Topics in Finite Groups: Homology Groups, Pi-product Graphs, Wreath Products and Cuspidal Characters

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TOPICS IN FINITE GROUPS: HOMOLOGY GROUPS, π-PRODUCT GRAPHS, WREATH PRODUCTS AND CUSPIDAL CHARACTERS

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This thesis was submitted by David Ward to The University of Manchester for the degree of Doctor of Philosophy on Friday 24$^{\text{th}}$ July, 2015.

Abstract: In this thesis we consider four topics relating to finite groups; homology of presheaves of abelian groups, π-product graphs, wreath products of cyclic p-groups and p-cuspidal characters.

Let G be a finite group, p a prime divisor of the order of G and $k := GF(p)$ - the field of p elements. In a series of papers in the 1980s, Mark Ronan and Stephen Smith formed the notion of a presheaf of abelian groups for G. This mirrored the topological definition of a presheaf and was built on an arbitrary simplicial complex having associated G-action. They then proceeded to define a chain complex on a presheaf such that the corresponding homology groups were kG-modules. We consider this theory and investigate if every irreducible kG-module can be achieved as a quotient of the zero-homology group of a universal panel-irreducible presheaf defined on the simplicial complex of parabolic subgroups of G. In general, we see that this is not the case. We also explicitly calculate the zero-homology groups of universal panel-irreducible presheaves defined on $\mathrm{Sym}(6)$, M_{11} and M_{22} over $GF(2)$, together with the irreducible quotients of the zero-homology groups of universal panel-irreducible presheaves defined on M_{12}, M_{23} and M_{24} over $GF(2)$.

Suppose that G is a group, X is a subset of G and π is a set of natural numbers. The π-product graph $\mathcal{P}_\pi(G, X)$ has X as its vertex set and distinct vertices are joined by an edge if the order of their product is contained in π. If X is a set of involutions, then $\mathcal{P}_\pi(G, X)$ is called a π-product involution graph. We study the connectivity and diameters of $\mathcal{P}_\pi(G, X)$ when G is a finite symmetric group and X is a G-conjugacy class of involutions, and fully determine when $\mathcal{P}_\pi(G, X)$ is connected. In the case that $\pi = \{4\}$, the diameter of all connected graphs $\mathcal{P}_\pi(G, X)$ is shown to equal 2. The connectivity of $\mathcal{P}_\pi(G, X)$ is also determined in the cases that X is a class of involutions with support of order 2^a for some $a \geqslant 3$ and $\pi = \{2^a\}$, X is a class of involutions having support of order $p^a - 1$ for some odd prime p, some $a \geqslant 1$ and $\pi = \{p^a\}$ and in the more general case that X is a class of involutions and $\pi = \{b\}$ where b is closely related to the order of the support of the involutions in X. Bounds on the diameters of the connected graphs are also obtained. In the case that X is the conjugacy class of involutions with support of order 8 and $G := \mathrm{Sym}(n)$ for $n \geqslant 10$, the diameter of $\mathcal{P}_{\{8\}}(G, X)$ is explicitly determined.

The π-product involution graph $\mathcal{P}_{\{4\}}(G, X)$ may also be viewed as a graph having vertex set X, and with two involutions $x, y \in X$ joined by an edge precisely when they generate $\mathrm{Dih}(8)$ - the dihedral group of order 8. A natural generalisation of the results for $\{4\}$-product involution graphs is to consider p-elements in place of involutions and the wreath product $C_p \wr C_p$ in place of $\mathrm{Dih}(8)$. Indeed, when do two conjugate p-elements of a given symmetric group generate a wreath product of two cyclic groups of order p? We give necessary and sufficient conditions for this in the case that our p-elements have full support. These conditions relate to given matrices that are of circulant or permutation type, and corresponding polynomials that represent these matrices. We also consider the case that the elements do not have full support, and see why generalising our results to such elements would not be a natural generalisation.

The concluding topic of this thesis is p-cuspidal characters of finite groups. Given a finite group G containing a subgroup X, an X-parabolic system of G may be defined. This consists of pairs of subgroups (P_J, Q_J) for all subsets J of a given indexing set such that certain conditions hold. These X-parabolic systems mirror some of the properties of parabolic subgroups and unipotent radicals for groups of Lie type, and allow an analogue of cuspidal characters to be defined for an arbitrary finite group. Such cuspidal characters are dependant on the choice of X-parabolic system. We consider systems defined by p-minimal parabolic systems, and hence obtain p-cuspidal characters of G for each prime divisor of the order of G. Theoretical results are developed reflecting known results for groups of Lie type. For each sporadic group G and each prime divisor of the order of G, we then calculate the p-cuspidal characters of G (with the exception of $G = \mathbb{B}$ and $p = 2$).

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Overview

Since the axiomatisation of the abstract group in the late nineteenth century, mathematicians have used many methods for studying groups. The axiomatisation brought the areas of permutation groups, abelian groups and classical groups under a universal umbrella from which universal properties could be established in the greatest of generality. In subsequent years, the individual areas of group theory have been treated in both universal and individual contexts. Properties and notions readily evident in one area could then be considered in other areas, with varying degrees of success. This is especially evident in the generalisation of ideas concerning groups of Lie type to arbitrary finite groups and vice versa. In this thesis, we consider two such notions; presheaves of abelian groups and cuspidal characters. We also see how considering a specific graph-theoretic approach to involutions in symmetric groups leads to a matrix-orientated approach to p-elements.

The thesis has four main chapters, which are preceded in Chapter 2 by a summary of the mathematical background material on which the subsequent chapters build, together with a review of the notation used throughout the thesis. In Chapter 3 we consider the work of Mark Ronan and Stephen Smith on the homology groups of presheaves of abelian groups [RS85], [RS86], [RS89], [Ron89]. Ronan and Smith were motivated by the topological definition of a presheaf, and used parabolic subgroups in place of open sets. They further required an associated group action. Equipped with such an action, their *presheaves* had nice combinatorial properties which allow a homological approach to be used in considering them. With such an approach, the homology groups are modules for the given group. Moreover, Ronan and Smith were able to define the notion of a *universal presheaf*, such that any "nice" presheaf could be considered as a quotient of a universal presheaf.

Full details of the work of Ronan and Smith is given in Sections 3.1-3.6. All results in these sections, except for Corollary 3.6.8 and Theorem 3.6.9, are due to Ronan and Smith. Corollary 3.6.8 is an easy deduction from previous results, whilst Theorem 3.6.9 is a corrected version of a comment appearing in [RS89]. In Sections 3.1-3.6 we direct the reader to the literature for proofs of known results where possible. However, we provide proofs for results when no such references exist.

Having given a survey of the work of Ronan and Smith, we pose a question about irreducible quotients of zero-homology groups of universal panel-irreducible presheaves in Section 3.7. A partial answer is given in this section, and the question is reconsidered in Section 3.9. These sections sandwich our main work in this area, which is given in Section 3.8, namely the calculation of the zero-homology groups of universal panel-irreducible presheaves of the symmetric group Sym(6) and the Mathieu groups M_{11} and M_{22} together with calculations of the irreducible quotients of such zero-homology groups for M_{12}, M_{23} and M_{24} all over the field of two elements, $GF(2)$. The zero-homology groups of universal panel-irreducible presheaves of M_{24} over $GF(2)$ were considered by Ronan and Smith in [RS89, Section 4]. However, this was over a different geometry to that used in this thesis. For the sake of brevity, some calculations of vertex terms of universal panel-irreducible presheaves appear separately as Appendix A.

In Chapter 3, our universal panel-irreducible presheaves are defined on the simplicial complex of parabolic subgroups of a given group. However, presheaves may be defined on any simplicial complex with associated group action. One such simplicial complex is given by the Brown complex $|\mathcal{S}_p(G)|$ of a group G, which for a given prime p consists of all chains of non-trivial p-subgroups of G, ordered by inclusion. Let G be the symmetric group of degree n and suppose that X is the conjugacy class of all involutions in G whose support have order $2m$ for some $m \leqslant n/2$. We may consider all simplices of $|S_2(G)|$ consisting of a single subgroup isomorphic to the dihedral group, Dih(8), of order 8 together with all simplices consisting of a single subgroup generated by an element of X. A natural question to ask is when do two such subgroups of order 2 generate a subgroup isomorphic to Dih(8) (or equivalently when does the product of the two generating involutions have order 4)? This is the main topic of chapter 4.

Generalising these ideas, we may define the π-product graph, $\mathcal{P}_\pi(G,X)$, for an arbitrary group G, a subset $X \subseteq G$ and for π a set of positive integers to be the graph having vertex set X and with two vertices joined by an edge precisely when the order of their product is contained in π. In all of the cases that we consider, our vertex set X will consist of involutions, and hence the resulting π-product graph (also know as a π-product involution graph) is undirected. Taking π to be the set of all odd natural numbers and X a G-conjugacy class, $\mathcal{P}_\pi(G,X)$ becomes the local fusion graph $\mathcal{F}(G,X)$ which has featured in [Bal13] and [BGR13].

Chapter 4 contains a survey of π-product graphs of symmetric groups in the case that X is a conjugacy class of involutions and π is a singleton set and is heavily based on the preprint [RW14]. The work relies heavily on the notion of an x-graph, a computationally-pleasing graph introduced by Bates, Bundy, Parker and Rowley whilst considering the commuting involution graphs of symmetric groups in [BBPR03b]. We see that π-product graphs are one of a bountifulness of graphs that may be associated to groups. Further examples are given by local fusion graphs, power graphs and commuting graphs.

The main result of Chapter 4 is that if G is a symmetric group and X is a conjugacy class of involutions of G, then the $\{4\}$-product graph $\mathcal{P}_{\{4\}}(G,X)$ is either disconnected or it is connected and has diameter equal to 2. The proof of this result involves a large number of x-graphs and these appear as Appendix B.

The contents of Chapter 4 may be generalised in a number of different ways, and one such way is considered in Chapter 5. Indeed, rather than taking X to be a conjugacy class of involutions within a given symmetric group, we consider the case that X is a conjugacy class of elements of order p. In such a situation, we no longer have the versatile x-graph at our disposal. Generalising x-graphs from a graph-theoretical viewpoint is impractical and would result in weighted directed graphs. In Chapter 5 we see that we can define certain matrices to be - in some senses - the analogue to x-graphs in this more general setting.

The matrices defined may be used in computations, and by considering the dihedral group Dih(8) to be the wreath product, $C_2 \wr C_2$, of cyclic groups of order 2, we may generalise the results of Chapter 4. In the case that X contains elements of full support, we ask when do two elements of X generate the wreath product $C_p \wr C_p$? We see that in such cases, the aforementioned matrices take the form of block matrices involving both permutation and circulant matrices. A definitive answer to our question may then be given by considering these matrices, the determinants of their blocks and the representer polynomials of the circulant blocks. We also see that considering such an approach when elements of X do not have full support is unrealistic.

To gain a greater understanding of the matrices and associated polynomials arising in Chapter 5 the computer algebra system MAGMA may be used. Further information about MAGMA can be found in the online handbook [Com] or in [BC06], [CPB08] or [CP08]. MAGMA code that may be beneficial to the reader is given in Appendix C.

We note that Chapter 5 has appeared in its entirety in the preprint [War14] and an abridged version is also available as [Warar]

The final main chapter is of a slightly different sapidity. In a style mirroring that of Chapter 3 we take a concept from groups of Lie type and seek to form an analogue in a more general setting. The concept in question is that of being a cuspidal character. This is a property that a character of a group of Lie type in characteristic zero may satisfy, and in that setting, cuspidal characters may be seen - in some senses - as the building blocks which contain all irreducible characters.

We begin Chapter 6 by formulating an X-parabolic system of a group G having subgroup X. We see that these X-parabolic systems are the cognate of the building of a group of Lie type, upon which the cuspidal property is based, and hence they allow us to define cuspidal characters in a more general setting. This is followed by a brief résumé of some elementary definitions and properties of cuspidal characters of groups of Lie type.

Having established the corresponding properties of cuspidal characters in the finite setting, we then proceed to give a full survey of the p-cuspidal characters for each of the 26 sporadic simple groups and for each prime divisor, p, of the group order. These results are obtained via assimilating known knowledge about the fusion within subgroups of each of the sporadic groups and combining this with information from the $\mathbb{ATLAS}$ and direct calculations involving MAGMA. We fully determine all p-cuspidal characters of each sporadic group G, with the exception of the case when $(G, p) = (\mathbb{B}, 2)$. In this case, it is likely that there are no 2-cuspidal characters of $\mathbb{B}$. However, at the time of writing, we have been unable to prove this.

The thesis concludes in Chapter 7 with a few comments on possible extensions of and future areas of research related to the topics contained in Chapters 3-6.

Chapter 2

Background Material

The aim of this chapter is to introduce the general background material that is required in subsequent chapters. For general group-theoretic definitions and results, we refer the reader to [Asc00], [Gor80], [Isa08] or [Suz82].

We begin by introducing some general notation in Section 2.1 that will be used throughout the thesis. Further notation from the $\mathbb{ATLAS}$ *of Finite Groups* [CCN$^+$09] (subsequently referred to as *the* $\mathbb{ATLAS}$) is introduced in Section 2.3, together with a summary of the required results on finite simple groups. This is preceded in Section 2.2 with background material on p-groups and is followed in Sections 2.4, 2.5 and 2.6 with material on buildings, minimal parabolic systems and geometries.

Throughout the thesis, there are many references to the computer algebra system MAGMA. Further information about MAGMA can be found in the online handbook [Com] or in [BC06], [CPB08] or [CP08]. In a number of places throughout this thesis we have also used information contained in the $\mathbb{ATLAS}$. Such information has been used without proof.

2.1 General Notation and Conventions

Throughout the thesis, unless otherwise stated, G will denote a finite group and k will denote a field of characteristic $\mathrm{char}\, k$. The Galois field of order p^n is denoted by $GF(p^n)$. We will use the notation $|G|$ and $\mathrm{ord}(g)$ for the order of G and the order of the element $g \in G$ respectively. If H is a subgroup, proper subgroup or normal subgroup of G, then the respective notation $H \leqslant G$, $H < G$ and $H \trianglelefteq G$ is used. The index of H in G is denoted by $[G : H]$. If p is a prime, we denote the set of Sylow p-subgroups of G by $Syl_p(G)$. The trivial group is denoted by 1, the cyclic group of order r is denoted by C_r and the symmetric and alternating groups on n symbols are denoted by $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ respectively.

Given elements $g, h \in G$, we use g^h to denote the conjugate of g by h and g^G to denote the G-conjugacy class of g. The commutator of g with h is given by $[g, h] := g^{-1}h^{-1}gh$. The centralizer of g in G is denoted by $C_G(g)$ whilst if H is a subgroup of G, the normalizer of H in G is denoted by $N_G(H)$. If G acts on a set Δ and $\sigma \in \Delta$, we denote the stabilizer in G of σ by $\mathrm{Stab}(\sigma)$ and the G-orbit of σ by σ^G.

For a given group G, we denote the center of G and the derived subgroup of G by $Z(G)$ and $G' = [G, G]$ respectively.

For a ring R, if M is an R-module and N is an R-submodule of M, we use the notation $N \leqslant M$. If S is a subring of R, we denote the restriction of M to S by M_S. Conversely, if L is an S-module, then we denote the induction of L to R by $\mathrm{Ind}\, L_S^R$ or L_S^R. Similarly, if χ and ψ are characters of H and G respectively with $H \leqslant G$, then we denote the restriction of ψ to H and the induction of χ to G by $\psi|_H$ or ψ_H and χ_H^G, $\chi_H \uparrow^G$, χ^G or $\chi \uparrow^G$ respectively. The set of all irreducible complex characters of G is denoted by $\mathrm{Irr}(G)$, and the inner product on complex characters is denoted $(\cdot, \cdot)$.

For a given group G and field k, if there is a unique irreducible kG-module of dimension i, then we shall denote this by i_{kG}, or simply i_G if k is clear from the context. The zero kG-module will be denoted by 0_{kG} or 0. Given a reducible kG-module M having composition series

$$0 = M_{n+1} \leqslant M_n \leqslant \cdots \leqslant M_2 \leqslant M_1 = M$$

and composition factors $N_i := M_i/M_{i+1}$, we shall sometimes denote M by $N_1/N_2/\cdots/N_n$.

The notation $X := \ldots$ should be read as X *is defined to be* $\ldots$. Finally, throughout the thesis we shall use the convention that functions are composed from left to right. Thus for functions $f : X \to Y$ and $g : Y \to Z$, we denote the image of $x \in X$ under the successive actions of f and g by xfg.

2.2 p-Groups

Let p be a prime. A group P is called a *p-group* if every element of G has order p^r for some $r \geqslant 0$. For finite groups this is equivalent to saying that P has order p^r for some $r \geqslant 0$. An abelian p-group, P, of exponent p is called an *elementary abelian p-group* and we call $\log_p |P|$ the *rank* of P. We write p^a for the elementary abelian p-group of rank a, namely C_p^a. An arbitrary p-group of order p^a will be denoted by $[p^a]$. For the rest of this section, we will assume that P is finite.

For an arbitrary p-group, P, we will often use the notation of [RS84]. Thus we assume that P has a composition series

$$1 = P_0 \lhd P_1 \lhd \cdots \lhd P_n = P$$

having elementary abelian quotients (also known as *sections*) P_i/P_{i-1} of rank r_i for $i = 1, \ldots, n$. The notation $p^{r_1+\cdots+r_n}$ is then used to represent P.

An important class of p-groups are the *extra-special* p-groups. Before describing these, we recall that the *Frattini subgroup*, $\Phi(G)$, of a finite group G is the intersection of all maximal subgroups of G.

Definition 2.2.1. *[Gor80] A p-group P is called* special *if either P is elementary abelian or P is of nilpotency class 2 and $P' = Z(P) = \Phi(P)$ is elementary abelian. A special p-group, P, of nilpotency class 2 satisfying $|P'| = p$ is called* extra-special.

There are two isomorphism classes of extra-special groups of order p^3.

Theorem 2.2.2. *[Gor80, Theorem 5.5.1] A non-abelian p-group P of order p^3 is extra-special and is isomorphic to one of the following groups if $p = 2$:*

(i) $D := \mathrm{Dih}(8) = \langle x, y | x^4 = y^2 = 1, x^y = x^3 \rangle$ *- the dihedral group of order 8; or*

(ii) $Q := \langle x, y, z | x^2 = y^2 = z, z^2 = 1, y^{-1}xy = x^{-1} \rangle$ *- the quaternion group.*

Meanwhile, if $p > 2$, then P is isomorphic to one of:

(i) $M := \langle x, y, z | x^p = y^p = z^p = 1, [x, z] = [y, z] = 1, \text{ and } [x, y] = z \rangle$*; or*

(ii) $N := \langle x, y | x^{p^2} = y^p = 1, x^y = x^{1+p} \rangle$.

We see that when $p = 2$, all extra-special groups of order 8 have exponent 4, whilst if $p > 2$, then the group M has exponent p and the group N has exponent p^2. The groups D, M, N and Q may be used to classify all extra-special p-groups. To do so, we require a further definition.

Definition 2.2.3. *[Suz82] The group G is called the* central product *of two subgroups H and K if H and K commute element-wise and $G = HK$.*

The central product of H and K is denoted $H \circ K$. However, for the sake of brevity, we shall denote the central product of i copies of N (respectively D) and j copies of M (respectively Q) by $N^i M^j$ (respectively $D^i Q^j$).

Theorem 2.2.4. *[Gor80, Theorem 5.5.2] An extra-special p-group P is the central product of $r \geqslant 1$ nonabelian subgroups of order p^3. Moreover, we have*

(i) If p is odd, P is isomorphic to $N^k M^{r-k}$, while if $p = 2$, P is isomorphic to $D^k Q^{r-k}$ for some k. In either case, $|P| = p^{2r+1}$.

(ii) If p is odd and $k \geqslant 1$, $N^k M^{r-k}$ is isomorphic to NM^{r-1}, and the groups M^r and NM^{r-1} are not isomorphic.

(iii) If $p = 2$, then $D^k Q^{r-k}$ is isomorphic to DQ^{r-1} if k is odd and to Q^r if k is even, and the groups Q^r and DQ^{r-1} are not isomorphic.

We follow the conventions of [CCN$^+$09] and denote the extra-special 2-groups DQ^{r-1} and Q^r by 2_+^{1+2r} and 2_-^{1+2r} if r is odd and by 2_-^{1+2r} and 2_+^{1+2r} if r is even. We also denote M^r by p_+^{1+2r} and NM^{r-1} by p_-^{1+2r}.

For any group G and any prime p, we may construct a p-group.

Definition 2.2.5. *[DH92] Let G be a group and p be a prime. The* p-core *of G, $O_p(G)$, is defined to be*

$$O_p(G) := \langle N | N \trianglelefteq G, N \text{ is a } p\text{-group} \rangle.$$

Clearly, $O_p(G)$ is the largest normal p-subgroup of G. In Chapters 3 and 6, we shall use the p-cores of the parabolic subgroups of an arbitrary group (defined in Section 2.5) as the analogue of the unipotent radicals of the parabolic subgroups of a finite group of Lie type. Indeed, in Chapter 3 we shall refer to the p-cores of certain groups as unipotent radicals (as this mirrors the motivating examples of presheaves from [RS85]).

Another tool that we shall use in Chapter 3 is that of local subgroups.

Definition 2.2.6. *[Isa08] Let G be a group and p be a prime. A subgroup H of G is called* p-local *if H is of the form $H = N_G(P)$, where P is some nonidentity p-subgroup of G. A subgroup is called* local *if it is p-local for some prime p*

We shall see that one can use representations on a local level for a given group G - representations that are already known - to obtain representations for the group G itself.[1]

2.3 Finite Simple Groups and $\mathbb{ATLAS}$ Notation

2.3.1 Finite Simple Groups

A group G is called simple if it has no non-trivial proper normal subgroups. Simple groups are seen as the building blocks of finite group theory, since every finite group admits a composition series, which - by the Jordan-Hölder Theorem - has unique composition factors up to reordering. It follows that an understanding of finite simple groups is crucial to an understanding of many areas of finite group theory.

During the twentieth century, a project aimed at classifying all finite simple groups was undertaken, and this resulted in the Classification Theorem for Finite Simple Groups.

Theorem 2.3.1 (The Classification Theorem for Finite Simple Groups)**.** *[Wil09] Every finite simple group is isomorphic to one of the following:*

(i) *a cyclic group C_p of prime order p;*

(ii) *an alternating group $\mathrm{Alt}(n)$, for $n \geqslant 5$;*

(iii) *a classical group:*

linear:	$PSL_n(q)$, $n \geqslant 2$, *except* $PSL_2(2)$ *and* $PSL_2(3)$;
unitary:	$PSU_n(q)$, $n \geqslant 3$, *except* $PSU_3(2)$;
symplectic:	$PSp_{2n}(q)$, $n \geqslant 2$, *except* $PSp_4(2)$;
orthogonal:	$P\Omega_{2n+1}(q)$, $n \geqslant 3$, q *odd;*
	$P\Omega^+_{2n}(q)$, $n \geqslant 4$;
	$P\Omega^-_{2n}(q)$, $n \geqslant 4$

where q is a power p^a of a prime p;

[1] As an aside, we note that a subgroup P of G for which $P = O_p(N_G(P))$ is called a *radical subgroup* or a *p-radical subgroup*.

(iv) an exceptional group of Lie type:

$$G_2(q), q \geqslant 3; F_4(q); E_6(q); {}^2E_6(q); {}^3D_4(q); E_7(q); E_8(q)$$

where q is a prime power, or

$${}^2B_2(2^{2n+1}), n \geqslant 1; {}^2G_2(3^{2n+1}), n \geqslant 1; {}^2F_4(2^{2n+1}), n \geqslant 1$$

or the Tits group ${}^2F_4(2)'$;

(v) one of 26 sporadic simple groups:

the five Mathieu groups:	M_{11}, M_{12}, M_{22}, M_{23}, M_{24};
the seven Leech lattice groups:	Co_1, Co_2, Co_3, McL, HS, Suz, J_2;
the three Fischer groups:	Fi_{22}, Fi_{23}, Fi'_{24};
the five Monstrous groups:	$\mathbb{M}$, $\mathbb{B}$, Th, HN, He;
the six pariahs:	J_1, J_3, J_4, $O'N$, Ly, Ru.

The only repetitions within the groups in Theorem 2.3.1 are defined by the following six isomorphisms (see [Wil09]):

$$\begin{array}{ccccc} PSL_2(4) & \cong & PSL_2(5) & \cong & \mathrm{Alt}(5); \\ & & PSL_2(7) & \cong & PSL_3(2); \\ & & PSL_2(9) & \cong & \mathrm{Alt}(6); \\ & & PSL_4(2) & \cong & \mathrm{Alt}(8); \\ & & PSU_4(2) & \cong & PSp_4(3). \end{array} \tag{2.3.1}$$

A good overview of the finite simple groups listed in Theorem 2.3.1 and in (2.3.1) may be found in [Wil09].

The study of finite simple groups is an ongoing area of active research, and the quest to gain a greater understanding of certain groups or families of groups continues. The methods of Chapter 3 will hopefully allow a greater understanding of the representation theory of the sporadic groups over finite fields to be achieved, whilst analysis of the p-cuspidal characters of these groups found in Chapter 6 may lead to a greater understanding of certain geometries associated to these groups.

2.3.2 $\mathbb{ATLAS}$ Notation

To an arbitrary finite group G, we may associate a *shape*. This gives certain information about G via extensions of groups, and is the notation used throughout this thesis and in the $\mathbb{ATLAS}$. As we shall see, a shape of a group does not define the group up to isomorphism. However, it does give useful information about the group's structure. We begin by defining the two types of extensions that we will encounter.

Definition 2.3.2. *[Isa08] Let G, H and N be groups. If there exists a subgroup $N_0 \leqslant G$ with $N_0 \cong N$ and $G/N_0 \cong H$, then we call G* an extension of H by N, *denoted $G = N.H$ or $G = NH$.*

A special type of extension is a split extension.

Definition 2.3.3. *[Isa08] Let G, H and N be groups. If there exist subgroups $N_0, H_0 \leqslant G$ with $N_0 \cong N$, $H_0 \cong H$, $G/N_0 \cong H$ and $N_0 \cap H_0 = 1$, then we call G* a split extension of H by N, *denoted $G = N : H$.*

If $G = N : H$, then we also call H a *complement to N in G.*

Another way to view extensions is via the corresponding short exact sequence. Indeed, we have that $G = N.H$ if there exists a short exact sequence

$$1 \longrightarrow N_0 \longrightarrow G \xrightarrow{q} H \longrightarrow 1 \tag{2.3.2}$$

for some normal subgroup $N_0 \cong N$ of G. The extension is then a split extension if the exact sequence (2.3.2) is right split (that is there exists a homomorphism $r : H \to G$ such that $rq = id_H$).

If G is a split extension of H by N, then we also refer to G as the *(internal) semidirect product of N with H.* An alternative formulation of the semidirect product may be produced using a group of operators.

Definition 2.3.4. *[DH92] Let G and H be groups, and suppose that for each $g \in G$ and $h \in H$ an element $g^h \in G$ is defined such that:*

(i) the map $g \mapsto g^h$ is an automorphism of G; and

(ii) $g^{(hk)} = (g^h)^k$

for all $g \in G$ and all $h, k \in H$. Then we say that H is a group of operators for G, *or that G is an* H-group. *If a homomorphism $\phi : H \to Aut(G)$ is specified and an action of H on G is defined by $g^h = g(h\phi)$, we call H a* group of operators for G via ϕ.

The aforementioned alternative formulation of the semidirect product is called the (external) semidirect product.

Definition 2.3.5. *[DH92] Let H be a group of operators for a group G, and define a binary operation on the Cartesian product $X = G \times H$ by:*

$$(g_1, h_1)(g_2, h_2) = (g_1(g_2)^{h_1^{-1}}, h_1h_2) \qquad \textit{for all } (g_1, h_1), (g_2, h_2) \in G \times H.$$

We call X the (external) semidirect product of G with H via ϕ *(where $\phi : H \to Aut(G)$ determines the H-action on G). When the H-action (and hence ϕ) is clear from the context, we suppress reference to ϕ.*

It is easily shown that external semidirect products are internal semidirect products and vice versa. Thus we can use whichever formulation is most amiable for a given situation. We also use the notation $G := N \rtimes H$ in the case that G is the semidirect product of N with H.

A specific type of semidirect product is given by the wreath product.

Definition 2.3.6. *[DM96] Let K and H be groups and suppose H acts on the nonempty set Γ. Then the* wreath product *of K by H with respect to this action is defined to be the semidirect product* $\mathrm{Fun}(\Gamma, K) \rtimes H$ *where H acts on the group* $\mathrm{Fun}(\Gamma, K)$ *via*

$$f^x(\gamma) := f(\gamma^{x^{-1}}) \qquad \textit{for all } f \in \mathrm{Fun}(\Gamma, K), \gamma \in \Gamma \textit{ and } x \in H.$$

We denote this group by $K \text{ wr}_\Gamma H$ (or $K \wr H$), and call the subgroup

$$B := \{(f, 1) | f \in \text{Fun}(\Gamma, K)\} \cong \text{Fun}(\Gamma, K)$$

the base group *of the wreath product.*

Since all the sets that we encounter are finite, we will usually identify $\text{Fun}(\Gamma, K)$ with $\prod_{\gamma \in \Gamma} K_\gamma$ - the direct product of $|\Gamma|$ copies of K - with $(k_{\gamma_1}, \ldots, k_{\gamma_{|\Gamma|}}) \in \prod_{\gamma \in \Gamma} K_\gamma$ corresponding to the function $f : \Gamma \to K$ given by $f : \gamma_i \mapsto k_{\gamma_i}$.

In the case that Γ is infinite, the above construction is known as the *unrestricted wreath product of K by H*. If the functions in $\text{Fun}(\Gamma, K)$ are taken to have finite support (or equivalently, we take the direct sum $\bigoplus_{\gamma \in \Gamma} K_\gamma$ instead of the direct product), then we obtain a different group known as the *restricted wreath product of K by H*. In the case that the set Γ in Definition 2.3.6 is equal to H, and the action of H on Γ is given by the *regular action* (so for $x, h \in H$ the action of h on x is given by $x \cdot h = xh$), then the resulting wreath product is known as the *regular wreath product of K by H*.

Example 2.3.7. *Let G be the regular wreath product of C_p by C_p. The base group of G consists of p copies of C_p, which we may generate by the elements*

$$a_i := ((i-1)p + 1, \ldots, (i-1)p + p) \in \text{Sym}(\{(i-1)p + 1, \ldots, (i-1)p + p\})$$

for $i = 1, \ldots, p$. Thus we may embed the base group within $\text{Sym}(p^2)$. The action of the second cyclic group on this base group is given by the regular action, and hence may be given by conjugation by the element

$$x := (1, p+1, \ldots, p(p-1)+1)(2, p+2, \ldots, p(p-1)+2) \cdots (p, 2p, \ldots, p(p-1)+p) \in \text{Sym}(p^2).$$

Thus we see that

$$G \cong \langle a_i, x | i = 1, \ldots, p\rangle = \langle a_1, x\rangle \leqslant \text{Sym}(p^2).$$

This is the representation of $C_p \wr C_p$ that we will use frequently in Chapter 5. •

We now return to the notion of the shape of a group G. The aim is to describe some of the structure of G, and is best illustrated with an example.

Example 2.3.8. *Let $G := \text{Sym}(6)$ and let S be a self-normalizing Sylow 2-subgroup of G. By using MAGMA, we see that there are two isomorphic subgroups of G containing S as a maximal subgroup (see Example 2.5.3). Let P be one of these subgroups. We see that - using the notation of Section 2.2 - $O_2(P) \cong 2^3$. Moreover, the quotient $P/O_2(P)$ is isomorphic to $\text{Sym}(3)$. Thus we say that P has shape $2^3.\text{Sym}(3)$.* •

If the group G has shape X, then we write $G \sim X$. Thus in the above example, $P \sim 2^3.\text{Sym}(3)$.

We conclude this section by noting that we will also follow 𝔸𝕋𝕃𝔸𝕊-notation when referring to characters of finite groups over the complex numbers in Chapter 6. Indeed, for a given group G the character χ_i will equal the corresponding character of G given in the character table in [CCN+09].

2.4 Complexes and Buildings

Let G be a group of Lie type defined over a field k. There is a natural module V associated to G, and we may consider this to be a vector space of dimension n over k. The (non-zero) subspaces of V have dimensions $1, \ldots, n$, and we may construct a *flag* of subspaces of V of rank r for each $1 \leqslant r \leqslant n$:

$$\mathcal{F} : 0 \neq V_{i_1} \subsetneq V_{i_2} \subsetneq \cdots \subsetneq V_{i_r}.$$

By constructing the stabilizer of all such flags, we obtain a structure known as a building, first introduced by Jacques Tits.[2] Before giving the definition of a building as formulated by Tits, we first recall a few basic ideas relating to complexes and simplexes. We begin by defining a simplex.

Definition 2.4.1. *[Suz82] A partially ordered set X is called a* simplex of rank d *if X is isomorphic to the partially ordered set formed from all the subsets of a set of d elements with respect to the containment relation.*

Closely related to simplexes are complexes.

Definition 2.4.2. *[Suz82] Let Δ be a set endowed with a partial order which will be denoted by $A \supset B$. The set Δ is said to be a* complex *if the following two conditions are satisfied:*

(a) For any two elements A and B of Δ, there is a greatest lower bound *(an element C such that $C \subseteq A, C \subseteq B$, and C is the largest element which satisfies these conditions - C is denoted $A \cap B$).*

(b) For any $A \in \Delta$, the subset of Δ which consists of the elements contained in A forms a simplex.

For example, let V be any vector space of dimension $d+1$ over a field k, and let Δ denote the set of all subspaces of V. Define a partial order $\leqslant$ on Δ as follows; for all $A, B \in \Delta$,

$$A \leqslant B \Longleftrightarrow B \subseteq A.$$

Given two elements $A, B \in \Delta$, the subspace of V generated by A and B is the greatest lower bound of A and B. Moreover, since Δ consists of all subspaces of V, the subset of Δ consisting of the elements contained in some $A \in \Delta$ (with respect to $\leqslant$) is naturally isomorphic to the partially ordered set on $\dim_k V - \dim_k A$ elements. Thus Δ is a complex.

As with most mathematical structures, we have the concept of a subcomplex, and for certain complexes we may also associate a rank.

Definition 2.4.3. *[Suz82]* (i) *A subset Γ of a complex Δ is said to be a* subcomplex *of Δ if*

$$A \in \Gamma, B \subseteq A \Longrightarrow B \in \Gamma$$

(ii) *Given a complex Δ and $d \in \mathbb{N}$, if any element is contained in a maximal element which is a simplex of rank d, then we call Δ a* complex of rank d.

[2]Tits gave a rigorous introduction to buildings in his book *Buildings of Spherical Type and Finite BN-pairs* [Tit86]

Returning to our example, where Δ consisted of the complex of subspaces of a given vector space V partially ordered by reverse inclusion, we see that for any two maximal elements α, β of Δ - so two subspaces of V of dimension 1 over k - there is an element of Δ of rank $d-1$, namely the subspace γ such that $\gamma \leqslant \alpha$ and $\gamma \leqslant \beta$. We conclude that in some sense α and β are *close*. This idea is made rigorous by defining adjacency and connectivity within a complex.

Definition 2.4.4. *[Suz82] Let Δ be a complex of rank d. Two maximal elements A and B are said to be* adjacent *if $A \neq B$ and if there is an element, C, of rank $d-1$ such that $C \subseteq A$ and $C \subseteq B$. If A' and B' are arbitrary elements of Δ, then we say that A' and B' are* connected *if there is a finite sequence $\{C_i\}_{i=1}^m$ of maximal elements C_i of Δ such that $A' \subseteq C_1$, $B' \subseteq C_m$, and C_i and C_{i+1} are adjacent for each i. We say that Δ is* connected *if any $A', B' \in \Delta$ are connected.*

Before defining a building, we must first give one final technical definition.

Definition 2.4.5. *[Suz82] Let Γ be a complex of rank d. If any element A of rank $d-1$ is contained in exactly two maximal elements, then Γ is said to be a* thin *complex. If there are at least three maximal elements containing any element of rank $d-1$, then Γ is called a* thick *complex.*

We are now in a position to define a building.

Definition 2.4.6. *[Suz82] A complex Δ of rank d is said to be a* building of rank d *if there is a collection $\mathcal{A}$ of subcomplexes of Δ such that $(\Delta, \mathcal{A})$ satisfies the following conditions:*

(B1) Δ is a thick complex.

(B2) Every element of $\mathcal{A}$ is a connected, thin complex of rank d.

(B3) For any two elements A and B of Δ, there is an element of $\mathcal{A}$ which contains both A and B.

(B4) Let A and A' be two elements of Δ. If Σ and Σ' are elements of $\mathcal{A}$ which contain both A and A', then there is an isomorphism ϕ from Σ onto Σ' which satisfies $\phi(B) = B$ for all $B \subseteq A$ as well as $\phi(B') = B'$ for all $B' \subseteq A'$.

A subcomplex of Δ which is a member of $\mathcal{A}$ is called an apartment*. A maximal element of Δ is called a* chamber*.*

Let us consider again the case of a finite group G of Lie type, with natural module V of dimension $d+1$ over some field k. For the sake of example, we consider $G := GL_{d+1}(k)$. Let $\mathcal{P}(V)$ denote the set of all subspaces of V,[3] and let $\Delta(\mathcal{P})$ denote the set of all flags of subspaces of V. Let $\Sigma = \{a_0, a_1, \ldots, a_d\}$ denote a basis for V. If

$$\mathcal{F} : 0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d = V$$

is a flag of subspaces of V such that for each V_i there is a subset Σ_i of Σ such that Σ_i is a basis for V_i, then we say that Σ *supports* $\mathcal{F}$. Let $\Sigma_{\Delta(\mathcal{P})}$ denote the set of all flags of subspaces of V supported

[3]Previously we had denoted this set by Δ. However, it actually corresponds to the *projective geometry*, $\mathcal{P}(V)$, of V.

by Σ. Then it can be proved that $\Delta(\mathcal{P})$ forms a complex under inclusion, which has the structure of a building, whose apartments are the complexes $\Sigma_{\Delta(\mathcal{P})}$ for each basis Σ of V (see [Suz82] for full details).

For each flag $\mathcal{F}$ in $\Delta(\mathcal{P})$, let $G_{\mathcal{F}} := \mathrm{Stab}_G(\mathcal{F})$. Let $\Delta := \{G_{\mathcal{F}} | \mathcal{F} \in \Delta(\mathcal{P})\}$, and partially order Δ by reverse inclusion, denoted $\sqsubseteq$. If

$$\mathcal{F}_1 : 0 \subsetneq V_{i_0} \subsetneq V_{i_1} \subsetneq \cdots \subsetneq V_{i_j} \subsetneq \cdots \subsetneq V_{i_n}; \qquad \text{and}$$
$$\mathcal{F}_2 : 0 \subsetneq V_{i_0} \subsetneq V_{i_1} \subsetneq \cdots \subsetneq V_{i_{j-1}} \subsetneq V_{i_{j+1}} \subsetneq \cdots \subsetneq V_{i_n},$$

are elements of $\Delta(\mathcal{P})$, then $\mathcal{F}_2$ is contained in $\mathcal{F}_1$. It follows that

$$G_{\mathcal{F}_1} = \mathrm{Stab}_G(\mathcal{F}_1) \subseteq \mathrm{Stab}_G(\mathcal{F}_2) = G_{\mathcal{F}_2},$$

and hence that $G_{\mathcal{F}_2} \sqsubseteq G_{\mathcal{F}_1}$. It follows that Δ and $\Delta(\mathcal{P})$ are isomorphic as complexes. For an apartment $\mathcal{A}$ of $\Delta(\mathcal{P})$, let $G_{\mathcal{A}}$ denote the subcomplex of Δ corresponding to the stabilizers in G of elements of $\mathcal{A}$. Since properties (B1)-(B4) hold for the building $\Delta(\mathcal{P})$, they also hold for the isomorphic complex Δ, with apartments given by the $G_{\mathcal{A}}$ where $\mathcal{A}$ is an apartment for $\Delta(\mathcal{P})$. We call Δ the *building of* G.

We conclude this section with a brief aside relating to abstract simplicial complexes.

Definition 2.4.7. *[AB08] An* (abstract) simplicial complex *with vertex set $\mathcal{V}$ is a nonempty collection Δ of finite subsets of $\mathcal{V}$ (called* simplices*) such that every singleton $\{v\}$ is a simplex and every subset of a simplex A is a simplex (called a* face *of A). The cardinality r of A is called the* rank *of A, and $r-1$ is called the* dimension *of A. A* subcomplex *of Δ is a subset Δ' that contains, for each of its elements A, all the faces of A; thus Δ' is a simplicial complex in its own right, with vertex set equal to some subset of $\mathcal{V}$.*

We will also require the following associated definition in Chapter 3:

Definition 2.4.8. *[AB08] Let Δ be a simplicial complex, and let σ be a simplex of Δ. We define the* star of σ*, denoted* $\mathrm{St}\,\sigma$*, to be the subset of Δ consisting of the simplices of Δ having σ as a face.*

If Δ is the building of a group G of Lie type with natural module V of dimension n over k, then we set $\mathcal{V} := \{\mathrm{Stab}_G(\mathcal{F}) | \mathcal{F} \in \Delta(\mathcal{P}) \text{ has rank } 1\}$ to be our vertex set, stabilizers of flags of rank 2 to be edges,..., and stabilizers of flags of rank n to be chambers. For simplices $\sigma, \tau \in \Delta$, we define τ to be a face of σ if $\sigma \subseteq \tau$. With this convention, we see that the building Δ has the structure of a simplicial complex.

