $\tau = \tau(S)$, and assume that $S' \notin \gamma_3(a,S)$. Then, by (2.8), there exists $x' \in \Gamma_0(S') \cap \{a,c\}^\perp$ such that $a + x' \in \beta_1(a,S)$. Lemma 3.3 implies that $x'^\tau \neq x'$ and from part (iv) $c^\tau = c$. For $x \in \Gamma_0(S) \cap \{a,c\}^\perp$ we now have τ fixing $c, x' \in \Delta_2^2(x)$ with τ not fixing x' which, using Lemmas 3.11 and 3.5, yields $c \in \Delta_2^1(a)$. From this contradiction we conclude that $S' \in \gamma_3(a,S)$, as claimed.

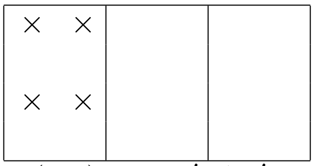
We now use the standard MOG labelling as described in Section 2 and will next show that

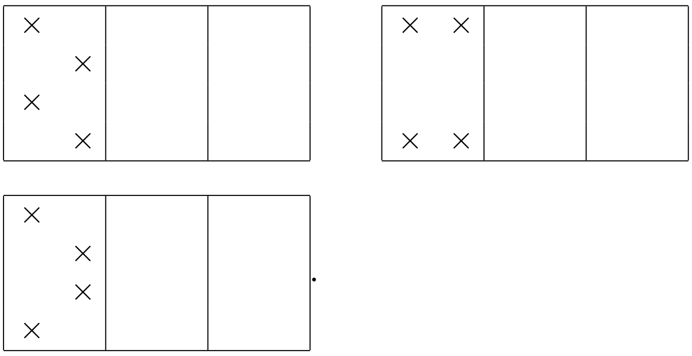
(4.7.7) for each $S' \in \mathcal{S}_d(a,c)$, S' has a tetrad containing $\{0,\infty\}$.

Assume that $S' \in \mathcal{S}_d(a,c)$ has no tetrad containing $\{0,\infty\}$ and let s be the tetrad of S' containing $\{\infty\}$. Then there are 8 possibilities for s as follows:





Without loss assume s is  ; the other 7 cases may be dealt with similarly. If $T' \in \mathcal{S}_d(a,c)$ and $T' \neq S'$, then since $T' \in \gamma_3(a,S)$ for each $S \in \mathcal{S}_b(a,c)$ by (4.7.6) and $T' \in \gamma_3(a,S')$ we see that there are three possibilities for the tetrad, t, in T' which contains $\{\infty\}$, these being



However for each of the three possibilities we see that $s \oplus t$ is a tetrad of a sextet in $\mathcal{S}_b(a,c)$. Since $\mathcal{S}_d(a,c)$ is a sextet line, this yields $\mathcal{S}_d(a,c) \cap \mathcal{S}_b(a,c) \neq \emptyset$, contrary to (4.7.5). Therefore S' has a tetrad containing $\{0,\infty\}$.

Using a similar argument to that in (4.7.7) we see that for each $S' \in \mathcal{S}_d(a,c)$, S' has a tetrad containing the following 2-element subsets: $\{3,15\}$, $\{14,8\}$, $\{20,18\}$,

$\{17,4\}$, $\{16,10\}$, $\{11,13\}$, $\{7,2\}$. Therefore X'_a, Y'_a, Z'_a must appear in the following list of 4 sextets.

$$
\begin{array}{|cc|cc|cc|}
\hline
\star & \square & \square & \star & \circ & \circ \\
\star & \square & \square & \star & + & + \\
\times & \cdot & \cdot & \times & \circ & + \\
\times & \cdot & \cdot & \times & + & \circ \\
\hline
\end{array}
\qquad
\begin{array}{|cc|cc|cc|}
\hline
\star & \cdot & \square & \times & \circ & + \\
\star & \cdot & \square & \times & + & \circ \\
\times & \square & \cdot & \star & \circ & \circ \\
\times & \square & \cdot & \star & + & + \\
\hline
\end{array}
$$

$$
\begin{array}{|cc|cc|cc|}
\hline
\star & \cdot & \star & \cdot & + & + \\
\star & \cdot & \star & \cdot & \circ & \circ \\
\times & \square & \times & \square & \circ & + \\
\times & \square & \times & \square & + & \circ \\
\hline
\end{array}
\qquad
\begin{array}{|cc|cc|cc|}
\hline
\star & \cdot & \times & \square & + & \circ \\
\star & \cdot & \times & \square & \circ & + \\
\times & \square & \star & \cdot & \circ & \circ \\
\times & \square & \star & \cdot & + & + \\
\hline
\end{array}
$$

It may be checked directly that taking the symmetric difference of any pair of these sextets we obtain one of the sextets in $\mathcal{S}_b(a,c) = \{X_a, Y_a, Z_a\}$. (By the symmetric difference of two of the above sextets we mean the sextet whose tetrads are given by the symmetric difference of intersecting tetrads of these two sextets). Thus we have $\mathcal{S}_d(a,c) \cap \mathcal{S}_b(a,c) \neq \emptyset$, again contradicting (4.7.5). This is the desired contradiction and so we have established part (v).

Finally we come to part (vi). Combining Theorem 3.6(i), (2.3)(ii), part (v) and Lemma 4.6(i) gives, for $b \in \{a,c\}^{\perp}$,

$$|\Delta_2^2(a)| = \frac{4|\alpha_1(b, b+a)||\Delta_1(a)|}{|\{a,c\}^{\perp}|} = \frac{4.1008.15180}{18} = 2^7.3.5.7.11.23,$$

as claimed. At last the proof of Theorem 4.7 is complete.

Because of Theorem 4.7(iii) we may, and shall, use $\mathcal{S}(a,c) = \{X_a, Y_a, Z_a\}$ to denote the sextet line in Γ_a when $c \in \Delta_2^2(a)$. Below we summarize, for later use, the concrete description of $\{a,c\}^{\perp}$ obtained in the previous theorem.

$$
\underset{X_a}{
\begin{array}{|cc|cc|cc|}
\hline
\star & \circ & \times & \cdot & + & \square \\
\star & \circ & \times & \cdot & + & \square \\
\star & \circ & \times & \cdot & + & \square \\
\star & \circ & \times & \cdot & + & \square \\
\hline
\end{array}}
\qquad
\underset{Y_a}{
\begin{array}{|cc|cc|cc|}
\hline
\star & \star & \cdot & \cdot & \times & \times \\
\star & \star & \cdot & \cdot & \times & \times \\
\circ & \circ & \square & \square & + & + \\
\circ & \circ & \square & \square & + & + \\
\hline
\end{array}}
\qquad
\underset{Z_a}{
\begin{array}{|cc|cc|cc|}
\hline
\star & \circ & \times & \cdot & + & \square \\
\star & \circ & \times & \cdot & + & \square \\
\circ & \star & \cdot & \times & \square & + \\
\circ & \star & \cdot & \times & \square & + \\
\hline
\end{array}}
$$

$$\begin{array}{|cc|cc|cc|}\hline + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ \hline\end{array}$$
$a+x_1$

$$\begin{array}{|cc|cc|cc|}\hline + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ + & + & - & - & - & - \\ + & + & - & - & - & - \\ \hline\end{array}$$
$a+y_1$

$$\begin{array}{|cc|cc|cc|}\hline + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ + & + & - & \circ & - & \circ \\ + & + & - & \circ & - & \circ \\ \hline\end{array}$$
$a+z_1$

$$\begin{array}{|cc|cc|cc|}\hline + & + & \circ & - & - & \circ \\ + & + & \circ & - & - & \circ \\ + & + & \circ & - & - & \circ \\ + & + & \circ & - & - & \circ \\ \hline\end{array}$$
$a+x_2$

$$\begin{array}{|cc|cc|cc|}\hline + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ + & + & - & - & \circ & \circ \\ + & + & - & - & \circ & \circ \\ \hline\end{array}$$
$a+y_2$

$$\begin{array}{|cc|cc|cc|}\hline + & + & \circ & - & - & \circ \\ + & + & \circ & - & - & \circ \\ + & + & - & \circ & \circ & - \\ + & + & - & \circ & \circ & - \\ \hline\end{array}$$
$a+z_2$

$$\begin{array}{|cc|cc|cc|}\hline \circ & - & + & + & \circ & - \\ \circ & - & + & + & \circ & - \\ \circ & - & + & + & \circ & - \\ \circ & - & + & + & \circ & - \\ \hline\end{array}$$
$a+x_3$

$$\begin{array}{|cc|cc|cc|}\hline \circ & \circ & + & + & \circ & \circ \\ \circ & \circ & + & + & \circ & \circ \\ - & - & + & + & - & - \\ - & - & + & + & - & - \\ \hline\end{array}$$
$a+y_3$

$$\begin{array}{|cc|cc|cc|}\hline - & \circ & + & + & - & \circ \\ - & \circ & + & + & - & \circ \\ \circ & - & + & + & \circ & - \\ \circ & - & + & + & \circ & - \\ \hline\end{array}$$
$a+z_3$

$$\begin{array}{|cc|cc|cc|}\hline \circ & - & + & + & - & \circ \\ \circ & - & + & + & - & \circ \\ \circ & - & + & + & - & \circ \\ \circ & - & + & + & - & \circ \\ \hline\end{array}$$
$a+x_4$

$$\begin{array}{|cc|cc|cc|}\hline \circ & \circ & + & + & - & - \\ \circ & \circ & + & + & - & - \\ - & - & + & + & \circ & \circ \\ - & - & + & + & \circ & \circ \\ \hline\end{array}$$
$a+y_4$

$$\begin{array}{|cc|cc|cc|}\hline - & \circ & + & + & \circ & - \\ - & \circ & + & + & \circ & - \\ \circ & - & + & + & - & \circ \\ \circ & - & + & + & - & \circ \\ \hline\end{array}$$
$a+z_4$

$$\begin{array}{|cc|cc|cc|}\hline \circ & - & \circ & - & + & + \\ \circ & - & \circ & - & + & + \\ \circ & - & \circ & - & + & + \\ \circ & - & \circ & - & + & + \\ \hline\end{array}$$
$a+x_5$

$$\begin{array}{|cc|cc|cc|}\hline \circ & \circ & \circ & \circ & + & + \\ \circ & \circ & \circ & \circ & + & + \\ - & - & - & - & + & + \\ - & - & - & - & + & + \\ \hline\end{array}$$
$a+y_5$

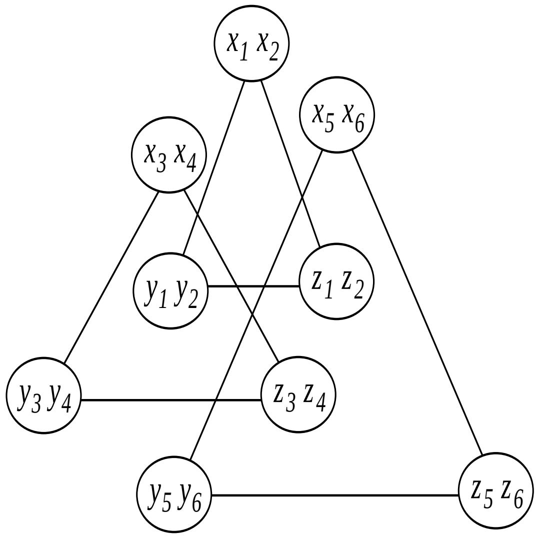
$$\begin{array}{|cc|cc|cc|}\hline \circ & - & \circ & - & + & + \\ \circ & - & \circ & - & + & + \\ - & \circ & - & \circ & + & + \\ - & \circ & - & \circ & + & + \\ \hline\end{array}$$
$a+z_5$

$$\begin{array}{|cc|cc|cc|}\hline \circ & - & - & \circ & + & + \\ \circ & - & - & \circ & + & + \\ \circ & - & - & \circ & + & + \\ \circ & - & - & \circ & + & + \\ \hline\end{array}$$
$a+x_6$

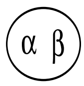
$$\begin{array}{|cc|cc|cc|}\hline \circ & \circ & - & - & + & + \\ \circ & \circ & - & - & + & + \\ - & - & \circ & \circ & + & + \\ - & - & \circ & \circ & + & + \\ \hline\end{array}$$
$a+y_6$

$$\begin{array}{|cc|cc|cc|}\hline - & \circ & \circ & - & + & + \\ - & \circ & \circ & - & + & + \\ \circ & - & - & \circ & + & + \\ \circ & - & - & \circ & + & + \\ \hline\end{array}$$
$a+z_6$

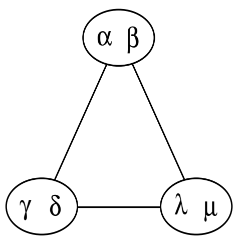
Diagrammatically we may think of the points in $\{a,c\}^{\perp}$ in the following way.



The points in



are such that $\{\alpha, \beta, a\}$ and $\{\alpha, \beta, c\}$ are both sparse triangles. The joining lines in the picture show $\{a, c\}^{\perp}$ from c's point of view. That is to say



means that $\{\alpha, \beta, \gamma, \delta, \lambda, \mu\}$ are the points in $\Gamma_0(S) \cap \{a, c\}^{\perp}$ for some $S \in \mathcal{S}(c, a)$.

Theorem 4.8 Let $c \in \Delta_2^3(a)$.
(i) $\{a, c\}^{\perp} = \Gamma_0(l)$ for some $l \in \Gamma_1$, and so $|\{a, c\}^{\perp}| = 5$.
(ii) $|\Delta_2^3(a)| = 2^{11}.3^2.7.11.23$.

Proof (i) By definition there exists $b \in \{a, c\}^{\perp}$ with $b + c \in \alpha_0(b, b + a)$, whence Lemma 4.6(i) implies

(4.8.1) for every $x \in \{a, c\}^{\perp}$, $x + c \in \alpha_0(x, x + a)$.

From Lemma 4.4 there exists a unique line $l \in \Gamma_1(b)$ with $l \in \alpha_3(b, b+a) \cap \alpha_3(b, b+c)$ and $\Delta_1(b) \cap \{a,c\}^\perp = \Gamma_0(l) \setminus \{b\}$. We assume there exists $d \in \{a,c\}^\perp$ with $d \notin \Gamma_0(l)$ and argue for a contradiction. By (4.8.1) and Lemma 4.4 again we may find $k \in \Gamma_1(d)$ with $k \in \alpha_3(d, d+a) \cap \alpha_3(d, d+c)$ and $\Gamma_1(d) \cap \{a,c\}^\perp = \Gamma_0(k) \setminus \{d\}$. Clearly $k \neq l$. Let X (respectively Y) be the unique plane in $\Gamma_2(a,l)$ (respectively $\Gamma_2(a,k)$).