2.5 Minimal Parabolic Systems

In Chapter 3 we shall see how Mark Ronan and Stephen Smith used the building of a finite Chevalley group G to construct representations for G over a finite field k using structures known as *presheaves*. Naturally, they wished to construct such presheaves for more general groups as indicated in [Ron84] and [RS85]. They proceeded to formulate a construction on an arbitrary simplicial complex with associated G-action (full details of which are given in Chapter 3). If G is a finite group, then a rich source of such simplicial complexes arise from minimal parabolic systems.

The theory of minimal parabolic systems was first introduced by Ronan and Stroth [RS84] and was later considered by Lempken, Parker and Rowley [LPR98]. Throughout this section, we assume that G is a finite group, p is a prime and that p divides $|G|$.

Definition 2.5.1. *[LPR98] Let $S \in Syl_p(G)$ and let $B := N_G(S)$. A* minimal parabolic subgroup *of G is a subgroup $P \leqslant G$ such that B is contained in a unique maximal subgroup of P. We denote the set of all minimal parabolic subgroups of G with respect to B by $\mathcal{M}(G, B)$.*

We note that Ronan and Stroth imposed the additional requirement that $O_p(P) \neq 1$. However - like Lempken, Parker and Rowley - we do not require this.

Given such a set of minimal parabolic subgroups for G, it is natural to consider certain subsets of $\mathcal{M}(G, B)$ that generate G.

Definition 2.5.2. *[LPR98]*

(i) A set $\mathcal{S} = \{P_1, \ldots, P_n\} \subseteq \mathcal{M}(G, B)$ is called a minimal parabolic system for G of rank n *or a* p-minimal parabolic system for G of rank n *if $G = \langle \mathcal{S} \rangle$ and no proper subset of $\mathcal{S}$ generates G.*

(ii) We call subgroups of the form $P := \langle P_{i_1}, \ldots, P_{i_m} \rangle$ parabolic subgroups *of G and denote them by $P_{i_1 \cdots i_m}$. The* rank *of P is defined to be the minimal such m.*

A full exposition of the p-minimal parabolic systems of the sporadic groups is given in [RS84].

Example 2.5.3. *Let $G := \mathrm{Sym}(6)$ and let $S \sim 2^{3+1}$ be a self-normalizing Sylow 2-subgroup of G. There are two isomorphic (but not conjugate) proper subgroups of G containing S, having shape $2^3.\mathrm{Sym}(3)$. It follows that G has a unique 2-minimal parabolic system consisting of these two subgroups.* •

The parabolic subgroups associated to a minimal parabolic system of G are self-normalizing as we now prove.

Lemma 2.5.4. *Let G be a finite group, $S \in Syl_p(G)$, $B = N_G(S)$ and let $\mathcal{S} = \{P_1, \ldots, P_n\}$ be a minimal parabolic system for G with respect to B. Then the parabolic subgroups of G formed from $\mathcal{S}$ are self-normalizing.*

Proof. Let P be a parabolic subgroup of G formed from $\mathcal{S}$, and let $g \in N_G(P)$. As P contains S and $P^g = P$, we deduce that $S^g \leqslant P^g = P$, and hence $S^g \in Syl_p(P)$. By Sylow's Theorems, the Sylow p-subgroups of P are conjugate to S in P. Thus there exists $h \in P$ such that $S^g = S^h$, and hence $S^{gh^{-1}} = S$. Consequently $gh^{-1} \in N_G(S) = B \leqslant P$. As $h \in P$, it follows that $g \in P$ and so $N_G(P) \leqslant P$. Since the reverse inclusion clearly holds, we deduce that $P = N_G(P)$ and P is self-normalizing. □

Parabolic Subgroup, P	Rank of P	$\lvert O_2(P)\rvert$	Shape of $O_2(P)$	$P/O_2(P)$
P_5	1	2^4	2^4	$\mathrm{Sym}(5)$
P_{5a}		2^4	2^4	$\mathrm{Sym}(5)^\sigma$
$P_{11a} := \langle P_1, P_{1a}\rangle$	2	2^5	2^{3+2}	$\mathrm{Sym}(3) \times \mathrm{Sym}(3)$
$P_{12} := \langle P_1, P_2\rangle$		2^4	2^4	$L_3(2)$
$P_{13} := \langle P_1, P_3\rangle$		2^4	2^4	$L_3(2)$
$P_{14} := \langle P_1, P_4\rangle$		2^4	2^4	$\mathrm{Alt}(6)$
$P_{1a2a} := \langle P_{1a}, P_{2a}\rangle$		2^4	2^4	$L_3(2)^\sigma$
$P_{1a3} := \langle P_{1a}, P_3\rangle$		2^4	2^4	$L_3(2)^\sigma$
$P_{1a4} := \langle P_{1a}, P_4\rangle$		2^4	2^4	$\mathrm{Alt}(6)^\sigma$
$P_{22a34} := \langle P_2, P_{2a}, P_3, P_4\rangle$		2^6	2^{4+2}	$(3 \times 3)2$
$P_{55a} := \langle P_5, P_{5a}\rangle$		1	1	$M_{21}2$
$P_{122a345} := \langle P_1, P_2, P_{2a}, P_3, P_4, P_5\rangle$	3	2^4	2^4	$2^4\,\mathrm{Alt}(7)$
$P_{145a} := \langle P_1, P_4, P_{5a}\rangle$		1	1	M_{22}
$P_{11a3} := \langle P_1, P_{1a}, P_3\rangle$		2	2	$2\,\mathrm{Alt}(8)$
$P_{11a4} := \langle P_1, P_{1a}, P_4\rangle$		1	1	$U_4(3)$
$P_{1a45} := \langle P_{1a}, P_4, P_5\rangle$		1	1	M_{22}^σ
$P_{1a22a345a} := \langle P_{1a}, P_2, P_{2a}, P_3, P_4, P_{5a}\rangle$		2^4	2^4	$2^4\,\mathrm{Alt}(7)^\sigma$

Table 2.1: The shapes of the 2-parabolic subgroups of McL that do not contain a Sylow 2-subgroup as a maximal subgroup.

By considering the parabolic subgroups associated to given minimal parabolic systems, we see that the set of all minimal parabolic systems for G can be partitioned into two sets; geometric minimal parabolic systems and non-geometric minimal parabolic systems.

Definition 2.5.5. *[RS84] A minimal parabolic system $\mathcal{S}$ is called* geometric *if for any two associated parabolic subgroups P and Q, $P \cap Q$ is also a parabolic subgroup. If $\mathcal{S}$ is not geometric, then we call it* non-geometric*.*

Example 2.5.6. *Let $G := McL$ and let S be a self-normalizing Sylow 2-subgroup of G. There are eight 2-minimal parabolic subgroups of G, which we denote by P_1, $P_{1a} := P_1^\sigma$, P_2, $P_{2a} := P_2^\sigma$, P_3, P_4, P_5 and $P_{5a} := P_5^\sigma$, where σ is the non-trivial outer-automorphism of G. We see that S is maximal in all 2-minimal parabolic subgroups excluding P_5 and P_{5a}. These latter two subgroups have a unique maximal subgroup containing S, namely P_4.*

The Hasse diagram of the associated parabolic subgroups is presented in [RS84, p79]. The 2-minimal parabolic subgroups P_1 and P_{1a} both have shape $2^{4+2}.\mathrm{Sym}(3)$, whilst the other 2-minimal parabolic subgroups containing S as a maximal subgroup are pairwise isomorphic and also have shape $2^{4+2}.\mathrm{Sym}(3)$ (although they are not isomorphic to P_1). We summarise the shapes of the other parabolic subgroups of G in Table 2.1. There are 16 minimal parabolic systems for G associated to the given minimal parabolic subgroups. Of these, 9 are geometric minimal parabolic systems and 7 are non-geometric. We summarise these systems in Table 2.2. •

For a given p-minimal parabolic system $\mathcal{S} := \{P_1, \ldots, P_m\}$ for a group G with respect to the normalizer, B, of a Sylow p-subgroup and for any $g \in G$, we see that $\mathcal{S}^g = \{P_1^g, \ldots, P_m^g\}$ is a

Geometric Minimal Parabolic Systems	Non-Geometric Minimal Parabolic Systems
$\{P_1, P_{1a}, P_2\}$	$\{P_1, P_{2a}, P_{5a}\}$
$\{P_1, P_{1a}, P_{2a}\}$	$\{P_1, P_5, P_{5a}\}$
$\{P_1, P_{1a}, P_5\}$	$\{P_{1a}, P_2, P_5\}$
$\{P_1, P_{1a}, P_{5a}\}$	$\{P_{1a}, P_5, P_{5a}\}$
$\{P_1, P_2, P_{5a}\}$	$\{P_2, P_5, P_{5a}\}$
$\{P_{1a}, P_{2a}, P_5\}$	$\{P_{2a}, P_5, P_{5a}\}$
$\{P_1, P_3, P_{5a}\}$	$\{P_3, P_5, P_{5a}\}$
$\{P_{1a}, P_3, P_5\}$	
$\{P_1, P_{1a}, P_3, P_4\}$	

Table 2.2: The geometric and non-geometric 2-minimal parabolic systems for the McLaughlin group, McL.

p-minimal parabolic system for G with respect to B^g. Define $P_\varnothing := B$ and

$$\Delta := \bigcup_{g \in G} \{P_J^g : J \subsetneq \{1, \ldots, m\}\}.$$

We see that Δ is a simplicial complex. Indeed, to mirror the setting of buildings, we define the *vertices* of Δ to be parabolic subgroups of the form P_J^g for some $g \in G$ and some $J \subseteq \{1, \ldots, m\}$ of cardinality $m - 1$ and use reverse inclusion. Thus *edges* are parabolic subgroups P_J^g corresponding to some $J \subseteq \{1, \ldots, m\}$ of cardinality $m - 2$, and so on. Finally, we define conjugates of $P_\varnothing = B$ to be the *chambers* of Δ.

The group G has a natural action on Δ given by conjugation. We shall see in Chapter 3 that we are able to define presheaves on Δ, and thus obtain representations for G.

2.6 Geometries and their Diagrams

In many areas of mathematics, we may partition a given set of elements into subsets, each subset containing elements that are in some sense of the same *type*. For example, if we consider the set of simplices of a 3-dimensional Euclidean tetrahedron, T, then we naturally partition the set into 0-, 1- and 2-dimensional simplices known as *vertices*, *edges* and *faces* of T. These ideas lead to the concept of an incidence geometry.

Definition 2.6.1. *[Bue81] Let Δ be a set. An* incidence geometry Γ over Δ *is a triple $\Gamma = (S, *, t)$, where S is a set (the* elements *of Γ), $*$ is a symmetric and reflexive relation defined on S (the* incidence relation *of Γ) and t is a mapping of S onto Δ (the* type function *of Γ) such that:*

*(TF) The restriction of t to every maximal set of pairwise incident elements is a bijection onto Δ (*transversality property*).*

For brevity, when the incident relation and type function are clear from the context we will simply refer to the geometry S . The transversality property (TF) is sometimes relaxed and replaced by the property:

(TF') If $x, y \in S$ $(x \neq y)$ with $x * y$, then $t(x) \neq t(y)$.

Clearly (TF) implies (TF′), however the converse is not true in general. Indeed we can easily extend any geometry Γ by adding a single element α to the set S and defining the type of α to be an element not contained in Δ. Moreover, we may define α in such a way that it is not incident to any other element of S. Then clearly (TF′) will still hold, however (TF) will not hold. Indeed, any maximal set, $\mathcal{M}$, of pairwise incident elements of $S \cup \{\alpha\}$ will either be $\{\alpha\}$ or will not contain α, and hence the restriction of t to $\mathcal{M}$ will not be a bijection onto $\Delta \cup \{t(\alpha)\}$.

An important property of a geometry is its rank. If $\Gamma = (S, *, t)$ is a geometry over Δ, then the *rank* of Γ is defined to be $|\Delta|$. Property (TF) illustrates the importance of sets of pairwise incident elements of a geometry, and we define a *flag*, F, of Γ to be a (possibly empty) set of pairwise incident elements of S. Moreover, the *type* of F is defined to be the image of F under t, whilst the *cotype* of F is defined to be $\Delta \backslash t(F)$. Clearly a flag, F, together with the restriction of $*$ and t to F forms a geometry over $t(F)$. The rank of this geometry is just $|F|$.

Given a flag of a geometry, we may construct a further geometry in the following way:

Definition 2.6.2. *[Bue81] The* residue *of a flag F in a geometry $\Gamma = (S, *, t)$ (over Δ) is the geometry $\Gamma_F = (S_F, *_F, t_F)$ over $t(S_F) = \Delta \backslash t(F)$ defined by*

- *S_F is the set of all elements of Γ not in F, incident with all elements of F;*
- *$*_F$ is the restriction of $*$ to S_F;*
- *t_F is the restriction of t to S_F.*

Returning to our example of the Euclidean tetrahedron, T, if we set F to be a flag consisting of an incident face and vertex, then S_F consists of the two edges of T that are incident to the given face and vertex.

Given a geometry $\Gamma = (S, *, t)$ over a set Δ, we may construct the incidence graph of Γ whose vertex set is S and where $x, y \in S$ are joined by an edge if and only if $x * y$. This is an easy graph to construct from a theoretical point of view, but it can be computationally cumbersome. It is standard to represent a geometry by the *diagram of the geometry*. This is constructed as follows; for each pair of distinct elements $i, j \in \Delta$, let F be a flag of Γ with residue Γ_F of type $\{i, j\}$. Construct the incidence graph G_F of Γ_F. We define the i-diameter of G_F, denoted $d_{i,j}$, to be the maximal diameter at an element of type i, the j-diameter of G_F, denoted $d_{j,i}$, to be the maximal diameter at an element of type j, and the girth of G_F to be $2g_{i,j}$ (so $g_{i,j} = g_{j,i}$). The diagram of the geometry Γ has vertex set Δ, and between each pair of incident vertices $i, j \in \Delta$, there is an undirected edge from i to j labelled $d_{i,j}$, $g_{i,j}$, $d_{j,i}$.

Example 2.6.3. Let Γ be the geometry of vertices, edges and faces of the Euclidean tetrahedron, T, as described above. By labelling the vertices of T by $v_1, \ldots, v_4$, the edges by $e_1, \ldots, e_6$ and the faces by $f_1, \ldots, f_4$ we may represent T diagrammatically as in Figure 2.1. Let $F = F_{v_i}$ be a flag consisting of a single vertex. The residue Γ_F consists of the three edges and three faces incident to v_i, and the incidence graph of Γ_F can be seen to have edge-diameter and face-diameter equal to 3, and girth equal to 6. In a similar way if $F = F_{e_i}$ is a flag consisting of a single edge, then the incidence graph of Γ_F has vertex-diameter and face-diameter equal to 2, and girth equal to

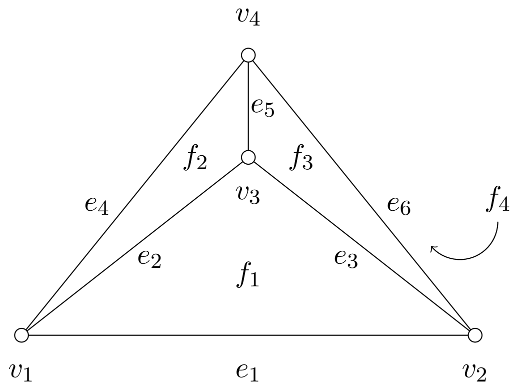


Figure 2.1: The Euclidean tetrahedron T from Example 2.6.3.

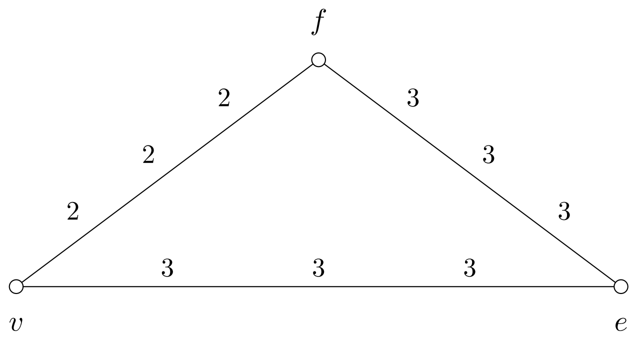


Figure 2.2: The diagram of the geometry defined on the Euclidean tetrahedron T from Example 2.6.3.

4. Finally if $F = F_{f_i}$ is a flag consisting of a single face, then the incidence graph of the residue Γ_F has vertex-diameter and edge-diameter equal to 3, and girth equal to 6. The diagram of the geometry Γ is given in Figure 2.2, where we use v, e and f to represent the three types of elements of T; vertex, edge and face respectively. •

Example 2.6.3 illustrates a common situation. If for each pair of distinct types $i, j \in \Delta$, we have $d_{i,j} = d_{j,i} = g_{i,j}$, then we may simplify the diagram of our geometry to create the *Buekenhout diagram of the geometry* as follows; we represent each type in Δ by a circle, and we join $i, j \in \Delta$ by $g_{i,j} - 2$ edges if $g_{i,j} \leqslant 4$ (noting that $2 \leqslant g_{i,j}$ by definition), and if $g_{i,j} > 4$, we join i and j by a single edge and label the edge (n) where $n = g_{i,j} - 2$. Using this convention, we see that the Buekenhout diagram of the tetrahedral geometry from Example 2.6.3 is much simpler, and takes the form shown in Figure 2.3.

Our motivation for considering geometries arises from minimal parabolic systems. Indeed, given a minimal parabolic system $\mathcal{S} = \{P_1, \ldots, P_n\}$ for a group G with respect to a Sylow normalizer B, then we may form a geometry as follows; take S to be the set containing all G-conjugates of the maximal parabolic subgroups $P_{\{1,\ldots,n\}\setminus\{i\}}$, $\Delta := \{1, \ldots, n\}$, define $t : S \twoheadrightarrow \Delta$ by $t : P^g_{\{1,\ldots,n\}\setminus\{i\}} \mapsto i$, and define the incidence relation $*$ by $P^g_{\{1,\ldots,n\}\setminus\{i\}} * P^h_{\{1,\ldots,n\}\setminus\{j\}}$ precisely when $i \neq j$ and there exists some $k \in G$ for which $B^k \subseteq P^g_{\{1,\ldots,n\}\setminus\{i\}} \cap P^h_{\{1,\ldots,n\}\setminus\{j\}}$. Since the transversality property clearly

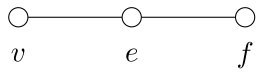


Figure 2.3: The Buekenhout diagram of the geometry defined on the Euclidean tetrahedron T from Example 2.6.3.

holds, we conclude that $(S, *, t)$ is a geometry.[4] The rank of the resulting geometry is equal to the rank of the minimal parabolic system $\mathcal{S}$. In Chapter 3 we shall consider rank 2 and rank 3 geometries arising from 2-minimal parabolic systems for the Mathieu Groups.

[4]We note that our formulation of the geometry S differs from that used in [RS84]. Ronan and Stroth defined two maximal parabolic subgroups of differing types to be incident if their intersection was also a parabolic subgroup. Thus they only considered the geometries defined by *geometric* minimal parabolic systems. We have chosen a broader formulation of our geometries to allow all minimal parabolic systems to be considered.

Chapter 3

Homology of Presheaves of Abelian Groups

The subject of representation theory for finite groups has a long and rich history stretching back to the nineteenth century. One of the first mathematicians to consider the subject was William Burnside (although we note that he referred to a representation of a group as a *group of linear representations* in has famous treatise *Theory of Groups of Finite Order* [Bur55]). By the beginning of the twentieth century representation theory was starting to be seen as a powerful tool in proving group-theoretic results, such as Burnside's $p^a q^b$-Theorem. Further applications came to fruition later in the century, and representation theory lies at the heart of parts of the proof of the classification theorem for finite simple groups. This led to a comprehensive study of representations of finite groups over the complex numbers, and the publication of the $\mathbb{ATLAS}$ of character tables for many finite groups. [CCN$^+$09]

Whilst considerable focus was paid to representation theory over the complex numbers, much time and effort was also devoted to modular representation theory. As early as 1902, Leonard Dickson had published a paper considering modular representations [Dic02], and over the course of the twentieth century, much work was undertaken on the subject. However, although many results have been formulated about modular representations and a modular atlas has been published [ABL$^+$], the exact structure of many of the irreducible kG-modules for a given finite group G and finite field k remain unknown.

The theory of modular representations has been well developed for finite Chevalley groups. For such a group, G, we have already seen that it is possible to define a simplicial complex consisting of the parabolic subgroups of G, known as the *building of G*. This simplicial complex can then be used to determine the irreducible kG-modules for a finite field k. In the 1980s, a number of mathematicians considered how the idea of the building of a group could be extended from the Chevalley groups to an arbitrary finite group. This led to the 1984 paper of Mark Ronan and Gernoth Stroth [RS84], where they defined the notion of minimal parabolic subgroups and minimal parabolic systems of a finite group G with respect to the normalizer of a given Sylow p-subgroup of G. They considered such a situation for each of the 26 sporadic simple groups, and for the fields $GF(p)$ where $p = 2, 3, 5$. Mark Ronan and Stephen Smith then used such minimal parabolic systems to define a *presheaf of abelian groups*.[1] By defining chain spaces and boundary maps on such presheaves, they were then able to form homology groups. These homology groups are kG-modules, and via their construction, considerable information is known about their structure. The two mathematicians published a series of three joint papers on the subject between 1985 and 1989 ([RS85], [RS86], [RS89]), in addition to a further paper in 1989 published by Ronan ([Ron89]). Their work culminated in the calculation of the zero-homology groups of presheaves defined over the field $GF(2)$ for the groups $L_4(2)$, $\mathrm{Alt}(7)$ and the triple cover of $Sp_4(2)$ in addition to calculations of quotients of the zero-homology groups for presheaves of M_{24}.

Ronan and Smith went on to prove that for a finite Chevalley group G, there is a 1-1 correspondence between irreducible kG-modules V and presheaves $\mathcal{F}_V$, known as the *fixed-point presheaf of the module* V [RS85, Section 3].

We begin this Chapter in Section 3.1 by introducing the basic motivation behind the subject and the underlying definitions introduced by Ronan and Smith. This is followed in Section 3.2 by a brief discussion about two important classes of presheaves; *constant presheaves* and *fixed-point presheaves* and in Section 3.3 by the formulation of the homology and cohomology groups of a given presheaf. The Euler characteristic is also introduced here. In Section 3.4 we consider the *universal* construction of a presheaf, from a minimal substructure known as a Π-stalk. Such presheaves admit all other chamber-generated presheaves as quotients, and hence they are of great interest. Closely connected to universal presheaves are *dual presheaves*, and these are introduced in Section 3.5. These were first introduced by Ronan in [Ron89], and their importance lies in a theorem of Ronan relating the homology groups of a presheaf to the cohomology groups of the dual presheaf. This result is given - in addition to many other known results - in Section 3.6 as Theorem 3.6.10. In Section 3.7, we briefly consider the question of whether for a finite group G and finite field k, every irreducible kG-module can occur as the quotient of the zero-homology group of a universal panel-irreducible presheaf. We see that in general this does not occur. The focus of the chapter then changes in Section 3.8 from theoretical to computational, as the zero-homology groups over $GF(2)$ of presheaves of the symmetric group $\mathrm{Sym}(6)$ and the Mathieu groups M_{11} and M_{22} are explicitly calculated, whilst those of the Mathieu group M_{12} are partially calculated. In addition, the irreducible quotients of the zero-homology groups of presheaves of the Mathieu groups M_{23} and M_{24} over $GF(2)$ are determined. These calculations are preceded by a discussion of a method known as *geometric spanning*, which can be used to obtain upper bounds on the dimensions of zero-homology groups. The chapter concludes in Section 3.9 with a few final remarks on the aforementioned question concerning irreducible kG-modules arising as quotients of zero-homology groups of universal panel-irreducible presheaves.

3.1 Presheaves and Coordinate Systems

Let X be an arbitrary topological space. Denote by $\mathscr{X}$ the category of open subsets of X with morphisms consisting of inclusion maps, and let Ab denote the category of abelian groups.

[1] Ronan and Smith referred to presheaves as sheaves throughout their papers [RS85], [RS86] and [RS89].

Definition 3.1.1. *[Bre97] A* presheaf $\mathscr{F}$ *(of abelian groups) on* X *is a contravariant functor from* $\mathscr{X}$ *to Ab. Thus for each* U *an open subset of* X*, we associate an abelian group* $\mathscr{F}(U)$*, and for each inclusion map* $i : U \hookrightarrow V$ *we associate a homomorphism of abelian groups* $\tau_{U,V} : \mathscr{F}(V) \to \mathscr{F}(U)$ *such that*

1. $\tau_{U,U} = \mathrm{id}_U$ *for all open subsets* $U \subseteq X$*; and*

2. *for* $U \subseteq V \subseteq W$ *open subsets of* X*, we have* $\tau_{U,W} = \tau_{V,W} \circ \tau_{U,V}$.

An example of a presheaf on X is given by the constant presheaf. Let A be an arbitrary abelian group. The *constant presheaf* of A on X is defined by setting $\mathscr{F}(U) = A$ for all open subsets $U \subseteq X$, and for open subsets $U \subseteq V$, define $\tau_{U,V} = \mathrm{id}_A$. It is clear that this satisfies the given conditions, and hence $\mathscr{F}$ is a well-defined presheaf.

Ronan and Smith extended this notion of presheaf from topological spaces to simplicial complexes. First they defined a coefficient system as follows:

Definition 3.1.2. *[RS85] Let* Δ *be a simplicial complex. A* coefficient system $\mathscr{F}$ *for* Δ *is a set of abelian groups* $\{\mathscr{F}_\sigma \mid \sigma$ *is a simplex of* $\Delta\}$ *together with connecting homomorphisms* $\varphi_{\sigma\tau} : \mathscr{F}_\sigma \to \mathscr{F}_\tau$ *whenever* τ *is a face of* σ*, such that if* γ *is also a face of* τ*, then*

$$\varphi_{\sigma\gamma} = \varphi_{\sigma\tau} \circ \varphi_{\tau\gamma}. \tag{3.1.1}$$

Since the parabolic subgroups associated to a geometric minimal parabolic system of a group G form a simplicial complex Δ, Ronan and Smith imposed further conditions on a coefficient system $\mathscr{F}$ for Δ, mirroring those properties found in such a minimal parabolic system. They required that each $\mathscr{F}_\sigma$ is a vector space over a finite field k, and that there exists a G-action on the formal direct sum of the terms of $\mathscr{F}$, corresponding to a permutation action of G on Δ. For $g \in G$, the restriction of the g-action to $\mathscr{F}_\sigma$, denoted $\tilde{g}_\sigma$, is a k-homomorphism from $\mathscr{F}_\sigma$ to $\mathscr{F}_{\sigma g}$ (the term at the simplex σg). The focus of Ronan and Smith's work was on kG-modules, so it is natural to further assume that the G-action commutes with the connecting maps. This means that the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}_\sigma & \xrightarrow{\tilde{g}_\sigma} & \mathscr{F}_{\sigma g} \\ {\scriptstyle\varphi_{\sigma\tau}}\downarrow & & \downarrow{\scriptstyle\varphi_{\sigma g \tau g}} \\ \mathscr{F}_\tau & \xrightarrow{\tilde{g}_\tau} & \mathscr{F}_{\tau g} \end{array} \tag{3.1.2}$$

whenever τ is a face of σ. In such a situation, we call the coefficient system $\mathscr{F}$ a *presheaf on* Δ.[2] If for each simplex $\sigma \in \Delta$, the presheaf term $\mathscr{F}_\sigma$ is spanned by the images of the presheaf terms of the chambers of Δ containing σ under the corresponding connecting maps,

$$\mathscr{F}_\sigma = \langle \mathscr{F}_c \varphi_{c\sigma} | c \text{ is a chamber and } \sigma \text{ is a face of } c \rangle \qquad \text{for all } \sigma \in \Delta,$$

then we call $\mathscr{F}$ a *chamber-generated presheaf*.

[2] We note that for the sake of brevity, Ronan and Smith referred to presheaves as sheaves. However, we will not be adopting this convention.

Let Δ be a simplicial complex with associated G-action for some group G, and let $\mathscr{F}$ be a presheaf defined on Δ. If $\sigma \in \Delta$ is a simplex, then we denote the stabilizer of σ in G by G_σ. For $g \in G_\sigma$, we have $\tilde{g}_\sigma : \mathscr{F}_\sigma \to \mathscr{F}_{\sigma g} = \mathscr{F}_\sigma$. It follows that $\mathscr{F}_\sigma$ has a kG_σ-module structure. We call a presheaf $\mathscr{F}$ *irreducible* if the terms of $\mathscr{F}$ at every simplex of Δ are irreducible modules. Similarly, we call $\mathscr{F}$ *panel-irreducible* if the terms of $\mathscr{F}$ at every panel and every chamber of Δ are irreducible modules.

By defining connecting maps $\varphi_{\sigma\sigma} = \mathrm{id}_\sigma$ for all simplices $\sigma \in \Delta$, we see that a coefficient system $\mathscr{F}$ on Δ with G-action satisfying (3.1.2) does indeed generalise the notion of a presheaf for a topological space.

As with most algebraic structures, we may define subpresheaves and quotient presheaves in the natural way. Let $\mathscr{F}$ be a presheaf defined on a simplicial complex Δ , with connecting maps $\varphi_{\sigma\tau}$ whenever τ is a face of σ. A presheaf $\mathscr{G}$ on Δ is called a *subpresheaf* of $\mathscr{F}$ if $\mathscr{G}_\sigma \subseteq \mathscr{F}_\sigma$ for each simplex $\sigma \in \Delta$, and whenever τ is a face of σ, the connecting map in $\mathscr{G}$ is given by $\varphi_{\sigma\tau}|_{\mathscr{G}_\sigma}$. Given a subpresheaf $\mathscr{G}$ of $\mathscr{F}$, we then define the *quotient presheaf* $\mathscr{F}/\mathscr{G}$, whose term at the simplex σ is the quotient space $\mathscr{F}_\sigma/\mathscr{G}_\sigma$, and whose connecting maps are given by the corresponding quotient maps.

It is also natural to consider mappings between presheaves. Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves defined on a simplicial complex Δ, with connecting maps $\varphi_{\sigma\tau}$ and $\theta_{\sigma\tau}$ respectively (whenever τ is a face of σ), and with G-actions denoted by $\tilde{g}$ and $\hat{g}$ respectively (for each g in the defining group G). A *morphism* from $\mathscr{F}$ to $\mathscr{G}$ is a set of k-linear maps $m := \{m_\sigma : \mathscr{F}_\sigma \to \mathscr{G}_\sigma\}$ for each simplex $\sigma \in \Delta$, such that the following two diagrams commute:

(i) if τ is a face of σ, then

$$\begin{tikzcd} \mathscr{F}_\sigma \arrow[r, "m_\sigma"] \arrow[d, "\varphi_{\sigma\tau}"'] & \mathscr{G}_\sigma \arrow[d, "\theta_{\sigma\tau}"] \\ \mathscr{F}_\tau \arrow[r, "m_\tau"] & \mathscr{G}_\tau \end{tikzcd} \quad ; \text{ and} \tag{3.1.3}$$

(ii) for each $\sigma \in \Delta$ and each $g \in G$,

$$\begin{tikzcd} \mathscr{F}_\sigma \arrow[r, "m_\sigma"] \arrow[d, "\tilde{g}"'] & \mathscr{G}_\sigma \arrow[d, "\hat{g}"] \\ \mathscr{F}_{\sigma g} \arrow[r, "m_{\sigma g}"] & \mathscr{G}_{\sigma g} \end{tikzcd} \quad . \tag{3.1.4}$$

We call the morphism $m : \mathscr{F} \to \mathscr{G}$ *injective* if each k-linear map m_σ is injective, *surjective* if each m_σ is surjective, and we call m an *isomorphism* if it is both injective and surjective. From a categorical point-of-view, we see that morphisms of presheaves generalise the notion of natural transformations between presheaves (when viewed as functors). We note that if $m : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, then each $m_\sigma \in m$ is a kG_σ-homomorphism.

Before continuing with the theory of presheaves, we consider the notion of isomorphic presheaves further. If $m : \mathscr{F} \to \mathscr{G}$ is an isomorphism of presheaves, then using the commutativity of (3.1.3) (resp. (3.1.4)), we see that the connecting maps (resp. G-action) in $\mathscr{G}$ are uniquely determined by the connecting maps (resp. G-action) in $\mathscr{F}$. The fact that commutativity holds in (3.1.3) is crucial to the notion of an isomorphism. Indeed, let Δ be the simplicial complex of parabolic subgroups of some group G. For each parabolic subgroup $\sigma = P_a \in \Delta$, define $\mathscr{F}_\sigma = \mathscr{G}_\sigma = kP_a$. Define connecting maps in $\mathscr{F}$ by setting $\varphi_{\sigma\tau}$ to be the inclusion map, and in $\mathscr{G}$ by setting $\theta_{\sigma\tau}$ to be the zero map, whenever τ is a face of σ. It is easy to check that $\mathscr{F}$ and $\mathscr{G}$ with their connecting maps $\varphi_{\sigma\tau}$ and $\theta_{\sigma\tau}$ respectively are presheaves (where the G-action in each presheaf is given by multiplication in kG). However, $\mathscr{F}$ and $\mathscr{G}$ are clearly not isomorphic presheaves.

3.2 Classes of Presheaves

We now consider two important classes of presheaves that we will use frequently throughout our subsequent work; constant presheaves and fixed-point presheaves. We define constant presheaves for arbitrary simplicial complexes. However, we will only define fixed-point presheaves on the simplicial complex of parabolic subgroups of a group G defined by a given geometric minimal parabolic system. Unless otherwise stated, for the rest of this chapter k will denote a finite field of characteristic p.

3.2.1 Constant Presheaves

Let V be a kG-module. In Section 3.1 we saw how to define the constant presheaf of V on a topological space X. Given a simplicial complex Δ (with associated G-action) we can follow a similar construction.

Definition 3.2.1. *[RS85] The* constant presheaf *of V on Δ, denoted $\mathcal{K}_V$, is the presheaf on Δ with*

(i) $(\mathcal{K}_V)_\sigma := V$ for every simplex $\sigma \in \Delta$;

(ii) for τ a face of σ we define $\varphi_{\sigma\tau}$ to be the identity map on V; and

(iii) the G-action in $\mathcal{K}_V$ is defined by the G-action in the module V.

It is clear from the definition that constant presheaves are chamber-generated.

Suppose that $n : V \to W$ is a kG-module isomorphism. By setting $m_\sigma = n$ for each $\sigma \in \Delta$ we see that $m := \{m_\sigma | \sigma \in \Delta\}$ is an isomorphism of presheaves from $\mathcal{K}_V$ to $\mathcal{K}_W$. Hence isomorphic kG-modules give rise to isomorphic constant presheaves.

3.2.2 Fixed-Point Presheaves

Let G be a group and let Δ be the simplicial complex of parabolic subgroups of G defined by a geometric minimal parabolic system. Let V be a kG-module. For each parabolic subgroup $\sigma \in \Delta$ we may consider the unipotent radical $U_\sigma := O_p(G_\sigma)$ of G_σ (as defined in Chapter 2). Let V_σ be the fixed-point subspace of V under U_σ:

$$V_\sigma := \{v \in V | v * u = v \text{ for all } u \in U_\sigma\}.$$

Since U_σ is a normal subgroup of G_σ, it follows that V_σ is a kG_σ-module (more precisely, it is a kG_σ-submodule of $V|_{kG_\sigma}$). Moreover, if $\sigma \subseteq \tau$, then $G_\sigma \subseteq G_\tau$. Since U_σ, U_τ are normal p-subgroups of each Sylow p-subgroup of G_σ, it follows that U_τ is a normal p-subgroup of G_σ. Since U_σ is the largest such subgroup, we have that $U_\tau \subseteq U_\sigma$ and hence $V_\sigma \subseteq V_\tau$.

Definition 3.2.2. *[RS85] The* fixed-point presheaf *of V on Δ, denoted $\mathcal{F}_V$, is defined by setting:*

(i) $(\mathcal{F}_V)_\sigma := V_\sigma$ for each $\sigma \in \Delta$;

(ii) for τ a face of σ, setting $\varphi_{\sigma\tau}$ to be the inclusion map from V_σ to V_τ; and

(iii) for $g \in G$, define the action of g on $\mathcal{F}_V$ to be the action defined within V.

Since the connecting maps in $\mathcal{F}_V$ are inclusion maps, properties (3.1.1) and (3.1.2) clearly hold, and we have a well-defined presheaf. Clearly for a kG-module V, we have $\mathcal{F}_V \subseteq \mathcal{K}_V$ and equality holds when V is the trivial kG-module.

When G is a finite Chevalley group, we have the following result.

Theorem 3.2.3. *[Smi82] Let G be a finite Chevalley group over the finite field k, let P be a parabolic subgroup of G having unipotent radical U and complement L and let V be a finite-dimensional irreducible kG-module. Then the fixed-point subspace V^U affords an irreducible kL-module.*

An immediate consequence of this is that for finite Chevalley groups, the fixed-point presheaf $\mathcal{F}_V$ is irreducible.

We note that fixed-point presheaves are not-necessarily chamber-generated, as illustrated by the following example.

Example 3.2.4. *Let $G := McL$. The 2-minimal parabolic subgroups and systems of G were introduced in Example 2.5.6. We note that the 22-dimensional irreducible $GF(2)McL$-module V gives rise to a fixed-point space $\mathcal{F}_B \cong 1 \oplus 1 \oplus 1$. From this, we deduce that the kP-modules*

$$\langle \mathcal{F}_c \varphi_{cP} | c \text{ is a chamber of } P \rangle$$

for $P = P_{11a4}, P_{15a}, P_{1a5}, P_{55a}$ have dimensions 21, 21, 21 and 20 respectively over $GF(2)$. However, the 2-cores of each of these maximal 2-parabolic subgroups is trivial. Hence the presheaf terms $\mathcal{F}_{P_{11a4}}$, $\mathcal{F}_{P_{15a}}$, $\mathcal{F}_{P_{1a5}}$ and $\mathcal{F}_{P_{55a}}$ are all equal to V, and thus have dimension 22 over $GF(2)$. It follows that for any 2-minimal parabolic system of McL containing P_{11a4}, P_{15a}, P_{1a5} or P_{55a}, the fixed-point presheaf $\mathcal{F}_V$ is not chamber-generated. •

3.3 Homology of Presheaves

A simplicial complex, Δ, consists of simplices, which we may partition into sets according to their rank. If a group G acts on Δ naturally, then it will preserve the rank of simplices. Hence for a presheaf defined on Δ, we may think of the presheaf terms at simplices of equal rank to be - in some senses - similar. We make this idea precise through the notion of chain spaces for a presheaf, from which we may define homology groups. These homology groups will turn out to be kG-modules and are of great interest.

For the remainder of this chapter, we will assume that each simplicial complex Δ has a natural G-action for some group G.

3.3.1 Homology Groups

Let Δ be a simplicial complex of dimension n having underlying vertex set $\{v_i\}_{i=0}^m$ for some $m \in \mathbb{N}$ and let $\mathscr{F}$ be a presheaf defined on Δ. For each $r = 0, \ldots, n$ denote by $C_r(\Delta, \mathscr{F})$ the formal direct sum of the presheaf terms at r-simplices. As the action of G on $\mathscr{F}$ preserves the simplicial dimension, each $C_r(\Delta, \mathscr{F})$ is a kG-module. We call $C_r(\Delta, \mathscr{F})$ the *r-th chain space of $\mathscr{F}$ with respect to* Δ.

The connecting maps in $\mathscr{F}$ map direct summands of $C_r(\Delta, \mathscr{F})$ into $C_{r-1}(\Delta, \mathscr{F})$ for each $r = 1, \ldots n$. We use these maps to define boundary maps $\partial_r : C_r(\Delta, \mathscr{F}) \to C_{r-1}(\Delta, \mathscr{F})$. Indeed, let σ be an r-simplex. Then $\sigma = \{v_{i_0}, \ldots, v_{i_r}\} \subseteq \{v_0, \ldots, v_m\}$ with $i_j < i_{j+1}$ for all $j = 0, \ldots, r-1$. Define the *type* of σ to be $v_{i_0} < \cdots < v_{i_r}$ and define $\sigma_j := \{v_{i_0}, \ldots, v_{i_{j-1}}, v_{i_{j+1}}, \ldots, v_{i_r}\}$ for each $j = 0, \ldots, r$. Thus σ_j is a maximal face of σ of type $v_{i_0} < \cdots < v_{i_{j-1}} < v_{i_{j+1}} < \cdots < v_{i_r}$. We define $\partial_{\sigma\sigma_j} := (-1)^j \varphi_{\sigma\sigma_j}$ and then define $\partial_\sigma := \sum_{j=0}^r \partial_{\sigma\sigma_j}$. Finally, summing over all simplices of dimension r, we obtain a map $\partial_r := \sum_{\dim \sigma = r} \partial_\sigma : C_r(\Delta, \mathscr{F}) \to C_{r-1}(\Delta, \mathscr{F})$. We define ∂_0 to be the zero map on $C_0(\Delta, \mathscr{F})$.