(4.8.2) If $x_1, x_2 \in \Gamma_0(l)$ with $x_1 \neq x_2$, then $a + x_1 \in \alpha_3(a, a+x_2)$. Similarly if $y_1, y_2 \in \Gamma_0(k)$ with $y_1 \neq y_2$, then $a + y \in \alpha_3(a, a+y_2)$.

This is a consequence of the fact that $l \in \alpha_3(b, b+a)$, $k \in \alpha_3(d, d+a)$ and Lemma 3.2.

By (4.8.2) we get

(4.8.3) Each of the 15 octads containing a tetrad of X lies in exactly one of the trios $\{a + x | x \in \Gamma_0(l)\}$, with a similar statement for Y and $\{a + y | y \in \Gamma_0(k)\}$.

(4.8.4) $Y \in \gamma_3(a, X)$.

If $Y \in \gamma_0(a,X) \cup \gamma_1(a,X)$, then we can find an octad O, containing a tetrad of X, which cuts the 6 tetrads of Y in 3.1^5. Let $x \in \Gamma_0(l)$ be such that O is an octad of the trio $a + x$. Then by definition $a + x \in \beta_1(a, Y)$, whence $\tau(Y) \notin G_x$ from Lemma 3.3. Notice that $\tau(Y) \in G_c$ because $\tau(Y) \in Q(x)$ for all $x \in \Gamma_0(k)$ and if $c^{\tau(Y)} \neq c$, $c + c^{\tau(Y)} = c + x$ for every $x \in \Gamma_0(k)$, contrary to Lemma 3.5. Hence $\{c, x, x^{\tau(Y)}\}$ is a triangle and so Lemma 3.11 implies $\Gamma_2(c, x, x^{\tau(Y)}) \neq \emptyset$. However $\tau(Y) \in Q(a)$ and so $a + x = a + x^{\tau(Y)}$, which gives $\Gamma_2(a,c) \neq \emptyset$. Using Theorem 4.3(ii) we have $\Gamma_2(a,b,c) \neq \emptyset$, contrary to $b + c \in \alpha_0(b, b+a)$, and (4.8.4) is proved.

Denote the unique trio in $\Gamma_1(a, X, Y)$ by m. We first show that either $\Gamma_0(m) \cap \Gamma_0(l) = \emptyset$ or $\Gamma_0(m) \cap \Gamma_0(k) = \emptyset$. Assume that $x \in \Gamma_0(m) \cap \Gamma_0(l)$ and $y \in \Gamma_0(m) \cap \Gamma_0(k)$. If $x \neq y$, then $\{c, x, y\}$ is a triangle. Moreover $x + y = m = x + a$, whence $\Gamma_2(a,c) \neq \emptyset$ by Lemma 3.11 against $c \in \Delta_2^3(a)$. Therefore $x = y$. However, (4.8.1) implies that $x + c \in \alpha_0(x, x+a)$ and so we get a contradiction to Lemma 4.4 because l and k are two distinct lines in $\Gamma_1(x)$ lying in $\alpha_3(x, x+a) \cap \alpha_3(x, x+c)$ such that $\Delta_1(x) \cap \{a,c\}^\perp = \Gamma_0(l) \setminus \{x\} = \Gamma_0(k) \setminus \{x\}$. Without loss we assume $\Gamma_0(m) \cap \Gamma_0(k) = \emptyset$. By (4.8.4) and (2.7)(i) we may choose X and Y to be, respectively,

$$\begin{array}{|cc|cc|cc|}
\hline
\star & \circ & \times & \cdot & + & \square \\
\star & \circ & \times & \cdot & + & \square \\
\star & \circ & \times & \cdot & + & \square \\
\star & \circ & \times & \cdot & + & \square \\
\hline
\end{array}
\text{ and }
\begin{array}{|cc|cc|cc|}
\hline
\star & \star & \cdot & \cdot & \times & \times \\
\star & \star & \cdot & \cdot & \times & \times \\
\circ & \circ & \square & \square & + & + \\
\circ & \circ & \square & \square & + & + \\
\hline
\end{array}$$

with m being the standard trio. Using (4.8.3) we can choose $e \in \Gamma_0(k)$ such that $a+e$ contains the left-hand octad of m, and $a+e \neq m$. If $\Gamma_0(m) \cap \Gamma_0(l) = \emptyset$, then by (4.8.3) again we can find $f \in \Gamma_0(l)$ with $a+f \neq m$ such that $a+f$ contains the right-hand octad of m. This forces $a+e \in \alpha_1(a, a+f)$ and so $e \in \Delta_2^2(f)$. We now have a contradiction to Lemma 4.6(ii) because $e, f \in \{a,c\}^\perp$ and $c \in \Delta_2^3(a)$. Hence we may assume $\Gamma_0(m) \cap \Gamma_0(l) \neq \emptyset$. In this case we choose $f \in \Gamma_0(l)$ such that $a+f$ is a trio incident with (the octad) $\begin{array}{|cc|cc|cc|}
\hline
 & & \times & & \times & \\
 & & \times & & \times & \\
 & & \times & & \times & \\
 & & \times & & \times & \\
\hline
\end{array}$. From (4.8.3) and $\Gamma_0(m) \cap \Gamma_0(l) \neq \emptyset$ we have that $a+f$ does not contain the left-hand octad of m and we again obtain that $a+e \in \alpha_1(a, a+f)$, which leads to a contradiction. This completes the proof of part (i).

(ii) By Lemma 4.6(i) for every $x \in \{a,c\}^\perp$, $x+c \in \alpha_0(x, x+a)$. Hence if $b \in \{a,c\}^\perp$ (2.3)(ii), Theorem 3.6(i) and part (i) yield

$$|\Delta_2^3(a)| = \frac{4|\alpha_0(b, b+a)||\Delta_1(a)|}{|\{a,c\}^\perp|} = \frac{4.2688.15180}{5} = 2^{11}.3^2.7.11.23.$$

5. ADJACENCY IN $\Delta_2(a)$, A BEGINNING

This section is wholly devoted to the study of the set $\Delta_1(c) \cap \Delta_2(a)$ for $c \in \Delta_2^1(a)$. In turn we consider the sets $\Delta_1(c) \cap \Delta_2^i(a)$ for $i = 1,2,3$. We recall that for each $c \in \Delta_2^1(a)$, $X(a,c)$ is the unique plane in $\Gamma_2(a,c)$.

Lemma 5.1 Let $c, d \in \Delta_2^1(a)$ with $d \in \Delta_1(c)$. Assume $X(a,c) \neq X(a,d)$. Then $\{a,c\}^\perp \cap \{a,d\}^\perp \neq \emptyset$, and $X(a,c) \in \gamma_3(a, X(a,d))$.

Proof We first show that $X(a,c) \in \gamma_3(a, X(a,d))$ by assuming otherwise and arguing for a contradiction. By (2.7), $X(a,c) \in \gamma_0(a, X(a,d)) \cup \gamma_1(a, X(a,d))$ whence there exists $l \in \Gamma_1(a, X(a,d))$ such that $l \in \beta_1(a, X(a,c))$. From (2.16) there are three points in $\Gamma_0(l) \cap \Delta_1(d)$, and we let $\{a,x\} = \Gamma_0(l) \cap \Delta_2^1(d)$. Let $\tau =$

$\tau(X(a,c))$. By Lemma 3.3 $x^\tau \in \Gamma_0(l) \cap \Delta_1(d)$ and so $d^\tau \neq d$. Recall that $\tau \in Q(a) \cap Q(c)$. Hence $d + c = d + d^\tau$ and $X(a,d)^\tau = X(a,d)$. Therefore $d + c = d + d^\tau \in \Gamma_1(X(a,d))$ by Lemma 3.10 and thus $c \in \Gamma_0(X(a,d))$. But then $X(a,c) = X(a,d)$ by Theorem 4.3(i), contrary to $X(a,c) \neq X(a,d)$. Since $X(a,c) \in \gamma_3(a, X(a,d))$ there exists $m \in \Gamma_1(a, X(a,c), X(a,d))$. By (2.16) c and d are each collinear with three points of $\Gamma_0(m)$ and so $\Gamma_0(m) \cap \Delta_1(c) \cap \Delta_1(d) \neq \emptyset$ because $|\Gamma_0(m)| = 5$. This completes the proof of the lemma.

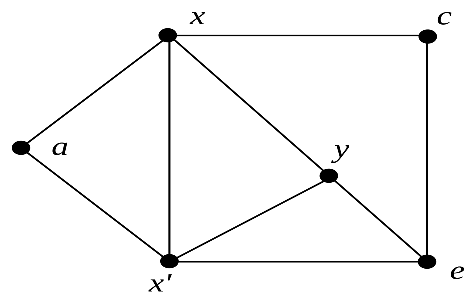
Theorem 5.2 Let $c \in \Delta_2^1(a)$ and $e \in \Delta_2^3(a)$. Then $e \notin \Delta_1(c)$.

Proof We assume the theorem is false and argue for a contradiction. So we have $c \in \Delta_2^1(a)$ and $e \in \Delta_2^3(a)$ with $e \in \Delta_1(c)$. Let X denote the unique plane in $\Gamma_2(a)$ such that $\{a,e\}^\perp \subseteq \Gamma_0(X)$ and put $\tau = \tau(X(a,c))$.

(5.2.1) $X \in \gamma_3(a, X(a,c)) \cup \{X(a,c)\}$.

If (5.2.1) does not hold then $X \in \gamma_0(a, X(a,c)) \cup \gamma_1(a, X(a,c))$. By considering the intersection matrices for the tetrads of X and $X(a,c)$ we see that there are exactly 8 octads of X intersecting the tetrads of $X(a,c)$ in 3.1^5 and every trio incident with X contains exactly 0 or 2 of these 8 octads. By Lemma 4.4 for any $x, x' \in \{a,e\}^\perp$ with $x \neq x'$ the trios $a + x$ and $a + x'$ are incident with X and do not contain a common octad, whence we may choose $b, b' \in \{a,e\}^\perp$ such that $a + b' \in \beta_2(a, X(a,c)) \cup \beta_3(a, X(a,c))$ and $a + b \in \beta_1(a, X(a,c))$. From Lemma 3.3 we have $b^\tau \neq b$ and $(b')^\tau = b'$. Thus $e^\tau \neq e$ otherwise $\{e, b, b^\tau\}$ is a triangle which by Lemma 3.11 implies $\Gamma_2(e, b, b^\tau) \neq \emptyset$ and then $\Gamma_2(e,a) \neq \emptyset$ from Lemma 3.11, against $e \in \Delta_2^3(a)$. We now have a triangle $\{e, e^\tau, b'\}$ which, by Lemma 3.11, must lie in the residue of a plane Y. Since $\tau \in Q(c)$, $e + e^\tau = e + c$, whence $c, b' \in \Gamma_0(Y)$ and so $b' \in \Delta_1(c) \cup \Delta_2^1(c)$. If $b' \in \Delta_2^1(c)$, Lemma 5.1 implies that $Y \in \gamma_3(c, X(c,a)) \cup \{X(c,a)\}$. However we then have $\tau (= \tau(X(c,a))) \in Q(Y)$ which contradicts the fact that $e \in \Gamma_0(Y)$ and $e^\tau \neq e$. We therefore conclude that $b' \in \Delta_1(c)$ and consequently $a + b \in \Gamma_1(a, X, X(a,c))$. This contradicts the fact that $X \in \gamma_0(a, X(a,c)) \cup \gamma_1(a, X(a,c))$ and so (5.2.1) is proved.

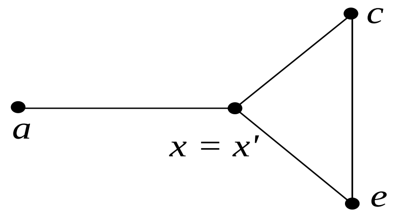
By (5.2.1) and Lemma 4.4 we can choose $x \in \{a,c\}^\perp$ and $x' \in \{a,e\}^\perp$ such that either $a + x \in \alpha_2(a, a + x')$ or $a + x = a + x'$. First assume that for every $x \in \{a,c\}^\perp$ and $x' \in \{a,e\}^\perp$, $a + x \neq a + x'$. Choose $x \in \{a,c\}^\perp$ and $x' \in \{a,e\}^\perp$ so that $a + x \in \alpha_2(a, a + x')$. From Lemma 3.8 x' is collinear with three points of $\Gamma_0(a + x)$ and since c is collinear with three points of $\Gamma_0(a + x) \setminus \{x\}$ we may suppose that x is chosen so that $x \in \Delta_1(x')$ and $a+x \in \alpha_2(a, a+x')$ (y will be introduced shortly).



By assumption $x' \notin \{a,c\}^{\perp}$, $x \notin \{a,e\}^{\perp}$, whence $d(x,e) = 2 = d(x',c)$. Since $x'+x \in \alpha_2(x', x'+a)$ by Lemma 3.9 and $x'+e \in \alpha_0(x', x'+a)$ by Lemma 4.6(i), we must have $x'+e \in \alpha_0(x', x'+x) \cup \alpha_1(x', x'+x)$. Therefore $e \in \Delta_2^2(x) \cup \Delta_2^3(x)$. If $e \in \Delta_2^3(x)$, then since all points of $\{e,x\}^{\perp}$ are collinear from Lemma 4.4 we obtain the contradiction that $c \in \Delta_1(x')$. Hence $e \in \Delta_2^2(x)$, whereupon $c \in \Delta_2^2(x')$ from Lemma 4.6(ii). From Theorem 4.7(ii) there exists a sextet line $\mathcal{S}(x,e)$ in Γ_x and a sextet $X_x \in \mathcal{S}(x,e) \cap \Gamma_2(x')$ such that $S := \{x,e\}^{\perp} \cap \Delta_1(x') \cap \Gamma_0(X_x)$ is a set of five pairwise collinear points. Let $y \in S$. Then e is collinear with exactly three points of $\Gamma_0(x'+y)$ and so $y \notin \Delta_1(a)$ because the five points of $\{a,e\}^{\perp}$ are incident with a unique line. If $y \in \Delta_2^1(a)$, then in $\Gamma_{x'}$ the trios $x'+y$ and $x'+a$ have a common octad which cuts the octads of $x'+e$ in $4, 2^2$. Hence $x'+e \in \alpha_i(x', x'+y)$ for $i = 0,1$ which contradicts the fact that $y \in \Delta_1(e)$. Furthermore $y \notin \Delta_2^3(a)$, otherwise Lemma 4.4 implies that a is collinear with all points of $\Gamma_0(x'+x)$, which is not the case by Lemma 3.8 because $x'+x \in \alpha_2(x', x'+a)$. Thus we deduce,

(5.2.2) $y \in \Delta_2^2(a)$ for every $y \in S$.