Since each ∂_r is a linear combination of connecting maps of the presheaf $\mathscr{F}$, and these connecting maps commute with the G-action on $\mathscr{F}$, we see that ∂_r commutes with the G-action. Moreover, since the connecting maps are homomorphisms, each map ∂_r is a kG-homomorphism. We call the maps ∂_r (for $r = 0, 1, \ldots, n$) *boundary maps*. The kernel of ∂_r is called the *space of r-cycles* and is denoted $Z_r(\Delta, \mathscr{F})$. The image of ∂_r, denoted $B_{r-1}(\Delta, \mathscr{F})$, is called the *space of r-boundaries*. We set $B_n(\Delta, \mathscr{F})$ to be the 0-space.

Lemma 3.3.1. *The chain spaces and boundary maps as defined above form a chain complex*

$$0 \xrightarrow{0} C_n(\Delta, \mathscr{F}) \xrightarrow{\partial_n} C_{n-1}(\Delta, \mathscr{F}) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(\Delta, \mathscr{F}) \xrightarrow{\partial_0} 0$$

Proof. Let $r \in \{2, \ldots n\}$. It suffices to show that $(\partial_r \partial_{r-1})|_{\mathscr{F}_\sigma} = 0$ for an arbitrary r-simplex $\sigma \in \Delta$. Indeed, assume that σ has type $a_0 < \cdots < a_r$. Let σ_i denote its maximal face of type $a_0 < \cdots < a_{i-1} < a_{i+1} < \cdots < a_r$ and for $i \neq j$ let σ_{ij} denote the maximal face of σ_i of type $a_0 < \cdots < a_{i-1} < a_{i+1} < \cdots < a_{j-1} < a_{j+1} < \cdots < a_r$ or $a_0 < \cdots < a_{j-1} < a_{j+1} < \cdots < a_{i-1} <$

$a_{i+1} < \cdots < a_r$ depending on whether $i < j$ or $i > j$ respectively. It follows that $\sigma_{ij} = \sigma_{ji}$. Thus

$$
\begin{aligned}
(\partial_r \partial_{r-1})|_{\mathscr{F}_\sigma} &= \partial_\sigma \partial_{r-1} \\
&= \sum_{i=0}^{r} \left((-1)^i \varphi_{\sigma\sigma_i} \left(\sum_{\substack{j=0 \\ j<i}}^{r} (-1)^j \varphi_{\sigma_i\sigma_{ij}} - \sum_{\substack{j=0 \\ j>i}}^{r} (-1)^j \varphi_{\sigma_i\sigma_{ij}} \right) \right) \\
&= \sum_{i=0}^{r} \left((-1)^i \varphi_{\sigma\sigma_i} \left(\sum_{\substack{j=0 \\ j<i}}^{r} (-1)^j \varphi_{\sigma_i\sigma_{ij}} \right) \right) - \sum_{i=0}^{r} \left((-1)^i \varphi_{\sigma\sigma_i} \left(\sum_{\substack{j=0 \\ j>i}}^{r} (-1)^j \varphi_{\sigma_i\sigma_{ij}} \right) \right) \\
&= \sum_{i=0}^{r} \sum_{\substack{j=0 \\ j<i}}^{r} (-1)^{i+j} \varphi_{\sigma\sigma_i} \varphi_{\sigma_i\sigma_{ij}} - \sum_{i=0}^{r} \sum_{\substack{j=0 \\ j>i}}^{r} (-1)^{i+j} \varphi_{\sigma\sigma_i} \varphi_{\sigma_i\sigma_{ij}} \\
&= \sum_{i=0}^{r} \sum_{\substack{j=0 \\ j<i}}^{r} (-1)^{i+j} \varphi_{\sigma\sigma_i} \varphi_{\sigma_i\sigma_{ij}} - \sum_{j=0}^{r} \sum_{\substack{i=0 \\ i>j}}^{r} (-1)^{i+j} \varphi_{\sigma\sigma_j} \varphi_{\sigma_j\sigma_{ij}} \\
&= \sum_{i=0}^{r} \sum_{\substack{j=0 \\ j<i}}^{r} (-1)^{i+j} \varphi_{\sigma\sigma_i} \varphi_{\sigma_i\sigma_{ij}} - \sum_{j=0}^{r} \sum_{\substack{i=0 \\ j<i}}^{r} (-1)^{i+j} \varphi_{\sigma\sigma_j} \varphi_{\sigma_j\sigma_{ij}} \\
&= \sum_{i=0}^{r} \sum_{\substack{j=0 \\ j<i}}^{r} (-1)^{i+j} \varphi_{\sigma\sigma_{ij}} - \sum_{j=0}^{r} \sum_{\substack{i=0 \\ j<i}}^{r} (-1)^{i+j} \varphi_{\sigma\sigma_{ij}} = 0,
\end{aligned}
$$

where we have used property (3.1.1) to deduce that $\varphi_{\sigma\sigma_i}\varphi_{\sigma_i\sigma_{ij}} = \varphi_{\sigma\sigma_j}\varphi_{\sigma_j\sigma_{ij}} = \varphi_{\sigma\sigma_{ij}}$. Thus by our comments above the result holds. □

Definition 3.3.2. *[RS85] Let $\mathscr{F}$ be a presheaf defined on Δ. We define the* homology groups*, $H_r(\Delta, \mathscr{F})$, of $\mathscr{F}$ to be*

$$H_r(\Delta, \mathscr{F}) := \frac{Z_r(\Delta, \mathscr{F})}{B_r(\Delta, \mathscr{F})} \qquad \textit{for } r = 0, \ldots n.$$

These homology groups are well-defined by Lemma 3.3.1. When the simplicial complex Δ is understood, we drop all references to Δ, and denote our chain spaces, cycle spaces, boundary spaces and homology groups by $C_r(\mathscr{F})$, $Z_r(\mathscr{F})$, $B_r(\mathscr{F})$ and $H_r(\mathscr{F})$ respectively.

It is natural to consider presheaves up to isomorphism. To do so, the following result will be useful:

Lemma 3.3.3. *Let $\mathscr{F}$ and $\mathscr{G}$ be isomorphic presheaves defined on a simplicial complex Δ. Then $H_0(\mathscr{F})$ and $H_0(\mathscr{G})$ are isomorphic kG-modules.*

Proof. Assume that $\mathscr{F}$ and $\mathscr{G}$ are isomorphic via an isomorphism $m := \{m_\sigma : \mathscr{F}_\sigma \to \mathscr{G}_\sigma | \sigma \in \Delta\}$. Since m is a presheaf morphism it commutes with the connecting maps of $\mathscr{F}$ and $\mathscr{G}$. The boundary maps ∂_i and δ_i (of $\mathscr{F}$ and $\mathscr{G}$ respectively) are linear combinations of connecting maps, hence also commute with m. Moreover, if we consider m as acting on the chain spaces $C_i(\mathscr{F})$, we have a chain map between the chain complexes $(C_i(\mathscr{F}), \partial_i)$ and $(C_i(\mathscr{G}), \delta_i)$:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_2(\mathscr{F}) & \xrightarrow{\partial_2} & C_1(\mathscr{F}) & \xrightarrow{\partial_1} & C_0(\mathscr{F}) & \xrightarrow{\partial_0} & 0 \\ & & \downarrow m & & \downarrow m & & \downarrow m & & \\ \cdots & \longrightarrow & C_2(\mathscr{G}) & \xrightarrow{\delta_2} & C_1(\mathscr{G}) & \xrightarrow{\delta_1} & C_0(\mathscr{G}) & \xrightarrow{\delta_0} & 0 \end{array} \tag{3.3.5}$$

By diagram chasing in (3.3.5) we see that m maps $\operatorname{im}\partial_1$ bijectively onto to $\operatorname{im}\delta_1$. Moreover, as m is a presheaf morphism, it commutes with the action of the group algebra kG, and hence is a kG-isomorphism from $C_0(\mathscr{F})$ to $C_0(\mathscr{G})$ which restricts to a kG-isomorphism from $\operatorname{im}\partial_1$ to $\operatorname{im}\delta_1$. It follows that m induces a kG-isomorphism from $H_0(\mathscr{F})$ to $H_0(\mathscr{G})$ as required. □

The argument above can evidently be generalised to show that if $\mathscr{F}$ and $\mathscr{G}$ are isomorphic presheaves, then $H_i(\mathscr{F})$ and $H_i(\mathscr{G})$ will be isomorphic kG-modules for all i. We also note that this is a necessary but not sufficient condition. Indeed, let Δ be the simplicial complex of parabolic subgroups associated to the unique 2-minimal parabolic system of $\mathrm{Sym}(6)$ given in Example 2.5.3. Define the presheaf $\mathscr{F}$ on Δ to have 0-dimensional terms at simplices of type P_1 and P_2 and a 1-dimensional irreducible module at presheaf terms of type B. Then clearly $H_0(\mathscr{F}) \cong H_0(\mathcal{K}_0) \cong 0$, but $\mathscr{F} \ncong \mathcal{K}_0$.

We conclude this subsection by considering a presheaf $\mathscr{F}$ defined on Δ. For each vertex $\sigma \in \Delta$, let φ_σ denote the restriction of the quotient map $\varphi : C_0(\mathscr{F}) \to H_0(\mathscr{F})$ to $\mathscr{F}_\sigma$. For a simplex $\gamma \in \Delta$, if γ is not a vertex, then choose a vertex σ of γ. We define $\varphi_\gamma := \varphi_{\gamma\sigma}\varphi_\sigma$. The definition of φ_γ appears to depend on the choice of vertex σ, but this is not the case. Indeed, let τ be another vertex of γ. Since Δ is a simplicial complex, the vertices σ and τ lie in a common 1-simplex β where $\partial_\beta = \pm(\varphi_{\beta\sigma} - \varphi_{\beta\tau})$. Since β is a 1-simplex, the image of ∂_β is contained in the image of ∂_1, which in turn is contained in the $\ker\varphi$. We deduce that $\varphi_{\beta\sigma}\varphi_\sigma - \varphi_{\beta\tau}\varphi_\tau = 0$. Consequently, using property (3.1.1) for connecting maps we have

$$\varphi_{\gamma\sigma} \circ \varphi_\sigma = (\varphi_{\gamma\beta} \circ \varphi_{\beta\sigma}) \circ \varphi_\sigma = (\varphi_{\gamma\beta} \circ \varphi_{\beta\tau}) \circ \varphi_\tau = \varphi_{\gamma\tau} \circ \varphi_\tau,$$

and our maps φ_γ for $\gamma \in \Delta$ are well-defined. We will use these maps to define universal presheaves in Section 3.4.

3.3.2 Cohomology Groups

For a presheaf $\mathscr{F}$ defined on a simplicial complex Δ over the field k, the chain spaces $C_i(\Delta, \mathscr{F})$ may be used to define a co-chain complex. Indeed we take $C^i(\Delta, \mathscr{F}) := \mathrm{Hom}(C_i(\Delta, \mathscr{F}), k)$ and for $f \in C^i(\Delta, \mathscr{F})$, define $f\delta^i := \partial_{i+1} f$. This gives rise to the co-chain complex

$$0 \longrightarrow C^0(\Delta, \mathscr{F}) \xrightarrow{\delta^0} C^1(\Delta, \mathscr{F}) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} C^n(\Delta, \mathscr{F}) \longrightarrow 0.$$

It follows that we may define the r-th cohomology group of a given presheaf.

Definition 3.3.4. *[Ron89] Let $\mathscr{F}$ be a presheaf defined on Δ. We define the* r-th cohomology group*, $H^r(\Delta, \mathscr{F})$, of $\mathscr{F}$ to be*

$$H^r(\Delta, \mathscr{F}) := \begin{cases} \ker \delta^0 & \text{if } r = 0; \\ \frac{\ker \delta^r}{\operatorname{im} \delta^{r-1}} & \text{if } r = 1, \dots n-1; \text{ and} \\ \frac{C^n(\Delta, \mathscr{F})}{\operatorname{im} \delta^{n-1}} & \text{if } r = n. \end{cases}$$

As with homology groups, when the simplicial complex Δ is clear, we will suppress it in our notation and we shall respectively write $C^r(\mathscr{F})$ and $H^r(\mathscr{F})$ for $C^r(\Delta, \mathscr{F})$ and $H^r(\Delta, \mathscr{F})$.

We shall see in Section 3.6 that there is a close relation between the zero-th and n-th homology groups of a presheaf $\mathscr{F}$ and the respective n-th and zero-th cohomology groups of its dual presheaf $\mathscr{F}^*$ (as defined in Section 3.5).

3.3.3 Euler Characteristic

Closely associated to chain spaces and homology groups is the notion of the Euler characteristic of a presheaf. We assume that a geometric minimal parabolic system of rank $n+1$ for a group G (and hence a simplicial complex Δ) has been fixed.

Definition 3.3.5. *The* Euler characteristic *of the presheaf $\mathscr{F}$, denoted $\chi(\mathscr{F})$, is the alternating sum*

$$\chi(\mathscr{F}) := \sum_{i=0}^{n} (-1)^i \dim H_i(\mathscr{F})$$

The Euler characteristic can also be defined in terms of dimensions of the chain spaces of the presheaf $\mathscr{F}$ as we now prove.

Lemma 3.3.6. *For a presheaf $\mathscr{F}$ we have*

$$\chi(\mathscr{F}) = \sum_{i=0}^{n} (-1)^i \dim H_i(\mathscr{F}) = \sum_{i=0}^{n} (-1)^i \dim C_i(\mathscr{F}).$$

Proof. This is a simple calculation. We note that $\dim H_i(\mathscr{F}) = \dim Z_i(\mathscr{F}) - \dim B_i(\mathscr{F})$ for all $i = 0, \dots, n$. Moreover, as each boundary map $\partial_i : C_i(\mathscr{F}) \to C_{i-1}(\mathscr{F})$ is a kG-homomorphism, we have $C_i(\mathscr{F})/\ker \partial_i \cong \operatorname{im} \partial_i$ and hence

$$\dim C_i(\mathscr{F}) = \dim \operatorname{im} \partial_i + \dim \ker \partial_i = \dim B_{i-1}(\mathscr{F}) + \dim Z_i(\mathscr{F}).$$

It follows that

$$\begin{aligned} \chi(\mathscr{F}) &= \sum_{i=0}^{n} (-1)^i \dim H_i(\mathscr{F}) \\ &= \sum_{i=0}^{n} (-1)^i \left(\dim Z_i(\mathscr{F}) - \dim B_i(\mathscr{F}) \right) \\ &= \dim Z_0(\mathscr{F}) + (-1)^{n+1} \dim B_n(\mathscr{F}) + \sum_{i=1}^{n} (-1)^i \left(\dim Z_i(\mathscr{F}) + \dim B_{i-1}(\mathscr{F}) \right) \\ &= \dim C_0(\mathscr{F}) + 0 + \sum_{i=1}^{n} (-1)^i \dim C_i(\mathscr{F}) = \sum_{i=0}^{n} (-1)^i \dim C_i(\mathscr{F}) \end{aligned}$$

as required. □

As the chain spaces $C_i(\mathscr{F})$ are formal direct sums of presheaf terms, their dimensions are easily determined. Lemma 3.3.6 allows us to effortlessly calculate the Euler characteristic of $\mathscr{F}$, and thus we obtain a useful relationship between the dimensions of the homology groups of $\mathscr{F}$. This is a method we will use frequently in Section 3.8, and is very powerful when used in conjunction with the dual presheaf $\mathscr{F}^*$ of a given presheaf $\mathscr{F}$ (as defined in Section 3.5).

3.4 Universal Presheaves

So far we have considered presheaves which are characterised by explicitly defining all terms and connecting maps. However, it is possible to construct a presheaf from a minimal amount of data, namely the presheaf terms at a chamber B and at given panels of each type that are faces of B. This leads to the notion of *universal presheaves*, which as their name suggests satisfy a universal property.

Our motivating simplicial complexes arise from geometric minimal parabolic systems. Consequently we will assume that Δ is the standard n-simplex, having vertices $\{v_0, \ldots, v_n\}$ and we set $I = \{0, \ldots, n\}$. For the sake of brevity, if $J = \{i_0, \ldots, i_\ell\} \subseteq I$, with $i_j < i_{j+1}$ for all j, then we say that the simplex $\{v_{i_0}, \ldots, v_{i_\ell}\}$ has type J. Let Π be a subset of the power set of I that is closed under taking subsets; if $J \subseteq K \in \Pi$, then $J \in \Pi$.

Definition 3.4.1. *[RS86] A* Π-presheaf *is a collection of k-vector spaces $\mathscr{F}_\sigma$ for each σ of cotype $J \in \Pi$, connecting maps $\varphi_{\sigma\tau}$ whenever τ is a face of σ, and g-action for each $g \in G$ on the $\mathscr{F}_\sigma$ such that conditions* (3.1.1) *and* (3.1.2) *hold for all simplices of Δ of cotype $J \in \Pi$.*

There may be many presheaves $\mathscr{G}$ whose terms at simplices of cotype $J \in \Pi$ form the same Π-presheaf $\mathscr{F}$. We call such presheaves *extensions of* $\mathscr{F}$, and for each such $\mathscr{G}$ we call $\mathscr{F}$ the Π-*restriction of* $\mathscr{G}$. Ronan and Smith proved that all extensions of a Π-presheaf $\mathscr{F}$ that are generated by the terms at simplices of cotype $J \in \Pi$ are quotients of some universal extension.

Theorem 3.4.2. *[RS86, Theorem 2.1] If Π is a non-empty subset of the power set of I that is closed under taking subsets, then each Π-presheaf $\mathscr{F}$ admits an extension $\mathscr{F}'$ with the following universal property. Suppose $\mathscr{G}$ is a presheaf on Δ (with connecting maps $\psi_{\sigma\tau}$) satisfying:*

(i) the Π-restriction of $\mathscr{G}$ is isomorphic to $\mathscr{F}$; and

(ii) if cotype $\tau \notin \Pi$, then $\mathscr{G}_\tau$ is generated by the $(\mathscr{G}_\sigma)\psi_{\sigma\tau}$.

Then $\mathscr{G}$ is a quotient of $\mathscr{F}'$.

We call the presheaf $\mathscr{F}'$ from Theorem 3.4.2 a *universal presheaf*. Below we describe the presheaf terms, connecting maps and G-action for $\mathscr{F}'$. We refer the reader to [RS86, p141] for a proof of Theorem 3.4.2 (that is a proof that these terms, maps and actions give a well-defined presheaf $\mathscr{F}'$ and that $\mathscr{F}'$ is indeed universal). The presheaf $\mathscr{F}'$ is defined inductively.

Let J be a subset of I not contained in Π such that every proper subset of J is contained in Π. Let σ be a simplex of cotype J. By assumption for each simplex ρ having σ as a face, the presheaf term $\mathscr{F}_\rho$ has been defined. Thus $\mathscr{F}|St\sigma$ is a well-defined presheaf on the star of σ. We define $\mathscr{F}'_\sigma := H_0(\mathscr{F}|St\sigma)$. If σ is a face of ρ, then we have a map

$$\varphi_\rho : \mathscr{F}'_\rho = (\mathscr{F}|St\sigma)_\rho \longrightarrow H_0(\mathscr{F}|St\sigma) = \mathscr{F}'_\sigma,$$

as defined in Section 3.3, and this map satisfies $\varphi_\tau = \varphi_{\tau\rho}\varphi_\rho$ whenever ρ is a face of τ. We define $\varphi_{\rho\sigma} := \varphi_\rho$. Finally for each $g \in G$, the presheaves $\mathscr{F}|_{St\sigma}$ and $\mathscr{F}|_{St(\sigma g)}$ are isomorphic via the action of g in $\mathscr{F}$. This induces an action $\tilde{g} : \mathscr{F}'_\sigma \to \mathscr{F}'_{\sigma g}$, which we take to be the action of g on $\mathscr{F}'_\sigma$. Continuing in this way, we define the presheaf $\mathscr{F}'$.

It is possible to generalise the construction above by considering only terms $\mathscr{F}_\sigma$ defined at certain faces of single (fixed) chamber c. We then extend these terms to a full Π-presheaf $\mathscr{F}$, using multiplication within G to define the other Π-presheaf terms. In such a case, condition (3.1.2) is not relevant, since we only have a single chamber c. It is replaced by the following condition:

(**3.4.3.**)*[RS86, Hypothesis 2.2] If $J \subseteq K \in \Pi$ and σ_J, σ_K are faces of c of cotype J and K respectively, then $\mathscr{F}_{\sigma_J}$ is a kG_J-module and $\varphi_{\sigma_J\sigma_K}$ is a kG_J-module homomorphism.*

We now give an analogue of Definition 3.4.1 in this more general setting.

Definition 3.4.4. *[RS86] A* Π-stalk at c *is a system of coefficients $\mathscr{F}_{\sigma_J}$ for $J \in \Pi$ satisfying the composition property* (3.1.1) *and property* (3.4.3).

We note that a Π-presheaf $\mathscr{F}$, will restrict to a Π-stalk at any one of its chambers, by considering only the presheaf terms corresponding to faces of that chamber. In fact the converse also holds, as proved by Ronan and Smith:

Theorem 3.4.5. *[RS86, Theorem 2.3] Every Π-stalk is the restriction (to the faces of c) of a unique Π-presheaf.*

An immediate consequence of this theorem is that every Π-stalk can be extended to a unique Π-presheaf, which in turn may be extended to a universal presheaf. As with Theorem 3.4.2, we illustrate the construction of such a Π-presheaf, and refer the reader to [RS86, pp142-144] for a proof that this construction is well-defined and unique.

Continuing the notation of property (3.4.3), assume that $\{\mathscr{F}_{\sigma_J} | J \in \Pi\}$ is a Π-stalk with connecting maps $\varphi_{\sigma_J\sigma_K}$. Let T_J denote a right transversal of G_J in G. We may choose a transversal such that the elements of T_J are indexed by the coefficient spaces of cotype J, so that for each $\sigma_J h$ of cotype J, there is a unique element $g_{\sigma_J h} \in T_J$ such that $\sigma_J g_{\sigma_J h} = \sigma_J h$. For $\sigma_J 1 = \sigma_J$, we take $g_{\sigma_J 1} := 1$. Having obtained such a transversal, we define each $\mathscr{F}_\sigma$ corresponding to a simplex of cotype J to be a space k-isomorphic to $\mathscr{F}_{\sigma_J}$ via a given isomorphism $\hat{g}_\sigma : \mathscr{F}_{\sigma_J} \to \mathscr{F}_\sigma$. For ease of notation, we take $\hat{g}_{\sigma_J} = \mathrm{id}_{\mathscr{F}_{\sigma_J}}$.

To define the G-action on the direct sum of these coefficient spaces, we use the isomorphisms above to map each coefficient space $\mathscr{F}_\sigma$ of cotype J back to $\mathscr{F}_{\sigma_J}$, apply a G-action on $\mathscr{F}_{\sigma_J}$, and then apply an isomorphism to the resulting image. More precisely, if $a \in G$ and σ is a simplex of cotype J, then we have a k-homomorphism $\hat{g}_\sigma : \mathscr{F}_{\sigma_J} \to \mathscr{F}_\sigma$. Since T_J is a right transversal of G_J in G, there exists a unique element $h \in G_J$ satisfying

$$g_\sigma \cdot a = h \cdot g_{(\sigma a)}. \tag{3.4.6}$$

As $h \in G_J$, and $\mathscr{F}_{\sigma_J}$ is a kG_J-module (by (3.4.3)), we have a well-defined action of h on $\mathscr{F}_{\sigma_J}$. It follows that we may define the action, $\tilde{a}$, of a on the direct sum of our coefficient spaces of cotype J, so that $\tilde{a}$ completes the following commutative diagram:

$$\begin{tikzcd} \mathscr{F}_{\sigma_J} \arrow[r, "\hat{g}_\sigma"] \arrow[d, "h"'] & \mathscr{F}_\sigma \arrow[d, "\tilde{a}"] \\ \mathscr{F}_{\sigma_J} \arrow[r, "\hat{g}_{\sigma a}"] & \mathscr{F}_{\sigma a} \end{tikzcd} . \tag{3.4.7}$$

Thus the action of $a \in G$ on $\mathscr{F}_\sigma$ is given by $\tilde{a} := (\hat{g}_\sigma)^{-1} h \hat{g}_{\sigma a}$.

To define the connecting maps of the Π-presheaf $\mathscr{F}$, we use a similar approach as above; we map everything back to our Π-stalk and use the G-action we have just defined. Indeed let $\sigma \subseteq \tau$ be simplices of cotypes $J \subseteq K \in \Pi$ respectively, and let $\sigma_J \subseteq \tau_K$ be faces of c of cotypes $J \subseteq K \in \Pi$ respectively. Take any $a \in G$ such that $\tilde{a} : \mathscr{F}_{\sigma_J} \to \mathscr{F}_\sigma$ and $\tilde{a} : \mathscr{F}_{\tau_K} \to \mathscr{F}_\tau$. As above, we define $\varphi_{\sigma\tau} : \mathscr{F}_\sigma \to \mathscr{F}_\tau$ to be the homomorphism that completes the commutative diagram

$$\begin{tikzcd} \mathscr{F}_{\sigma_J} \arrow[r, "\tilde{a}"] \arrow[d, "\varphi_{\sigma_J \tau_K}"'] & \mathscr{F}_\sigma \arrow[d, "\varphi_{\sigma\tau}"] \\ \mathscr{F}_{\tau_K} \arrow[r, "\tilde{a}"] & \mathscr{F}_\tau \end{tikzcd} ,$$

meaning that $\varphi_{\sigma\tau} := (\tilde{a})^{-1} \varphi_{\sigma_J \tau_K} \tilde{a} = (a^{-1})^{\sim} \varphi_{\sigma_J \tau_K} \tilde{a}$.

Theorems 3.4.2 and 3.4.5 illustrate why it suffices to restrict our attention to universal presheaves. This will be the main focus of our attention in Sections 3.7-3.9.

We conclude this section by reconsidering chamber-generated presheaves. As mentioned in Section 3.1, a presheaf $\mathscr{F}$ is called chamber-generated if for each $\sigma \in \Delta$, the presheaf term $\mathscr{F}_\sigma$ is spanned by the images $\mathscr{F}_c \varphi_{c\sigma}$ for all chambers c containing σ. Clearly such presheaves are nice to work with. Indeed, since the connecting maps of a presheaf $\mathscr{F}$ are k-homomorphisms that commute with the G-action, we see that there is a homomorphism

$$\bigoplus_{c \text{ a chamber containing } \sigma} \varphi_{c\sigma} : (\mathscr{F}_c)_c^\sigma \to \mathscr{F}_\sigma.$$

Thus $\mathscr{F}_\sigma$ is a quotient of the induced module $(\mathscr{F}_c)_c^\sigma$.

If we form the universal presheaf from a chamber-generated Π-presheaf or Π-stalk, it follows that the terms of the universal presheaf, not defined by the Π-presheaf/stalk, are homology groups. Hence we obtain the following result:

Lemma 3.4.6. *If $\mathscr{F}$ is a Π-stalk or a Π-presheaf, and each term $\mathscr{F}_\sigma$ of $\mathscr{F}$ is spanned by the images $\mathscr{F}_c \varphi_{c\sigma}$ for all chambers c containing σ, then the universal presheaf obtained using Theorems 3.4.2 and 3.4.5 will be a chamber-generated presheaf.*

3.5 Dual Presheaves

In Section 3.4 we saw how one could define a presheaf inductively using the presheaf terms at a single chamber, a single panel of each type contained in this chamber and the associated connecting maps. We shall now see that given a panel-irreducible chamber-generated universal presheaf $\mathscr{F}$ defined on the simplicial complex Δ of parabolic subgroups of some group G with respect to some prime divisor p of the order of G, we may construct a *dual presheaf* as originally introduced in [Ron89]. Throughout this section we will set $k := GF(p)$.

Consider the panel-terms of $\mathscr{F}$ and appealing to Lemma 2.5.4, we see that chamber-generatedness ensures that for a fixed panel $P \in \Delta$, the panel term $\mathscr{F}_P$ will be generated by the set

$$\{\mathscr{F}_{B^g}\varphi_{B^gP} | g \in P\} = \{\mathscr{F}_{B^g}\varphi_{B^gP} | g \in T \text{ for } T \text{ a transversal of } B \text{ in } P\}.$$

It follows that $\mathscr{F}_P$ will be a quotient of the induced module $(\mathscr{F}_B)_B^P$, and hence we have a short exact sequence

$$0 \longrightarrow M_P \xrightarrow{\theta_P} (\mathscr{F}_B)_B^P \xrightarrow{\Psi_P} \mathscr{F}_P \longrightarrow 0, \tag{3.5.8}$$

for some kP-module M_P. Applying $\mathrm{Hom}(-,k)$ to (3.5.8), we obtain the short exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathscr{F}_P, k) \xrightarrow{\Psi'_P} \mathrm{Hom}((\mathscr{F}_B)_B^P, k) \xrightarrow{\theta'_P} \mathrm{Hom}(M_P, k) \longrightarrow 0. \tag{3.5.9}$$

Denote the rank of Δ by n.

Definition 3.5.1. *Let $\mathscr{F}$ be a universal panel-irreducible presheaf constructed on Δ from a $\{\varnothing, \{0\}, \{1\}, \ldots, \{n-1\}\}$-stalk $\overline{\mathscr{F}}$. The* dual presheaf *is the universal presheaf $\mathscr{F}^*$ constructed from the $\{\varnothing, \{0\}, \{1\}, \ldots, \{n-1\}\}$-stalk $\overline{\mathscr{F}^*}$ having chamber term $(\overline{\mathscr{F}^*})_B := \mathrm{Hom}(\mathscr{F}_B, k)$ and panel terms $(\overline{\mathscr{F}^*})_{P_i} := \mathrm{Hom}(M_{P_i}, k)$ and having connecting maps θ'_P (as defined in (3.5.9)).*

We illustrate this with an example.

Example 3.5.2. *As seen in Example 2.5.3, the symmetric group $\mathrm{Sym}(6)$ admits a unique 2-minimal parabolic system given by $\{P_1 \sim 2^3.\mathrm{Sym}(3), P_2 \sim 2^3.\mathrm{Sym}(3)\}$. Let $k := GF(2)$. There are two non-isomorphic irreducible kP_i-modules for $i = 1, 2$ having dimension 1 and 2 over k. We denote these by 1_i and 2_i respectively. In addition, we have a unique irreducible kB-module, 1_B, of dimension 1 over k. These modules give rise to four universal panel-irreducible presheaves. We denote by $\lambda_{i,j}$ the universal panel-irreducible presheaf satisfying $(\lambda_{i,j})_{P_1} = i_1$, $(\lambda_{i,j})_{P_2} = j_2$ and $(\lambda_{i,j})_B = 1_B$. The presheaf terms of the dual presheaves $\lambda^*_{i,j}$ for $i, j = 1, 2$ are given in Table 3.1.* •

We conclude this section by considering the case that G is a finite group, p is a prime divisor of $|G|$ and $k := GF(p)$. Given an irreducible kG-module M, we see that every element of $\mathrm{Hom}(M, k)$ is uniquely defined by the preimage of $1 \in k$. Thus for $m \in M\backslash\{0\}$ we may define $\varphi_m \in \mathrm{Hom}(M, k)$ by $\varphi_m : m \mapsto 1$, whilst taking $\varphi_0 := 0 \in \mathrm{Hom}(M, k)$. This gives a well-defined epimorphism from M to $\mathrm{Hom}(M, k)$ and hence $M \cong_{kG} \mathrm{Hom}(M, k)$. We shall use this fact without further mention when explicitly calculating homology groups in Section 3.8.

Presheaf, $\lambda^*_{i,j}$	$(\lambda_{i,j})^*_B$	$(\lambda_{i,j})^*_{P_1}$	$(\lambda_{i,j})^*_{P_2}$
$\lambda^*_{1,1}$	$\mathrm{Hom}(1_B, k)$	$\mathrm{Hom}(2_1, k)$	$\mathrm{Hom}(2_2, k)$
$\lambda^*_{1,2}$	$\mathrm{Hom}(1_B, k)$	$\mathrm{Hom}(2_1, k)$	$\mathrm{Hom}(1_2, k)$
$\lambda^*_{2,1}$	$\mathrm{Hom}(1_B, k)$	$\mathrm{Hom}(1_1, k)$	$\mathrm{Hom}(2_2, k)$
$\lambda^*_{2,2}$	$\mathrm{Hom}(1_B, k)$	$\mathrm{Hom}(1_1, k)$	$\mathrm{Hom}(1_2, k)$

Table 3.1: The dual presheaves of the universal panel-irreducible presheaves over $GF(2)$ defined on the unique 2-minimal parabolic system of $\mathrm{Sym}(6)$.

3.6 Summary of Known Results

Ronan and Smith first defined a presheaf of abelian groups on the building Δ of a finite Chevalley group in their 1985 paper [RS85]. In this paper, they looked at basic properties of such presheaves. They proceeded this paper with a further paper on universal presheaves [RS86] in 1986, before using their results to calculate/partially calculate the zero-homology groups for presheaves associated to $L_4(2)$, the C_3 geometry of $\mathrm{Alt}(7)$, the triple cover of $\mathrm{Sym}(6)$ and the Mathieu group M_{24}, which they published in 1989 (see [RS89]). In this section we present many of the results from these papers.

We give all results for the simplicial complex Δ of parabolic subgroups arising from a minimal parabolic system of a given group G (mirroring the notion of a building). However, we note that many of the results hold in a more general setting.

Lemma 3.6.1. *[RS85, (1.1)] If W is a kG-module, and $\mathcal{K}_W$ is the constant presheaf on W, then $H_0(\mathcal{K}_W) \cong W$.*

Proof. As Δ is connected, the copies of W in $\mathcal{K}_W$ at the vertices of Δ are all identified in $H_0(\mathcal{K}_W)$. □

When H is a subgroup of G, Frobenius-Nakayama reciprocity relates the restriction of kG-modules to the induction of kH-modules.

Theorem 3.6.2. *[Ben91, Frobenius-Nakayama Reciprocity, 2.8.3] If k is a field, $H \leqslant G$ are groups, M is a kG-module and N is a kH-module, then there is a natural isomorphism*

$$\mathrm{Hom}_{kH}(N, M|_{kH}) \cong \mathrm{Hom}_{kG}(N^{kG}_{kH}, M).$$

When considering presheaves on our simplicial complex Δ, we consider all parabolic subgroups. It follows that the process of induction is replaced by taking the zero-homology group.

Theorem 3.6.3. *[RS85, (1.2)] Let W be a kG-module, and $\mathscr{F}$ a presheaf. Then*

$$\mathrm{Hom}_k(\mathscr{F}, \mathcal{K}_W) \cong_k \mathrm{Hom}_{kG}(H_0(\mathscr{F}), W).$$

For a presheaf $\mathscr{F}$, this result allows us to determine certain irreducible quotients of $H_0(\mathscr{F})$.

Theorem 3.6.4. *[RS85, (1.3)] Let W be a kG-module, and suppose $\mathscr{F}$ is a subpresheaf of $\mathcal{K}_W$, whose terms generate W. Then W is a quotient of $H_0(\mathscr{F})$.*

If W is an irreducible kG-module, then the terms of any non-zero subpresheaf of $\mathcal{K}_W$ will generate W. Thus we obtain the following result.

Corollary 3.6.5. *If $\mathscr{F}$ is a subpresheaf of $\mathcal{K}_W$ for some irreducible kG-module W, then W is a quotient of $H_0(\mathscr{F})$.*

Conversely, if W is not irreducible, then the terms of any non-zero subpresheaf $\mathscr{F}$ of $\mathcal{K}_W$ will generate a submodule V of W. Since all modules we encounter are finite-dimensional k-vector spaces, we may extend V to form a composition series for W, in which V will be one of the terms:

$$0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_i = V \subseteq W_{i+1} \subseteq \cdots \subseteq W_n = W.$$

By Theorem 3.6.4, V is a quotient of $H_0(\mathscr{F})$. It follows that V/W_{i-1} is also a quotient of $H_0(\mathscr{F})$. Thus we have the following corollary to Theorem 3.6.4:

Corollary 3.6.6. *If $\mathscr{F}$ is a subpresheaf of $\mathcal{K}_W$ for some kG-module W, then there is a composition factor V of W which is a quotient of $H_0(\mathscr{F})$.*

There is also a relationship between surjections of presheaves and surjections of the homology groups of presheaves.

Proposition 3.6.7. *[RS86, (1.4)] A surjection $p : \mathscr{F} \to \mathscr{G}$ of presheaves induces a surjection $\hat{p} : H_0(\mathscr{F}) \to H_0(\mathscr{G})$ in homology groups.*

Combining Proposition 3.6.7 and Theorem 3.4.2 we obtain the following result.

Corollary 3.6.8. *Let $\mathscr{U}$ be the universal presheaf defined by a Π-presheaf or Π-stalk $\mathscr{F}$, whose existence is asserted by Theorem 3.4.2. If $\mathscr{G}$ is a presheaf (with connecting maps $\psi_{\sigma\tau}$) satisfying:*

(i) the Π-restriction of $\mathscr{G}$ is isomorphic to $\mathscr{F}$; and

(ii) if cotype $\tau \notin \Pi$, then $\mathscr{G}_\tau$ is generated by the $(\mathscr{G}_\sigma)\psi_{\sigma\tau}$,

then $H_0(\mathscr{G})$ is a quotient of $H_0(\mathscr{U})$.

Proof. By Theorem 3.4.2, the presheaf $\mathscr{G}$ is a quotient of $\mathscr{U}$, and thus there is a surjection $p : \mathscr{U} \to \mathscr{G}$. This surjection of presheaves induces a surjection in homology $\hat{p} : H_0(\mathscr{U}) \to H_0(\mathscr{G})$ by Proposition 3.6.7 and hence $H_0(\mathscr{G})$ is a quotient of $H_0(\mathscr{U})$. □

This corollary illustrates the importance of universal presheaves in determining the zero-homology groups of arbitrary presheaves.

A final application of Proposition 3.6.7 is found in the proof of the following result:

Theorem 3.6.9. [3] *Let $\mathscr{F}$ be a presheaf. An irreducible module V is a quotient of $H_0(\mathscr{F})$ if and*

[3] *Theorem 3.6.9 is a variation on a remark made by Ronan and Smith in their 1989 paper [RS89]. In the introduction to that paper they stated that*

> *The analogue of Frobenius reciprocity [11; (1.2)] asserts that a module V is a quotient of H_0 of a presheaf $\mathcal{F}$ precisely when $\mathcal{F}$ is a subsheaf of the constant sheaf $\mathcal{K}_V$.*

This statement is actually incorrect, and a counterexample to this situation arises in the presheaf λ_3 of Table (2.1) in the same paper. The presheaf λ_3 has a 5-dimensional term at vertices of a given type, but $H_0(\lambda_3)$ is a 4-dimensional module. However, Stephen Smith has kindly confirmed the correct formulation of the result as found in Theorem 3.6.9, which we have then been able to prove [Smi12]. The presheaf λ_3 mentioned above does satisfy the hypothesis of the theorem, as there is a map into the constant presheaf of the 4-dimensional irreducible module. However, this map is not faithful.

only if there is a surjective morphism of presheaves from $\mathscr{F}$ onto a (non-zero) subpresheaf of the constant presheaf, $\mathcal{K}_V$, of V.

Proof. Let $\mathscr{F}$ be a presheaf defined on a simplicial complex Δ and suppose that V is a quotient of $H_0(\mathscr{F})$ via a quotient map $q : H_0(\mathscr{F}) \twoheadrightarrow V$. For each $\sigma \in \Delta$, there is a map $\varphi_\sigma : \mathscr{F}_\sigma \to H_0(\mathscr{F})$ (as described in Section 3.3). Set $m_\sigma := \varphi_\sigma q$ for each $\sigma \in \Delta$. Since the map φ_σ is the composition of a connecting map in the presheaf $\mathscr{F}$ and the quotient map $C_0(\mathscr{F}) \to H_0(\mathscr{F})$, it commutes with the G-action. The map q also commutes with the G-action, and thus so does the map m_σ for each $\sigma \in \Delta$. If τ is a face of σ, then the connecting map $\theta_{\sigma\tau}$ in the constant presheaf $\mathcal{K}_V$ is the inclusion map. It follows that

$$\varphi_{\sigma\tau} m_\tau = \varphi_{\sigma\tau}\varphi_\tau q = \varphi_\sigma q = m_\sigma = m_\sigma \theta_{\sigma\tau}.$$

We conclude that $m := \{m_\sigma : \sigma \in \Delta\}$ is a morphism from $\mathscr{F}$ into the constant presheaf $\mathcal{K}_V$, and hence $\mathscr{F}$ is mapped onto a subpresheaf of $\mathcal{K}_V$, namely the image of m.

Conversely, suppose that $\mathscr{F}$ is mapped surjectively onto a subpresheaf, $\mathscr{G}$, of the constant presheaf $\mathcal{K}_V$. Since V is irreducible, Corollary 3.6.5 asserts that V is a quotient of $H_0(\mathscr{G})$. Moreover, as $\mathscr{F}$ is mapped surjectively onto $\mathscr{G}$, the homology group $H_0(\mathscr{G})$ is a quotient of $H_0(\mathscr{F})$ by Proposition 3.6.7. It follows that V is a quotient of $H_0(\mathscr{F})$. □

The final results that we give in this section were formulated by Ronan in his paper on dual presheaves [Ron89].

Theorem 3.6.10. *Let $\mathscr{F}$ be a presheaf defined on the simplicial complex of parabolic subgroups associated to a minimal parabolic system of rank $n+1$ of a group G, and let $\mathscr{F}^*$ be the dual presheaf of $\mathscr{F}$. Then*

(i) [Ron89, Theorem 2] $H_0(\mathscr{F})$ and $H^n(\mathscr{F}^*)$ are isomorphic as kG-modules.

(ii) [Ron89, Corollary] $H_n(\mathscr{F})$ and $H^0(\mathscr{F}^*)$ are isomorphic as kG-modules.

We shall sometimes refer to Theorem 3.6.10 as *Ronan's Duality Theorem.*

Since the dimension of the i-th cohomology group and the corresponding i-th homology group are equal as k-vector spaces (as they are dual vector spaces), we see that for a given presheaf $\mathscr{F}$ we have

$$\dim_k H_0(\mathscr{F}) = \dim_k H^n(\mathscr{F}^*) = \dim_k H_n(\mathscr{F}^*); \qquad \text{and} \tag{3.6.10}$$

$$\dim_k H_n(\mathscr{F}) = \dim_k H^0(\mathscr{F}^*) = \dim_k H_0(\mathscr{F}^*). \tag{3.6.11}$$

We will use (3.6.10) and (3.6.11) in conjunction with the Euler characteristic of a given presheaf (as formulated in Section 3.3) to obtain bounds on the possible dimensions of our homology groups. This is a very effective way to obtain such bounds when the rank of the minimal parabolic system is small.

3.7 Homology Groups of Universal Presheaves

We have previously seen in Theorem 3.2.3 that if G is a finite Chevalley group over the finite field k, and V is a finite-dimensional irreducible kG-module, then the fixed-point presheaf $\mathcal{F}_V$ is irreducible. It follows from Corollary 3.6.5 that V is a quotient of $H_0(\mathcal{F}_V)$. In particular, every irreducible kG-module is the quotient of the zero-homology group of a universal panel-irreducible presheaf. We might hope that this generalises and ask the following question:

Question 3.7.1. *If G is a group, p a prime dividing $|G|$, $k = GF(p)$, Δ is the simplicial complex of parabolic subgroups associated to a p-minimal parabolic system of G and V is an irreducible kG-module, then does there exist a universal panel-irreducible presheaf $\mathscr{F}$ defined on Δ such that V is a quotient of $H_0(\mathscr{F})$?*

In general, there is no guarantee that such a presheaf $\mathscr{F}$ will exist, as we now illustrate.

Example 3.7.2. *Let $G := \mathrm{Sym}(10)$ and let $B = S \sim 2^{5+2+1} \in Syl_2(G)$ be a self-normalizing Sylow subgroup. There is a unique 2-minimal parabolic system of G with respect to S, given by $\{P_1 \sim 2^5.\mathrm{Sym}(5), P_2 \sim 2^5.3^2.\mathrm{Dih}(8)\}$. Taking V to be the 26-dimensional irreducible $GF(2)G$-module, we see that there is no $v \in V$ such that the kB-module $\langle v\rangle_{kB}$ and the kP_1-module $\langle v\rangle_{kP_1}$ are both irreducible. It follows that the presheaf $\mathcal{K}_V$ does not admit an irreducible subpresheaf, and hence there does not exist a panel-irreducible presheaf $\mathscr{F}$ for which V is a quotient of $H_0(\mathscr{F})$.*

We note that in this case B is a maximal subgroup of P_2, but it is not maximal in P_1. Indeed we may take $B < H_1 < P_1$ with $H_1 \sim 2^{5+2}.\mathrm{Sym}(3)$. We see that there is a unique element $v \in V$ such that the modules $\langle v\rangle_{kB}$, $\langle v\rangle_{kH_1}$ and $\langle v\rangle_{kP_2}$ are all irreducible. Hence, considering the minimal parabolic system $\{H_1, P_2\}$ of $\mathrm{Sym}(8) \times 2 \cong \langle H_1, P_2\rangle$, we may form a panel-irreducible presheaf $\mathscr{F}$ on this system for which a composition factor of V is a quotient of $H_0(\mathscr{F})$. •

In the above example, to prove that no such panel-irreducible presheaf existed, we considered subpresheaves of the constant presheaf $\mathcal{K}_V$ whose panel and chamber terms were generated by a given element of V. Clearly, panel-irreducibility means that it suffices to check subpresheaves of the fixed-point presheaf $\mathcal{F}_V$.

Although in Example 3.7.2 the maximality of the chamber in the minimal parabolic subgroups appeared to be the issue, we shall see in Section 3.8 that maximality alone is not a sufficient condition for Question 3.7.1 to have an affirmative answer. Indeed, we shall exhibit two simplicial complexes - namely $\Delta_{1,3}$ and $\Delta_{2,3}$ - of 2-parabolic subgroups associated to the Mathieu group M_{11} for which there is no universal panel-irreducible presheaf, $\mathscr{F}$, having the irreducible $GF(2)M_{11}$-module $10_{M_{11}}$ as a quotient of $H_0(\mathscr{F})$.

3.8 Calculation of Homology Groups

In Sections 3.1-3.7 we looked at things from a purely theoretical viewpoint. In the current section, we will explicitly calculate the zero-homology groups of some universal panel-irreducible presheaves. In general this is extremely difficult, and at best we can hope to obtain bounds on the dimensions and possible irreducible quotients of such homology groups. This mirrors the goals of Ronan and Smith in [RS89]. Before giving explicit calculations in Subsection 3.8.2, we describe the method used throughout [RS89] known as *geometric spanning*.

3.8.1 Geometric Spanning

Given a simplicial complex of parabolic subgroups Δ associated to some minimal parabolic system $\mathcal{S} = \{P_1, \ldots, P_{n+1}\}$ of a finite group G, we recall that the vertices of Δ are the conjugates of the parabolic subgroups of the form $P_{I\setminus\{i\}}$ for some $i \in I = \{1, \ldots, n+1\}$. We call a conjugate of $P_{I\setminus\{i\}}$, a *vertex of type* i. Suppose we are given a vertex $\sigma = P^g_{I\setminus\{j\}}$ of type j and let $\gamma := P_{I\setminus\{i\}}$ for some $i \neq j$. Let $\mathscr{F}$ be a chamber-generated presheaf defined on Δ with connecting maps $\varphi_{\alpha\tau}$ whenever τ is a face of α. Thus

$$\mathscr{F}_\alpha = \langle \mathscr{F}_{B^h}\varphi_{B^h\alpha} | B^h \subseteq \alpha\rangle \qquad \text{for each } \alpha \in \Delta,$$

where B is the normalizer of the Sylow p-subgroup that defines the minimal parabolic system $\mathcal{S}$. For each chamber $B^h \subseteq \sigma$, we have $B^h \subseteq \gamma^h$. Moreover, as γ^h (resp. σ) is the unique vertex of type i (resp. j) that is a face of B^h, and $P^h_{I\setminus\{i,j\}}$ is the unique 1-simplex of type $\{i, j\}$ that is a face of B^h, we have that both γ^h and σ are faces of $P^h_{I\setminus\{i,j\}}$.

Since $\mathscr{F}$ is chamber generated, there exists $v_1, \ldots, v_r$ in chamber-terms $\mathscr{F}_{B^{g_1}}, \ldots, \mathscr{F}_{B^{g_r}}$ respectively (where $B^{g_l} \subseteq \sigma$ for each $l = 1, \ldots, r$) such that $\mathscr{F}_\sigma$ is generated by $v_1\varphi_{B^{g_1}\sigma}, \ldots, v_r\varphi_{B^{g_r}\sigma}$. Now setting $\mu_l := P^{g_l}_{I\setminus\{i,j\}}$, we have that

$$v_l\varphi_{B^{g_l}\sigma} = v_l\varphi_{B^{g_l}\mu_l}\varphi_{\mu_l\sigma} \qquad \text{for each } l = 1, \ldots, r.$$

Defining $w_l := v_l\varphi_{B^{g_l}\mu_l} \in \mathscr{F}_{\mu_l}$ for each l, the presheaf term $\mathscr{F}_\sigma$ is generated by $w_1\varphi_{\mu_1\sigma}, \ldots, w_r\varphi_{\mu_r\sigma}$. Consider $w_1 \in \mathscr{F}_{\mu_1}$. As $\mu_1 = P^{g_1}_{I\setminus\{i,j\}}$ is a 1-simplex of type $\{i, j\}$, it has precisely two faces, namely σ and γ^{g_1}. Considering the image of w_1 under the boundary map $\partial_1 : C_1(\mathscr{F}) \to C_0(\mathscr{F})$ we have

$$w_1\partial_1 = \pm(w_1\varphi_{\mu_1\sigma} - w_1\varphi_{\mu_1\gamma^{g_1}}),$$

where the sign is positive if $j < i$ and negative if $i < j$. As $w_1\varphi_{\mu_1\sigma}$ and $w_1\varphi_{\mu_1\gamma^{g_1}}$ are both contained in $C_0(\mathscr{F})$, their images in the homology group $H_0(\mathscr{F})$ are equal:

$$w_1\varphi_{\mu_1\sigma} + \operatorname{Im}\partial_1 = w_1\varphi_{\mu_1\gamma^{g_1}} + \operatorname{Im}\partial_1.$$

We may use the same methodology for each $l = 1, \ldots, r$ to see that the image of the presheaf term $\mathscr{F}_\sigma$ in the homology group $H_0(\mathscr{F})$ is contained in the span of the images of the presheaf terms of $\mathscr{F}$ at vertices of type i.

Repeating this process for all vertices of type $j \neq i$ we see that $H_0(\mathscr{F})$ is spanned by the images of all presheaf terms at vertices of type i. Since i was chosen arbitrarily, we see that $H_0(\mathscr{F})$ is spanned by the images of presheaf terms of $\mathscr{F}$ at vertices of any given type.

Using further applications of this methodology, we can determine subsets of vertices of a given type such that the images of the presheaf terms at these vertices generate the zero-homology group.

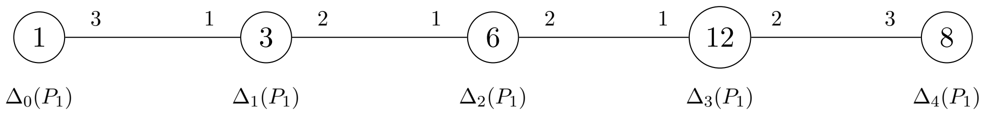


Figure 3.1: The collapsed adjacency diagram of the graph Γ from Example 3.8.1 with respect to P_1.

This method is best illustrated by an example. For the sake of brevity, from this point forward we will often talk about presheaf terms spanning the zero-homology group. However strictly speaking, we mean that the images of the presheaf terms span the zero-homology group.

Example 3.8.1. *Let $G := \mathrm{Sym}(6)$. We saw in Example 2.5.3 that there is a unique 2-minimal parabolic system of G given by $\{P_1 \sim 2^3.\mathrm{Sym}(3), P_2 \sim 2^3.\mathrm{Sym}(3)\}$, having associated simplicial complex, Δ, of parabolic subgroups. Defining Γ to be the graph having vertex set the minimal parabolic subgroups of G, and with P_1^g and P_2^h incident if they share a common chamber, we may calculate the collapsed adjacency diagram of Γ with respect to P_1. This is given in Figure 3.1. The girth of Γ is 8. We note that the collapsed adjacency diagram of Γ with respect to P_2 is identical to the collapsed adjacency diagram with respect to P_1. Since Γ may be seen to be an incidence geometry, we will refer to conjugates of P_1 and P_2 as* points *and* lines *respectively.*

Let $k = GF(2)$ and let $\mathscr{F}$ denote the universal panel-irreducible presheaf defined on Δ having 1-dimensional irreducible kB- and kP_2-modules at the simplices B and P_2 and a 2-dimensional irreducible kP_1-module at P_1.[4] *Define $p_0 := P_1$ and denote the lines in $\Delta_1(p_0)$ by l_0, l_1, l_2, the points in $\Delta_2(p_0)$ by $p_1, \ldots, p_6$ and the lines in $\Delta_3(p_0)$ by $l_3, \ldots, l_{14}$ such that the subgraph of Γ spanned by these vertices is as given in Figure 3.2. From Figures 3.1 and 3.2 we see that the images of $p_1, \ldots, p_6$ in $H_0(\mathscr{F})$ form a spanning set, since the images of all lines will be contained in this span. It follows that the images of $l_3, \ldots, l_{14}$ also span $H_0(\mathscr{F})$. Considering each point not occurring in Figure 3.2, we note that as the girth of Γ is 8, each point must be incident to one line from each of the sets $\{l_3, l_4, l_5, l_6\}$, $\{l_7, l_8, l_9, l_{10}\}$ and $\{l_{11}, l_{12}, l_{13}, l_{14}\}$. Since the image of a point is spanned by the images of any two of its incident lines, we conclude that the images of $l_3, \ldots, l_{10}$ span $H_0(\mathscr{F})$, and hence so do the images of $p_1, \ldots, p_4$.*

Finally, denote the non-zero vector in $\mathscr{F}_{l_i}$ by v_i for $i = 3, \ldots, 14$ and let $p_j \in \Delta_4(p_0)$ be incident to l_9. Without loss of generality we may assume that p_j is also incident to l_3 and l_{11}. Thus in $H_0(\mathscr{F})$ the relation

$$v_9 = v_3 + v_{11} \tag{3.8.12}$$

holds. Let p_k be the other point of $\Delta_4(p_0)$ that is incident to l_{11}. Without loss of generality, p_k is also incident to l_5 and l_7. This gives

$$v_{11} = v_5 + v_7 \tag{3.8.13}$$

[4] *In Example 3.5.2 we denoted $\mathscr{F}$ by $\lambda_{2,1}$.*

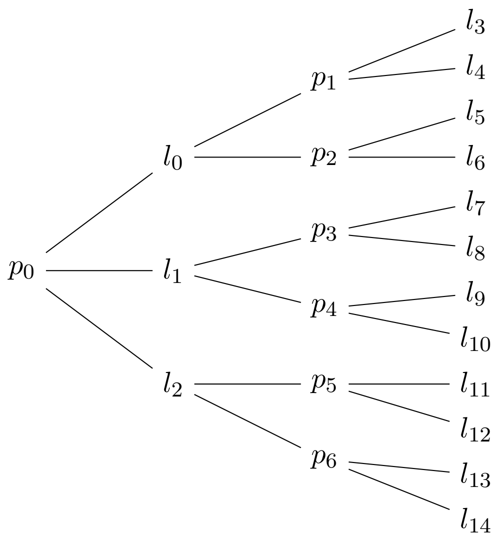


Figure 3.2: The subgraph of Γ spanned by the points $p_0, \ldots, p_6$ and the lines $l_0, \ldots, l_{14}$ from Example 3.8.1.

in $H_0(\mathscr{F})$. *Combining* (3.8.12) *and* (3.8.13) *we obtain*

$$v_9 = v_3 + v_5 + v_7$$

in $H_0(\mathscr{F})$ *and hence* $\mathscr{F}_{l_9} \subseteq \mathrm{span}\{\mathscr{F}_{p_1}, \mathscr{F}_{p_2}, \mathscr{F}_{p_3}\}$. *An analogous argument may be used to show that* $\mathscr{F}_{l_{10}} \subseteq \mathrm{span}\{\mathscr{F}_{p_1}, \mathscr{F}_{p_2}, \mathscr{F}_{p_3}\}$. *Thus* $H_0(\mathscr{F})$ *is spanned by* $\{\mathscr{F}_{p_1}, \mathscr{F}_{p_2}, \mathscr{F}_{p_3}\}$.

We conclude by noting that

$$H_0(\mathscr{F}) = \mathrm{span}\{\mathscr{F}_{p_1}, \mathscr{F}_{p_2}, \mathscr{F}_{p_3}\} = \mathrm{span}\{\mathscr{F}_{l_0}, \mathscr{F}_{l_1}, \mathscr{F}_{l_3}, \mathscr{F}_{l_5}, \mathscr{F}_{l_7}\},$$

whence

$$\dim(H_0(\mathscr{F})) = \dim(\mathrm{span}\{\mathscr{F}_{l_0}, \mathscr{F}_{l_1}, \mathscr{F}_{l_3}, \mathscr{F}_{l_5}, \mathscr{F}_{l_7}\}) \leqslant 5. \tag{3.8.14}$$

We shall utilise the bound in (3.8.14) *later, to explicitly determine the structure of* $H_0(\mathscr{F})$. •

We conclude this subsection by noting that geometric spanning arguments are most powerful when dealing with groups of small order.

3.8.2 Explicit Calculations

The explicit calculation of the zero-homology groups of universal panel-irreducible presheaves is typically extremely complex. Often the best that we can hope to obtain is a bound on the dimension of the homology group and/or a list of possible irreducible quotients. We illustrate this complexity by calculating the zero-homology groups of the universal panel-irreducible presheaves of the symmetric group Sym(6) and the Mathieu groups M_{11} and M_{22} and partially calculating the zero-homology groups of the universal panel-irreducible presheaves of the Mathieu group M_{12}. All of these presheaves are defined on rank 2 minimal parabolic systems. We also calculate the irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on M_{23} and M_{24} over $GF(2)$, noting that the minimal parabolic systems on which they are defined have rank 3. Some of our calculations are heavily based on the geometric spanning method introduced in Subsection 3.8.1, combined with the notion of the dual presheaf and Theorem 3.6.9.

Sym(6)

We continue the notation of Example 3.5.2 for the universal panel-irreducible presheaves of Sym(6) over $GF(2)$. There are four irreducible $GF(2)\,\mathrm{Sym}(6)$-modules having dimensions 1, 4, 4 and 16. We denote these by 1, 4_a, 4_b and 16 respectively.

$\lambda_{1,1}$

The presheaf $\lambda_{1,1}$ is precisely the constant presheaf $\mathcal{K}_1$. Thus $H_0(\mathscr{F}) \cong 1$ by Lemma 3.6.1.

$\lambda_{2,1}$

In Example 3.8.1 we obtained the bound

$$\dim(H_0(\lambda_{2,1})) \leqslant 5. \tag{3.8.15}$$

Since $H_0(\lambda_{2,1})$ is spanned by the images at simplices of any given type, we see that $H_0(\lambda_{2,1})$ is simultaneously a quotient of $((\lambda_{2,1})_B)_B^{\mathrm{Sym}(6)}$, $((\lambda_{2,1})_{P_1})_{P_1}^{\mathrm{Sym}(6)}$ and $((\lambda_{2,1})_{P_2})_{P_2}^{\mathrm{Sym}(6)}$. There are three such quotients of dimension at most 5, namely 0, 4_b and $4_b/1$.

Consider the $GF(2)\,\mathrm{Sym}(6)$-module $4_b/1$. It contains two vectors v_1 and v_2 such that the modules $\langle v_i\rangle_{kB}$, $\langle v_i\rangle_{kP_1}$ and $\langle v_i\rangle_{kP_2}$ are all irreducible (for $i = 1, 2$). Of these vectors, one vector generates the universal panel-irreducible subpresheaf $\lambda_{1,1}$ of $\mathcal{K}_{4_b/1}$, whilst the other generates the universal panel-irreducible subpresheaf $\lambda_{2,1}$. Thus we obtain $\lambda_{2,1} \subseteq \mathcal{K}_{4_b/1}$ and since $(\lambda_{2,1})_{P_1}$ is 2-dimensional, the terms of $\lambda_{2,1}$ must generate $4_b/1$. Thus $4_b/1$ is a quotient of $H_0(\lambda_{2,1})$ by Theorem 3.6.4. Combining this with (3.8.15) we obtain $H_0(\lambda_{2,1}) \cong 4_b/1$.

$\lambda_{1,2}$

As $\lambda_{1,2}^* \cong \lambda_{2,1}$, Ronan's Duality Theorem asserts that $\dim(H_0(\lambda_{1,2})) = \dim(H_1(\lambda_{2,1}))$. Considering the Euler characteristic of $\lambda_{2,1}$ we obtain

$$\dim(H_0(\lambda_{1,2})) = \dim(H_1(\lambda_{2,1})) = \dim(H_0(\lambda_{2,1})) - \chi(\lambda_{2,1}) = 5 - 0 = 5.$$

Mirroring the approach for $\lambda_{2,1}$, we see that there is a unique 5-dimensional quotient of $((\lambda_{1,2})_B)_B^{\mathrm{Sym}(6)}$, $((\lambda_{1,2})_{P_1})_{P_1}^{\mathrm{Sym}(6)}$ and $((\lambda_{1,2})_{P_2})_{P_2}^{\mathrm{Sym}(6)}$, namely $4_a/1$. Thus $H_0(\lambda_{1,2}) \cong 4_a/1$.

$\lambda_{2,2}$

The presheaves $\lambda_{2,2}$ and $\lambda_{1,1}$ are dual to each other. Thus by Ronan's Duality Theorem

$$\dim(H_0(\lambda_{2,2})) = \dim(H_1(\lambda_{1,1})) = \dim(H_0(\lambda_{1,1})) - \chi(\lambda_{1,1}) = 1 - (-15) = 16.$$

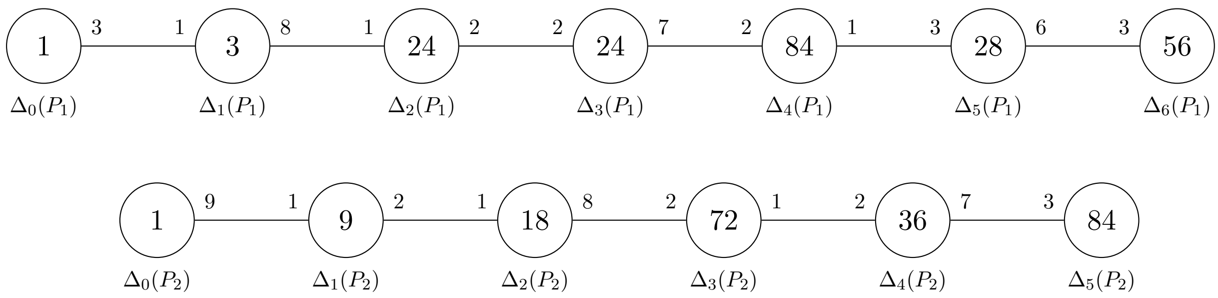


Figure 3.3: The collapsed adjacency diagrams of Γ_{12} with respect to P_1 (top) and P_2 (bottom).

Considering the irreducible $GF(2)\,\mathrm{Sym}(6)$-module of dimension 16, we see that it contains a vector v satisfying $\langle v\rangle_{kB} \cong 1_B$, $\langle v\rangle_{kP_1} \cong 2_{P_1}$ and $\langle v\rangle_{kP_2} \cong 2_{P_2}$. Thus we may realise $\lambda_{2,2}$ as a subpresheaf of $\mathcal{K}_{16}$ and so 16 is an irreducible quotient of $H_0(\lambda_{2,2})$ by Theorem 3.6.9. We conclude that $H_0(\lambda_{2,2}) \cong 16$.

M_{11}

We now consider the Mathieu group M_{11}, which admits three 2-minimal parabolic systems, namely

$$\mathcal{S}_1 := \{P_1 \sim 2_-^{1+2}.\,\mathrm{Sym}(3), P_2 \sim 3^2SD_{16}\}, \qquad \mathcal{S}_2 := \{P_1 \sim 2_-^{1+2}.\,\mathrm{Sym}(3), P_3 \sim \mathrm{Alt}(6).2\} \qquad \text{and}$$
$$\mathcal{S}_3 := \{P_2 \sim 3^2SD_{16}, P_3 \sim \mathrm{Alt}(6).2\}.$$

Denote the simplicial complexes of parabolic subgroups associated to $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ by $\Delta_{1,2}$, $\Delta_{1,3}$ and $\Delta_{2,3}$ respectively.

The self-normalizing Sylow 2-subgroup, B, of M_{11} admits a unique irreducible $GF(2)B$-module of dimension 1, whilst the minimal parabolic subgroups P_1, P_2 and P_3 admit irreducible $GF(2)P_1$-, $GF(2)P_2$- and $GF(2)P_3$- modules of dimensions 1 and 2, 1 and 8 and 1, 8 and 16 respectively. We denote by $\lambda_{i,j}$ the universal panel-irreducible presheaf defined on $\mathcal{S}_1$ having an irreducible 1-dimensional module at chambers, and irreducible i- and j-dimensional $GF(2)P_1$- and $GF(2)P_2$-modules at simplices of type P_1 and P_2 respectively. Similarly, denote by $\mu_{i,j}$ (respectively $\gamma_{i,j}$) the universal panel-irreducible presheaf defined on $\mathcal{S}_2$ (respectively $\mathcal{S}_3$) having irreducible i- and j-dimensional $GF(2)P_1$- and $GF(2)P_3$-modules (respectively $GF(2)P_2$- and $GF(2)P_3$-modules) at simplices of type P_1 and P_3 (respectively P_2 and P_3).

To apply geometric spanning arguments to calculate the zero-homology groups of the $\lambda_{i,j}$, $\mu_{i,j}$ and $\gamma_{i,j}$, we first define the graphs $\Gamma_{i,j}$ for $(i,j) = (1,2), (1,3), (2,3)$. Indeed we take the vertex set of $\Gamma_{i,j}$ to be the union of all conjugates of P_i and all conjugates of P_j in M_{11}, and we join P_i^g and P_j^h precisely when they contain a common chamber. The collapsed adjacency diagrams of Γ_{12}, Γ_{13} and Γ_{23} are given in Figures 3.3, 3.4 and 3.5. The respective diameters of $\Gamma_{1,2}$, $\Gamma_{1,3}$ and $\Gamma_{2,3}$ are 6, 4 and 3, and the respective girths are 6, 4 and 4.

There are four isomorphism classes of irreducible $GF(2)M_{11}$-modules, having dimensions 1, 10, 32 and 44. Clearly $\lambda_{1,1}$, $\mu_{1,1}$ and $\gamma_{1,1}$ are the constant presheaves of the 1-dimensional irreducible

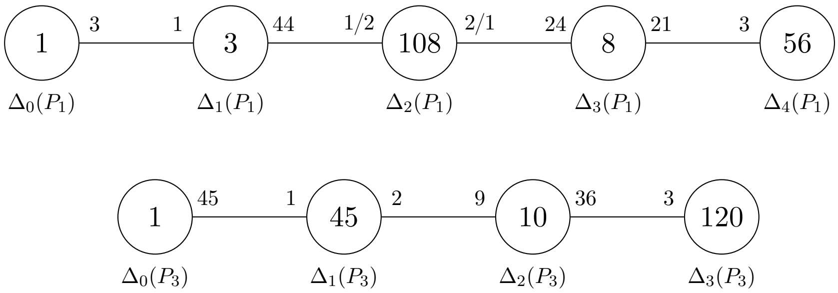


Figure 3.4: The collapsed adjacency diagrams of Γ_{13} with respect to P_1 (top) and P_3 (bottom).

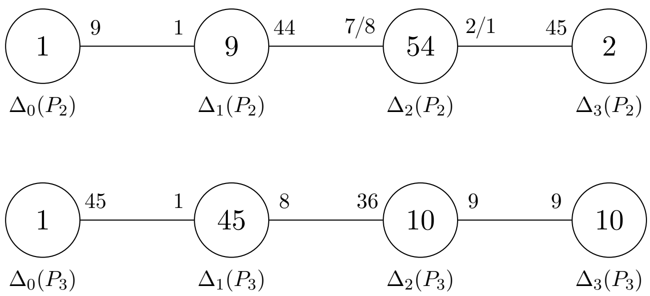


Figure 3.5: The collapsed adjacency diagrams of Γ_{23} with respect to P_2 (top) and P_3 (bottom).

$GF(2)M_{11}$-module and hence $H_0(\lambda_{1,1}) \cong H_0(\mu_{1,1}) \cong H_0(\gamma_{1,1}) \cong 1$. Moreover, these are the only universal panel-irreducible presheaves for which 1 can be an irreducible quotient.

Considering the 10-dimensional irreducible $GF(2)M_{11}$-module $10_{M_{11}}$, we see that there are two vectors in $10_{M_{11}}$ that generate irreducible kB-, kP_i- and kP_j-modules for distinct $i, j = 1, 2, 3$. The first vector gives rise to irreducible kB-, kP_1- and kP_2-modules of dimension 1, 1 and 8, whilst the second gives rise to irreducible kB-, kP_1- and kP_2-modules of dimension 1, 2 and 1. It follows from Theorem 3.6.9 that $10_{M_{11}}$ is an irreducible quotient of precisely $H_0(\lambda_{1,8})$ and $H_0(\lambda_{2,1})$. Thus, as stated in Section 3.7, there are no universal panel-irreducible presheaves defined on $\Delta_{1,3}$ or $\Delta_{2,3}$, for which $10_{M_{11}}$ is a quotient of their zero-homology group.

In a similar way, calculations in MAGMA show that the 32-dimensional module, $32_{M_{11}}$, is a quotient of the zero-homology group of precisely three universal panel-irreducible presheaves, these being $\lambda_{2,8}$, $\mu_{2,16}$ and $\gamma_{8,16}$ (each arising from the unique element of $32_{M_{11}}$ that generates an irreducible $GF(2)B$-module). We note that such explicit calculations are not required in the case of presheaves defined on $\Delta_{1,2}$ and $\Delta_{1,3}$ having a 1-dimensional term at panels of type P_1. Indeed, the zero-homology group of such presheaves would be a quotient of $(1_{P_1})_{P_1}^{M_{11}}$. However, no such quotient has $32_{M_{11}}$ as an irreducible quotient.

We now consider the $\lambda_{i,j}$, $\mu_{i,j}$ and $\gamma_{i,j}$ for $(i,j) \neq (1,1)$ in turn.

Presheaves Defined on $\Delta_{1,2}$

We note that $\lambda_{1,1}^* \cong \lambda_{2,8}$ and $\lambda_{1,8}^* \cong \lambda_{2,1}$.

$\lambda_{1,8}$ and $\lambda_{2,1}$

The zero-homology group $H_0(\lambda_{2,1})$ is spanned by the images of the presheaf terms at vertices of type P_2 and hence is a quotient of the induced module $(1_{P_2})_{P_2}^{M_{11}}$. It is clear from the collapsed adjacency diagram of $\Gamma_{1,2}$ with respect to P_2 - seen in Figure 3.3 - that $\dim(H_0(\lambda_{2,1})) \leqslant 53$ and we have seen that $10_{M_{11}}$ is a quotient of $H_0(\lambda_{2,1})$. Considering the possible quotients of $(1_{P_2})_{P_2}^{M_{11}}$ having dimensions between 10 and 53, we see that there are four such quotients, having dimensions 10, 11, 44 and 45. The quotients of dimension 11 and 45 have a 1-dimensional quotient, whilst the 44-dimensional quotient is irreducible. Thus by our knowledge of irreducible quotients of $H_0(\lambda_{2,1})$, we deduce that $H_0(\lambda_{2,1}) \cong 10_{M_{11}}$.

To calculate the dimension of $H_0(\lambda_{1,8})$ we use the Euler characteristic and Ronan's Duality Theorem. Indeed

$$\begin{aligned}\dim(H_0(\lambda_{1,8})) &= \chi(\lambda_{1,8}) + \dim(H_1(\lambda_{1,8}))\\ &= \chi(\lambda_{1,8}) + \dim(H_0(\lambda_{1,8}^*))\\ &= (1 \cdot 165 + 8 \cdot 55 - 1 \cdot 495) + 10 = 120.\end{aligned}$$

There are two isomorphism classes of 120-dimensional quotients of $(1_B)_B^{M_{11}}$ that do not have $1_{M_{11}}$ as a quotient. They have the structures

$$10/10/1/44/1/10/44 \quad \text{and} \quad 44 \oplus 10/1/10/44/1/10.$$

Consideration of the elements of $44|_{P_2}$ which generate a copy of 8_{P_2}, we see that $\lambda_{1,8}$ may be constructed as a subpresheaf of $\mathcal{K}_{44}$. It follows that $H_0(\lambda_{1,8}) \cong 44 \oplus 10/1/10/44/1/10$.

$\lambda_{2,8}$

The Euler characteristic of $\lambda_{2,8}$ is

$$\chi(\lambda_{2,8}) = \dim(C_0(\lambda_{2,8})) - \dim(C_1(\lambda_{2,8})) = (2 \cdot 165 + 8 \cdot 55) - 1 \cdot 495 = 275.$$

Appealing to Ronan's Duality Theorem we see that

$$\dim(H_0(\lambda_{2,8})) = \chi(\lambda_{2,8}) + \dim(H_1(\lambda_{2,8})) = \chi(\lambda_{2,8}) + \dim(H_0(\lambda^*_{2,8})) = 275 + 1 = 276.$$

There is a unique isomorphism class of 276-dimensional quotients of $(2_{P_1})_{P_1}^{M_{11}}$, and thus $H_0(\lambda_{2,8})$ must belong to this class. Hence

$$H_0(\lambda_{2,8}) \cong 32 \oplus 44/1/44/10/1 \oplus 44/1/10/1/44/44.$$

Presheaves Defined on $\Delta_{1,3}$

$\mu_{1,8}$ and $\mu_{2,8}$

The zero-homology groups of $\mu_{1,8}$ and $\mu_{2,8}$ are both quotients of the indecomposable induced $GF(2)M_{11}$-module $(8_{P_3})_{P_3}^{M_{11}} \cong 44/44$.

Let T denote a transversal in M_{11} of $N_{M_{11}}(P_3)$. From the collapsed adjacency diagram of $\Gamma_{1,3}$ with respect to P_3 (given in Figure 3.4), we see that

$$H_0(\mu_{1,8}) \cong \langle (\mu_{1,8})_{P_3^t} + \operatorname{im} \partial_1 | t \in T \backslash \{1\} \rangle.$$

We also note that the images of any two presheaf terms at vertices of type P_3 will have 9 non-zero vectors in common. Finally, the image of any presheaf term in the second disc of the adjacency diagram, will intersect the images of the other presheaf terms corresponding to the disc in 36 non-zero vectors. Enumerating the elements of T by $\{0, \ldots, 10\}$ with $t_0 := 1$, we may assume that $\dim((\mu_{1,8})_{P_3^{t_1}} + \operatorname{im} \partial_1) \leqslant 8$, $\dim((\mu_{1,8})_{P_3^{t_j}} + \operatorname{im} \partial_1) \leqslant 4$ for $j = 2, \ldots, 9$ and $\dim((\mu_{1,8})_{P_3^{t_{10}}} + \operatorname{im} \partial_1) \leqslant 2$. Hence

$$\dim(H_0(\mu_{1,8})) \leqslant 8 + 8 \cdot 4 + 2 = 42.$$

Since $H_0(\mu_{1,8})$ is a quotient of 44/44, we conclude that $H_0(\mu_{1,8}) \cong 0$.

Calculations in MAGMA show that $44_{M_{11}}$ admits $\mu_{2,8}$. We conclude that $H_0(\mu_{2,8}) \cong 44_{M_{11}}$, since clearly $\dim(H_0(\mu_{2,8})) \neq 88$.

$\mu_{1,16}$

As $1_{M_{11}}$, $10_{M_{11}}$ and $32_{M_{11}}$ are not quotients of $H_0(\mu_{1,16})$, it suffices to consider quotients featuring $44_{M_{11}}$. The restriction $(44_{M_{11}})|_{P_3}$ decomposes as a direct sum of 16_{P_3} and a 28-dimensional indecomposable module for which 16_{P_3} is not a submodule. Thus we may explicitly construct

16_{P_3} within $(44_{M_{11}})|_{P_3}$. Considering its elements, and the $GF(2)B$- and $GF(2)P_1$-modules that they generate, we see that there is a copy of $\mu_{2,16}$ sitting inside $\mathcal{K}_{44}$, but there is no subpresheaf isomorphic to $\mu_{1,16}$. Hence $44_{M_{11}}$ is not a quotient of $H_0(\mu_{1,16})$, meaning that $H_0(\mu_{1,16}) = 0$.

$\mu_{2,1}$

The homology group $H_0(\mu_{2,1})$ is spanned by the images of the 1-dimensional presheaf terms at vertices of type P_3, meaning that $\dim(H_0(\mu_{2,1})) \leqslant [G : P_3] = 11$. Hence, as $1_{M_{11}}$ and $10_{M_{11}}$ are not quotients of $H_0(\mu_{2,1})$, it follows that $H_0(\mu_{2,1}) = 0$.

$\mu_{2,16}$

We know that $32_{M_{11}}$ and $44_{M_{11}}$ are both quotients of $H_0(\mu_{2,16})$, but that $1_{M_{11}}$ and $10_{M_{11}}$ are not. Consider the induced module

$$(16_{P_3})_{P_3}^{M_{11}} \cong 32 \oplus 44/(44 \oplus 1/10/1)/44.$$

Here the summand 1/10/1 is a uniserial module.[5] There are nine quotients of $(16_{P_3})_{P_3}^{M_{11}}$ satisfying our condition on irreducible quotients. Calculations in MAGMA show that of these quotients, only $32 \oplus 44$ and $32 \oplus 44/1$ have a generating element that also generates an isomorphic copy of $\mu_{2,16}$. It follows from Theorem 3.6.4 that $H_0(\mu_{2,16}) \cong 32 \oplus 44/1$.

Presheaves Defined on $\Delta_{2,3}$

$\gamma_{1,8}$ and $\gamma_{1,16}$

Considering the collapsed adjacency diagram of $\Gamma_{2,3}$ with respect to P_2 - as given in Figure 3.5 - we see that the image of the presheaf terms in $\Delta_3(P_2)$ span a space of dimension at most

$$2 \cdot \dim((\gamma_{1,*})_{P_3}) - 1 \leqslant 2 \cdot 16 - 1 = 31.$$

Consequently, the images of the presheaf terms in $\Delta_2(P_2)$ span a space of dimension at most 31. However, the images of the presheaf terms in both $\Delta_1(P_2)$ and $\Delta_3(P_2)$ are contained in this span. Thus as the images of all presheaf terms of type P_3 are contained in this span, we conclude that $\dim(H_0(\gamma_{1,*})) \leqslant 31$. Hence as $1_{M_{11}}$ and $10_{M_{11}}$ are not quotients of $H_0(\gamma_{1,*})$, we deduce that $H_0(\gamma_{1,8}) = H_0(\gamma_{1,16}) = 0$.

$\gamma_{8,1}$

Since $H_0(\gamma_{8,1})$ is spanned by the images of presheaf terms at vertices of type P_3, it follows that $\dim(H_0(\gamma_{8,1})) \leqslant [G : P_3] = 11$. Thus as $1_{M_{11}}$ and $10_{M_{11}}$ are not quotients of $H_0(\gamma_{8,1})$, we conclude that $H_0(\gamma_{8,1}) = 0$

[5] We recall that a module is called *uniserial* if its submodules can be totally ordered. In our case the submodules are given by 0, 1, 10/1 and 1/10/1.

$\gamma_{8,8}$

The Euler characteristic of $\gamma_{8,8}$ is 33. Appealing to Lemma 3.3.6 it follows that $H_0(\gamma_{8,8})$ is non-zero. Since it is also a quotient of the indecomposable induced module $(8_{P_3})_{P_3}^{M_{11}} \cong 44/44$ and $\dim(H_0(\gamma_{8,8})) < 88$ clearly holds (for example using a geometric spanning argument), we deduce that $H_0(\gamma_{8,8}) \cong 44$.

$\gamma_{8,16}$

The Euler characteristic of $\gamma_{8,16}$ is 121. Consequently $\dim(H_0(\gamma_{8,16})) \geqslant 121$. There are 5-possible quotients of $(16_{P_3})_{P_3}^{M_{11}}$ satisfying this dimensional bound. Their dimensions are 121, 131, 132, 144 and 176.