Since $x+c, x+x' \in \alpha_2(x, x+a)$, in Γ_x the trios $x+c$ and $x+x'$ each share a common octad with the trio $x+a$. However $x' \in \Delta_2^2(c)$ implies that $x+x' \in \alpha_1(x, x+c)$, whence these two common octads are distinct octads of $x+a$. Since these two octads are disjoint and $x+x' \neq x+a \neq x+c$, by Theorem 4.7 we conclude that the two octads lie in the trio l incident with each sextet of $\mathcal{S}(x,e)$. Any trio is uniquely determined by two of its octads and so we have $x+a = l$. We now have a contradiction to (5.2.2) because $l \in \alpha_2(x, x+y)$ for every $y \in S$ and so $y \in \Delta_1(a) \cup \Delta_2^1(a)$ by Lemma 3.11. Therefore we deduce that there exists $x \in \{a,c\}^{\perp}$, $x' \in \{a,e\}^{\perp}$ with $a+x = a+x'$. If $x \notin \{a,e\}^{\perp}$ and $x' \notin \{a,c\}^{\perp}$, then $x \in \Delta_2^3(e)$ and $x' \notin \Delta_2^1(c)$ which contradicts Lemma 4.6(ii). Thus we may assume that $x = x'$, as pictured below.



In Γ_x, $x+a$ and $x+c$ share a common octad which cuts the octads of $x+e$ in 4.2^2 because $x+e \in \alpha_0(x,x+a)$. Therefore $x+e \notin \alpha_i(x,x+c)$ for $i=2,3$ which contradicts Lemma 3.11 and (2.9) because $e \in \Delta_1(c)$. This completes the proof of Theorem 5.2.

Lemma 5.3 Let $c \in \Delta_2^1(a)$. Then $|\Delta_1(c) \cap \Delta_2^1(a)| = 375$ with 15 points of $\Delta_1(c) \cap \Delta_2^1(a)$ lying in $\Gamma_0(X(a,c))$. Further, for $l \in \beta_0(c,X(c,a))$, $|\Delta_1(a) \cap \Gamma_0(l)| = 3$ and $|\Delta_2^1(a) \cap \Gamma_0(l)| = 2$.

Proof Assume $d \in \Delta_1(c) \cap \Delta_2^1(a)$, with $d \notin \Gamma_0(X(a,c))$. Then Lemma 5.1 implies that $X(a,c) \in \gamma_3(a,X(a,d))$ and $\{a,c\}^\perp \cap \{a,d\}^\perp \neq \emptyset$. Let $b \in \{a,c\}^\perp \cap \{a,d\}^\perp$. Since $b+c \in \alpha_2(b,b+a)$, (2.3) allows us to assume, without loss, that in Γ_b,

$$b+a = \begin{array}{|cc|cc|cc|} \hline + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ + & + & \circ & \circ & - & - \\ \hline \end{array} \text{ and } b+c = \begin{array}{|cc|cc|cc|} \hline + & + & \circ & \circ & \circ & \circ \\ + & + & \circ & \circ & \circ & \circ \\ + & + & - & - & - & - \\ + & + & - & - & - & - \\ \hline \end{array}.$$

The trio $b+d$ has an octad in common with $b+a$ because $b+d \in \alpha_2(b,b+a)$. If this octad is not the left-hand octad of $b+a$, then $X(a,c) \in \gamma_3(b,X(a,d))$ forces $b+d \in \alpha_1(b,b+c)$. However $\{b,c,d\}$ is a triangle whence $\Gamma_2(b,c,d) \neq \emptyset$ which contradicts (2.9). Therefore $b+d$ contains the left-hand octad of $b+a$ and we can choose $b+d$ to be $\begin{array}{|cc|cc|cc|} \hline + & + & \circ & - & \circ & - \\ + & + & \circ & - & \circ & - \\ + & + & - & \circ & - & \circ \\ + & + & - & \circ & - & \circ \\ \hline \end{array}$. Notice that $b+d \in \alpha_2(b,b+c)$. This forces $X(a,c)$ and $X(a,d)$, when viewed as sextets in Γ_b, to be respectively

$$\begin{array}{|cc|cc|cc|} \hline \star & \star & \cdot & \cdot & \times & \times \\ \star & \star & \cdot & \cdot & \times & \times \\ \circ & \circ & \square & \square & + & + \\ \circ & \circ & \square & \square & + & + \\ \hline \end{array} \text{ and } \begin{array}{|cc|cc|cc|} \hline \star & \circ & \times & \cdot & + & \square \\ \star & \circ & \times & \cdot & + & \square \\ \circ & \star & \cdot & \times & \square & + \\ \circ & \star & \cdot & \times & \square & + \\ \hline \end{array}.$$

(5.3.1) If $m = b+a$, then d and c are collinear with the same three points in $\Gamma_0(m)$.

Assume otherwise and argue for a contradiction. Let $\Gamma_0(m) = \{a, b, x_1, x_2, x_3\}$ where $\{b, x_1, x_2\} = \Delta_1(c) \cap \Gamma_0(m)$ and $\{b, x_1, x_3\} = \Delta_1(d) \cap \Gamma_0(m)$. Choose the sextet Y in Γ_b to be $\begin{array}{|cc|cc|cc|} \hline \star & \star & \times & \square & \circ & \square \\ \cdot & \square & \times & \cdot & \circ & \cdot \\ \square & + & \times & + & \circ & + \\ + & \cdot & \circ & \star & \times & \star \\ \hline \end{array}$. Then $b+d \in \beta_3(a,Y)$, $b+c \in \beta_3(a,Y)$ and $b+a \in \beta_1(a,Y)$. Since $\tau(Y) \in Q(b)$, $\tau(Y)$ fixes $b+d$, $b+c$ and $b+a$. By Lemma 3.3, $\tau(Y)$ fixes c and d but acts on $\Gamma_0(m)$ by interchanging two pairs of points in $\Gamma_0(m) \backslash \{b\}$. Since $\tau(Y)$ must fix $\Delta_1(c) \cap \Gamma_0(m)$ and $\Delta_1(d) \cap \Gamma_0(m)$ we obtain the required contradiction and (5.3.1) is proved.

Notice that given $b \in \{a,c\}^{\perp}$ there are 6 sextets of Γ_b in $\gamma_3(b, X(a,c))$ which are incident with the trio $b+a$. If we let $Y \in \gamma_3(b, X(a,c)) \cap \Gamma_2(b+a)$, there are two lines in $\Gamma_1(b,Y) \cap \alpha_2(b, b+c) \cap \alpha_2(b, b+a)$ and each line is incident with precisely two points of $\Delta_2^1(a) \cap \Delta_1(c)$. Let $S = \{d \in \Delta_2^1(a) | d \in \Delta_1(c), X(a,d) \neq X(a,c)\}$. Then by the above and (5.3.1)

$$|S| = \frac{|\{a,c\}^{\perp}||\gamma_3(b, X(a,c)) \cap \Gamma_2(b+a)||\Gamma_1(b,Y) \cap \alpha_2(b, b+c) \cap \alpha_2(b, b+a)|.2}{3}$$

$$= \frac{45.6.2.2}{3} = 360.$$

To complete the proof we need to show that there are exactly 15 points of $\Delta_1(c) \cap \Delta_2^1(a)$ which are incident with $X(a,c)$. Since $\Delta_1(c) \cap \Gamma_0(X(a,c)) \subseteq \Delta_1(a) \cup \Delta_2^1(a)$ and $|\{a,c\}^{\perp} \cap \Gamma_0(X(a,c))| = 45$ we conclude that

$$|\Delta_1(c) \cap \Delta_2^1(a) \cap \Gamma_0(X(a,c))| = |\Delta_1(c) \cap \Gamma_0(X(a,c))| - 45 = 60 - 45 = 15.$$

Lemma 5.4 Let $c \in \Delta_2^1(a)$ and $d \in \Delta_1(c)$. Then $\{a,c\}^{\perp} \cap \Delta_1(d) \neq \emptyset$ if and only if $c+d \notin \beta_1(c, X(c,a))$.