To obtain an upper bound for $\dim(H_0(\gamma_{8,16}))$ we use a geometric spanning argument. Indeed, let $t_1, \ldots, t_{10} \in M_{11}$ be such that the 2-disc, $\Delta_2(P_3)$, in Γ_{23} is equal to $\{P_3^{t_i} | i = 1, \ldots, 10\}$. For each $i = 1, \ldots, 10$, we consider the number of vectors in $(\gamma_{8,16})_{P_3^{t_i}} + \operatorname{im} \partial_1$ that are contained in $\bigcup_{j=i+1}^{10} (\gamma_{8,16})_{P_3^{t_j}} + \operatorname{im} \partial_1$. Since the image of each $(\gamma_{8,16})_{P_3^{t_i}}$ contains 9 vectors in the image of presheaf terms of elements of the 3-disc of P_3, and these in turn each give rise to an additional 8 vectors in $\bigcup_{j=1, j\neq i}^{10} (\gamma_{8,16})_{P_3^{t_j}} + \operatorname{im} \partial_1$, we see that there are 72 non-zero vectors in $(\gamma_{8,16})_{P_3^{t_i}} + \operatorname{im} \partial_1$ that are also contained in $\bigcup_{j=1, j\neq i}^{10} (\gamma_{8,16})_{P_3^{t_j}} + \operatorname{im} \partial_1$. Thus a 7-dimensional subspace of $(\gamma_{8,16})_{P_3^{t_i}} + \operatorname{im} \partial_1$ is contained in $\operatorname{span}\{(\gamma_{8,16})_{P_3^{t_j}} + \operatorname{im} \partial_1 | j = 1, \ldots, 10, j \neq i\}$. Working sequentially from $i = 1$ to $i = 10$ to calculate the number of vectors in $(\gamma_{8,16})_{P_3^{t_i}} + \operatorname{im} \partial_1$ that are contained in $\bigcup_{j=i+1}^{10} (\gamma_{8,16})_{P_3^{t_j}} + \operatorname{im} \partial_1$ we deduce that

$$\begin{aligned}
\dim(\operatorname{span}\{(\gamma_{8,16})_{P_3^{t_i}} + \operatorname{im} \partial_1 | i = 1, \ldots, 10\}) &\leqslant (16-7) + 4\cdot(16-6) + 2\cdot(16-5) + (16-4) + 2\cdot 16 \\
&= 115. \qquad (3.8.16)
\end{aligned}$$

Defining $t_0 := 1$, we may repeat the above argument with $P_3^{t_1}$ taking the role of $P_3^{t_0} = P_3$, to see that $(\gamma_{8,16})_{P_3} + \operatorname{im} \partial_1$ contains 72 vectors contained in $\operatorname{span}\{(\gamma_{8,16})_{P_3^{t_i}} + \operatorname{im} \partial_1 | i = 1, \ldots, 10\}$. Combining this with (3.8.16) we see that

$$\begin{aligned}
\dim(H_0(\gamma_{8,16})) &= \dim(\operatorname{span}\{(\gamma_{8,16})_{P_3^{t_i}} + \operatorname{im} \partial_1 | i = 0, \ldots, 10\}) \\
&\leqslant 115 + (16-7) = 124.
\end{aligned}$$

We conclude that $H_0(\gamma_{8,16})$ is the unique 121-dimensional quotient of $(16_{P_3})_{P_3}^{M_{11}}$, namely

$$H_0(\gamma_{8,16}) \cong 32 \oplus 44/1/44.$$

M_{12}

The Mathieu group M_{12} admits a unique 2-minimal parabolic system, given by

$$\{P_1 \sim 2^{1+4}.\operatorname{Sym}(3), P_2 \sim 4^2 2.\operatorname{Sym}(3)\},$$

which has associated simplicial complex Δ. Both minimal parabolic subgroups give rise to two classes of irreducible $GF(2)P_i$-modules, having dimensions 1 and 2, whilst the self-normalizing

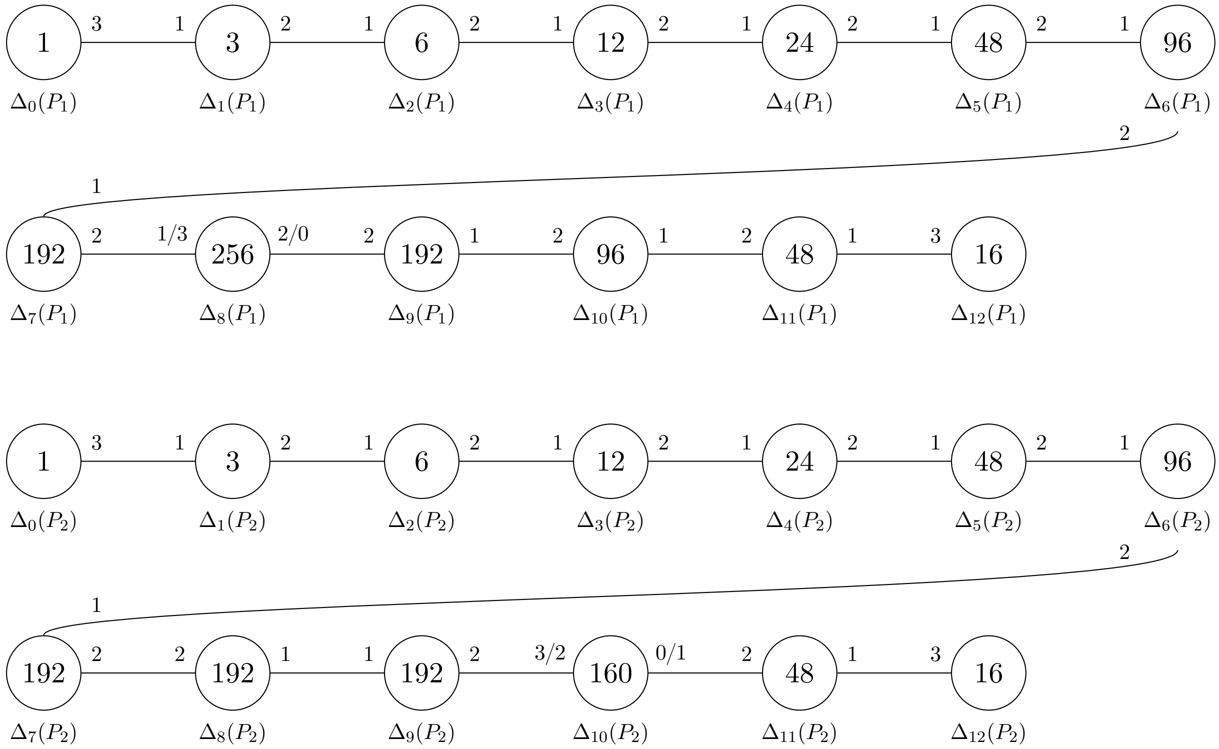


Figure 3.6: The collapsed adjacency diagrams of the graph Γ associated to M_{12} with respect to the 2-minimal parabolic subgroups P_1 (top) and P_2 (bottom).

Sylow 2-subgroups of M_{12} admit a unique class of irreducible modules having dimension 1 over $GF(2)$. Consequently, there are four universal panel-irreducible presheaves defined on Δ which we denote by $\lambda_{i,j}$ for $i, j = 1, 2$. Here the presheaf $\lambda_{i,j}$ has i- and j-dimensional modules at the minimal parabolic subgroups of type P_1 and P_2 respectively, and a 1-dimensional irreducible module at each chamber.

Define the graph Γ to have vertex set all vertices of Δ, and join a vertex P_1^t of type P_1 to a vertex P_2^s of type P_2 precisely when they intersect in a common chamber. The diameter and girth of Γ are 12 and 16 respectively and the collapsed adjacency diagrams of Γ with respect to P_1 and P_2 are given in Figure 3.6. These will be utilised in the geometric spanning arguments used to bound the dimensions of the homology groups $H_0(\lambda_{1,2})$ and $H_0(\lambda_{2,1})$ below.

A consequence of the 2-minimal parabolic subgroups having shape $O_2(P_i).\mathrm{Sym}(3)$ is that the 4 universal panel-irreducible presheaves are in dual pairs. Indeed, $\lambda_{1,1}^* \cong \lambda_{2,2}$ and $\lambda_{1,2}^* \cong \lambda_{2,1}$. We shall use this in our calculations below. We shall also utilise the fact that there are five classes of irreducible $GF(2)M_{12}$-modules, having dimensions 1, 10, 32, 44 and 144. Clearly the 1-dimensional irreducible can only be a quotient of $H_0(\lambda_{1,1})$ by Theorem 3.6.9. Indeed, $H_0(\lambda_{1,1}) \cong 1_{M_{12}}$ by Lemma 3.6.1. Moreover, applying the restriction methods used with presheaves of M_{11} together with Theorem 3.6.4, we may fully determine the irreducible quotients of the zero-homology groups of the remaining presheaves. We see that $10_{M_{12}}$ is the unique irreducible quotient of $H_0(\lambda_{1,2})$,

$32_{M_{12}}$ and $44_{M_{12}}$ are both quotients of $H_0(\lambda_{2,1})$ and $144_{M_{12}}$ is the unique irreducible quotient of $H_0(\lambda_{2,2})$.

For the presheaf $\lambda_{i,j}$, we see from the collapsed adjacency diagrams in Figure 3.6 that - irrespective of the values of i, j - the homology group $H_0(\lambda_{i,j})$ is spanned by

$$\{(\lambda_{i,j})_P + \operatorname{im}\partial_1 | P \in \Delta_8(P_1)\}.$$

It is also spanned by

$$\{(\lambda_{i,j})_P + \operatorname{im}\partial_1 | P \in \Delta_8(P_2) \cup \Delta_9(P_2)\}.$$

We shall now apply geometric spanning arguments to obtain upper bounds for the dimensions of the zero-homology groups of $\lambda_{1,2}$ and $\lambda_{2,1}$ in addition to commenting on the dimension of $H_0(\lambda_{2,2})$.

$\lambda_{1,2}$ and $\lambda_{2,1}$

The dual presheaves $\lambda_{1,2}$ and $\lambda_{2,1}$ satisfy $\chi(\lambda_{1,2}) = \chi(\lambda_{2,1}) = 0$, and hence by Ronan's Duality Theorem we deduce that $\dim(H_0(\lambda_{1,2})) = \dim(H_0(\lambda_{2,1}))$.

Consider the presheaf $\lambda_{2,1}$. As previously noted, by analysing the collapsed adjacency diagram of Γ with respect to P_2, we see that $H_0(\lambda_{2,1})$ is spanned by

$$\{(\lambda_{2,1})_P + \operatorname{im}\partial_1 | P \in \Delta_8(P_2) \cup \Delta_9(P_2)\}.$$

For each $P \in \Delta_8(P_2)$ there exists some $Q \in \Delta_9(P_2)$ which is adjacent to P in Γ. It follows that $(\lambda_{2,1})_P + \operatorname{im}\partial_1 \subseteq (\lambda_{2,1})_Q + \operatorname{im}\partial_1$, and hence $H_0(\lambda_{2,1})$ is spanned by $\{(\lambda_{2,1})_P + \operatorname{im}\partial_1 | P \in \Delta_9(P_2)\}$. Since each $Q \in \Delta_9(P_2)$ is adjacent to two elements of $\Delta_{10}(P_2)$ and the presheaf terms at elements of $\Delta_9(P_2)$ and $\Delta_{10}(P_2)$ are 2- and 1-dimensional respectively, we conclude that $H_0(\lambda_{2,1})$ is actually spanned by $\{(\lambda_{2,1})_P + \operatorname{im}\partial_1 | P \in \Delta_{10}(P_2)\}$.

Define

$$\Delta_{10}^a(P_2) := \{P \in \Delta_{10}(P_2) \mid \Delta_1(P) \cap \Delta_{11}(P_2) \neq \varnothing\},$$

and set $\Delta_{10}^b(P_2) := \Delta_{10}(P_2) \backslash \Delta_{10}^a(P_2)$. We have that $|\Delta_{10}^a(P_2)| = 96$ and $\left|\Delta_{10}^b(P_2)\right| = 64$. The image of the presheaf terms of $\lambda_{2,1}$ at elements of $\Delta_{10}^a(P_2)$ are contained in the span of $D_{11} := \{(\lambda_{2,1})_P + \operatorname{im}\partial_1 | P \in \Delta_{11}(P_2)\}$. However, each element of D_{11} is spanned by the image of the presheaf term at an element of $\Delta_{12}(P_2)$ and the image of a term at an element of $\Delta_{10}^a(P_2)$. Consequently

$$\begin{aligned}\dim(D_{11}) &\leqslant |\Delta_{11}(P_2)| + \dim(\operatorname{span}\{(\lambda_{2,1})_P + \operatorname{im}\partial_1 | P \in \Delta_{12}(P_2)\}) \\ &\leqslant 48 + 16 = 64.\end{aligned}$$

We conclude that

$$\begin{aligned}\dim(H_0(\lambda_{2,1})) &= \dim(\operatorname{span}\{(\lambda_{2,1})_P + \operatorname{im}\partial_1 | P \in \Delta_{10}(P_2)\}) \\ &\leqslant \dim(\operatorname{span}\{(\lambda_{2,1})_P + \operatorname{im}\partial_1 | P \in \Delta_{10}^a(P_2)\}) \\ &\qquad + \dim(\operatorname{span}\{(\lambda_{2,1})_P + \operatorname{im}\partial_1 | P \in \Delta_{10}^b(P_2)\}) \\ &\leqslant \dim(\operatorname{span} D_{11}) + \left|\Delta_{10}^b(P_2)\right| \\ &\leqslant 64 + 64 = 128.\end{aligned}$$

Presheaf, $\lambda_{i,j}$	Irreducible quotients of $H_0(\lambda_{i,j})$	Notes on $H_0(\lambda_{i,j})$
$\lambda_{1,1}$	1	$H_0(\lambda_{1,1}) \cong 1$
$\lambda_{1,2}$	10	$76 \leqslant \dim(H_0(\lambda_{1,2})) = \dim(H_0(\lambda_{2,1})) \leqslant 128$
$\lambda_{2,1}$	32, 44	$76 \leqslant \dim(H_0(\lambda_{2,1})) = \dim(H_0(\lambda_{1,2})) \leqslant 128$
$\lambda_{2,2}$	144	$\dim(H_0(\lambda_{2,2})) = 496$

Table 3.2: Dimensional-bounds and irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves of M_{12} over $GF(2)$.

Finally, as $32_{M_{12}}$ and $44_{M_{12}}$ are quotients of $H_0(\lambda_{2,1})$ we see that

$$76 \leqslant \dim(H_0(\lambda_{1,2})) = \dim(H_0(\lambda_{2,1})) \leqslant 128.$$

$\lambda_{2,2}$

As $\lambda_{2,2} \cong \lambda_{1,1}^*$, Ronan's Duality Theorem asserts that

$$\begin{aligned}\dim(H_0(\lambda_{2,2})) &= \chi(\lambda_{2,2}) + \dim(H_1(\lambda_{2,2}))\\ &= \chi(\lambda_{2,2}) + \dim(H_0(\lambda_{1,1})) = 495 + 1 = 496.\end{aligned}$$

Unfortunately, it is unrealistic to fully determine the structure of $H_0(\lambda_{i,j})$ for $(i,j) \in \{(1,2),(2,1),(2,2)\}$. Obtaining bounds on the dimension and the irreducible quotients of the homology group is the best that one can usually hope for. We summarise our findings in Table 3.2.

M_{22}

The unique 2-minimal parabolic system for M_{22} is given by

$$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_2 \sim 2^4.\,\mathrm{Sym}(5)\}.$$

There are six classes of irreducible $GF(2)M_{22}$-modules which we denote by $1_{M_{22}}$, $10_{M_{22}}$, $\overline{10}_{M_{22}}$, $34_{M_{22}}$, $98_{M_{22}}$ and $140_{M_{22}}$ (where the number denotes the dimension of the module). Furthermore, setting B to be a self-normalizing Sylow 2-subgroup, we note that there are one, two and three classes of irreducible $GF(2)B$-, $GF(2)P_1$- and $GF(2)P_2$- modules. We denote these by 1_B, 1_{P_1} and 2_{P_1} and 1_{P_2}, 4_{P_2} and $\overline{4}_{P_2}$ respectively. It follows that there are six universal panel-irreducible presheaves defined on M_{22} over $GF(2)$, given in the usual notation by $\lambda_{i,j}$ for $(i,j) \in \{(1,1),(1,4),(1,\overline{4}),(2,1),(2,4),(2,\overline{4})\}$.

For each irreducible $GF(2)M_{22}$-module V, we may - as in previous cases - consider the restriction of V to P_1 and P_2 to allow us to determine those $\lambda_{i,j}$ for which V is a quotient of $H_0(\lambda_{i,j})$. The modules $1_{M_{22}}$, $10_{M_{22}}$, $\overline{10}_{M_{22}}$, $98_{M_{22}}$ and $140_{M_{22}}$ are each quotients of the zero-homology group of a unique presheaf, these being $\lambda_{1,1}$, $\lambda_{1,4}$, $\lambda_{2,1}$, $\lambda_{2,4}$ and $\lambda_{2,\overline{4}}$ respectively. Moreover, $34_{M_{22}}$ is not a quotient of any zero-homology group. It follows that $H_0(\lambda_{1,\overline{4}}) = 0$. In addition, we trivially have that $H_0(\lambda_{1,1}) \cong 1_{M_{22}}$.

To obtain upper bounds on the dimensions of the zero-homology groups of the remaining four presheaves, we define Γ to be the graph having conjugates of P_1 and P_2 as its vertices and with

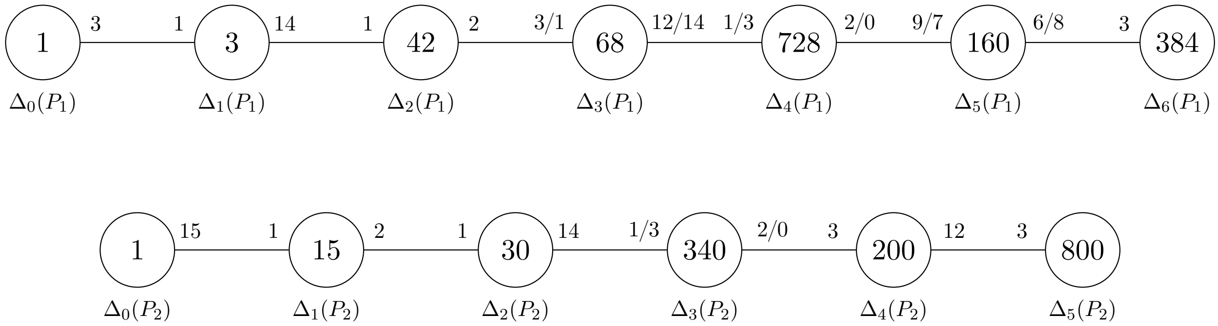


Figure 3.7: The collapsed adjacency diagrams of the graph Γ associated to M_{22} with respect to the 2-minimal parabolic subgroups P_1 (top) and P_2 (bottom).

$\{P_1^{t_1}, P_2^{t_2}\}$ forming an edge of Γ precisely when $P_1^{t_1} \cap P_2^{t_2} = B^{t_B}$ for some $t_B \in M_{22}$. The diameter and girth of Γ are both equal to 6 and the collapsed adjacency diagrams of Γ with respect to P_1 and P_2 are given in Figure 3.7.

We now consider the four remaining presheaves in turn.

$\lambda_{1,4}$

By constructing quotients of the induced module $(4_{P_2})_{P_2}^{M_{22}}$ in MAGMA, we obtain seven possibilities for $H_0(\lambda_{1,4})$. Indeed, the homology group has $10_{M_{22}}$ as a unique irreducible quotient, and there are seven isomorphism classes of quotients of $(4_{P_2})_{P_2}^{M_{22}}$ that do likewise. These are 10, $10/\overline{10}$, $10/\overline{10}/1$, $10/\overline{10}/1/34$, $10/\overline{10}/1/34/1$, $10/\overline{10}/1/34/1/10$ and $10/\overline{10}/1/34/1/10/\overline{10}$, all of which are uniserial modules. Considering the restriction of each module to B, P_1 and P_2, we see that the only modules whose constant presheaves admit panel-irreducible subpresheaves are 10, $10/\overline{10}$ and $10/\overline{10}/1$. Thus by Theorem 3.6.4, we conclude that $H_0(\lambda_{1,4}) \cong 10/\overline{10}/1$.

$\lambda_{2,1}$

The induced module $(1_{P_2})_{P_2}^{M_{22}}$ is 231-dimensional, and hence its quotients may be explicitly constructed in MAGMA. The homology group $H_0(\lambda_{2,1})$ is one such quotient, having $\overline{10}$ as a unique irreducible quotient. Considering all such quotients of $(1_{P_2})_{P_2}^{M_{22}}$ with this property, we see that there are four possibilities for $H_0(\lambda_{2,1})$, namely $\overline{10}$, $\overline{10}/1$, $\overline{10}/10$ and $\overline{10}/10/1$ all of which are indecomposable (with $\overline{10}$, $\overline{10}/1$ and $\overline{10}/10$ also being uniserial modules). Using elements of each module to generate subpresheaves of the constant presheaves $\mathcal{K}_{\overline{10}}$, $\mathcal{K}_{\overline{10}/1}$, $\mathcal{K}_{\overline{10}/10}$ and $\mathcal{K}_{\overline{10}/10/1}$ and appealing to Theorem 3.6.4 we see that $\overline{10}$ and $\overline{10}/1$ are quotients of $H_0(\lambda_{2,1})$, but $\overline{10}/10$ and $\overline{10}/10/1$ are not. Consequently, $H_0(\lambda_{2,1}) \cong \overline{10}/1$.

$\lambda_{2,4}$ and $\lambda_{2,\overline{4}}$

For the presheaves $\lambda_{2,4}$ and $\lambda_{2,\overline{4}}$ we begin by using a geometric spanning argument. From the collapsed adjacency diagram of the graph Γ associated to M_{22} with respect to P_2 we see that the image of each presheaf term is contained in $\Delta_4(P_2) + \operatorname{im}\partial_1$ with the possible exception of the 40 elements of $\Delta_3(P_2)$ having no neighbour in $\Delta_4(P_2)$. The image of these latter terms will be at most 2-dimensional, whilst the images of the terms in $\Delta_4(P_2)$ will be at most 4-dimensional. However, the image of each element of $\Delta_5(P_2)$ gives rise to a relation between the images of three non-zero vectors of the presheaf terms at elements of $\Delta_4(P_2)$. At worst, these 800 relations would lead to the images of 67 elements of $\Delta_4(P_2)$ being zero. In practice, they will account for a much larger decrease in the possible dimension of the zero-homology groups. We conclude that a rather crude bound on the dimension of the zero-homology groups of $\lambda_{2,4}$ and $\lambda_{2,\overline{4}}$ is given by

$$\dim(H_0(\lambda_{2,4})), \dim(H_0(\lambda_{2,\overline{4}})) \leqslant 200 \cdot 4 + 40 \cdot 2 - 67 \cdot 4 = 612. \tag{3.8.17}$$

The homology group $H_0(\lambda_{2,4})$ is a quotient of $(4_{P_2})_{P_2}^{M_{22}}$, which has a unique irreducible quotient, namely $98_{M_{22}}$. There are 481 classes of such quotients satisfying the dimensional bound from (3.8.17). We may consider these classes in MAGMA and appeal to Theorem 3.6.4 and Proposition 3.6.7. However, we do not need to consider all classes explicitly. Indeed for a given quotient W, if we determine that it does admit $\lambda_{2,4}$ as a subpresheaf of $\mathcal{K}_W$, then we may rule out all classes for which W is a not a quotient. Conversely, if W does not admit $\lambda_{2,4}$, then we may rule out all classes for which W is a quotient. We deduce that

$$H_0(\lambda_{2,4}) \cong 98/1/\left(\frac{34}{(1/10) \oplus (\overline{10}/1)} \bigoplus (10/\overline{10}/1)\right)/34/1.$$

We follow an analogous approach to calculate $H_0(\lambda_{2,\overline{4}})$, which is by definition a quotient of $(\overline{4}_{P_2})_{P_2}^{M_{22}}$. Using the bound from (3.8.17) together with the fact that $140_{M_{22}}$ is the unique irreducible quotient of $H_0(\lambda_{2,\overline{4}})$, we are left with 38 possible classes for $H_0(\lambda_{2,\overline{4}})$. By using MAGMA as above, we may calculate that

$$H_0(\lambda_{2,\overline{4}}) \cong 140/(1 \oplus 1/34/1/10/34).$$

As we have seen with M_{22}, even for small groups it is usually extremely difficult to determine the exact dimension and structure of the zero-homology group of a universal panel-irreducible presheaf. In the case of M_{23} and M_{24} below, we merely determine the irreducible quotients of the homology groups by considering the restriction of modules to minimal parabolic subgroups.

M_{23}

The second largest Mathieu group, M_{23}, admits seven 2-minimal parabolic subgroups with respect to a self-normalizing Sylow 2-subgroup B. In the notation of Ronan and Stroth (see [RS84]) these are $P_i \sim 2^{4+2}.\operatorname{Sym}(3)$ for $i = 1, \ldots, 5$ and $P_j \sim 2^4.\operatorname{Sym}(5)$ for $j = 6, 7$, where $P_5 = P_6 \cap P_7$. These minimal parabolic subgroups give rise to seven 2-minimal parabolic systems for M_{23}, namely $\{P_1, P_3, P_7\}$, $\{P_2, P_3, P_7\}$, $\{P_3, P_4, P_7\}$ and $\{P_i, P_6, P_7\}$ for $i = 1, 2, 3, 4$. The systems $\{P_1, P_3, P_7\}$ and $\{P_3, P_4, P_7\}$ are geometric, whilst the remaining systems are non-geometric.

There are ten irreducible $GF(2)M_{23}$-modules, which we denote by their dimensions as $1_{M_{23}}$, $11_{M_{23}}$, $\overline{11}_{M_{23}}$, $44_{M_{23}}$, $\overline{44}_{M_{23}}$, $120_{M_{23}}$, $220_{M_{23}}$, $\overline{220}_{M_{23}}$, $252_{M_{23}}$ and $1792_{M_{23}}$. For $i = 1, 2, 3, 4$ there

$GF(2)M_{23}$-Module	Element of Module, v	$\langle v\rangle\vert_B$	$\langle v\rangle\vert_{P_1}$	$\langle v\rangle\vert_{P_2}$	$\langle v\rangle\vert_{P_3}$	$\langle v\rangle\vert_{P_4}$	$\langle v\rangle\vert_{P_6}$	$\langle v\rangle\vert_{P_7}$
$1_{M_{23}}$	v_1	1_B	1_{P_1}	1_{P_2}	1_{P_3}	1_{P_4}	1_{P_6}	1_{P_7}
$11_{M_{23}}$	v_{11}	1_B	1_{P_1}	1_{P_2}	1_{P_3}	1_{P_4}	1_{P_6}	4_{P_7}
$\overline{11}_{M_{23}}$	$v_{\overline{11}a}$	1_B	1_{P_1}	1_{P_2}	2_{P_3}	1_{P_4}	4_{P_6}	1_{P_7}
	$v_{\overline{11}b}$	1_B	2_{P_1}	2_{P_2}	N/A	2_{P_4}	1_{P_6}	1_{P_7}
$44_{M_{23}}$	v_{44}	1_B	1_{P_1}	1_{P_2}	N/A	1_{P_4}	4_{P_6}	1_{P_7}
$\overline{44}_{M_{23}}$	$v_{\overline{44}a}$	1_B	2_{P_1}	2_{P_2}	2_{P_3}	2_{P_4}	4_{P_6}	1_{P_7}
	$v_{\overline{44}b}$	1_B	1_{P_1}	1_{P_2}	N/A	1_{P_4}	N/A	1_{P_7}
$120_{M_{23}}$	v_{120a}	1_B	1_{P_1}	2_{P_2}	N/A	2_{P_4}	N/A	$\overline{4}_{P_7}$
	v_{120b}	1_B	2_{P_1}	2_{P_2}	N/A	2_{P_4}	1_{P_6}	4_{P_7}
	v_{120c}	1_B	1_{P_1}	1_{P_2}	2_{P_3}	1_{P_4}	4_{P_6}	4_{P_7}
$220_{M_{23}}$	v_{220a}	1_B	N/A	2_{P_2}	2_{P_3}	N/A	$\overline{4}_{P_6}$	$\overline{4}_{P_7}$
	v_{220b}	1_B	1_{P_1}	2_{P_2}	2_{P_3}	2_{P_4}	N/A	$\overline{4}_{P_7}$
	v_{220c}	1_B	2_{P_1}	2_{P_2}	2_{P_3}	1_{P_4}	N/A	$\overline{4}_{P_7}$
$\overline{220}_{M_{23}}$	$v_{\overline{220}a}$	1_B	2_{P_1}	1_{P_2}	N/A	2_{P_4}	N/A	$\overline{4}_{P_7}$
	$v_{\overline{220}b}$	1_B	2_{P_1}	2_{P_2}	N/A	2_{P_4}	4_{P_6}	1_{P_7}
	$v_{\overline{220}c}$	1_B	2_{P_1}	2_{P_2}	2_{P_3}	1_{P_4}	N/A	$\overline{4}_{P_7}$
$252_{M_{23}}$	v_{252a}	1_B	1_{P_1}	1_{P_2}	N/A	1_{P_4}	N/A	1_{P_7}
	v_{252b}	1_B	2_{P_1}	1_{P_2}	N/A	2_{P_4}	N/A	$\overline{4}_{P_7}$
	v_{252c}	1_B	2_{P_1}	2_{P_2}	2_{P_3}	2_{P_4}	4_{P_6}	4_{P_7}
	v_{252d}	1_B	1_{P_1}	1_{P_2}	N/A	1_{P_4}	N/A	4_{P_7}
$1792_{M_{23}}$	v_{1792a}	1_B	2_{P_1}	2_{P_2}	N/A	2_{P_4}	4_{P_6}	4_{P_7}

Table 3.3: The elements of the irreducible $GF(2)M_{23}$-modules which admit universal panel-irreducible presheaves.

are two classes of irreducible $GF(2)P_i$-modules, 1_{P_i} and 2_{P_i}, whilst for $j = 6, 7$ there are three classes of irreducible $GF(2)P_j$-modules, namely 1_{P_j}, 4_{P_J} and $\overline{4}_{P_j}$.

Every 2-minimal parabolic system of M_{23} features P_7. Consequently, for each irreducible $GF(2)M_{23}$-module, V, we may consider the elements of V which generate irreducible submodules of the restriction of V to P_7. For each such element, we may then consider the submodules of $V|_B$ and $V|_{P_i}$ that the element generates for $i = 1, 2, 3, 4, 6, 7$. In Table 3.3, we list such elements that generate irreducible $GF(2)B$-modules and irreducible $GF(2)P_i$-modules for every P_i featuring in one of the 2-minimal parabolic systems of M_{23}. Entries of the form "N/A" indicate that the given module is reducible.

The seven 2-minimal parabolic systems give rise to an abundance of universal panel-irreducible presheaves for M_{23} over $GF(2)$. For ease of notation, we denote the presheaf defined on the minimal parabolic system $\{P_i, P_j, P_k\}$ having a $*_{dim}$-dimensional irreducible $GF(2)P_*$-module at panels of type P_* by $\lambda^{(ijk)}_{i_{dim}, j_{dim}, k_{dim}}$. An analysis of the vertex terms of these presheaves is given in Appendix A. Using the information in Table 3.3 we may fully determine the irreducible quotients of the zero-homology group of each presheaf. We summarise our findings in Tables 3.4-3.10. In these tables, entries of the form "None" indicate that the given homology group is equal to $0_{M_{23}}$.

In most of the cases mentioned above, the irreducible quotients may be determined by taking an irreducible $GF(2)M_{23}$-module V and finding the socle of its restriction to P_7.[6] By considering all

[6]We recall that the *socle* of a module V, denoted $\mathrm{soc}(V)$, is the sum of all minimal submodules of V. [Lam99]

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$
$\lambda^{(137)}_{1,1,1}$	$1_{M_{23}}$
$\lambda^{(137)}_{1,1,4}$	$11_{M_{23}}$
$\lambda^{(137)}_{1,1,\overline{4}}$	None
$\lambda^{(137)}_{1,2,1}$	$\overline{11}_{M_{23}}$
$\lambda^{(137)}_{1,2,4}$	$120_{M_{23}}$
$\lambda^{(137)}_{1,2,\overline{4}}$	$220_{M_{23}}$
$\lambda^{(137)}_{2,1,1}$	None
$\lambda^{(137)}_{2,1,4}$	None
$\lambda^{(137)}_{2,1,\overline{4}}$	None
$\lambda^{(137)}_{2,2,1}$	$\overline{44}_{M_{23}}$
$\lambda^{(137)}_{2,2,4}$	$252_{M_{23}}$
$\lambda^{(137)}_{2,2,\overline{4}}$	$220_{M_{23}}$, $\overline{220}_{M_{23}}$

Table 3.4: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_1, P_3, P_7\}$ of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$
$\lambda^{(167)}_{1,1,1}$	$1_{M_{23}}$
$\lambda^{(167)}_{1,1,4}$	$11_{M_{23}}$
$\lambda^{(167)}_{1,1,\overline{4}}$	None
$\lambda^{(167)}_{1,4,1}$	$\overline{11}_{M_{23}}$, $44_{M_{23}}$
$\lambda^{(167)}_{1,4,4}$	$120_{M_{23}}$
$\lambda^{(167)}_{1,4,\overline{4}}$	None
$\lambda^{(167)}_{1,\overline{4},1}$	None
$\lambda^{(167)}_{1,\overline{4},4}$	None
$\lambda^{(167)}_{1,\overline{4},\overline{4}}$	None
$\lambda^{(167)}_{2,1,1}$	$\overline{11}_{M_{23}}$
$\lambda^{(167)}_{2,1,4}$	$120_{M_{23}}$
$\lambda^{(167)}_{2,1,\overline{4}}$	None
$\lambda^{(167)}_{2,4,1}$	$\overline{44}_{M_{23}}$, $\overline{220}_{M_{23}}$
$\lambda^{(167)}_{2,4,4}$	$252_{M_{23}}$, $1792_{M_{23}}$
$\lambda^{(167)}_{2,4,\overline{4}}$	None
$\lambda^{(167)}_{2,\overline{4},1}$	None
$\lambda^{(167)}_{2,\overline{4},4}$	None
$\lambda^{(167)}_{2,\overline{4},\overline{4}}$	None

Table 3.5: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_1, P_6, P_7\}$ of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$
$\lambda^{(237)}_{1,1,1}$	$1_{M_{23}}$
$\lambda^{(237)}_{1,1,4}$	$11_{M_{23}}$
$\lambda^{(237)}_{1,1,\bar{4}}$	None
$\lambda^{(237)}_{1,2,1}$	$\overline{11}_{M_{23}}$
$\lambda^{(237)}_{1,2,4}$	$120_{M_{23}}$
$\lambda^{(237)}_{1,2,\bar{4}}$	None
$\lambda^{(237)}_{2,1,1}$	None
$\lambda^{(237)}_{2,1,4}$	None
$\lambda^{(237)}_{2,1,\bar{4}}$	None
$\lambda^{(237)}_{2,2,1}$	$\overline{44}_{M_{23}}$
$\lambda^{(237)}_{2,2,4}$	$252_{M_{23}}$
$\lambda^{(237)}_{2,2,\bar{4}}$	$220_{M_{23}}$, $\overline{220}_{M_{23}}$

Table 3.6: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_2, P_3, P_7\}$ of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$
$\lambda^{(267)}_{1,1,1}$	$1_{M_{23}}$
$\lambda^{(267)}_{1,1,4}$	$11_{M_{23}}$
$\lambda^{(267)}_{1,1,\bar{4}}$	None
$\lambda^{(267)}_{1,4,1}$	$\overline{11}_{M_{23}}$, $44_{M_{23}}$
$\lambda^{(267)}_{1,4,4}$	$120_{M_{23}}$
$\lambda^{(267)}_{1,4,\bar{4}}$	None
$\lambda^{(267)}_{1,\bar{4},1}$	None
$\lambda^{(267)}_{1,\bar{4},4}$	None
$\lambda^{(267)}_{1,\bar{4},\bar{4}}$	None
$\lambda^{(267)}_{2,1,1}$	$\overline{11}_{M_{23}}$
$\lambda^{(267)}_{2,1,4}$	$120_{M_{23}}$
$\lambda^{(267)}_{2,1,\bar{4}}$	None
$\lambda^{(267)}_{2,4,1}$	$\overline{44}_{M_{23}}$, $\overline{220}_{M_{23}}$
$\lambda^{(267)}_{2,4,4}$	$252_{M_{23}}$, $1792_{M_{23}}$
$\lambda^{(267)}_{2,4,\bar{4}}$	None
$\lambda^{(267)}_{2,\bar{4},1}$	None
$\lambda^{(267)}_{2,\bar{4},4}$	None
$\lambda^{(267)}_{2,\bar{4},\bar{4}}$	$220_{M_{23}}$

Table 3.7: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_2, P_6, P_7\}$ of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$
$\lambda^{(347)}_{1,1,1}$	$1_{M_{23}}$
$\lambda^{(347)}_{1,1,4}$	$11_{M_{23}}$
$\lambda^{(347)}_{1,1,\overline{4}}$	None
$\lambda^{(347)}_{1,2,1}$	None
$\lambda^{(347)}_{1,2,4}$	None
$\lambda^{(347)}_{1,2,\overline{4}}$	None
$\lambda^{(347)}_{2,1,1}$	$\overline{11}_{M_{23}}$
$\lambda^{(347)}_{2,1,4}$	$120_{M_{23}}$
$\lambda^{(347)}_{2,1,\overline{4}}$	$220_{M_{23}}$, $\overline{220}_{M_{23}}$
$\lambda^{(347)}_{2,2,1}$	$\overline{44}_{M_{23}}$
$\lambda^{(347)}_{2,2,4}$	$252_{M_{23}}$
$\lambda^{(347)}_{2,2,\overline{4}}$	$220_{M_{23}}$

Table 3.8: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_3, P_4, P_7\}$ of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$
$\lambda^{(367)}_{1,1,1}$	$1_{M_{23}}$
$\lambda^{(367)}_{1,1,4}$	$11_{M_{23}}$
$\lambda^{(367)}_{1,1,\overline{4}}$	None
$\lambda^{(367)}_{1,4,1}$	None
$\lambda^{(367)}_{1,4,4}$	None
$\lambda^{(367)}_{1,4,\overline{4}}$	None
$\lambda^{(367)}_{1,\overline{4},1}$	None
$\lambda^{(367)}_{1,\overline{4},4}$	None
$\lambda^{(367)}_{1,\overline{4},\overline{4}}$	None
$\lambda^{(367)}_{2,1,1}$	None
$\lambda^{(367)}_{2,1,4}$	None
$\lambda^{(367)}_{2,1,\overline{4}}$	None
$\lambda^{(367)}_{2,4,1}$	$\overline{11}_{M_{23}}$, $\overline{44}_{M_{23}}$
$\lambda^{(367)}_{2,4,4}$	$120_{M_{23}}$, $252_{M_{23}}$
$\lambda^{(367)}_{2,4,\overline{4}}$	None
$\lambda^{(367)}_{2,\overline{4},1}$	None
$\lambda^{(367)}_{2,\overline{4},4}$	None
$\lambda^{(367)}_{2,\overline{4},\overline{4}}$	$220_{M_{23}}$

Table 3.9: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_3, P_6, P_7\}$ of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$
$\lambda^{(467)}_{1,1,1}$	$1_{M_{23}}$
$\lambda^{(467)}_{1,1,4}$	$11_{M_{23}}$
$\lambda^{(467)}_{1,1,\bar{4}}$	None
$\lambda^{(467)}_{1,4,1}$	$\overline{11}_{M_{23}}$, $44_{M_{23}}$
$\lambda^{(467)}_{1,4,4}$	$120_{M_{23}}$
$\lambda^{(467)}_{1,4,\bar{4}}$	None
$\lambda^{(467)}_{1,\bar{4},1}$	None
$\lambda^{(467)}_{1,\bar{4},4}$	None
$\lambda^{(467)}_{1,\bar{4},\bar{4}}$	None
$\lambda^{(467)}_{2,1,1}$	$\overline{11}_{M_{23}}$
$\lambda^{(467)}_{2,1,4}$	$120_{M_{23}}$
$\lambda^{(467)}_{2,1,\bar{4}}$	None
$\lambda^{(467)}_{2,4,1}$	$\overline{44}_{M_{23}}$, $\overline{220}_{M_{23}}$
$\lambda^{(467)}_{2,4,4}$	$252_{M_{23}}$, $1792_{M_{23}}$
$\lambda^{(467)}_{2,4,\bar{4}}$	None
$\lambda^{(467)}_{2,\bar{4},1}$	None
$\lambda^{(467)}_{2,\bar{4},4}$	None
$\lambda^{(467)}_{2,\bar{4},\bar{4}}$	None

Table 3.10: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_4, P_6, P_7\}$ of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$
$\lambda_{1,1,1}$	$1_{M_{24}}$
$\lambda_{1,1,2}$	$\overline{11}_{M_{24}}$
$\lambda_{1,2,1}$	$44_{M_{24}}$, $\overline{44}_{M_{24}}$, $\overline{220}_{M_{24}}$
$\lambda_{1,2,2}$	$\overline{220}_{M_{24}}$, $\overline{320}_{M_{24}}$
$\lambda_{2,1,1}$	$11_{M_{24}}$, $120_{M_{24}}$
$\lambda_{2,1,2}$	$120_{M_{24}}$, $220_{M_{24}}$
$\lambda_{2,2,1}$	$252_{M_{24}}$, $320_{M_{24}}$, $1792_{M_{24}}$
$\lambda_{2,2,2}$	$1242_{M_{24}}$, $1792_{M_{24}}$

Table 3.11: The irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the unique 2-minimal parabolic system of M_{24}.

elements of $\mathrm{soc}(V|_{P_7})$, we may then determine which universal panel-irreducible presheaves admit V. In the case of $1792_{M_{23}}$, a more subtle approach is required as $\mathrm{soc}(1792_{M_{23}}|_{P_7})$ is 32-dimensional. However, we may decompose the socle as a direct sum of two semisimple submodules, the first generated by all minimal submodules isomorphic to 4_{P_7} and the second generated by all minimal submodules isomorphic to $\overline{4}_{P_7}$. The former is an 8-dimensional submodule, whilst the latter is 24-dimensional. By considering all elements of each of these summands, we then obtain the results.