Proof From Theorem 4.3(v), the orbits of G_{ac} on $\Gamma_1(c)$ are $\beta_i(c, X(c,a))$ for $i = 0,1,2,3$. If $c+d \in \beta_0(c, X(c,a))$, then $c+d \in \Gamma_1(X(c,a))$. By (2.16) $|\Gamma_0(c+d) \cap \{a,c\}^{\perp}| = 3$ and hence $\{a,c\}^{\perp} \cap \Delta_1(d) \neq \emptyset$ as required. Next suppose $c+d \in \beta_2(c, X(c,a))$. Thus there exists $l \in \Gamma_1(c, X(c,a))$ with $l \in \alpha_2(c, c+d)$ by (2.11) and by Lemma 3.8 $d \in \Delta_1(x)$ for three points x in $\Gamma_0(l)$. Since a is collinear with three points in $\Gamma_0(l) \backslash \{c\}$ we must have $\{a,c\}^{\perp} \cap \Delta_1(d) \neq \emptyset$. If $c+d \in$

$\beta_3(c, X(c,a))$ we can find $k \in \Gamma_1(c, X(c,a))$ with $k \in \alpha_3(c, c+d)$ from (2.11); then Lemma 3.8 and (2.16) give $\{a,c\}^{\perp} \cap \Delta_1(d) \neq \emptyset$. To conclude the proof we show that if $c+d \in \beta_1(c, X(c,a))$ then $\{a,c\}^{\perp} \cap \Delta_1(d) = \emptyset$. Assume that $c+d \in \beta_1(c, X(c,a))$ and that there exists $x \in \{a,c\}^{\perp} \cap \Delta_1(d)$. Then $c+x \in \Gamma_1(X(c,a))$ and $c+x \in \alpha_i(c, c+d)$ for $i = 2, 3$. However, by (2.11) $c+x \in \alpha_j(c, c+d)$ for $j = 0, 1$, a contradiction.

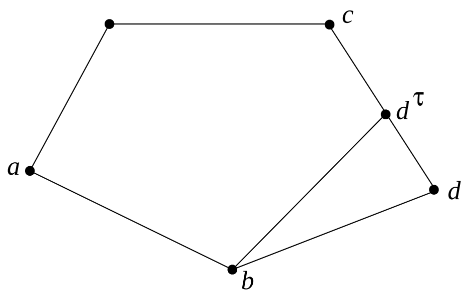
If $d \in \Delta_2^2(a)$ we let l_d denote the trio in $\Gamma_1(a)$ which is incident with S for every sextet $S \in \mathcal{S}(a,d)$.

Lemma 5.5 Let $c \in \Delta_2^1(a)$ and $d \in \Delta_2^2(a) \cap \Delta_1(c)$. Then either $\{a,c\}^{\perp} \cap \{a,d\}^{\perp} \neq \emptyset$ or $X(a,c) \in \Gamma_2(l_d)$.

Proof By Theorem 4.3(i) there is a unique sextet $X(a,c) \in \Gamma_2(a,c)$. Let $\mathcal{S}(a,d) = \{X, Y, Z\}$ be the sextet line in Γ_a introduced in Theorem 4.7. We first show

(5.5.1) for every $S \in \mathcal{S}(a,d)$, $S \in \gamma_3(a, X(a,c)) \cup \{X(a,c)\}$.

Assume $S \notin \gamma_3(a, X(a,c)) \cup \{X(a,c)\}$ and argue for a contradiction. Then $l \in \beta_1(a, X(a,c))$ for precisely 12 trios $l \in \Gamma_1(a,S)$ by (2.8) (i), (ii). Using the intersection matrices for $X(a,c)$ and S given in Section 2 we see that the three trios $l \in \Gamma_1(a,S)$ which do not lie in $\beta_1(a, X(a,c))$ contain the same octad. However every octad incident with S lies in some trio $a+x$ where $x \in \{a,d\}^{\perp}$ by Theorem 4.7, and so we may choose $b \in \{a,d\}^{\perp} \cap \Gamma_0(S)$ such that $a+b \notin \beta_1(a, X(a,c))$. Let $b' \in \{a,d\}^{\perp}$ with $a+b' \in \beta_1(a, X(a,c))$ and set $\tau = \tau(X(a,c))$. Then $(b')^{\tau} \neq b'$ by Lemma 3.3, whence $d^{\tau} \neq d$, otherwise $\{b', (b')^{\tau}, d\}$ is a triangle and Lemma 3.11 implies the impossible $\Gamma_2(a,d) \neq \emptyset$. Since $a+b \notin \beta_1(a, X(a,c))$ and $\tau \in Q(c)$ Lemma 3.3 gives us the following.

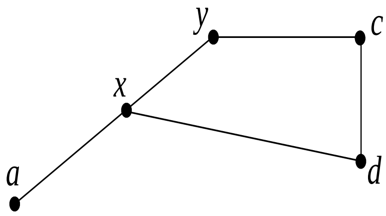


However, $b \notin \Delta_1(c)$ because $S \in \gamma_i(a, X(a,c))$ for $i = 0$ or 1 and so we must have $b \in \Delta_2^1(c)$. Therefore, by Lemma 5.1, there exists $b'' \in \{a,c\}^{\perp} \cap \{c,b\}^{\perp}$. Using

Lemma 4.6 and the fact that $S \in \gamma_i(a, X(a,c))$ for $i = 0$ or 1 we have that $b'' \in \Delta_2^1(d)$. By Theorem 4.7(i) $b + b''$ must lie in $\Gamma_1(S)$. We then have that $a + b'' \in \Gamma_1(S) \cap \Gamma_1(a, X(a,c))$ which contradicts the fact that $S \in \gamma_i(a, X(a,c))$ for $i = 0$ or 1. This proves (5.5.1)

(5.5.2) If $x \in \{a,d\}^\perp$ with $a + x \in \Gamma_1(X(a,c))$, then $\{a,c\}^\perp \cap \{a,d\}^\perp \neq \emptyset$.

By (2.16) there exists $y \in \{a,c\}^\perp$ with $a + y = a + x$. If $x \in \{a,c\}^\perp$ we are done; so we may suppose $x \in \Delta_2^1(c)$ and $x \neq y$.



However Theorem 4.3(ii) implies that $d \in \Gamma_0(X(c,x))$ and since $X(c,x) = X(c,a)$ we have $\Gamma_2(a,d) \neq \emptyset$, contrary to $d \in \Delta_2^2(a)$. Therefore (5.5.2) holds.

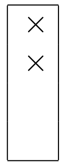
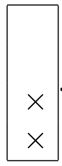
Suppose that $S = X(a,c)$ for some $S \in \mathcal{S}(a,d)$. Then there exists $x \in \{a,d\}^\perp \cap \Gamma_0(S)$ with $a + x \in \Gamma_1(X(a,c))$ and so (5.5.2) proves the result in this case. In view of (5.5.1) we may now assume $S \in \gamma_3(a, X(a,c))$ for all $S \in \mathcal{S}(a,d)$.

(5.5.3) In Γ_a, $X(a,c)$ is incident with an octad of l_d.

Without loss we may assume

$$X = \begin{array}{|cc|cc|cc|} \hline \star & \circ & \cdot & + & \times & \square \\ \star & \circ & \cdot & + & \times & \square \\ \star & \circ & \cdot & + & \times & \square \\ \star & \circ & \cdot & + & \times & \square \\ \hline \end{array} \text{ and } Y = \begin{array}{|cc|cc|cc|} \hline \star & \star & \cdot & \cdot & \times & \times \\ \star & \star & \cdot & \cdot & \times & \times \\ \circ & \circ & \square & \square & + & + \\ \circ & \circ & \square & \square & + & + \\ \hline \end{array},$$

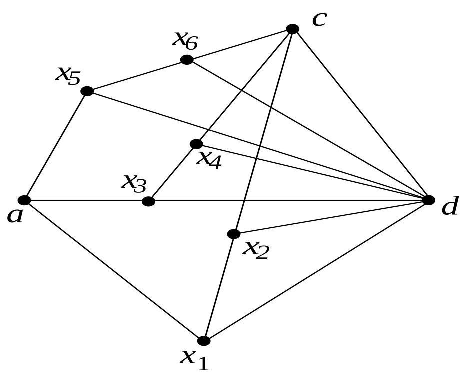
and so l_d is the standard trio. Let t be the tetrad of $X(a,c)$ containing the element $\begin{array}{|cc|cc|cc|} \hline \times & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ \hline \end{array}$. Since t cuts the tetrads of X and Y in $2^2.0^4$, either t is contained in the left-hand octad of l_d or t contains the duad $\begin{array}{|cc|cc|cc|} \hline \times & & & & & \\ \times & & & & & \\ & & & & & \\ & & & & & \\ \hline \end{array}$

and intersects one of the other five tetrads of X as  or . In the latter case we see, using the MOG, that t lies in a sextet incident with one of the three octads of l_d. Hence $X(a,c)$ is incident with an octad of l_d because $X(a,c)$ is the unique sextet containing t, and (5.5.3) is verified.

By (5.5.3) and Theorem 4.7 there exists $x \in \{a,d\}^{\perp}$ with $a+x \in \Gamma_1(X(a,c))$ and so $X(a,c) \notin \Gamma_2(l_d)$ implies that $\{a,c\}^{\perp} \cap \{a,d\}^{\perp} \neq \emptyset$. Now (5.5.2) completes the proof of the lemma.

Lemma 5.6 Let $c \in \Delta_2^1(a)$ and $l \in \Gamma_1(c)$ be such that $l \in \beta_2(c, X(c,a))$. If $d \in \Gamma_0(l) \cap \Delta_2^2(a)$, then, in Γ_d, $d+c$ is the trio incident with each sextet $S \in \mathcal{S}(d,a)$

Proof Let $X = X(a,c)$. By (2.11) there are three lines in $\Gamma_1(c,X) \cap \alpha_2(c,l)$ and 12 lines in $\Gamma_1(c,X) \cap \alpha_1(c,l)$. Let $\Gamma_1(c,X) \cap \alpha_2(c,l) = \{k_1,k_2,k_3\}$. From (2.16) and Lemma 3.8 there exists $x_1 \in \{a,c\}^{\perp} \cap \Gamma_0(k_1)$ with $x_1 \in \Delta_1(d)$. Since, in Γ_{x_1}, $x_1 + c \in \alpha_2(x_1, x_1+a) \cap \alpha_2(x_1, x_1+d)$, Theorem 4.7(i) implies that there exists a unique point $x_2 \in \Gamma_0(x_1+c) \setminus \{x_1\}$ with $x_2 \in \{a,d\}^{\perp}$. Similarly there exist $x_3, x_4 \in \Gamma_0(k_2)$ and $x_5, x_6 \in \Gamma_0(k_3)$ with $x_i \in \Delta_1(d) \cap \{a,c\}^{\perp}$ for $i = 3,4,5,6$ and $x_3 \neq x_4$, $x_5 \neq x_6$. Since $x_i \in \Gamma_0(X)$ for all $i = 1,2,3,4,5,6$ and $X \in \Gamma_2(a)$, then Theorem 4.7(ii) gives that $X \in \mathcal{S}(a,d)$.



Let k,m be the two lines in $\Gamma_1(x_1)$ incident with points in $\{a,d\}^{\perp} \setminus \{x_2,x_3,x_4,x_5,x_6\}$ and set $\Gamma_0(k) \cap \{a,d\}^{\perp} = \{x_1,y_1,y_2\}$ and $\Gamma_0(m) \cap \{a,d\}^{\perp} = \{x_1,z_1,z_2\}$. By Theorem 4.7(i) there exists $Y \in \Gamma_2(d)$ with $Y \in \Gamma_2(x_1+x_2,m,k)$ and since $c \in \Gamma_0(x_1+x_2)$ we have $c \in \Gamma_0(Y)$. However $\{y_1,y_2,z_1,z_2\} \cap \Gamma_0(X) = \emptyset$ and so $\{y_1,y_2,z_1,z_2\} \cap \{a,c\}^{\perp} = \emptyset$ which means that $\{y_1,y_2,z_1,z_2\} \subseteq \Delta_2^1(c)$. Therefore, in Γ_d, $d+x_i$, $d+y_i$, $d+z_i$ ($i=1,2$),each contain some octad of the trio $d+c$. Since the 6 trios $\{d+x_i, d+y_i, d+z_i | i=1,2\}$ are all incident with $Y \in \mathcal{S}(d,a)$. Theorem 4.7(i) and Lemma 4.6 imply they do not share a common octad and so the trio $d+c$ must be the trio incident with each sextet $S \in \mathcal{S}(d,a)$ which completes the proof of the lemma.

Lemma 5.7 Let $c \in \Delta_2^1(a)$ and $d \in \Delta_2^2(a) \cap \Delta_1(c)$. Then $\{a,c\}^{\perp} \cap \{a,d\}^{\perp} \neq \emptyset$.

Proof Assume the result is false and argue for a contradiction. From Lemmas 5.4 and 5.5 $c+d \in \beta_1(c, X(c,a))$ and $X(a,c) \in \Gamma_2(l_d)$. Let $\tau = \tau(X(a,c))(= \tau(X(c,a)))$. Since $X(a,c) \in \gamma_3(a,S)$ for all $S \in \mathcal{S}(a,d)$, τ fixes b for every $b \in \{a,d\}^{\perp}$ by Lemma 3.3. However τ does not fix d, by Lemma 3.3 again, because $c+d \in \beta_1(c, X(c,a))$ and so $d^{\tau} \in \Gamma_0(c+d) \backslash \{c,d\}$. If $b \in \{a,d\}^{\perp}$, then $\{d,b,d^{\tau}\}$ forms a sparse triangle because $c \notin \Delta_1(b)$, which implies that $d+c$ and $d+b$ have a common octad in Γ_d, for every $b \in \{a,d\}^{\perp}$. The only way this can happen is if $d+c$ is the trio incident with each $S \in \mathcal{S}(d,a)$. Fix $b \in \{a,d\}^{\perp}$ and let x be the unique point in $\{a,d\}^{\perp} \cap \Delta_1(b)$ with $b+x \in \alpha_2(b,b+a) \cap \alpha_2(b,b+d)$ (see Theorem 4.7(i)). We can choose a point $c' \in \Gamma_0(b+x)$ such that $c' \in \Delta_2^1(a) \cap \Delta_1(d)$ and $c' \neq c$. In $\Gamma_{c'}$, $c'+d$ has an octad in common with $c'+b$ and so $c'+d \in \beta_0(c', X(c',a)) \cup \beta_2(c', X(c',a))$. Since $d \notin \Delta_2^1(a)$ we must have $c'+d \in \beta_2(c', X(c',a))$. Using Lemma 5.6 gives that $d+c'$ is the unique trio incident with every sextet $S \in \mathcal{S}(d,a)$, whence $d+c' = d+c$. Furthermore, in Γ_a $X(a,c')$ must be a sextet in $\mathcal{S}(a,d)$ which means that $X(a,c) \in \gamma_3(X(a,c')) \cup \{X(a,c')\}$. Hence $\tau \in Q(X(a,c'))$ and so τ fixes c'. However we already know that τ moves d and we have a contradiction to Lemma 3.3 because $c+c' = c+d$. This completes the proof of Lemma 5.7.