M_{24}

The simple group M_{24} admits four 2-minimal parabolic subgroups. In the notation of Ronan and Stroth these are $P_i \sim 2^{6+3}.\mathrm{Sym}(3)$ for $i = 1, 2, 3, 4$. Here $P_1 \cong P_2$ and $P_3 \cong P_4$. These minimal parabolic subgroups give rise to a unique 2-minimal parabolic system $\{P_1, P_3, P_4\}$ of M_{24}, which is geometric. Since each minimal parabolic subgroup, P_i, has Sym(3) above its 2-core, there are two classes of irreducible $GF(2)P_i$-modules, 1_{P_i} and 2_{P_i}. There is also a unique class of irreducible $GF(2)B$-modules for a self-normalizing Sylow 2-subgroup B. These modules are 1-dimensional. It follows that there are eight universal panel-irreducible presheaves for M_{24}. We denote the presheaf having a 1-dimensional chamber term and i-, j- and k-dimensional panel terms at panels of type P_1, P_3 and P_4 by $\lambda_{i,j,k}$. An exploration of the vertex terms of these presheaves is given in Section A.2.

Aside 3.8.2. *We note that the presheaves $\lambda_{i,j,k}$ differ from those considered by Ronan and Smith in [RS89]. Indeed, since the minimal parabolic subgroup P_2 does not feature in a minimal parabolic system of M_{24}, Ronan and Smith defined $Q_i := \langle P_i, P_2\rangle$ for $i = 1, 2, 3, 4$ and considered presheaves defined on the simplicial complex having conjugates of Q_2 as its chambers, and conjugates of Q_i for $i = 1, 3, 4$ as its panels.*

There are thirteen classes of irreducible $GF(2)M_{24}$-modules, denoted by their dimensions as $1_{M_{24}}$, $11_{M_{24}}$, $\overline{11}_{M_{24}}$, $44_{M_{24}}$, $\overline{44}_{M_{24}}$, $120_{M_{24}}$, $220_{M_{24}}$, $\overline{220}_{M_{24}}$, $252_{M_{24}}$, $320_{M_{24}}$, $\overline{320}_{M_{24}}$, $1242_{M_{24}}$ and $1792_{M_{24}}$. For each such module, V, we mimic the calculations undertaken for M_{23} to determine which homology groups $H_0(\lambda_{i,j,k})$ admit V as a quotient. A summary of these calculations is given in Table 3.11.

To deduce that $1242_{M_{24}}$ admits a unique universal panel-irreducible presheaf, namely $\lambda_{2,2,2}$, we note that the restriction $1242_{M_{24}}|_{P_{34}}$ has three minimal submodules, these being isomorphic to

$1_{P_{34}}$, $3_{P_{34}}$ and $8_{P_{34}}$. The elements of $1242_{M_{24}}$ generating $3_{P_{34}}$ do not give rise to a panel-irreducible presheaf, whilst those generating $8_{P_{34}}$ give rise to $\lambda_{2,2,2}$. Consequently, the only remaining presheaf that may have $1242_{M_{24}}$ as a quotient of its zero-homology group will necessarily have 1-dimensional terms at panels of type P_3 and P_4. The only such possibility is $\lambda_{2,1,1}$. However, consideration of the elements of $1242_{M_{24}}$ that generate 2-dimensional $GF(2)P_1$-modules proves that this possibility cannot occur.

3.9 Homology Groups of Universal Presheaves Revisited

We saw in Section 3.8 that the normalizer of a Sylow subgroup being maximal in each minimal parabolic subgroup was not sufficient for an affirmative answer to Question 3.7.1. Indeed, as described in Subsection 3.8.2 there does not exist a universal panel-irreducible presheaf, $\mathscr{F}$, defined on the minimal parabolic systems $\mathcal{S}_2$ or $\mathcal{S}_3$ of M_{11} for which $10_{M_{11}}$ is a quotient of $H_0(\mathscr{F})$.

A natural evolvement is to ask if there are any types of p-minimal parabolic system, $\mathcal{S}$, of a group G for which every irreducible $GF(p)G$-module is admitted as a quotient of the zero-homology group of some universal panel-irreducible presheaf? From our calculations in Section 3.8 we see that this may be the case when $p = 2$, a Sylow 2-subgroup of G is self-normalizing and each 2-minimal parabolic subgroup, P, of G is of shape $O_2(P).\mathrm{Sym}(3)$. Indeed, we saw that this occurred for the unique 2-minimal parabolic systems of Sym(6), M_{12} and M_{24}. It also occurs for the 2-minimal parabolic systems of $L_4(2)$ and Alt(7), considered by Ronan and Smith in [RS89] and for the parabolic subgroups P_{13} and P_{34} of M_{23} (considered in Appendix A). This leads to the following question.

Question 3.9.1. *Suppose that G is a group of even order possessing a self-normalizing Sylow 2-subgroup. Moreover, assume that G admits a 2-minimal parabolic system $\mathcal{S} := \{P_i | i = 1, \ldots, n\}$ such that $P_i \sim O_2(P_i).\mathrm{Sym}(3)$ for all $i = 1, \ldots, n$. For each $GF(2)G$-module V, does there exist a universal panel-irreducible presheaf, $\mathscr{F}_V$, defined on $\mathcal{S}$ such that V is a quotient of $H_0(\mathscr{F}_V)$?*

If Question 3.9.1 has a positive answer, it would likely rely on the fact that if B is the self-normalizing Sylow 2-subgroup of G and P is a 2-minimal parabolic subgroup containing B, then there is a unique class of irreducible $GF(2)B$-modules and every non-trivial proper quotient of $(1_B)_B^P$ is irreducible. Thus it would suffice to prove that for every irreducible $GF(2)G$-module, V, there exists a non-zero element $v \in V$ satisfying $\langle v \rangle_{kB} \cong 1_B$ and $\langle v \rangle_{kP} \neq (1_B)_B^P$ for every minimal parabolic subgroup P.

Chapter 4

π-Product Graphs in Symmetric Groups

There is a cornucopia of combinatorial and geometric structures which are associated with groups. These range from graphs to posets and topological spaces such as simplicial complexes. An example of the latter type arises in a finite group G where for a prime p dividing the order of G we may define the poset of all non-trivial p-subgroups of G, denoted $\mathcal{S}_p(G)$, ordered by inclusion. This poset has a rich structure, as has its associated order complex $|\mathcal{S}_p(G)|$ known as the *Brown complex*, after being studied by - among others - Brown in his paper [Bro75]. An analogous order complex, called the *Quillen complex*, can be defined for the poset $\mathcal{A}_p(G)$ of all non-trivial elementary abelian p-subgroups of G. Indeed, Quillen showed in [Qui78] that the Brown and Quillen complexes are G-homotopy equivalent. Thévenez and Webb later showed that the complexes consisting of chains of normal series of p-subgroups, and chains of radical p-subgroups are also G-homotopy equivalent to the Brown and Quillen complexes (see [Thé87] and [TW91] for full details). In the case when G is a group of Lie type, the order complex $|\mathcal{S}_p(G)|$ is the same as the building of G.

The aforementioned subgroup complexes arise in many different areas. Brown was motivated by cohomology and the calculation of cohomology groups for discrete groups - the subject of [Bro76]. The complexes are also closely related to fusion in finite groups and the existence of strongly p-embedded subgroups in G is equivalent to the disconnectedness of $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ (further details of which can be found for example in [Asc00]). It is also possible to build modular representations of G by first defining such representations on stabilizers of the simplices of these complexes, following the constructions of Ronan and Smith in [Ron89], [RS85], [RS86] and [RS89]. A good survey of the versatility of such complexes can be found in [Smi11].

We mention a few graphs among the multitude of such structures that we may associate to a given group G. Let X be a subset of G. The commuting graph $\mathcal{C}(G, X)$ has vertex set X and distinct elements $x, y \in X$ are joined by an edge whenever $xy = yx$. The case when $X = G \backslash Z(G)$, first studied in [BF55], has been the focus of interest recently - see [BG13], [MP13] and [Par13]. When X is taken to be a G-conjugacy class of involutions, we get the so-called commuting involution graph, the subject of a number of papers (see [BBHR09], [BBPR03a], [BBPR03b], [BBPR04], [Eve11], [Per06] and [Sal11]).

If π is a set of natural numbers, then the π-product graph $\mathcal{P}_\pi(G, X)$ again has vertex set X, with distinct vertices $x, y \in X$ joined by an edge if the order of xy is in π. In the case when X is a G-conjugacy class of involutions, we note that $\mathcal{P}_{\{2\}}(G, X)$ is just a commuting involution graph. Taking π to be the set of all odd natural numbers and X a G-conjugacy class, $\mathcal{P}_\pi(G, X)$ becomes the local fusion graph $\mathcal{F}(G, X)$ which has featured in [Bal13] and [BGR13].

In the case when X is a set of involutions we refer to $\mathcal{P}_\pi(G, X)$ as a π-product involution graph. It is such graphs when X is a conjugacy class that we consider in this chapter for $G = \mathrm{Sym}(n)$, the symmetric group of degree n. We use the standard distance metric on $\mathcal{P}_\pi(G, X)$, which we denote by $d(\cdot, \cdot)$. For $x \in X$ and $i \in \mathbb{N}$ we denote the set of vertices distance i from x in $\mathcal{P}_\pi(G, X)$ by $\Delta_i(x)$. We also denote by $\Omega := \{1, \ldots, n\}$ the underlying set upon which $\mathrm{Sym}(n)$ acts.

We first consider the case when $\pi = \{4\}$. In other words, two distinct involutions $x, y \in X$ are joined by an edge whenever $\langle x, y\rangle \cong \mathrm{Dih}(8)$, the dihedral group of order 8. In considering this, we are in effect looking at a section of the poset $\mathcal{S}_2(\mathrm{Sym}(n))$. Our first result determines when $\mathcal{P}_{\{4\}}(G, X)$ is connected and in such cases, the diameter of $\mathcal{P}_{\{4\}}(G, X)$ is also determined.

Theorem 4.0.1. *Suppose $G = \mathrm{Sym}(n)$, $t = (1, 2)\cdots(2m-1, 2m) \in G$, and let X denote the G-conjugacy class of t.*

(i) The graph $\mathcal{P}_{\{4\}}(G, X)$ is disconnected if and only if one of the following holds:

(a) $n = 2m + 1$;

(b) $m = 1$;

(c) $(n, m) = (4, 2)$ or $(6, 3)$.

(ii) If $\mathcal{P}_{\{4\}}(G, X)$ is connected, then $\mathrm{Diam}(\mathcal{P}_{\{4\}}(G, X)) = 2$.

In (i)(a) of Theorem 4.0.1 we observe that $\mathcal{P}_{\{4\}}(G, X)$ consists of n copies of $\mathcal{P}_{\{4\}}(\mathrm{Sym}(2m), Y)$ where Y consists of all involutions of cycle type 2^m. This corresponds to the n possible fixed points of the involutions of X. Cases (i)(b) and (i)(c) result in totally disconnected graphs.

For symmetric groups, the diameters of the connected π-product involution graphs have been determined when $\pi = \{2\}$ - that is the commuting involution graphs - and $\pi = \mathbb{N}_{\mathrm{odd}}$ (=the set of all odd natural numbers) - the local fusion graphs. In the former case the diameter is bounded above by 3 except for three small cases when the diameter is 4. Moreover, the diameter can be 3 infinitely often. In the latter case, the connected local fusion graphs for symmetric groups always have diameter 2. So, from this perspective, $\mathcal{P}_\pi(G, X)$ for $\pi = \{4\}$ and $\pi = \mathbb{N}_{\mathrm{odd}}$ are bed fellows. However, this apparent similarity does not extend to the case that $\pi = \{2^a\}$ for some $a \geqslant 3$. Indeed, we shall derive the following result.

Theorem 4.0.2. *Suppose that $G = \mathrm{Sym}(n)$, $2m = 2^a \leqslant n$ for some $a \geqslant 3$, $t = (1, 2)(3, 4)\cdots(2m-1, 2m)$ and X is the G-conjugacy class of t. Then*

(i) $\mathcal{P}_{\{2m\}}(G, X)$ is connected if and only if $n \geqslant 2m + 2$; and

(ii) if $\mathcal{P}_{\{2m\}}(G, X)$ is connected, then

$$\min\{m, \lceil n/2 - m\rceil\} \leqslant \mathrm{Diam}(\mathcal{P}_{\{2m\}}(G, X)) \leqslant 2m - 1$$

(where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x). Taking $n = 4m$ in this theorem gives an infinite family of π-product involution graphs whose diameter is unbounded.

Specialising to the case $m = 4$ (so $2m = 8$) we can give precise values for the diameter of $\mathcal{P}_{\{8\}}(G, X)$ in our next theorem.

Theorem 4.0.3. *Suppose $G = \mathrm{Sym}(n)$, $t = (1,2)(3,4)(5,6)(7,8)$ and let X be the G-conjugacy class of t. Then*

(i) for $10 \leqslant n \leqslant 14$, $\mathrm{Diam}(\mathcal{P}_{\{8\}}(G, X)) = 3$; and

(ii) for $n \geqslant 15$, $\mathrm{Diam}(\mathcal{P}_{\{8\}}(G, X)) = 4$.

An analogous version of Theorem 4.0.2 also holds for any odd prime power.

Theorem 4.0.4. *Suppose that $G = \mathrm{Sym}(n)$, p is an odd prime and $q = p^a$ for some $a \geqslant 1$. Let $t = (1,2)\cdots(q-2, q-1)$ and X be the G-conjugacy class of t. Then*

(i) $\mathcal{P}_{\{q\}}(G, X)$ is connected if and only if $n \geqslant q$; and

(ii) if $\mathcal{P}_{\{q\}}(G, X)$ is connected, then

$$\min\{q-1, n+1-q\} \leqslant \mathrm{Diam}(\mathcal{P}_{\{q\}}(G, X)) \leqslant q - 1.$$

Our final result combines Theorems 4.0.2 and 4.0.4.

Theorem 4.0.5. *Suppose that $G = \mathrm{Sym}(n)$, and $p_1, \ldots, p_r$ are distinct primes with $p_i < p_{i+1}$ for $i = 1, \ldots, r-1$. Let $q = p_1^{a_1} \cdots p_r^{a_r}$ for some $a_1, \ldots, a_r \geqslant 1$ with $a_1 \geqslant 2$ if $p_1 = 2$ and set*

$$q_i = \begin{cases} p_i^{a_i} & \text{if } p_i = 2; \text{ and} \\ p_i^{a_i} - 1 & \text{otherwise,} \end{cases}$$

and $2m = q_1 \cdots q_r$. Assuming $2m \leqslant n$, let $t = (1,2)\cdots(2m-1, 2m)$ and X be the G-conjugacy class of t.

(i) The graph $\mathcal{P}_{\{q\}}(G, X)$ is connected if and only if

$$n \geqslant \begin{cases} q + 2 & \text{if } p_1 = 2; \text{ and} \\ q & \text{otherwise.} \end{cases}$$

(ii) If $\mathcal{P}_{\{q\}}(G, X)$ is connected, then

$$\mathrm{Diam}(\mathcal{P}_{\{q\}}(G, X)) \leqslant \max_i q_i$$

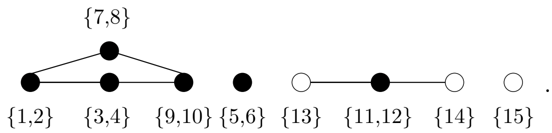
This chapter is arranged as follows. In Section 4.1 we introduce the notion of the *x-graph* of an element of X. These are graphs that encapsulate the $C_G(x)$-orbits of X and were first introduced by Bates, Bundy, Perkins and Rowley in [BBPR03b]. We present a number of their results, and relate the connected components of an x-graph to the disc $\Delta_1(t)$ for a fixed involution

t of X. Section 4.2 begins by considering combinations of connected components of x-graphs, and we show that Theorem 4.0.1 holds when restricted to the supports of such components. In particular we consider the case when our conjugacy class consists of elements of full support in Lemma 4.2.8. We then proceed to give a general proof of Theorem 4.0.1 at the end of this section. The chapter concludes in Section 4.3 with an analysis of π-product graphs when $\pi \neq \{4\}$. We begin by considering the case when $\pi = \{2^a\}$ for some $a \geqslant 3$. Calculations of the sizes of discs $\Delta_i(t)$ for certain π-product involution graphs are given and these give a direct proof of Theorem 4.0.3. This is followed by constructive proofs of Theorems 4.0.2, 4.0.4 and 4.0.5. Finally, we consider some smaller symmetric groups and calculate the sizes of discs of the π-product graphs $\mathcal{P}_\pi(G, X)$ when $\pi = \{6\}$ or $\{8\}$.

4.1 Preliminary Results

Throughout this chapter, we set $G = \mathrm{Sym}(n)$ and consider G as acting on a set of n letters (or points), $\Omega = \{1, \ldots, n\}$. Let $t \in G$ be a fixed involution and let X be the G-conjugacy class of t. For an element $g \in G$, we denote the set of fixed points of g on Ω by $\mathrm{fix}(g)$ and define the *support* of g to be $\mathrm{supp}(g) := \Omega \backslash \mathrm{fix}(g)$. For the sake of brevity, if $x_1, x_2, \ldots, x_r \in G$ we denote $\mathrm{supp}(x_1) \cup \mathrm{supp}(x_2) \cup \cdots \cup \mathrm{supp}(x_r)$ by $\mathrm{supp}(x_1, x_2, \ldots, x_r)$.

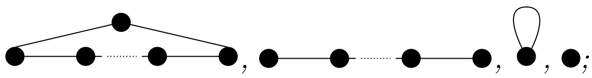
To study the graph $\mathcal{P}_{\{4\}}(G, X)$, we first introduce another type of graph known as an *x-graph*. Indeed, let $x \in X$. The x-graph corresponding to x, denoted $\mathcal{G}_x$, has vertex set given by the orbits of Ω under $\langle t \rangle$. Two vertices σ, γ are joined in $\mathcal{G}_x$ if there exists $\sigma_0 \in \sigma$ and $\gamma_0 \in \gamma$ such that $\{\sigma_0, \gamma_0\}$ is an orbit of Ω under $\langle x \rangle$. We call the vertices corresponding to transpositions of t *black vertices*, denoted ●, and those corresponding to fixed points of t *white vertices*, denoted ○. As an example, let $n = 15$, $t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ and $x = (1,7)(2,3)(4,10)(8,9)(11,13)(12,14)$. The x-graph $\mathcal{G}_x$ is given by



We note that we could swap the roles of t and x to produce another x-graph which we denote by $\mathcal{G}_t^x$. In general the x-graph $\mathcal{G}_x^y$ has vertices the orbits of Ω under $\langle y \rangle$, and edges defined by the orbits of Ω under $\langle x \rangle$.

The concepts of x-graphs were first introduced in [BBPR03b] as a tool for studying the commuting involution graphs of the symmetric groups. More recently they have also been used in the study of local fusion graphs for the symmetric groups (see [BGR13] for further details). The versatility of x-graphs in calculations arises from the simple observation that each black vertex has valency at most two and each white vertex has valency at most one. Consequently, we may fully determine the possible connected components of a given x-graph.

Lemma 4.1.1. *Let $x \in X$. The possible connected components of $\mathcal{G}_x$ are*

(i) 

(ii) ○—●···●—○, ●—●···●—○; *and*

(iii) ○—○, ○.

In the subsequent discussion, we will consider x-graphs up to isomorphism. It is implicit that such an isomorphism will preserve vertex colours. We also fix $t = (1, 2) \cdots (2m - 1, 2m) \in G$.

Bates, Bundy, Perkins and Rowley's interest in x-graphs stemmed from the following elementary result.

Lemma 4.1.2. *(i) Every graph with b black vertices of valency at most two, w white vertices of valency at most one and exactly b edges is the x-graph for some $x \in X$ (with $m = b$ and $n = 2b + w$).*

(ii) Let $x, y \in X$. Then x and y are in the same $C_G(t)$-orbit if and only if $\mathcal{G}_x$ and $\mathcal{G}_y$ are isomorphic graphs.

Proof. See Lemma 2.1 of [BBPR03b]. □

Part (i) of Lemma 4.1.2 is of particular interest, as it confirms that when employing a combinatorial approach using the connected components of x-graphs, we must consider all possible connected components given in Lemma 4.1.1. This approach will be used repeatedly in the proof of Theorem 4.0.1.

An immediate consequence of the definition of $\mathcal{G}_x$ is that the number of black vertices is equal to the number of edges. Consequently the number of connected components of the form ●—●···●—● containing at least one black vertex must be equal to the number of connected components of the form ○—●···●—○ and ○—○.

Lemma 4.1.1 allows a combinatorial approach to be used when considering conjugate involutions. Indeed, given a connected component C_i of $\mathcal{G}_x$, we may define Ω_i to be the union of all vertices of C_i. We may then define the i-part of t, denoted t_i, to be the product of those transpositions of t that occur in $\mathrm{Sym}(\Omega_i)$. We define x_i similarly. By analysing the structure of the connected components given in Lemma 4.1.1 it is possible to relate the order of tx to the x-graph $\mathcal{G}_x$.

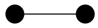
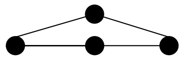
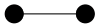
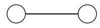
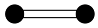
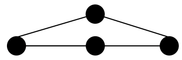
Lemma 4.1.3. *Suppose that $x \in X$ and that $C_1, \ldots, C_k$ are the connected components of $\mathcal{G}_x$. Denote the number of black vertices, white vertices and cycles in C_i by b_i, w_i and c_i respectively. Then*

(i) the order of tx is the least common multiple of the orders of $t_i x_i$ (for $i = 1, \ldots, k$); and

(ii) the order of $t_i x_i$ is $(2b_i + w_i)/(1 + c_i)$ for each $i = 1, \ldots, k$.

Proof. See Proposition 2.2 of [BBPR03b]. □

We have the following immediate corollary to Lemmas 4.1.1 and 4.1.3.

Corollary 4.1.4. *For $\mathcal{P}_{\{4\}}(G, X)$ the disc $\Delta_1(t)$ consists of all $x \in X$ whose x-graphs have at least one connected component of the form* ,  *or*  *and all other components have the form* , , , , , ,  *or* .

Proof. The element x lies in $\Delta_1(t)$ precisely when tx has order 4. The result then follows from Lemmas 4.1.1 and 4.1.3. $\square$

We conclude this section by noting that we can define an x-graph for any two (not-necessarily conjugate) involutions. This we will do frequently in Section 4.2. However, in such a situation it is no longer the case that the number of edges of $\mathcal{G}_x$ is equal to the number of black vertices.

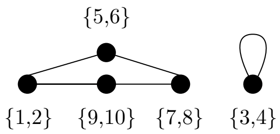
4.2 Proof of Theorem 4.0.1

In this section, we prove Theorem 4.0.1. Note that for $m \geqslant 2$ and $t = (1, 2) \cdots (2m-1, 2m)$, the involution $x = (1, 3)(2, 4)(5, 6) \cdots (2m-1, 2m) \in X$ satisfies $d(t, x) \geqslant 2$. Thus it suffices to prove when $\mathcal{P}_{\{4\}}(G, X)$ is connected, that for all $x \in X$ we have $d(t, x) \leqslant 2$. To do this we consider pairs or triples of connected components C_i, C_j and C_k of $\mathcal{G}_x$ and the corresponding parts t_i, t_j, t_k, x_i, x_j, x_k of t and x. We then construct an element $y_{ijk} \in H$, where $H := \mathrm{Sym}(\mathrm{supp}(t_i, t_j, t_k, x_i, x_j, x_k))$, which is H-conjugate to $t_i t_j t_k$ and such that the x-graphs $\mathcal{G}_{y_{ijk}}^{t_i t_j t_k}$ and $\mathcal{G}_{x_i x_j x_k}^{y_{ijk}}$ have connected components featuring in Corollary 4.1.4.

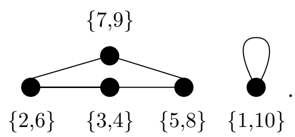
We begin by proving a few preliminary results, dealing with the case $n = 2m$.

Lemma 4.2.1. *Let $m \geqslant 5$, $n = 2m$ and suppose that $x \in X$ is such that $\mathcal{G}_x$ is connected. Then there exists $y \in X$ such that $d(t, y) = d(y, x) = 1$.*

Proof. Without loss of generality we may assume that $x = (1, 2m)(2, 3) \cdots (2m-2, 2m-1)$. If $m = 5$, then taking $y = (1, 10)(2, 6)(3, 4)(5, 8)(7, 9)$ we see that $\mathcal{G}_y$ and $\mathcal{G}_x^y$ are given respectively by

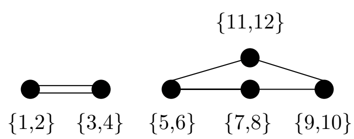


and

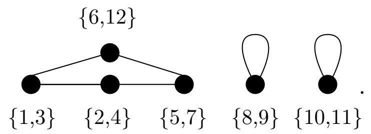


.

If $m = 6$, we take $y = (1, 3)(2, 4)(5, 7)(6, 12)(8, 9)(10, 11)$. Then $\mathcal{G}_y$ and $\mathcal{G}_x^y$ are, respectively



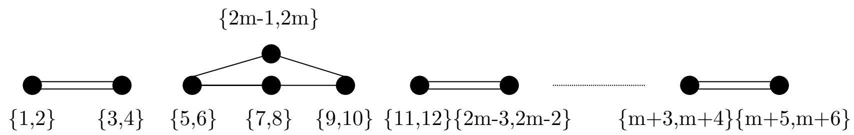
and



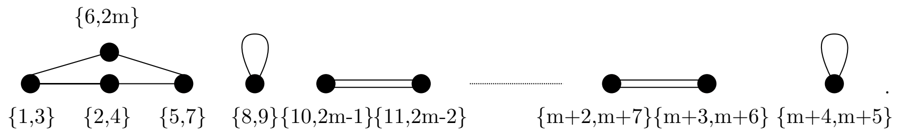
In the general case when $m \geqslant 7$, we take

$$y = (1,3)(2,4)(5,7)(6,2m)(8,9)(10,2m-1)(11,2m-2)\cdots(m+4,m+5).$$

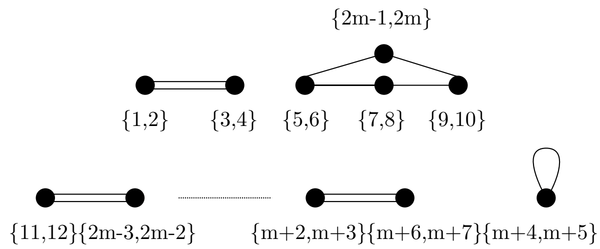
The exact nature of the associated x-graphs is dependent on the parity of m. If m is even, then $\mathcal{G}_y$ is given by



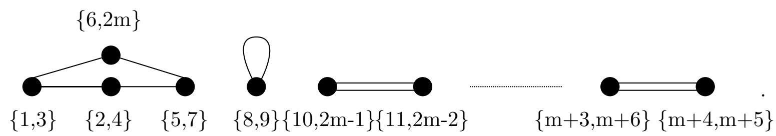
and $\mathcal{G}_x^y$ is given by



If m is odd, the graphs $\mathcal{G}_y$ and $\mathcal{G}_x^y$ are, respectively



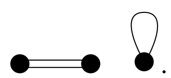
and



In all cases, the given graphs satisfy the conditions of Corollary 4.1.4, whence $d(t,y) = d(y,x) = 1$. □

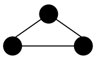
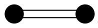
The proof of Lemma 4.2.1 illustrates a general feature that the actual x-graphs constructed may vary depending on the parity and values of the given parameters (such as the parameter m above). However, in using Corollary 4.1.4 we are only interested in the connected components of the x-graph. Thus for the sake of brevity, in all future proofs we will only describe the connected components of each x-graph, relegating the x-graphs themselves to Appendix B.

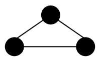
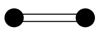
Lemma 4.2.2. *Suppose that $m = 3$, $n = 6$ and $x \in X$. If $\mathcal{G}_x$ is connected, then there exists $y \in X$ such that the x-graphs $\mathcal{G}_y$ and $\mathcal{G}_x^y$ are isomorphic to*

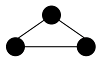


Proof. Without loss of generality, we may assume that $x = (1,6)(2,3)(4,5)$. Then $y = (1,2)(3,6)(4,5)$ is the required element. $\square$

Lemma 4.2.3. *Let $m = 4$ and $n = 8$. Suppose that $x \in X \backslash \{t\}$ has a disconnected x-graph, $\mathcal{G}_x$. Then there exists $y \in X$ such that $d(t,y) = d(y,x) = 1$.*

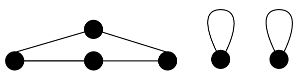
Proof. If $\mathcal{G}_x$ has a connected component of the form , then we may assume that $x = (1,6)(2,3)(4,5)(7,8)$. The element $y = (1,8)(2,4)(3,6)(5,7)$ is then the desired y. The other possibilities occur when $\mathcal{G}_x$ has one or two connected components of the form , corresponding respectively to $x = (1,3)(2,4)(5,6)(7,8)$ and $x = (1,3)(2,4)(5,7)(6,8)$. The y satisfying the lemma for both such x is $y = (1,8)(2,3)(4,5)(6,7)$. $\square$

Lemma 4.2.4. *Suppose that $m \geqslant 5$, $n = 2m$ and that $\mathcal{G}_x$ consists entirely of components of the form  and . Then there exists $y \in X$ such that $d(t,y) = d(y,x) = 1$.*

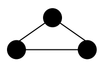
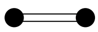
Proof. We consider three separate cases. First assume that $\mathcal{G}_x$ contains at least two components, $\mathcal{C}_i$ and $\mathcal{C}_j$, of the form . Without loss, we may take $t_i = (1,2)(3,4)(5,6)$, $t_j = (7,8)(9,10)(11,12)$, $x_i = (1,6)(2,3)(4,5)$ and $x_j = (7,12)(8,9)(10,11)$. Defining $y_{ij} \in \mathrm{Sym}(\mathrm{supp}(t_i,t_j,x_i,x_j))$ to be

$$y_{ij} = (1,2)(3,4)(5,7)(6,12)(8,9)(10,11),$$

we see that both $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ are isomorphic to

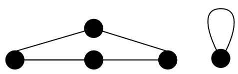
.

Denote the remaining parts of t and x by t_k and x_k. Applying Lemmas 4.2.2 and 4.2.3 to t_k and x_k produces an element $y_k \in \mathrm{Sym}(\mathrm{supp}(t_k,x_k))$ such that $y := y_{ij}y_k$ is the desired element of X.

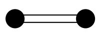
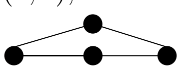
In the case that $\mathcal{G}_x$ contains a unique component, $\mathcal{C}_i$, of the form , then there exists at least one component, say $\mathcal{C}_j$, of the form . Taking $t_i = (1,2)(3,4)(5,6)$, $t_j = (7,8)(9,10)$, $x_i = (1,6)(2,3)(4,5)$ and $x_j = (7,9)(8,10)$, then the element $y_{ij} \in \mathrm{Sym}(\mathrm{supp}(t_i,t_j,x_i,x_j))$ given by

$$y_{ij} = (1,2)(3,10)(4,6)(5,8)(7,9)$$

results in x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ which are isomorphic to

.

Denoting the remaining part of t by t_k and setting $y := y_{ij}t_k \in X$ we have that $d(t,y) = d(y,x) = 1$ as required.

Finally, assume that all connected components of $\mathcal{G}_x$ are of the form  and let $\mathcal{C}_i$ and $\mathcal{C}_j$ be two such components. Without loss of generality we have that $t_i = (1,2)(3,4)$, $t_j = (5,6)(7,8)$, $x_i = (1,3)(2,4)$ and $x_j = (5,7)(6,8)$, and as above denote the remaining part of t by t_k. Setting $y_{ij} = (1,5)(2,7)(3,8)(4,6)$, we see that both $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ consist of a single connected component of isomorphism type . Hence, $y := y_{ij}t_k$ is our desired element of X. $\square$

Lemmas 4.2.1, 4.2.2, 4.2.3 and 4.2.4 combine to prove Theorem 4.0.1 in the case when $n = 2m$.

Corollary 4.2.5. *If $n = 2m$, then Theorem 4.0.1 holds.*

Proof. Let $x \in X$. If $\mathcal{G}_x$ has connected components containing precisely 4 black vertices then we leave the parts of t and x corresponding to such components alone. We then apply Lemma 4.2.1 to any connected component containing at least 5 black vertices, and Lemma 4.2.2 to any connected component containing 3 black vertices to obtain the desired result. Otherwise all connected components have at most 3 black vertices. Applying Lemmas 4.2.3 and 4.2.4 to a pair of components containing a total of 4, 5 or 6 black vertices, Lemma 4.2.2 to any remaining connected components containing 3 black vertices, and leaving all other connected components invariant gives the result. □

Before presenting the proof of Theorem 4.0.1 we give a further three intermediate results.

Lemma 4.2.6. *Let $x \in X$. Suppose that $\mathcal{G}_x$ has connected components C_i and C_j of the given forms. Then there exists $y_{ij} \in H$, where $H := \mathrm{Sym}(\mathrm{supp}(t_i, t_j, x_i, x_j))$, which is H-conjugate to t_it_j and such that the connected components of the x-graphs $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ satisfy the conditions of Corollary 4.1.4.*

(i) C_i: $\bullet - \bullet \cdots \bullet - \bullet$ *(with $q \geqslant 3$ black vertices),*

C_j: $\circ - \bullet \cdots \bullet - \circ$ *(with $r \geqslant 0$ black vertices);*

(ii) C_i: $\circ - \bullet \cdots \bullet - \circ$ *(with $q \geqslant 2$ black vertices), $C_j : \varnothing$;*

(iii) C_i: $\bullet - \bullet \cdots \bullet - \circ$ *(with $q \geqslant 2$ black vertices), C_j: $\circ$;*

(iv) C_i: $\bullet - \bullet \cdots \bullet - \circ$ *(with $q \geqslant 1$ black vertices),*

C_j: $\circ - \bullet \cdots \bullet - \circ$ *(with $r \geqslant 1$ black vertices); and*

(v) C_i and C_j are both of the form $\bullet - \bullet \cdots \bullet - \circ$ *(with $q, r \geqslant 1$ black vertices respectively).*

Proof. For each case, without loss of generality we give explicit formulations of the t_i and x_i. For ease of notation, where parameters q and r have been defined we set $v = 2(q + r)$.

In case (i) assume that $t_i = (1,2)\cdots(2q-1,2q)$, $t_j = (2q+1,2q+2)\cdots(v-1,v)(v+1)(v+2)$, and $x_i = (1)(2,3)\cdots(2q-2,2q-1)(2q)$. We now consider three possibilities. If $r = 0$, then we may assume that $x_j = (2q+1,2q+2)$ and we take

$$y_{ij} := (1,2q)(2)(2q-1)(3,2q-2)\cdots(q,q+1)(2q+1,2q+2).$$

If $r = 1$, then taking $x_j = (2q+1,v+1)(2q+2,v+2)$ we define

$$y_{ij} := (1,2q)(2)(2q-1)(3,2q-2)\cdots(q,q+1)(v+1,v+2)(2q+1,2q+2).$$

Finally, if $r > 1$, then we assume that $x_j = (2q+1,v+1)(2q+2,2q+3)\cdots(v-2,v-1)(v,v+2)$ and define

$$y_{ij} := (1,2q)(2)(2q-1)(3,2q-2)\cdots(q,q+1)(v+1,v+2)$$
$$(2q+1,v)\cdots(2q+r,2q+r+1).$$

We see that the x-graph $\mathcal{G}^{t_it_j}_{y_{ij}}$ has connected components of the form , and - depending on the values of q and r - also and . Similarly $\mathcal{G}^{y_{ij}}_{x_ix_j}$ has connected components of the form , and in some cases also and as required (see Figures B.1-B.8).

For (ii) we may set $t_i = (1,2)\cdots(2q-1,2q)(2q+1)(2q+2)$ and $x_i = (1,2q+2)(2,3)\cdots(2q,2q+1)$. Then the element

$$y_i = (1)(2,2q-1)(3,2q-2)\cdots(q,q+1)(2q)(2q+1,2q+2)$$

results in the x-graph $\mathcal{G}^{t_i}_{y_i}$ having connected components , and in some cases and - depending on the value and parity of q. The graph $\mathcal{G}^{y_i}_{x_i}$ has connected components and possibly and (see Figures B.9 and B.10).

Considering case (iii), we take $t_i = (1,2)\cdots(2q-1,2q)(2q+1)$, $x_i = (1)(2,3)\cdots(2q,2q+1)$ and $t_j = x_j = (2q+2)$. If $q=2$, define

$$y_{ij} = (1,4)(2)(3)(5,6),$$

whilst if $q \geqslant 3$ define

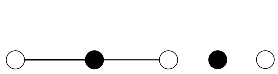
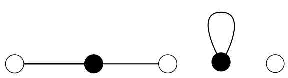
$$y_{ij} = (1,2q)(2)(2q-1)(3,2q-2)\cdots(q,q+1)(2q+1,2q+2).$$

Then the permutation y_{ij} gives the desired x-graphs. Indeed, $\mathcal{G}^{t_it_j}_{y_{ij}}$ has connected components of the form and , $\mathcal{G}^{y_{ij}}_{x_ix_j}$ has components of the form and (with the black vertex omitted if $q=2$) and both x-graphs may also have connected components of the form and depending on the value and parity of q (see Figures B.11-B.13).

Turning to (iv), if $q=1$, then without loss of generality we have that $t_i = (1,2)(v+1)$, $t_j = (3,4)\cdots(v-1,v)(v+2)(v+3)$, $x_i = (1)(2,v+1)$ and $x_j = (3,v+2)(4,5)\cdots(v-2,v-1)(v,v+3)$ (take $x_j = (3,6)(4,7)$ if $r=1$). When $r=1$, define

$$y_{ij} = (1)(2)(3,5)(4,7)(6).$$

The x-graphs $\mathcal{G}^{t_it_j}_{y_{ij}}$ and $\mathcal{G}^{y_{ij}}_{x_ix_j}$ are isomorphic to

 and 

respectively as required. If $r>1$, then

$$y_{ij} = (1)(2)(3,v-2)\cdots(r+1,r+2)(v-1,v+2)(v,v+1)(v+3)$$

is our desired element. Indeed in this case $\mathcal{G}^{t_it_j}_{y_{ij}}$ has connected components of the form , and in addition to components of the form and/or (depending on the value of

r), whilst $\mathcal{G}_{x_ix_j}^{y_{ij}}$ has components of the forms ○ and ○——●——○ in addition to components of the form  and/or ●═══● (depending on the value of r - see Figures B.14-B.16 for full details).

If $q > 1$, then we define $t_i = (1,2)\cdots(2q-1,2q)(v+1)$, $t_j = (2q+1,2q+2)\cdots(v-1,v)(v+2)(v+3)$, $x_i = (1)(2,3)\cdots(2q-2,2q-1)(2q,v+1)$ and $x_j = (2q+1,v+2)(2q+2,2q+3)\cdots(v-2,v-1)(v,v+3)$. Our desired element is then

$$y_{ij} = (1)(2,2q-1)\cdots(q,q+1)(2q)(2q+1,v-2)\cdots$$
$$\cdots(2q+r-1,2q+r)(v-1,v+2)(v,v+1)(v+3).$$

It follows that the connected components of $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ have the form ○——●——○ and ○ and possibly ●═══● and , whilst $\mathcal{G}_{y_{ij}}^{t_it_j}$ has an additional connected component of the form ●——● (see Figures B.17-B.20).

For case (v) we assume without loss of generality that $q \geqslant r$. We consider the subcases $q = r = 1$, $q > r = 1$, $q = r > 1$ and $q > r > 1$ in turn. If $q = r = 1$, then we take $t_i = (1,2)(5)$, $t_j = (3,4)(6)$, $x_i = (1)(2,5)$ and $x_j = (3)(4,6)$. Defining

$$y_{i,j} = (1,3)(5,6)(2)(4),$$

we see that $\mathcal{G}_{y_{ij}}^{t_it_j}$ has isomorphism type ●——● ○——○ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ has isomorphism type ○——●——○ ● as required (see Figure B.21).

If $q > r = 1$, then setting $t_i = (1,2)\cdots(2q-1,2q)(v+1)$, $t_j = (2q+1,2q+2)(v+2)$, $x_i = (1)(2,3)\cdots(2q,v+1)$ and $x_j = (2q+1)(2q+2,v+2)$ we define

$$y_{ij} = (1{,}2(q-1))(2,2(q-1)-1)\cdots(q-1,q)(2q-1,2q+1)$$
$$(2q)(2q+2)(v+1,v+2).$$

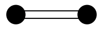
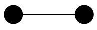
Consequently $\mathcal{G}_{y_{ij}}^{t_it_j}$ has connected components of the form ●——●, ○——○ and ●═══● and/or  (depending on the value of q), whilst $\mathcal{G}_{x_ix_j}^{y_{ij}}$ has components of the form ○——●——○, ●——● and ●═══● and/or  (see Figures B.22 and B.23).