Theorem 5.8 Let $c \in \Delta_2^1(a)$ and $d \in \Delta_2^2(a) \cap \Delta_1(c)$. Then the following hold.

(i) $c+d \in \beta_2(c, X(c,a)) \cup \beta_3(c, X(c,a))$.

(ii) If $l \in \beta_2(c, X(c,a))$, then $|\Gamma_0(l) \cap \Delta_2^1(a)| = 3$ and $|\Gamma_0(l) \cap \Delta_2^2(a)| = 2$.

(iii) If $l \in \beta_3(c, X(c,a))$, then $|\Gamma_0(l) \cap \Delta_2^1(a)| = 1$ and $|\Gamma_0(l) \cap \Delta_2^2(a)| = 4$.

(iv) $|\Delta_1(c) \cap \Delta_2^2(a)| = 360 + 2880 = 3240$.

Proof(i) Part (i) is a consequence of Lemmas 5.4 and 5.7 together with the fact that $d \notin \Gamma_0(X(c,a))$.

(ii) From Lemma 5.3 there are 360 points lying in

$$R := (\Delta_1(c) \cap \Delta_2^1(a)) \backslash \Gamma_0(X(c,a)).$$

The G_{ac} orbits on $\Gamma_1(c)$ are $\beta_i(c, X(c,a))$ for $i = 0,1,2,3$ by Theorem 4.3(v) and $\{c+x | x \in R\}$ is a union of these orbits. By considering the orbit sizes given in (2.5) we conclude that $\{c+x|x \in R\} = \beta_2(c, X(c,a))$ and $|\Gamma_0(c+x) \cap \Delta_2^1(a)| = 3$ for each $x \in R$. Fix $x \in R$ and let $y \in \Gamma_0(c+x) \backslash \Delta_2^1(a)$. Since $c+x \in \beta_2(c, X(c,a))$ we can choose $b \in \{a,c\}^\perp$ with $c+b \in \alpha_2(c, c+x)$. From (2.16) and Lemma 3.8 there exists $b' \in \Gamma_0(c+b)$ with $b' \in \{a,y\}^\perp$ and hence $d(a,y) \leqslant 2$. If $y \in \Delta_1(a)$, then $c+x = c+y \in \Gamma_1(X(c,a))$, a contradiction, whence $d(a,y) = 2$. Since $y \notin \Delta_2^3(a)$ by Theorem 5.2 we deduce that $y \in \Delta_2^2(a)$ which means that $|\Gamma_0(c+x) \cap \Delta_2^2(a)| = 2$ and (ii) is proved.

(iii) Let $x \in \{a,c\}^\perp$. Then we can choose $k \in \Gamma_1(x)$ with $k \in \alpha_3(x, x+c) \cap \alpha_1(x, x+a)$. If $y \in \Gamma_0(k) \backslash \{k\}$, then $y \in \Delta_2^2(a) \cap \Delta_1(c)$ by Lemma 3.8 and the definition of $\Delta_2^2(a)$. Further, $c+y \in \alpha_3(c, c+x)$ from Lemma 3.9 and since $y \notin \Gamma_0(X(c,a))$ (2.11) implies that $c+y \in \beta_3(c, X(c,a))$. Therefore if $l \in \beta_3(c, X(c,a))$ we have $\Gamma_0(l) \cap \Delta_2^2(a) \neq \emptyset$ because $\beta_3(c, X(c,a))$ is a G_{ac}-orbit. Choose $d \in \Delta_2^2(a) \cap \Gamma_0(l)$ and let $b \in \{a,c\}^\perp \cap \{a,d\}^\perp$ (b exists by Lemma 5.7). Since $l \in \beta_3(c, X(c,a))$ we must have $c+b \in \alpha_3(c,l)$ and so b is collinear with all points of $\Gamma_0(l)$ by Lemma 3.8. Let $d' \in \Gamma_0(l) \backslash \{c,d\}$ and $Y \in \Gamma_2(b+c, b+d)$. Since $b+a \in \alpha_2(b, b+c) \cap \alpha_1(b, b+d)$, (2.11) implies that $b+a \in \beta_2(b,Y)$. Using (2.11) again we have $b+a \in \alpha_2(b,l)$ for exactly three lines $l \in \Gamma_1(b,Y)$, one of which is $b+c$. As trios in Γ_b these three lines each contain a common octad. However $b+d' \in \alpha_3(b, b+c)$ and so $b+d'$ has no octad in common with $b+c$. Therefore, using (2.11) again, we must have $b+a \in \alpha_1(b, b+d')$, whence $d' \in \Delta_2^2(a)$ and (iii) is proved.

(iv) This follows from parts (i) - (iii) using (2.5).

References

[$\mathbb{A}$] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *An Atlas of Finite Groups*, Oxford Univ. Press, London 1985.

[Bu] F. Buekenhout, *The basic diagram of a geometry*, Lecture Notes, Springer, 893(1981).

[Cu1] R.T. Curtis, *On the Mathieu group M_{24} and related topics*, Ph.D. thesis, Cambridge, 1972.

[Cu2] R.T. Curtis, *A new combinatorial approach to M_{24}*, Math. Proc. Cambridge Philos. Soc. 79 (1976), 25-42.

[Go] D. Gorenstein, *Finite Groups*, New York 1968.

[Iv1] A.A. Ivanov, *A presentation for J_4*,Proc. Lond. Math. Soc (3)64, 369-396.

[Iv2] A.A. Ivanov, *The Fourth Janko Group*, Oxford Mathematics Monograph, 2005.

[IvMe] A.A. Ivanov and U. Meierfrankenfeld, *A computer free construction of* J_4, J. Algebra 219 (1999), 564-596.

[Ja] Z. Janko, *A new simple group of order 86,775,570,046,077,562,880 which possesses* M_{24} *and the full covering group of* M_{22} *as subgroups*, J. Algebra 42 (1976), 564-596.

[Me] U. Meierfrankenfeld, unpublished calculations.

[MeSt] U.Meierfrankenfeld and G. Stroth, *Quadratic* $GF(2)$*-modules for sporadic simple groups and alternating groups*, Comm. in Alg. 18(1990), 2099-2140.

[No] S.P. Norton, *The construction of* J_4, Proc. Sym. Pure Math. 37(1980), 271-278.

[RoSm] M. A. Ronan and S.D. Smith, *2-local geometries for some sporadic groups*, Proceedings Symp. Pure Math. No. 37, Santa Cruz 1979 Proceedings (Cooperstein and Mason, Eds.), Amer. Math. Soc., Providence R.I., 1980.

[RW1] P.J. Rowley and L.A. Walker, *The maximal 2-local geometry for* J_4*, II*, preprint.

[RW2] P.J. Rowley and L.A. Walker, *The maximal 2-local geometry for* J_4*, III*, preprint.

[Su] M. Suzuki, *Group Theory I*, Die Grundlehren der Mathematischen Wissenshaften, Band 247, Berlin, Springer-Verlag, 1982.