When $r > 1$ we may assume that $t_i = (1,2)\cdots(2q-1,2q)(v+1)$, $t_j = (2q+1,2q+2)\cdots(v-1,v)(v+2)$, $x_i = (1)(2,3)\cdots(2q-2,2q-1)(2q,v+1)$ and $x_j = (2q+1)(2q+2,2q+3)\cdots(v-2,v-1)(v,v+2)$. Define

$$y_{ij} = (1,2q+1)(2,2q+2)\cdots(2q-1,v-1)(2q)(v)(v+1,v+2)$$

if $q = r$ and

$$y_{ij} = (1,2(q-r))(2,2(q-r)-1)\cdots(q-r,q-r+1)$$
$$(2(q-r)+1,2q+1)\cdots(2q-1,v-1)(2q)(v)(v+1,v+2)$$

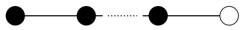
if $q \neq r$. We see that $\mathcal{G}_{y_{ij}}^{t_it_j}$ has connected components ○——○, ●——● and ●═══● in addition to  for some values of q and r. The x-graph $\mathcal{G}_{x_ix_j}^{y_{ij}}$ also has the desired properties having a component of the form , some components of the form  and possibly also ,  and/or  depending on the values of q and r and the parity of $q-r$ (see Figures B.24-B.26).

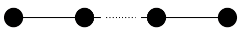
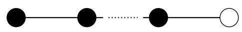
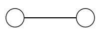
□

We note that in cases (ii) and (iv) above, t_it_j and x_ix_j have different cycle types. This is a fact which we will utilise in the proof of Theorem 4.0.1

In a similar vein to Lemma 4.2.6 we next consider collections of three connected components simultaneously.

Lemma 4.2.7. *Let $x \in X$. Suppose that $\mathcal{G}_x$ has connected components C_i, C_j and C_k of the given forms and define $H := \mathrm{Sym}(\mathrm{supp}(t_i,t_j,t_k,x_i,x_j,x_k))$. Then there exists $y_{ijk} \in H$ which is H-conjugate to $t_it_jt_k$ and such that the connected components of the x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ satisfy the conditions of Corollary 4.1.4:*

(i) C_i, C_j and C_k are each of the form  (having $q,r,s \geqslant 1$ black vertices respectively); and

(ii) C_i:  (with $q \geqslant 1$ black vertices),
C_j:  (with $r \geqslant 1$ black vertices), C_k: .

Proof. We follow the approach of the proof of Lemma 4.2.6 and construct the appropriate t_i and x_i. We also set $v = 2(q+r)$ and $w = 2(q+r+s)$.

For case (i) we may assume without loss of generality that $q \geqslant r \geqslant s \geqslant 1$ and set

$$t_i = (1,2)\cdots(2q-1,2q)(w+1),\ t_j = (2q+1,2q+2)\cdots(v-1,v)(w+2)$$
$$\text{and} \quad t_k = (v+1,v+2)\cdots(w-1,w)(w+3).$$

We also set

$$\begin{aligned} x_i &= (1)(2,3)\cdots(2q-2,2q-1)(2q,w+1), \\ x_j &= (2q+1)(2q+2,2q+3)\cdots(v-2,v-1)(v,w+2), \text{ and} \\ x_k &= (v+1)(v+2,v+3)\cdots(w-2,w-1)(w,w+3), \end{aligned}$$

taking $x_i = (1)(2,w+1)$ in the case when $q=1$, $x_j = (2q+1)(2q+2,w+2)$ when $r=1$ and $x_k = (v+1)(v+2,w+3)$ when $s=1$. There are three subcases to consider. If $q=r=s=1$, then taking $y_{ijk} = (1,4)(5,8)(6,7)(2)(3)(9)$ we see that the x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ are both isomorphic to

,

which has the desired form (see Figure B.27). If $s=1$, but $q \neq 1$, we set

$$y_{ijk} = (1,w+2)(2,w+1)(3,2q)(4,2q-1)\cdots(q+1,q+2)$$
$$(2q+1,v)\cdots(2q+r,2q+r+1)(v+1)(w)(w+3),$$

whilst if $s > 1$, we define

$$\begin{aligned} y_{ijk} = {} & (1, w+2)(2, w+1)(3, 2q)(4, 2q-1)\cdots(q+1, q+2) \\ & (2q+1, v)\cdots(2q+r, 2q+r+1) \\ & (v+2, w-1)\cdots(v+s, v+s+1)(v+1)(w)(w+3). \end{aligned}$$

It follows that the x-graph $\mathcal{G}_{y_{ijk}}^{t_i t_j t_k}$ has connected components of the form and ○ with further components of the form ● (if $s = 1$), (if $s > 1$) and and/or depending on the values of q, r and s. Meanwhile, $\mathcal{G}_{x_i x_j x_k}^{y_{ijk}}$ has connected components of the form , and ○, in addition to and/or depending on the parameters q, r, s (see Figures B.28-B.39 for full details).

Finally, we consider case (ii) and note that in this case $w = v+2$. Assume that $t_i = (1, 2)\cdots(2q-1, 2q)$, $t_j = (2q+1, 2q+2)\cdots(v-1, v)(v+1)$, $t_k = (v+2)(v+3)$, $x_i = (1)(2, 3)\cdots(2q-2, 2q-1)(2q)$, $x_j = (2q+1)(2q+2, 2q+3)\cdots(v, v+1)$ and $x_k = (v+2, v+3)$. If $q = r = 1$, then $x = (1)(2)(3)(4, 5)(6, 7)$, and defining

$$y_{ijk} = (1, 3)(2)(4)(5, 6)(7)$$

we see that the x-graphs $\mathcal{G}_{y_{ijk}}^{t_i t_j t_k}$ and $\mathcal{G}_{x_i x_j x_k}^{y_{ijk}}$ are of isomorphism type

and

respectively (see Figure B.40). Meanwhile, if $q = 1$ and $r > 1$, then $x = (1)(2)(3)(4, 5)\cdots(v, v+1)(v+2, v+3)$, and so setting

$$y_{ijk} = (1)(2, v-3)\cdots(r, r+1)(v-2)(v-1, v+2)(v, v+1)(v+3)$$

results in the x-graph $\mathcal{G}_{y_{ijk}}^{t_i t_j t_k}$ having connected components , , ○ and possibly and . Moreover, $\mathcal{G}_{x_i x_j x_k}^{y_{ijk}}$ has connected components ○, , and for some values of r also and/or (see Figures B.41 and B.42).

If $q = 2$, then we take

$$y_{ijk} = (1, v+1)(2, v+2)(3)(4)(2q+1, v)\cdots(2q+r, 2q+r+1)(v+3),$$

whilst if $q > 2$ we take

$$\begin{aligned} y_{ijk} = {} & (1, v+1)(2, v+2)(3)(4, 2q-1)\cdots(q+1, q+2)(2q) \\ & (2q+1, v)\cdots(2q+r, 2q+r+1)(v+3). \end{aligned}$$

Consequently, the x-graph $\mathcal{G}_{y_{ijk}}^{t_i t_j t_k}$ has connected components and ○ and possibly also ●, , and . Meanwhile, $\mathcal{G}_{x_i x_j x_k}^{y_{ijk}}$ has connected components , and ○ and in some cases also and/or (see Figures B.43-B.48 for full details). $\square$

Lemma 4.2.8. *Suppose that $m \geqslant 2$, $n \geqslant 7$ with $n \neq 2m+1$ and let $x \in X\backslash\{t\}$ be such that* $\mathrm{fix}(t) = \mathrm{fix}(x)$*. Then there exists $y \in X$ such that $d(t, y) = d(y, x) = 1$.*

Proof. By considering $t, x \in \mathrm{Sym}(\mathrm{supp}(t))$ and appealing to Corollary 4.2.5 we may assume that $m = 2$ or 3 and hence that $|\,\mathrm{fix}(t)| \geqslant 2$. If $\mathcal{G}_x$ contains a connected component of the form ●═●, then without loss of generality we have

$$t = (1,2)(3,4)(5,6)(7)\cdots(n) \quad \text{and} \quad x = (1,3)(2,4)(5,6)(7)\cdots(n)$$

(where the transposition $(5,6)$ is replaced by $(5)(6)$ if $m = 2$). We take $y = (1,4)(2)(3)(5,6)(7)\cdots(n-2)(n-1,n)$ (again replacing $(5,6)$ by $(5)(6)$ if $m = 2$). If $m = 3$ and $\mathcal{G}_x$ contains a cycle of three black vertices, then we may assume that

$$t = (1,2)(3,4)(5,6)(7)\cdots(n) \quad \text{and} \quad x = (1,6)(2,3)(4,5)(7)\cdots(n).$$

In this case, we set $y = (1)(2,3)(4)(5,6)(7,8)(9)\cdots(n)$. In all cases we have that $\mathcal{G}_y$ has one connected component of the form ●—●, $\mathcal{G}_x^y$ has one connected component of the form ○—●—○ and all other connected components of these x-graphs are of the form ○, ●, ○—○ and  (see Figures B.49-B.51). Thus $d(t, y) = d(y, x) = 1$ by Corollary 4.1.4. □

We are now in a position to prove Theorem 4.0.1 in the general case. For $x \in X$ we proceed by considering collections of connected components $\{C_i\}_{i \in I}$ of $\mathcal{G}_x$ for some set I, and then finding an element $y_I \in \mathrm{Sym}(\cup_{i \in I}\mathrm{supp}(C_i))$ that is conjugate to $t_I := \prod_{i \in I} t_i$ such that the connected components of $\mathcal{G}_{y_I}^{t_I}$ and $\mathcal{G}_{x_I}^{y_I}$ satisfy the conditions of Corollary 4.1.4 (where $x_I := \prod_{i \in I} x_i$). The product of all such y_I will then be our desired element of X.

Proof of Theorem 4.0.1:

Let $x \in X$.

(i) First assume that $n = 2m+1$. We observe that the product of two elements of X that fix distinct elements of Ω cannot have order 4. Thus $\mathcal{P}_{\{4\}}(G, X)$ consists of n copies of the $\{4\}$-product involution graph $\mathcal{P}_{\{4\}}(\mathrm{Sym}(2m), Y)$, where Y is the conjugacy class of $\mathrm{Sym}(2m)$ consisting of elements of cycle type 2^m.

If $m = 1$, then $\mathcal{P}_{\{4\}}(G, X)$ is clearly totally disconnected.

In the case that $(n, m) = (4, 2)$ (respectively $(n, m) = 6, 3)$), then any x-graph will contain 2 (respectively 3) black vertices and 2 (respectively 3) edges. It follows from Corollary 4.1.4 that $\mathcal{P}_{\{4\}}(G, X)$ is totally disconnected.

(ii) Assume that $m \neq 1$ and $(n, m) \neq (4, 2), (6, 3)$ or $(2m+1, m)$. We first consider the case that $(n, m) = (6, 2)$. If distinct involutions $t = (1,2)(3,4)$ and x of cycle type 2^2 do not have product of order 4, then the reader may check that the x-graph $\mathcal{G}_x$ will be isomorphic to one given in Table 4.1 and that the given element y satisfies $d(t, y) = d(y, x) = 1$. Hence $\mathcal{P}_{\{4\}}(G, X)$ is connected and $\mathrm{Diam}(\mathcal{P}_{\{4\}}(G, X)) = 2$.

By Corollary 4.2.5, as $n \neq 2m+1$, we may assume that $|\,\mathrm{fix}(t)| \geqslant 2$, and so $\mathcal{G}_x$ contains at least 2 white vertices. Moreover, by Lemma 4.2.8 we only need to consider the case when $\mathrm{fix}(t) \neq \mathrm{fix}(x)$.

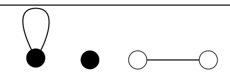
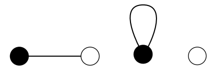
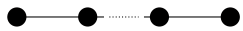
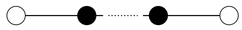
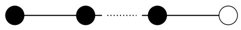
Representative $x \in X \backslash (\Delta_1(t) \cup \{t\})$	x-graph, $\mathcal{G}_x$	Representative $y \in \Delta_1(t) \cap \Delta_1(x)$
$(1,2)(5,6)$		$(3,6)(4,5)$
$(2,5)(4,6)$		$(1,3)(5,6)$
$(2,3)(4,5)$		$(1,4)(5,6)$
$(2,5)(3,4)$		$(1,5)(2,6)$
$(1,3)(2,4)$		$(1,4)(5,6)$

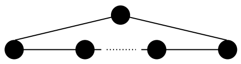
Table 4.1: Representatives of $X\backslash(\Delta_1(t) \cup \{t\})$ and their corresponding neighbours in $\Delta_1(t) \cap \Delta_1(x)$ for $n = 6$, $m = 2$.

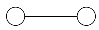
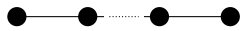
Let α, β, γ and δ denote the number of connected components (containing at least 1 black vertex and 1 edge) of $\mathcal{G}_x$ of the form

α: ;

β: ;

γ: ; and

δ: ,

and let ϵ denote the number of connected components of the form . For ease of reading, we shall refer to components of type α instead of components of the form . Similarly for β, γ, δ and ϵ. Note that $\alpha \leqslant \beta + \epsilon$, and as $\text{fix}(t) \neq \text{fix}(x)$ it follows that β, γ and ϵ are not all zero.

If $\gamma \geqslant 2$, then partitioning the components of type γ into pairs or triples we obtain a suitable y_I from Lemmas 4.2.6(v) and 4.2.7(i). Indexing the remaining connected components by J, a suitable y_J such that $t_J y_J$ and $x_J y_J$ have orders 1, 2 or 4 may be constructed using Lemmas 4.2.1, 4.2.2, and 4.2.6(i),(ii). In the forthcoming cases, when referring to the construction of y_J, it will be implicit that $t_J y_J$ and $x_J y_J$ have orders 1, 2 or 4.

If $\gamma = 1$ and $\beta \neq 0$, then we pair the unique component of type γ with one of type β to obtain an element y_I via Lemma 4.2.6(iv). In the case that $\alpha = \beta + \epsilon$, then consider one component of type α. Without loss we may assume that this corresponds to some $t_i = (1,2)\cdots(2r-1,2r)$ and some $x_i = (1)(2,3)\cdots(2r-2,2r-1)(2r)$. Setting $y_i = (1,2)(3,2r)(4,2r-1)(5,2r-2)\cdots(r+1,r+2)$ we have that $\text{ord}(t_i y_i), \text{ord}(y_i x_i) \in \{1,2,4\}$. An element y_J for the remaining components follows from Lemmas 4.2.1, 4.2.2 and 4.2.6(i),(ii).

If $\gamma = 1$, $\beta = 0$ and $\alpha \geqslant 1$, then $\epsilon \geqslant 1$. Hence we may use Lemma 4.2.7(ii) to construct the element y_I and Lemmas 4.2.1, 4.2.2 and 4.2.6(i) to obtain a suitable y_J.

If $\gamma = 1$ and $\alpha = \beta = 0$, then there must be a connected component of $\mathcal{G}_x$ consisting of a single vertex. Assume first that the connected component of type γ contains at least two black vertices. If there is an isolated white vertex in $\mathcal{G}_x$, then the existence of y_I follows from Lemma 4.2.6(iii). Conversely, if there is an isolated black vertex, then - as the number of black vertices equals the number of edges - there must be a connected component of type ϵ. Applying Lemma 4.2.7(ii) to this connected component, the connected component of type γ and an isolated black vertex results in our element y_I. Applying Lemmas 4.2.1 and 4.2.2 to our remaining components as appropriate gives our desired element y_J.

Now assume that our connected component of type γ contains precisely one black vertex. If all other white vertices are isolated, then $\mathcal{G}_x$ contains a connected component consisting of a cycle of $u \geqslant 1$ black vertices. We may consider one such cycle, an isolated white vertex and the connected component of type γ to correspond to those components indexed by I. Thus without loss of generality

$$\begin{aligned} t_I &= (1,2)\cdots(2u-1,2u)(2u+1,2u+2)(2u+3)(2u+4); \quad \text{and} \\ x_I &= (1,2u)(2,3)\cdots(2u-2,2u-1)(2u+1)(2u+2,2u+3)(2u+4), \end{aligned} \tag{4.2.1}$$

unless $u = 1$ in which case we let $t_I = (1,2)(3,4)(5)(6)$ and $x_I = (1,2)(3)(4,5)(6)$. Taking $y_I = (1)(2u)(2,2u-1)\cdots(u,u+1)(2u+1,2u+3)(2u+2,2u+4)$ (or $y_I = (1)(2)(3,5)(4,6)$ if $u = 1$) it follows that $\mathcal{G}_{y_I}^{t_I}$ has one connected component of the form ○—●—○, $\mathcal{G}_{x_I}^{y_I}$ has one connected component of the form ●—● and the remaining components of these x-graphs are of the form ●, ●, ●—●, ○—○ and ●═●, thus satisfying the conditions of Corollary 4.1.4 (see Figures B.52-B.54).

Conversely, if there exists a white vertex that is not isolated, then it will be in a component of type ϵ. Again, as the number of edges and black vertices must be equal, there exists an isolated black vertex. We take the connected component of type γ along with one of type ϵ and an isolated black vertex to be those indexed by I. The existence of y_I then follows from Lemma 4.2.7(ii). Finally applying Lemmas 4.2.1 and 4.2.2 as appropriate to the remaining connected components, we obtain an element y_J as required.

If $\gamma = 0$, but $\beta \neq 0$, then consider the connected components of type α. If there exist connected components of type α containing at least 3 black vertices, then we may pair these up with connected components of type β and ϵ and apply Lemma 4.2.6(i) to obtain our element y_I. If all connected components of type α contain at most 2 black vertices, then we simply apply Lemma 4.2.6(ii) (if required) to the connected components of type β to obtain y_I. Finally, applying Lemmas 4.2.1 and 4.2.2 to the remaining connected components ensures the existence of y_J.

If $\beta = \gamma = 0$ and $\alpha \neq 0$, then we apply Lemma 4.2.6(i) to the connected components of type α and ϵ (if required) to obtain y_I and Lemmas 4.2.1 and 4.2.2 to the remaining connected components to find a suitable y_J.

If $\alpha = \beta = \gamma = 0$, then $\epsilon \geqslant 1$ as by assumption $\mathrm{fix}(t) \neq \mathrm{fix}(x)$. As the number of edges of $\mathcal{G}_x$ equals the number of black vertices, there exists an isolated black vertex. Moreover, as $m \geqslant 2$, there are two possible cases. If every black vertex is isolated, then there exists m connected components of type ϵ. Take two such components and two isolated black vertices as the components corresponding to our indexing set I, and leave all other components of type ϵ alone. Without loss of generality, we may assume that the parts of t and x corresponding to I are

$$t_I = (1,2)(3,4)(5)(6)(7)(8) \quad \text{and} \quad x_I = (1)(2)(3)(4)(5,6)(7,8). \tag{4.2.2}$$

Let $y_I = (1,5)(2,6)(3)(4)(7)(8)$. Then the connected components of $\mathcal{G}_{y_I}^{t_I}$ and $\mathcal{G}_{x_I}^{y_I}$ satisfy the conditions of Corollary 4.1.4 (see Figure B.55).

Conversely, if there is only one isolated black vertex, then there exists a connected component which is a cycle of $u \geqslant 1$ black vertices. Thus taking the components indexed by I to be an isolated black vertex, a cycle of $u \geqslant 1$ black vertices and a component of type ϵ, and leaving all other components of type ϵ alone, we may take t_I and x_I to be

$$\begin{aligned} t_I &= (1,2)\cdots(2u-1,2u)(2u+1,2u+2)(2u+3)(2u+4); \quad \text{and} \\ x_I &= (1,2u)(2,3)\cdots(2u-2,2u-1)(2u+1)(2u+2)(2u+3,2u+4), \end{aligned} \tag{4.2.3}$$

or $t_I = (1,2)(3,4)(5)(6)$ and $x_I = (1,2)(3)(4)(5,6)$ if $u = 1$. Taking $y_I = (1)(2u)(2,2u-1)\cdots(u,u+1)(2u+1,2u+3)(2u+2,2u+4)$ (or $y_I = (1)(2)(3,5)(4,6)$ if $u = 1$), we see that the connected components of $\mathcal{G}_{y_I}^{t_I}$ and $\mathcal{G}_{x_I}^{y_I}$ satisfy the conditions of Corollary 4.1.4 (for full details see Figures B.56-B.58). Finally, applying Lemmas 4.2.1 and 4.2.2 to the remaining components of type δ (that is cycles of black vertices) gives the desired y_J.

Since all possible x-graphs have been analysed, this completes the proof of Theorem 4.0.1. □

4.3 The cases $\pi \neq \{4\}$:

We illustrate the exceptional nature of $\mathcal{P}_{\{4\}}(G,X)$ with a brief exploration of other π-product involution graphs. We begin by considering the case that $\pi = \{2m\}$ and $2m = 2^a$ for some $a \geqslant 3$. The simplest such case arises when $a = 3$. Thus for $G = \mathrm{Sym}(n)$, we consider the G-conjugacy class of $t = (1,2)(3,4)(5,6)(7,8)$, which we denote by X. As $\mathrm{supp}(t)$ has size 8, it suffices to consider $8 \leqslant n \leqslant 16$. We calculate the sizes of the discs $\Delta_i(t)$ of $\mathcal{P}_{\{8\}}(G,X)$ using the computer algebra package MAGMA. Theorem 4.0.3 is an immediate consequence of our calculations, which are summarised in Table 4.2. The situation observed is indicative of the situation which arises

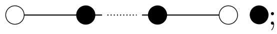
n	$\lvert\Delta_1(t)\rvert$	$\lvert\Delta_2(t)\rvert$	$\lvert\Delta_3(t)\rvert$	$\lvert\Delta_4(t)\rvert$	$\lvert X\rvert$	$\mathrm{Diam}(\mathcal{P}_{\{8\}}(G,X))$
8					105	Totally Disconnected
9					945	Totally Disconnected
10	384	4308	32		4725	3
11	1152	16076	96		17325	3
12	2304	49382	288		51975	3
13	3840	123974	7320		135135	3
14	5760	267014	42540		315315	3
15	8064	512630	154140	840	675675	4
16	10752	902012	431760	6825	1351350	4

Table 4.2: The sizes of the discs $\Delta_i(t)$ for $\mathcal{P}_{\{8\}}(G,X)$, where $G := \mathrm{Sym}(n)$ and X is the G-conjugacy class of $t := (1,2)(3,4)(5,6)(7,8)$.

when $\pi = \{2m\}$ and $2m = 2^a$ for some $a \geqslant 3$. This leads to the formulation of Theorem 4.0.2, a proof of which we now give.

Proof of Theorem 4.0.2:

(i) We note that as the support of t has size $2m$, any x in X - the G-conjugacy class of t - will have an x-graph, $\mathcal{G}_x$, containing m black vertices. If the order of tx is $2m$, then $\mathcal{G}_x$ must be of the form

(1) ; or

(2) ,

(where additional isolated white vertices may be present). In both cases, we see that $|\text{supp}(t) \cap \text{supp}(x)| = 2m - 2$. It follows that if $n = 2m$ or $2m + 1$, then $\mathcal{P}_{\{2m\}}(G, X)$ is totally disconnected. Thus assume that $n \geqslant 2m + 2$. Denote the m transpositions that x is comprised of by $x_1, \ldots, x_m$, where $\min \text{supp}(x_i) \leqslant \min \text{supp}(x_{i+1})$ for all $i = 1, \ldots, m - 1$. We will construct elements $y_i \in X$ for $i = 1, \ldots, m$ such that

$$y_i = x_1 \cdots x_i w_{i+1} \cdots w_m \tag{4.3.4}$$

for some transpositions $w_{i+1}, \ldots, w_m$ and such that y_i is connected to t in $\mathcal{P}_{\{2m\}}(G, X)$. At each stage, if y_i involves the transposition x_{i+1} we are done, so we will assume that this is not the case.

Assume first that $|\text{supp}(x_1) \cap \text{supp}(t)| = 2$. Then without loss we may assume that $x_1 = (1, 3)$. Define

$$y_1 = (1, 3)(4, 5)(6, 7) \cdots (2m - 2, 2m - 1)(2m + 1, 2m + 2). \tag{4.3.5}$$

Thus $\mathcal{G}_{y_1}$ has the form (2) (as seen in Figure B.59). If $|\text{supp}(x_1) \cap \text{supp}(t)| = 1$, then we may assume that $x_1 = (1, 2m + 1)$. Taking

$$y_1 = (1, 2m + 1)(2, 3)(4, 5) \cdots (2m - 4, 2m - 3)(2m - 2, 2m + 2), \tag{4.3.6}$$

it follows that $\mathcal{G}_{y_1}$ is isomorphic to (1) (see Figure B.60). Finally, if $|\text{supp}(x_1) \cap \text{supp}(t)| = 0$, then we assume that $x_1 = (2m + 1, 2m + 2)$. The element

$$y_1 = (2, 3)(4, 5) \cdots (2m - 2, 2m - 1)(2m + 1, 2m + 2) \tag{4.3.7}$$

results in an x-graph $\mathcal{G}_{y_1}$ of type (2) (as seen in Figure B.61). In all cases we see that t and y_1 are adjacent - and hence connected - vertices of $\mathcal{P}_{\{2m\}}(G, X)$.

Suppose that for some $1 \leqslant i < m$ an element y_i of the form (4.3.4) exists with y_i connected to t in $\mathcal{P}_{\{2m\}}(G, X)$. Note that y_i fixes at least two elements of Ω as $n \geqslant 2m + 2$. We denote these elements by f_1, f_2. Define

$$x_{j,1} := \min \text{supp}(x_j) \quad \text{and} \quad x_{j,2} := \max \text{supp}(x_j). \tag{4.3.8}$$

for $1 \leqslant j \leqslant i + 1$ and

$$w_{j,1} := \min \text{supp}(w_j) \quad \text{and} \quad w_{j,2} := \max \text{supp}(w_j). \tag{4.3.9}$$

for $i + 1 \leqslant j \leqslant m$. Set $\alpha = x_{i+1,1}$, $\beta = x_{i+1,2}$ and $w = w_{i+1} \cdots w_m$. Thus $x_{i+1} = (\alpha, \beta)$.

We follow an analogous approach to that used to define y_1. If $|\mathrm{supp}(x_{i+1}) \cap \mathrm{supp}(w)| = 2$, then without loss we have that $w_{i+1,1} = \alpha$ and $w_{i+2,1} = \beta$. First we construct an element $z_{i+1} \in X$ given by

$$\begin{aligned} z_{i+1} = (x_{1,1}, w_{m,2})(x_{1,2}, x_{2,1})(x_{2,2}, x_{3,1}) \cdots (x_{i-1,2}, x_{i,1}) \\ (x_{i,2}, \alpha)(w_{i+2,2}, w_{i+3,1}) \cdots (w_{m-1,2}, w_{m,1})(f_1, f_2) \end{aligned} \tag{4.3.10}$$

for $i > 1$ and

$$z_2 = (x_{1,1}, w_{m,2})(x_{1,2}, \alpha)(w_{3,2}, w_{4,1}) \cdots (w_{m-1,2}, w_{m,1})(f_1, f_2). \tag{4.3.11}$$

Consequently $\mathcal{G}_{z_{i+1}}^{y_i}$ has the form given in (2) (see Figures B.62 and B.63). The element

$$y_{i+1} = x_1 x_2 \cdots x_{i+1}(w_{i+1,2}, w_{i+2,2}) w_{i+3} \cdots w_m \tag{4.3.12}$$

results in an x-graph $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ of isomorphism type (1) (see Figures B.62 and B.63). We deduce that y_{i+1} is connected to y_i and hence to t, and that $d(y_i, y_{i+1}) \leqslant 2$.

In the case that $|\mathrm{supp}(x_{i+1}) \cap \mathrm{supp}(w)| = 1$, we may assume that $w_{i+1,1} = \alpha$ and that $\beta = f_1 \in \mathrm{fix}(y_i)$. Taking $z_2 = (\beta, w_{m,2})(w_{2,2}, w_{3,1}) \cdots (w_{m-1,2}, w_{m,1})(\alpha, f_2)$,

$$\begin{aligned} z_{i+1} = (\beta, x_{2,1})(x_{2,2}, x_{3,1}) \cdots (x_{i-1,2}, x_{i,1})(x_{i,2}, w_{m,2}) \\ (w_{i+1,2}, w_{i+2,1}) \cdots (w_{m-1,2}, w_{m,1})(\alpha, f_2) \end{aligned} \tag{4.3.13}$$

for $i > 1$ and

$$y_{i+1} = x_1 \cdots x_{i+1} w_{i+2} \cdots w_m \tag{4.3.14}$$

we see that $\mathcal{G}_{z_{i+1}}^{y_i}$ is of isomorphism type (1), whilst $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ is of isomorphism type (2) (see Figures B.64 and B.65). Thus $d(y_i, y_{i+1}) \leqslant 2$ and y_{i+1} is connected to t.

The final possibility is that $|\mathrm{supp}(x_{i+1}) \cap \mathrm{supp}(w)| = 0$. Consequently, defining $z_2 = (x_{1,1}, x_{2,1})(x_{1,2}, w_{m,2})(x_{2,2}, w_{3,1})(w_{3,2}, w_{4,1}) \cdots (w_{m-1,2}, w_{m,1})$,

$$\begin{aligned} z_{i+1} = (x_{1,1}, x_{i+1,1})(x_{1,2}, x_{2,1}) \cdots (x_{i-1,2}, x_{i,1})(x_{i,2}, w_{m,2}) \\ (x_{i+1,2}, w_{i+2,1})(w_{i+2,2}, w_{i+3,1}) \cdots (w_{m-1,2}, w_{m,1}) \end{aligned} \tag{4.3.15}$$

for $i > 1$ and

$$y_{i+1} = x_1 \cdots x_{i+1} w_{i+1} w_{i+2} \cdots w_{m-1} \tag{4.3.16}$$

we obtain x-graphs $\mathcal{G}_{z_{i+1}}^{y_i}$ and $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ of types (1) and (2) respectively (see Figures B.66 and B.67). We conclude that $d(y_i, y_{i+1}) \leqslant 2$, and hence y_{i+1} is connected to t in $\mathcal{P}_{\{2m\}}(G, X)$ as required.

(ii) If $\mathcal{P}_{\{2m\}}(G, X)$ is connected, then $n \geqslant 2m + 2$ by (i). Moreover, the above argument shows that $d(t, y_1) \leqslant 1$ and for $1 \leqslant i \leqslant m - 1$ we have $d(y_i, y_{i+1}) \leqslant 2$. Thus as $x = y_m$, we conclude that $\mathrm{Diam}(\mathcal{P}_{\{2m\}}(G, X)) \leqslant 2m - 1$. For the lower bound, we note that $x \in \Delta_1(t)$ precisely when $\mathcal{G}_x$ is of type (1) or (2). In particular $|\mathrm{supp}(t) \cap \mathrm{supp}(x)| = 2m - 2$. Arguing iteratively we deduce that if $d(t, x) \leqslant s$, then $|\mathrm{supp}(t) \cap \mathrm{supp}(x)| \geqslant 2m - 2s$. Since X contains an involution x satisfying $|\mathrm{supp}(t) \cap \mathrm{supp}(x)| = \max\{0, 4m - n\}$ we deduce that

$$\mathrm{Diam}(\mathcal{P}_{\{2m\}}(G, X)) \geqslant \min\{m, \lceil n/2 - m \rceil\}$$

as required. □

We note that it is also possible to define $y_1 \in X$ of the form (4.3.4) which is connected to t by a path of length 2 in $\mathcal{P}_{\{2m\}}(G, X)$. Indeed, if $x_1 = (1, 3)$, we set

$$\begin{aligned} z_1 =&(2,5)(4,2m)(6,7)\cdots(2m-2,2m-1)(2m+1,2m+2); \text{ and} \\ y_1 =&(1,3)(4,2m+1)(5,6)(7,8)\cdots(2m-1,2m). \end{aligned} \tag{4.3.17}$$

If $x_1 = (1, 2m + 1)$, then define

$$\begin{aligned} z_1 =&(2,3)(4,5)\cdots(2m-2,2m-1)(2m+1,2m+2); \text{ and} \\ y_1 =&(1,2m+1)(2,2m+2)(3,4)\cdots(2m-5,2m-4)(2m-3,2m). \end{aligned} \tag{4.3.18}$$

Finally, if $x_1 = (2m + 1, 2m + 2)$, then take

$$\begin{aligned} z_1 =&(1,2m+1)(2,3)(4,5)\cdots(2m-4,2m-3)(2m-2,2m+2); \text{ and} \\ y_1 =&(3,4)(5,6)\cdots(2m-1,2m)(2m+1,2m+2). \end{aligned} \tag{4.3.19}$$

In each case, the x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ are of isomorphism type (1) or (2) as required (see Figures B.68-B.70 for full details).

We now consider Theorem 4.0.4. A non-constructive proof of the connectivity of $\mathcal{P}_{\{q\}}(G, X)$ using Jordan's theorem is contained in the proof of [BGR13, Theorem 4.1]. Here we give a constructive proof in a similar vein to the proof of Theorem 4.0.2 above.

Proof of Theorem 4.0.4:

(i) We first note that the elements of the disc $\Delta_1(t)$ are precisely those elements $x \in X$ whose x-graph, $\mathcal{G}_x$, is of isomorphism type $\bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\circ$. Consequently, $\mathcal{P}_{\{q\}}(G, X)$ is totally disconnected if $n = q - 1$. Thus assume that $n \geqslant q$ and hence that $|\operatorname{fix}(t)| \geqslant 1$.

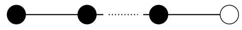
We proceed as in the proof of Theorem 4.0.2 and set $2m = q-1$. Let $x \in X$ be given and denote the transpositions of x as $x = x_1 x_2 \cdots x_m$. As in the proof of Theorem 4.0.2, we will construct elements $y_i \in X$ for $i = 1, \ldots m$ such that

$$y_i = x_1 \cdots x_i w_{i+1} \cdots w_m \tag{4.3.20}$$

for some transpositions w_j and such that y_i is connected to t in $\mathcal{P}_{\{q\}}(G, X)$. Mirroring the situation of the proof of Theorem 4.0.2, we may assume that x_{i+1} is not a transposition of y_i. We continue to use the notation $x_{j,1}, x_{j,2}$ and $w_{j,1}, w_{j,2}$ previously introduced in (4.3.8) and (4.3.9) respectively. For convenience we define $y_0 := t$ and for each y_i we consider the cases $\operatorname{supp}(y_i) = \operatorname{supp}(x)$ and $\operatorname{supp}(y_i) \neq \operatorname{supp}(x)$ separately.

Assume that $\operatorname{supp}(t) = \operatorname{supp}(x)$. Thus $|\operatorname{supp}(t) \cap \operatorname{supp}(x_1)| = 2$ and without loss of generality we may take $x_1 = (1, 3)$. Define z_1 and y_1 by

$$\begin{aligned} z_1 =&(2,3)(4,5)\cdots(2m,2m+1); \text{ and} \\ y_1 =&x_1(2,4)(5,6)\cdots(2m-1,2m). \end{aligned} \tag{4.3.21}$$

The x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ are both of isomorphism type  (see Figure B.71) and hence t and y_1 are connected in $\mathcal{P}_{\{q\}}(G,X)$ with $d(t,y_1) \leqslant 2$.

When $\text{supp}(t) \neq \text{supp}(x)$ we consider three subcases. If $|\text{supp}(t) \cap \text{supp}(x_1)| = 2$, then without loss we have $x_1 = (1,3)$ and we take

$$y_1 = x_1(4,5)(6,7)\cdots(2m,2m+1). \tag{4.3.22}$$

If $|\text{supp}(t) \cap \text{supp}(x_1)| = 1$, then we may assume that $x_1 = (1,2m+1)$ and thus take

$$y_1 = x_1(2,3)(4,5)\cdots(2m-2,2m-1). \tag{4.3.23}$$

In both cases, $\mathcal{G}_{y_1}$ is of the required form (see Figures B.72 and B.73). Consequently t and y_1 are adjacent in $\mathcal{P}_{\{q\}}(G,X)$. Finally, suppose that $|\text{supp}(t) \cap \text{supp}(x_1)| = 0$ and hence that $x_1 = (2m+1,2m+2)$. Defining

$$\begin{aligned} z_1 &= (1,2m+1)(2,3)\cdots(2m-2,2m-1); \text{ and} \\ y_1 &= (x_1)(1,2)(3,4)\cdots(2m-3,2m-2), \end{aligned} \tag{4.3.24}$$

we have that $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ are of the aforementioned isomorphism type (see Figure B.74), and so t and y_1 are connected in $\mathcal{P}_{\{q\}}(G,X)$. Moreover, $d(t,y_1) \leqslant 2$.

Now suppose that a y_i of the form (4.3.20) has been defined for some $i < m$ with y_i connected to t in $\mathcal{P}_{\{q\}}(G,X)$. First assume that $\text{supp}(y_i) = \text{supp}(x)$ and that $\alpha \in \text{fix}(y_i) \cap \text{fix}(x)$. Without loss we may assume that $x_{i+1,1} = w_{i+1,1}$ and $x_{i+1,2} = w_{i+2,1}$. Define $z_2 = (x_{1,2},w_{2,1})(w_{2,2},w_{3,1})\cdots(w_{m-1,2},w_{m,1})(w_{m,2},\alpha)$,

$$\begin{aligned} z_{i+1} =&(x_{1,2},x_{2,1})(x_{2,2},x_{3,1})\cdots(x_{i-1,2},x_{i,1})(x_{i,2},w_{i+1,1}) \\ &(w_{i+1,2},w_{i+2,1})\cdots(w_{m-1,2},w_{m,1})(w_{m,2},\alpha); \text{ and} \\ y_{i+1} =&x_1x_2\cdots x_{i+1}(w_{i+1,2},w_{i+2,2})w_{i+3}\cdots w_m \end{aligned} \tag{4.3.25}$$

for $i > 1$. We have that the x-graphs $\mathcal{G}_{z_{i+1}}^{y_i}$ and $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ have the required form and hence $d(y_i,y_{i+1}) \leqslant 2$.

It remains to consider the case that $\text{supp}(y_i) \neq \text{supp}(x)$. Let $f_i \in \text{fix}(y_i)\backslash\text{fix}(x)$ and $f_x \in \text{fix}(x)\backslash\text{fix}(y_i)$. In the case when $|\text{supp}(y_i) \cap \text{supp}(x_{i+1})| = 2$, assume that $x_{i+1} = (w_{i+1,1},w_{i+2,1})$ and set $z_2 = (x_{1,1},w_{m,2})(x_{1,2},w_{2,1})(w_{3,1},f_i)\ (w_{3,2},w_{4,1})\cdots(w_{m-1,2},w_{m,1})$,

$$\begin{aligned} z_{i+1} =&(x_{1,1},w_{m,2})(x_{1,2},x_{2,1})\cdots(x_{i-1,2},x_{i,1})(x_{i,2},w_{i+1,1})(w_{i+2,1},f_i) \\ &(w_{i+2,2},w_{i+3,1})\cdots(w_{m-1,2},w_{m,1}); \text{ and} \\ y_{i+1} =&x_1x_2\cdots x_{i+1}(w_{i+1,2},w_{i+2,2})w_{i+3}\cdots w_m. \end{aligned} \tag{4.3.26}$$

for $i > 1$. If $|\text{supp}(y_i) \cap \text{supp}(x_{i+1})| = 1$, then without loss we have $x_{i+1} = (w_{i+1,1},f_i)$. Hence we define $z_2 = (x_{1,1},w_{m,2})(x_{1,2},x_{2,1})(w_{3,1},f_i)(w_{3,2},w_{4,1})\cdots(w_{m-1,2},w_{m,1})$,

$$\begin{aligned} z_{i+1} =&(x_{1,1},w_{m,2})(x_{1,2},x_{2,1})\cdots(x_{i-1,2},x_{i,1})(x_{i,2},x_{i+1,1}) \\ &(w_{i+2,1},f_i)(w_{i+2,2},w_{i+3,1})\cdots(w_{m-1,2},w_{m,1}); \text{ and} \\ y_{i+1} =&x_1x_2\cdots x_{i+1}w_{i+2}\cdots w_{m-1}(w_{m,1},w_{i+1,2}). \end{aligned} \tag{4.3.27}$$

for $i > 1$. Finally, if x_{i+1} and y_i are disjoint we set $z_2 = (x_{1,1}, x_{2,1})(x_{1,2}, w_{2,1})(w_{2,2}, w_{3,1}) \cdots (w_{m-1,2}, w_{m,1})$,

$$\begin{aligned} z_{i+1} =& (x_{1,1}, x_{i+1,1})(x_{1,2}, x_{2,1}) \cdots (x_{i-1,2}, x_{i,1})(x_{i,2}, w_{i+1,1}) \\ & (w_{i+1,2}, w_{i+2,1}) \cdots (w_{m-1,2}, w_{m,1}); \text{ and} \\ y_{i+1} =& x_1 x_2 \cdots x_{i+1} w_{i+1} \cdots w_{m-1}. \end{aligned} \tag{4.3.28}$$

for $i > 1$. For each pair (z_{i+1}, y_{i+1}) the x-graphs $\mathcal{G}_{z_{i+1}}^{y_i}$ and $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ have isomorphism type $\bullet\!-\!\bullet \cdots \bullet\!-\!\circ$ (see Figures B.75-B.82 for details). Consequently $d(y_i, y_{i+1}) \leqslant 2$ and the elements t and y_{i+1} are connected in $\mathcal{P}_{\{q\}}(G, X)$.

(ii) By part (i) we have that $\operatorname{Diam} \mathcal{P}_{\{q\}}(G, x) \leqslant q - 1$. For the lower bound, we note that for $x \in X$ to be adjacent to t we have $|\operatorname{supp}(t) \cap \operatorname{supp}(x)| = |\operatorname{supp}(t)| - 1$. Taking $x \in X$ such that $|\operatorname{supp}(t) \cap \operatorname{supp}(x)|$ is minimal we have that $|\operatorname{supp}(t) \cap \operatorname{supp}(x)| = \max\{0, 2q - 2 - n\}$ and hence $d(t, x) \geqslant \min\{q - 1, n + 1 - q\}$ as required. □

Following a similar approach to that used when $p = 2$, we define $y_1 \in X$ of the form (4.3.20) such that there is a path in $\mathcal{P}_{\{q\}}(G, X)$ from t to y_1 of length 2. These paths will be used in the proof of Theorem 4.0.5. Such paths were defined in the above proof except when $\operatorname{supp}(t) \neq \operatorname{supp}(x)$ and $|\operatorname{supp}(t) \cap \operatorname{supp}(x_1)| = 1$ or 2. In the latter case, we may assume without loss that $x_1 = (1, 3)$ and define

$$\begin{aligned} z_1 =& (2, 3)(4, 5) \cdots (2m, 2m + 1); \text{ and} \\ y_1 =& (1, 3)(2, 4)(5, 6) \cdots (2m - 1, 2m). \end{aligned} \tag{4.3.29}$$

In the former case, we assume that $x_1 = (1, 2m + 1)$ and set

$$\begin{aligned} z_1 =& (2, 3)(4, 5) \cdots (2m, 2m + 1); \text{ and} \\ y_1 =& (1, 2m + 1)(3, 4)(5, 6) \cdots (2m - 1, 2m). \end{aligned} \tag{4.3.30}$$

Both cases give rise to x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ of the required form to show that there is a path from t to y_1 of length 2 in $\mathcal{P}_{\{q\}}(G, X)$ (see Figures B.83 and B.84).

The proofs of Theorems 4.0.2 and 4.0.4 are utilised in the proof of Theorem 4.0.5.

Proof of Theorem 4.0.5:

The x-graph $\mathcal{G}_x$ of any $x \in \Delta_1(t)$ must consist of a connected component of isomorphism type $\bullet\!-\!\bullet \cdots \bullet\!-\!\circ$ containing $q_i/2$ black vertices for each $i = 2, \ldots, r$. In addition, there will be connected components of types (1) or (2) from the proof of Theorem 4.0.2 if $p_1 = 2$, or a component of type $\bullet\!-\!\bullet \cdots \bullet\!-\!\circ$ containing $q_1/2$ black vertices if $p_1 \neq 2$. It follows that

$$|\operatorname{fix}(t)| \geqslant \begin{cases} r + 1 & \text{if } p_1 = 2; \text{ and} \\ r & \text{otherwise.} \end{cases} \tag{4.3.31}$$

We conclude that if

$$n \geqslant \begin{cases} q + 2 & \text{if } p_1 = 2; \text{ and} \\ q & \text{otherwise} \end{cases} \tag{4.3.32}$$

does not hold, then $\mathcal{P}_{\{q\}}(G,X)$ is totally disconnected. Thus assume that (4.3.32) holds.

Denote the the transpositions of x by

$$x = x_1^{(1)} \cdots x_{q_1/2}^{(1)} x_1^{(2)} \cdots x_{q_2/2}^{(2)} \cdots x_1^{(r)} \cdots x_{q_r/2}^{(r)}.$$

Moreover, if $\ell = \max\{q_i\}$, define $x_{q_j/2+1}^{(j)} = \cdots = x_{q_\ell/2}^{(j)} = 1$ for all $j = 1, \ldots, r$ where these permutations are not already defined.

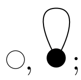
Using the proofs of Theorems 4.0.1, 4.0.2 and 4.0.4 and the comments following the proofs, we may construct $y_i \in X$ for $i = 0, \ldots, \ell/2$ with $y_0 := t$ satisfying

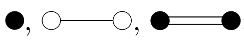
(i) there is a path in $\mathcal{P}_{\{q\}}(G,X)$ from y_i to y_{i+1} of length 2 for $i = 0, \ldots, \ell/2 - 1$; and

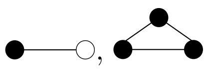
(ii) y_i contains the transpositions $x_1^{(j)}, \ldots, x_i^{(j)}$ for $j = 1, \ldots, r$.

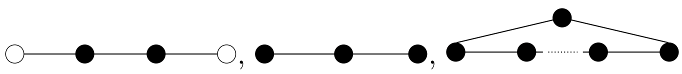
This is allowable as (4.3.31) holds. The result now follows immediately. $\square$

We briefly consider an example of a case when π consists of a composite number. The smallest such situation arises when $\pi = \{6\}$. If $t \in G$ is an involution and X is the G-conjugacy class of t, then any $x \in X$ will be adjacent to t in $\mathcal{P}_{\{6\}}(G,X)$ if the connected components of $\mathcal{G}_x$ consist of components of the form

(i) ;

(ii) ;

(iii) ; and

(iv)  (with 6 black vertices).

Moreover, either one component is of isomorphism type (iv), or there exists at least one component of type (ii) and one component of type (iii).

Finally, Table 4.3 gives the sizes of the discs $\Delta_i(t)$ of $\mathcal{P}_{\{6\}}(G,X)$ for the symmetric groups $G := \mathrm{Sym}(n)$ $(6 \leqslant n \leqslant 10)$, when X is the G-conjugacy class of an involution $t \in G$.

n	m	$\|\Delta_1(t)\|$	$\|\Delta_2(t)\|$	$\|\Delta_3(t)\|$	$\|X\|$	$\mathrm{Diam}(\mathcal{P}_{\{6\}}(G,X))$
6	2				45	Totally Disconnected
	3				15	Totally Disconnected
7	2	12	38	54	105	3
	3	12	60	32	105	3
8	2	48	158	3	210	3
	3	72	347		420	2
	4				105	Totally Disconnected
9	2	120	242	15	378	3
	3	216	1043		1260	2
	4	48	836	60	945	3
10	2	240	389		630	2
	3	624	2525		3150	2
	4	416	4308		4725	2
	5	160	784		945	2

Table 4.3: The sizes of the discs $\Delta_i(t)$ for $\mathcal{P}_{\{6\}}(G,X)$, where X is the G-conjugacy class of $t = (1,2)\cdots(2m-1,2m) \in G := \mathrm{Sym}(n)$.

Chapter 5

Conjugate p-elements of Full Support that Generate the Wreath Product $C_p \wr C_p$

In Chapter 4 we saw how the rich structure of the poset $\mathcal{S}_p(\mathrm{Sym}(n))$ could be used to characterise when two conjugate involutions generated the dihedral group $\mathrm{Dih}(8)$. There are two natural generalisations of this result. The first would consider when two conjugate involutions generate the dihedral group $\mathrm{Dih}(2^m)$ for $m > 3$. An analysis of x-graphs would be fundamental in such an approach, which would then become a case-by-case analysis of the possible situations that could arise - similar to that given in the proof of Theorem 4.0.1.

An alternative generalisation involves considering $\mathrm{Dih}(8)$ as the wreath product, $C_2 \wr C_2$, of two cyclic groups of order 2. From this viewpoint, a natural generalisation is to consider when two conjugate p-elements of $\mathrm{Sym}(n)$ generate the wreath product $C_p \wr C_p$. In this chapter, we consider conjugate p-elements of full support in $\mathrm{Sym}(n)$ that generate this wreath product. To analyse such situations, it would be desirable to form a generalisation of the x-graph. The natural generalisation results in non-planar directed graphs and hence we consider the adjacency matrix of such a graph. Indeed, given conjugate p-elements a and x of full support in $\mathrm{Sym}(n)$, we form suitable matrices A_x^a and A_a^x. Here a will be the standard p-element of G namely $a = (1, 2, \ldots, p)(p+1, \ldots, 2p) \cdots ((r-1)p+1, \ldots, rp)$ where $n = rp$ and we label the p-cycles forming a by $\alpha_i = (p(i-1)+1, \ldots, pi)$ for $i = 1, \ldots, r$. Similarly if x is a G-conjugate of a, then we may label its disjoint p-cycles by $\chi_1, \ldots, \chi_r$. The matrices A_x^a and A_a^x have (i, j) entries given by $|\mathrm{supp}(\alpha_i^x) \cap \mathrm{supp}(\alpha_j)|$ and $|\mathrm{supp}(\chi_i^a) \cap \mathrm{supp}(\chi_j)|$ respectively.

Throughout this chapter, p will be an odd prime. Our first result considers the case that $n = p^2$. We see that the matrices A_x^a and A_a^x do indeed encode data which we may use to determine when a and x generate the wreath product $W_p := C_p \wr C_p$. This encoding involves circulant matrices and their representer polynomials in addition to permutation matrices, all of which are defined in Section 5.1.

Theorem 5.0.1. *Let $G = \mathrm{Sym}(p^2)$, a be the standard p-element of G and let x be a conjugate of* *a. Then $\langle a, x\rangle \cong W_p$ precisely when the following two equivalent conditions hold.*

(i) $A_x^a = p \cdot Y_\sigma$ for some p-cycle $\sigma \in \mathrm{Sym}(p)$, $A_a^x = \mathrm{circ}(0, c_1, \ldots, c_{p-1})$ for some $c_i \in \mathbb{Z}_p$, $X = 1$ is a simple root of the representer polynomial $f_{A_a^x}(X) \in \mathbb{Z}_p[X]$ and $[a, a^x] = 1$.

(ii) $A_x^a = p \cdot Y_\sigma$ for some p-cycle $\sigma \in \mathrm{Sym}(p)$, $A_a^x = \mathrm{circ}(0, c_1, \ldots, c_{p-1})$ for some $c_i \in \mathbb{Z}_p$, $|\det(A_a^x)|_p = p^2$ and $[a, a^x] = 1$.

(or (i) and (ii) hold with the roles of a and x interchanged).

The second result concerns the case when $n = p^3$. It involves the notions of the block sum matrix, $BS(M)$, of a matrix M and of a reduced representer polynomial as given in Definitions 5.4.5 and 5.4.6.

Theorem 5.0.2. *Let $G = \mathrm{Sym}(p^3)$, a be the standard p-element of G and let x be a conjugate of a. Moreover, suppose that we cannot decompose $a = a_1 \cdots a_p$ and $x = x_1 \cdots x_p$ with a_i, x_i sitting inside a copy of $\mathrm{Sym}(p^2)$ for each i with one such pair satisfying the conditions of Theorem 5.0.1. Then $\langle a, x\rangle \cong W_p$ precisely when up to a renumbering of the α_i the following conditions hold*

(i) $[a, a^{x^i}] = 1$ for all $i = 1, \ldots, (p-1)/2$;

(ii) A_x^a is a block matrix having $p \times p$ blocks, all of which are zero except for those immediately above the leading diagonal. The non-zero $p \times p$ blocks are either equal to $p \cdot Y_\sigma$ for some $\sigma \in \mathrm{Sym}(p)$ or have every entry equal to 1. Moreover, A_x^a contains at least one block whose entries are all equal to 1; and

(iii) A_a^x is a block matrix having $p \times p$ blocks each of which is circulant and the diagonal blocks are all equal. Moreover, either

(c1) $X = 1$ is a root of multiplicity at least 2 of the representer polynomial $f_{BS(A_a^x)}(X) \in \mathbb{Z}_p[X]$ but is not a root of the representer polynomial $f'_{BS(A_a^x)}(X) \in \mathbb{Z}_{p^2}[X]$; or

(c2) $X = 1$ is not a root of the reduced representer polynomial $g'_{A_a^x}(X) \in \mathbb{Z}_{p^2}[X]$;

(or (i), (ii) and (iii) hold - up to a renumbering of the χ_i - with the roles of a and x interchanged).

Our final result combines Theorems 5.0.1 and 5.0.2 to consider the most general setting. Since $C_p \wr C_p$ is not a subgroup of $\mathrm{Sym}(n)$ for $n < p^2$, this occurs when $G = \mathrm{Sym}(rp)$ for some $r \geqslant p$.

Theorem 5.0.3. *Let $r \geqslant p$, $n = rp$ and $G = \mathrm{Sym}(n)$. If a is the standard p-element of G having G-conjugacy class $\mathcal{X}$ and $x \in \mathcal{X}$, then $\langle a, x\rangle \cong W_p$ precisely when for a suitable renumbering of the α_i and χ_j the following conditions hold*

(i) $[a, a^{x^i}] = 1$ for $i = 1, \ldots, (p-1)/2$;

(ii) A_x^a is a block diagonal matrix having blocks D_1, D_2 and D_3, where

- *$D_1 = p \cdot I$ for some identity matrix I;*

- *D_2 is a block diagonal matrix having $p \times p$ blocks, where each block has the form $p \cdot Y_\sigma$ for some p-cycle $\sigma \in \mathrm{Sym}(p)$; and*
- *D_3 is a block diagonal matrix having $p^2 \times p^2$ blocks having the form of the matrix A_x^a from Theorem 5.0.2.*

(iii) A_a^x is a block diagonal matrix having blocks E_1, E_2 and E_3, where

- *$E_1 = D_1$;*
- *E_2 is a block diagonal matrix of the same size as D_2, where each block is a $p \times p$ circulant matrix having row sum equal to p; and*
- *E_3 is a block diagonal matrix of the same size as D_3, where each block is a $p^2 \times p^2$ block matrix having $p \times p$ blocks each of which is circulant and the diagonal blocks are all equal.*

(iv) One of the following holds

- *The representer polynomial $f_B(X) \in \mathbb{Z}_p[X]$ of at least one of the circulant blocks, B, of E_2 has a simple root at $X = 1$;*
- *There is at least one block, C, of E_3 such that $X = 1$ is a root of multiplicity at least 2 over $\mathbb{Z}_p$ of the representer polynomial of the block sum matrix $BS(C)$ but $X = 1$ is not a root of the representer polynomial $f'_{BS(C)}(X) \in \mathbb{Z}_{p^2}[X]$;*
- *There is at least one block, C, of E_3 such that $X = 1$ is not a root of the reduced representer polynomial $g'_C(X) \in \mathbb{Z}_{p^2}[X]$.*

(or the conditions $(i) - (iv)$ hold with the roles of a and x interchanged)

This chapter is arranged as follows. In Section 5.1 we look at the matrices A_x^a and A_a^x and give some basic results about the wreath product $C_p \wr C_p$. We also introduce the notion of circulant matrices and illustrate a number of properties that they satisfy, which will be used in our subsequent work. Sections 5.2 and 5.3 consider the case that $n = p^2$. The former section leads up to a proof of the equivalence of $\langle a, x\rangle \cong W_p$ and part (i) of Theorem 5.0.1, whilst the latter proves the equivalence of parts (i) and (ii) of the same theorem. The case that $n = p^3$ is considered in Section 5.4 in which Theorem 5.0.2 is proved. Due to the technical nature of some of the proofs in Sections 5.2 and 5.4, both sections conclude with worked examples highlighting the key steps in the proofs. The chapter concludes in Section 5.5, where Theorem 5.0.3 is proved.

5.1 Preliminary Results

Throughout this chapter we fix a prime $p \neq 2$ and set $G := \mathrm{Sym}(pr)$ for some $r \geqslant p$. We shall also use the notation W_p to denote the wreath product $C_p \wr C_p$ of two cyclic groups of order p. We consider G to be acting on the set $\Omega := \{1, \ldots, pr\}$ and fix an element $a = \alpha_1\alpha_2 \cdots \alpha_r$ where $\alpha_i = (p(i-1)+1, p(i-1)+2, \ldots, pi)$, which we will sometimes refer to as the *standard p-element of G*. We denote the G-conjugacy class of a by $\mathcal{X} = a^G$. Given $x \in \mathcal{X}$, it will be useful to denote the disjoint p-cycles forming x by $\chi_1, \ldots, \chi_r$. The χ_i are defined recursively by setting χ_1 to be the p-cycle for which the orbit of 1 under $\langle \chi_1 \rangle$ is non-trivial. Assuming that χ_i has been defined for $1 \leqslant i \leqslant j$ we then define

$$t_j := \min \{t \in \Omega \mid \text{the orbit of } t \text{ under } \langle \chi_1, \ldots, \chi_j \rangle \text{ is trivial}\}$$

and define χ_{j+1} to be the p-cycle for which the orbit of t_j under $\langle \chi_{j+1} \rangle$ is non-trivial.

In Chapter 4, it was seen that when $\mathcal{X}$ was an arbitrary conjugacy class of involutions, then to determine when two conjugate involutions generate the wreath product $W_2 \cong \mathrm{Dih}(8)$, we may use graphs known as x-graphs. These graphs essentially considered the intersections of the supports of the transpositions in the decomposition of the involutions. Generalising this from a strictly graph-theoretic viewpoint is impractical. However, consideration of the adjacency matrix of such a generalisation does yield results.

Definition 5.1.1. *Let $x \in \mathcal{X}$ be given. We define the $r \times r$ matrices A_x^a and A_a^x by*

$$(A_x^a)_{i,j} := |\mathrm{supp}(\alpha_i^x) \cap \mathrm{supp}(\alpha_j)|$$

and

$$(A_a^x)_{i,j} := |\mathrm{supp}(\chi_i^a) \cap \mathrm{supp}(\chi_j)|.$$

As with x-graphs, since the element a is fixed, we suppress it in our notation and denote A_x^a by A_x. The determinant of a matrix B will be denoted by $\det(B)$ and the largest power of p dividing it will be denoted $|\det(B)|_p$.

We illustrate these notions with an example. Let $p = 5$ and let a be the standard 5-element of $G := \mathrm{Sym}(25)$. Taking

$$\begin{aligned} x &= \chi_1 \cdot \chi_2 \cdot \chi_3 \cdot \chi_4 \cdot \chi_5 \\ &= (1, 22, 15, 4, 5)(2, 8, 7, 19, 9)(3, 10, 14, 24, 11) \\ &\quad (6, 23, 25, 20, 13)(12, 18, 16, 21, 17) \end{aligned}$$

in the G-conjugacy class of a, we see that

$$\alpha_1^x = (1, 22, 8, 10, 5), \quad \alpha_2^x = (2, 14, 23, 19, 7), \quad \alpha_3^x = (3, 18, 6, 24, 4),$$
$$\alpha_4^x = (9, 13, 21, 12, 16) \quad \text{and} \quad \alpha_5^x = (11, 20, 17, 15, 25).$$

Consequently

$$A_x = \begin{pmatrix} 2 & 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}.$$

Similarly

$$\chi_1^a = (1,2,23,11,5), \quad \chi_2^a = (3,9,8,20,10), \quad \chi_3^a = (4,6,15,25,12),$$
$$\chi_4^a = (7,24,21,16,14), \quad \text{and} \quad \chi_5^a = (13,19,17,22,18).$$

This gives rise to the matrix

$$A_a^x = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

In Section 5.2 we will be concerned with two specific classes of matrices; permutation matrices and circulant matrices. Indeed, if $\sigma \in \mathrm{Sym}(n)$, then we denote by Y_σ the $n \times n$ permutation matrix defined by

$$(Y_\sigma)_{i,j} = \begin{cases} 1 & \text{if } i\sigma = j; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

In the case that $\sigma = (1, 2, \ldots, n)$, we set $\pi := Y_\sigma$. For a given $n \times n$-matrix A, multiplying A on the left by π cyclically shifts the rows of A up by a row, whilst multiplying by π on the right cyclically shifts the columns of A to the right by a column. Matrices that are invariant under conjugation by π are known as *circulant* matrices. Thus any circulant matrix C satisfies $C_{i,j} = C_{i+k,j+k}$ for any $k = 1, \ldots, n$ (where we replace $i + k$ (respectively $j + k$) by $i + k - n$ (respectively $j + k - n$) if $i + k > n$ (respectively $j + k > n$)). Consequently a circulant matrix is uniquely determined by its first row, and we denote the circulant matrix

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix},$$

by $C = \mathrm{circ}(c_0, c_1, \ldots, c_{n-1})$.

Associated to any circulant matrix $C = \mathrm{circ}(c_0, c_1, \ldots, c_{n-1})$, is the *representer* polynomial $f_C(X)$ in the indeterminant X, given by

$$f_C(X) = c_0 + c_1X + c_2X^2 + \cdots + c_{n-1}X^{n-1}.$$

Strictly speaking this is a polynomial with coefficients in the underlying ring of C. However, in this chapter the underlying ring of our circulant matrices will be the integers, and it will sometimes be preferable to consider $f_C(X)$ as an element of $\mathbb{Z}_p[X]$ or $\mathbb{Z}_{p^2}[X]$. The following result illustrates this correspondence.

Lemma 5.1.2. *Let* $X = \mathrm{circ}(x_0, \ldots, x_{p-1})$ *be an integer circulant matrix. Then* $\det(X) \equiv x_0 + \cdots + x_{p-1} \pmod{p}$.

Proof. Expanding $\det(X)$ gives a series of summands $\alpha_{a_0 \cdots a_{p-1}} x_0^{a_0} x_1^{a_1} \cdots x_{p-1}^{a_{p-1}}$ where $a_0 + \cdots + a_{p-1} = p$ and $a_i \in \mathbb{N}$. Clearly if $a_i = p$ for some i, then the coefficient $\alpha_{a_0 \cdots a_{p-1}}$ is equal to 1 and the corresponding summand of $\det(X)$ is x_i^p. Conversely, consider an occurrence of $x_0^{a_0} \cdots x_{p-1}^{a_{p-1}}$ in the expansion of $\det(X)$ where no a_j is equal to p. Suppose that $x_0^{a_0} \cdots x_{p-1}^{a_{p-1}}$ occurs from the entries $X_{i_1,j_1}, \ldots, X_{i_p,j_p}$ of X. Due to the circular nature of entries of X, it will also occur from the entries $X_{i_1+k,j_1+k}, \ldots, X_{i_p+k,j_p+k}$ for $k = 1, \ldots, p-1$ (where we set $X_{i+p,j}$, $X_{i,j+p}$ and $X_{i+p,j+p}$ equal to $X_{i,j}$ for $i, j = 1, \ldots, p$). Since each a_j is not equal to p, and $p \neq 2$ we conclude that

$$\{X_{i_1+s,j_1+s}, \ldots, X_{i_p+s,j_p+s}\} \neq \{X_{i_1+t,j_1+t}, \ldots, X_{i_p+t,j_p+t}\}$$

for any distinct $s, t = 0, \ldots, p-1$. Moreover, consideration of the signs of each of these occurrences of $x_0^{a_0} \cdots x_{p-1}^{a_{p-1}}$ shows that they must all be equal. Since there are p such occurrences, we conclude that if $a_j \neq p$ for $j = 0, \ldots, p-1$, then $\alpha_{a_0 \cdots a_{p-1}}$ is divisible by p. Consequently, applying Fermat's Little Theorem we obtain

$$\det(X) \equiv x_0^p + \cdots + x_{p-1}^p \equiv x_0 + \cdots + x_{p-1} \pmod{p}.$$

□

Next we consider the structure of $W_p \cong \Gamma \rtimes C_p$, where Γ is an elementary abelian p-group of rank p.

Lemma 5.1.3. *Let p be an odd prime and let G be a group having a normal elementary abelian p-subgroup Γ of index p in G. Suppose that Γ has rank p. If $a, x \in G$ satisfy $ord(a) = ord(x) = p$ and $\langle a, x \rangle = G$, then either a or x is contained in Γ.*

Proof. Let $a, x \in G \backslash \Gamma$ satisfy the given properties. Since G/Γ is a cyclic p-group, by replacing x by an appropriate power, we may assume without loss of generality that $a\Gamma = x\Gamma$ and hence that the action of a on Γ is equal to the action of x on Γ.

Let $\Delta := \langle a^{-1}x \rangle \times \langle a^{-2}x^2 \rangle \times \cdots \times \langle ax^{-1} \rangle$. We note that as $a\Gamma = x\Gamma$ each of $a^{-i}x^i \in \Gamma$ for $i = 1, \ldots, p-1$. Hence $\Delta \subseteq \Gamma$. Conversely, suppose that $y \in \Gamma$. As $y \in G = \langle a, x \rangle$, there exist $b_1, \ldots, b_{2n} \in \mathbb{Z}_p$ (for some n) such that

$$y = a^{b_1} x^{b_2} a^{b_3} x^{b_4} \cdots a^{b_{2n-1}} x^{b_{2n}}. \tag{5.1.1}$$

Note that as $G/\Gamma = \langle a\Gamma \rangle = \langle x\Gamma \rangle$, we have the relation $\sum_{i=1}^{2n} b_i \equiv 0 \pmod{p}$.

We now proceed to build y recursively. We assume that y is written in the form (5.1.1) such that no b_i is zero. In the case that $b_1 = 0$ or $b_{2n} = 0$, an analogous argument holds. Define $y_1 := a^{b_1} x^{-b_1}$. Thus y_1 agrees with y in the first position. Now assume that we have defined $y_i \in \Delta$ for some $i < 2n$ such that $y_i = a^{b_1} x^{b_2} \cdots a^{b_i} x^d$ or $y_i = a^{b_1} x^{b_2} \cdots x^{b_i} a^d$ depending on the parity of i. Here $d := -\sum_{j=1}^{i} b_j$. Thus y_i agrees with y on the first i entries. In the former case define $y_{i+1} := y_i x^{b_{i+1}-d} a^{d-b_{i+1}}$ and in the latter case define $y_{i+1} := y_i a^{b_{i+1}-d} x^{d-b_{i+1}}$. Thus $y_{i+1} \in \Delta$ and y_{i+1} agrees with y in the first $i+1$ positions. Continuing in this way we may define y_i for $i = 1, \ldots 2n$, where each $y_i \in \Delta$. However,

$$y_{2n} = a^{b_1} x^{b_2} a^{b_3} x^{b_4} \cdots a^{b_{2n-1}} x^{b_{2n}} a^{-\sum_{j=1}^{2n} b_i} = y,$$

and hence $y \in \Delta$. We conclude that $\Gamma = \Delta$, from which the result follows immediately. □

Applying Lemma 5.1.3 to W_p we obtain the following result that we will use in Section 5.2.

Corollary 5.1.4. *Let p be an odd prime and let $a, x \in W_p$ be elements of order p such that $\langle a, x\rangle = W_p$. Then either $[a, a^{x^i}] = 1$ for all $i \in \{1, \dots, p\}$ or $[x, x^{a^i}] = 1$ for all $i \in \{1, \dots, p\}$.*

Proof. Since W_p satisfies the hypothesis of Lemma 5.1.3, either a or x is contained in the base group of W_p, which is elementary abelian. □

We conclude this section with a remark.

Remark 5.1.5. *We may construct the matrices A_x and A_a^x in the computer algebra system MAGMA. The appropriate code for doing so, together with code for constructing arbitrary circulant matrices and for checking if a given matrix is circulant is given in Appendix C.*

5.2 The $n = p^2$ Case

In the next two sections, we prove Theorem 5.0.1. This theorem reflects the fact that in the wreath product $C_p \wr C_p$, the p copies of C_p are permuted in a p-cycle, and elements within each copy of C_p have a circular orbit.

The proof of Theorem 5.0.1 can be split into two separate cases. In Subsection 5.2.1, we will prove the equivalence of $\langle a, x\rangle \cong W_p$ with part (i) of Theorem 5.0.1. This is proved in Proposition 5.2.2. Due to the technical nature of the proofs in this subsection, we follow this in Subsection 5.2.2 with a worked example mirroring the proofs. The equivalence of parts (i) and (ii) of Theorem 5.0.1 is then proved in Section 5.3 as Theorem 5.3.2.

5.2.1 The Results

We begin by formulating a result relating the conjugation action of x on a to the rank of an elementary abelian p-group.

Lemma 5.2.1. *Let $G = \mathrm{Sym}(p^2)$, a be the standard p-element of G and let x be a conjugate of a. If $\alpha_1^{x^{j-1}} = \alpha_{1\sigma^{j-1}}^{e_{1\sigma^{j-1}}}$ for some p-cycle $\sigma \in \mathrm{Sym}(p)$, some non-zero $e_1, \dots, e_p \in \mathbb{Z}_p$ and for all $j = 1, \dots, p$, then the elementary abelian p-group $\Gamma := \langle a^{x^j} | j = 0, \dots, p-1\rangle$ has rank p if and only if $e_1^{-1} + \dots + e_p^{-1} \not\equiv 0 \pmod p$.*

Proof. For ease of notation we will assume without loss of generality that $\sigma = (1, 2, \dots, p)$. Moreover, for simplicity of subsequent arguments we define $e_{mp+i} := e_i$ for all $m \in \mathbb{Z}$. Let $d_1, \dots, d_p \in \mathbb{Z}_p$ be such that

$$\prod_{i=1}^{p} (a^{x^i})^{d_i} = 1. \tag{5.2.2}$$

As $\alpha_1^{x^{j-1}} = \alpha_{1\sigma^{j-1}}^{e_1\sigma^{j-1}} = \alpha_j^{e_j}$, it follows that $a^x = \prod_{j=1}^p \alpha_{j+1}^{e_j^{-1}e_{j+1}}$. Iterating this we have that $a^{x^2} = \prod_{j=1}^p \alpha_{j+1}^{e_{j-1}^{-1}e_{j+1}}$ and in general $a^{x^i} = \prod_{j=1}^p \alpha_{j+1}^{e_{j-i+1}^{-1}e_{j+1}}$. Thus we may write (5.2.2) as

$$\prod_{i=1}^{p}\prod_{j=1}^{p} \alpha_{j+1}^{(e_{j-i+1}^{-1}e_{j+1})d_i} = \prod_{i=1}^{p}\left(\prod_{j=1}^{p} \alpha_{j+1}^{e_{j-i+1}^{-1}e_{j+1}}\right)^{d_i} = 1. \tag{5.2.3}$$

It follows that for a fixed $\ell \in \{1, \ldots, p\}$, the exponent of α_ℓ in (5.2.3) is congruent to 0 (mod p). Thus we have

$$\sum_{i=1}^{p} (e_{\ell-i}^{-1} e_\ell) d_i \equiv 0 \pmod{p}. \tag{5.2.4}$$

The equations in (5.2.4) give rise to a homogeneous system of linear equations in the d_i with coefficient matrix given by $A = (a_{\ell,i})$, where $a_{\ell,i} = e_{\ell-i}^{-1} e_\ell$. This matrix is invertible if and only if the matrix $B = (b_{\ell,i})$ is invertible, where $b_{\ell,i} = e_{\ell-i}^{-1}$. We may conclude that there is a non-trivial solution to (5.2.2) meaning that the group Γ has rank less than p, precisely when $\det(B) \equiv 0 \pmod{p}$. However, $B = \operatorname{circ}(e_p^{-1}, e_{p-1}^{-1}, \ldots, e_1^{-1})$ and so appealing to Lemma 5.1.2 we deduce that this occurs precisely when $e_1^{-1} + \cdots + e_p^{-1} \equiv 0 \pmod{p}$. □

Lemma 5.2.1 underpins the proof of the equivalence of $\langle a, x \rangle \cong W_p$ and part (i) of Theorem 5.0.1, as it allows us to relate the coefficients of the circulant matrix - A_a^x or A_x - with the base group of the wreath product W_p.

Proposition 5.2.2. *Let $G = \operatorname{Sym}(p^2)$, a be the standard p-element of G and let x be a conjugate of a. The following are equivalent*

(i) $\langle a, x \rangle \cong W_p$;

(ii) $A_x = p \cdot Y_\sigma$ *for some p-cycle $\sigma \in \operatorname{Sym}(p)$, $A_a^x = \operatorname{circ}(0, c_1, \ldots, c_{p-1})$ for some $c_i \in \mathbb{Z}_p$, $X = 1$ is a simple root of the representer polynomial $f_{A_a^x}(X) \in \mathbb{Z}_p[X]$ and $[a, a^x] = 1$ (or with the roles of a and x interchanged).*

Proof. Throughout the proof we write $a = \alpha_1 \cdots \alpha_p$ and $x = \chi_1 \cdots \chi_p$ as in Section 5.1. Assume that $\langle a, x \rangle \cong W_p$. By Corollary 5.1.4 either $[a, a^{x^i}] = 1$ for all $i \in \{1, \ldots, p\}$ or $[x, x^{a^i}] = 1$ for all $i \in \{1, \ldots, p\}$. Without loss of generality assume that $[a, a^{x^i}] = 1$ for all $i = 1, \ldots, p$. An analogous argument can be used in the case that $[x, x^{a^i}] = 1$.

As $[a, a^{x^i}] = 1$, either $\operatorname{supp}(\alpha_i^x) = \operatorname{supp}(\alpha_{i\sigma})$ for each $i = 1, \ldots, p$ and some p-cycle $\sigma \in \operatorname{Sym}(p)$, or $|\operatorname{supp}(\alpha_i^x) \cap \operatorname{supp}(\alpha_j)| = 1$ for all i, j. If the former case holds, then $A_x = p \cdot Y_\sigma$. If the latter case holds, then in particular $|\operatorname{supp}(\alpha_1^x) \cap \operatorname{supp}(\alpha_1)| = 1$. It follows that in the disjoint cycle decomposition of x, there exists a cycle whose support contains two elements of $\{1, \ldots, p\}$. By taking appropriate powers of a and x, we deduce that $\operatorname{ord}(a^i x^j) < p^2$ for some $1 \leqslant i, j \leqslant p-1$ and hence as $\langle a, x \rangle \cong W_p$, it follows that $\operatorname{ord}(a^i x^j) = p$. Hence

$$1 = (a^i) \cdot (a^i)^{x^{-j}} \cdot (a^i)^{x^{-2j}} \cdots (a^i)^{x^{(1-p)j}}.$$

Thus as the a^{x^m} pairwise commute, they must form an elementary abelian subgroup of order at most p^{p-1}. Moreover, as x acts on this subgroup via conjugation, it would follow that $|\langle a, x \rangle| \leqslant p^p < p^{p+1} = |W_p|$ and hence $\langle a, x \rangle \ncong W_p$. We conclude that the latter of the two cases cannot hold, and hence $A_x = p \cdot Y_\sigma$. Consequently $\operatorname{supp}(\alpha_i^x) = \operatorname{supp}(\alpha_{i\sigma})$ for each $i \in \{1, \ldots, p\}$.

Define $b_i := \min \operatorname{supp}(\chi_i)$ and write x as a $p \times p$ array as

$$\begin{matrix} (b_1, b_1\chi_1, b_1\chi_1^2, \ldots, b_1\chi_1^{p-1}) \\ (b_2, b_2\chi_2, b_2\chi_2^2, \ldots, b_2\chi_2^{p-1}) \\ \vdots \\ (b_p, b_p\chi_p, b_p\chi_p^2, \ldots, b_p\chi_p^{p-1}) \end{matrix} . \tag{5.2.5}$$

As $[a, a^{x^i}] = 1$ for all $i \in \{1, \ldots, p\}$, we see that $\alpha_i^x \in \langle \alpha_{i\sigma} \rangle$ for each $i \in \{1, \ldots, p\}$ and hence the columns of (5.2.5) correspond to powers of the α_i. Consequently, $A_a^x = \operatorname{circ}(0, c_1, \ldots, c_{p-1})$ for some $c_1, \ldots, c_p \in \mathbb{N}$ with the sum of the c_i's equal to p. In particular $X = 1$ is a root of the representer polynomial $f_{A_a^x}(X) \in \mathbb{Z}_p[X]$.

Define $\Gamma := \langle a^{x^j} | j = 1, \ldots, p \rangle$. Thus Γ is an elementary abelian p-group of rank p. Moreover $\alpha_1^{x^{j-1}} = \alpha_{1\sigma^{j-1}}^{e_{1\sigma^{j-1}}}$ for some non-zero $e_1, \ldots, e_p \in \mathbb{Z}_p$. Define a_i for $i = 1, \ldots, p$ recursively by $a_1 := \min\{i \mid c_i \neq 0\}$ and

$$a_i := \min \left\{ \ell \;\middle|\; \sum_{j=1}^{\ell} j \cdot c_j > \sum_{j=1}^{i-1} a_j \right\}$$

for $i > 1$.

To understand the correspondence between the a_i, c_i and e_i, we again consider (5.2.5). We see that the a_i give the number of rows descended in (5.2.5) to go from the entry in the first row of a column to its image under a, while each c_j gives the number of times that j rows are descended within a column. Thus $\sum_{i=0}^{p-1} i \cdot c_i = \sum_{i=1}^{p} a_i$. Meanwhile, the e_i give the exponent to which α_1 is raised under the image of successive powers of x, and hence these are precisely the inverses of the a_j.

Consequently, as multisets we have that

$$\{a_i \mid i = 1, \ldots, p\} = \{e_i^{-1} \mid i = 1, \ldots, p\}.$$

Appealing to Lemma 5.2.1 we see that, as Γ has rank p, then $a_1 + \cdots + a_p \not\equiv 0 \pmod p$. Consequently

$$f'_{A_a^x}(1) = \sum_{i=0}^{p-1} i \cdot c_i = \sum_{i=1}^{p} a_i \not\equiv 0 \pmod p,$$

and the representer polynomial of A_a^x has a simple root at $X = 1$ as required.

Conversely, assume that the conditions in (ii) hold. As $A_x = p \cdot Y_\sigma$, it follows that $\operatorname{supp}(\alpha_i^x) = \operatorname{supp}(\alpha_{i\sigma})$. Moreover, as $[a, a^x] = 1$ we have $\alpha_i^x \in \langle \alpha_{i\sigma} \rangle$. Defining $\Gamma := \langle a^{x^i} | i = 1, \ldots, p \rangle$ we deduce that Γ is an elementary abelian p-group upon which $\langle x \rangle$ acts, and $\Gamma \cap \langle x \rangle = \{1\}$. Thus it suffices to prove that $\operatorname{rank}(\Gamma) = p$. However, an analogous argument to that used above involving the a_i, c_i and e_i shows that this is equivalent to proving that $f'_{A_a^x}(1) \not\equiv 0 \pmod p$, which holds as $f_{A_a^x}(X)$ has a simple root at $X = 1$.

$\square$

We note that all of the conditions in Proposition 5.2.2 are necessary. Indeed, the necessity of the structure of A_x and A_a^x was evident in the proof of the proposition. However, if $p = 5$ and

$$\begin{aligned} x =&(1,6,11,16,21)(2,7,14,18,25)(3,9,15,17,23)\\ &(4,8,12,20,24)(5,10,13,19,22), \end{aligned}$$

then the matrices A_x and A_a^x are given by

$$A_x = 5 \cdot \pi \qquad \text{and} \qquad A_a^x = \operatorname{circ}(0,2,1,1,1)$$

both of which satisfy the conditions in the proposition. However, in this case $[a, a^x] \neq 1$. Indeed, $\langle a, x\rangle$ is a group of order $3,888,000,000$ and exponent 150.

5.2.2 A Worked Example

To illustrate what is happening in the proofs of Lemma 5.2.1 and Proposition 5.2.2, we give an explicit example. Indeed, let $p = 5$ and consider $G = \mathrm{Sym}(25)$, having standard 5-element $a = (1,2,3,4,5)\cdots(21,22,23,24,25) \in G$. Using the computer algebra system MAGMA we may take a random element, x, of the G-conjugacy class of a that satisfies $\langle a, x\rangle \cong W_5$, namely

$$\begin{aligned} x =&(1,6,11,16,21)(2,9,12,19,22)(3,7,13,17,23)\\ &(4,10,14,20,24)(5,8,15,18,25). \end{aligned}$$

We now consider the methodology from the proofs of Lemma 5.2.1 and Proposition 5.2.2 in this specific case.

Lemma 5.2.1

We have that

$$\begin{aligned} \alpha_1^x &= (6,9,7,10,8) = \alpha_2^3, & \alpha_1^{x^2} &= (11,12,13,14,15) = \alpha_3,\\ \alpha_1^{x^3} &= (16,19,17,20,18) = \alpha_4^3, & \alpha_1^{x^4} &= (21,22,23,24,25) = \alpha_5,\\ & & \alpha_1^{x^5} &= (1,2,3,4,5) = \alpha_1. \end{aligned}$$

Thus - using the notation from Lemma 5.2.1 - $\sigma = (1,2,3,4,5)$ and $(e_1,e_2,e_3,e_4,e_5) = (1,3,1,3,1)$. Conjugating a by successive powers of x gives

$$\begin{aligned}
a^x &= \alpha_1^1 \cdot \alpha_2^3 \cdot \alpha_3^2 \cdot \alpha_4^3 \cdot \alpha_5^2 &&= \alpha_1^{e_5^{-1}e_1} \cdot \alpha_2^{e_1^{-1}e_2} \cdot \alpha_3^{e_2^{-1}e_3} \cdot \alpha_4^{e_3^{-1}e_4} \cdot \alpha_5^{e_4^{-1}e_5},\\
a^{x^2} &= \alpha_1^2 \cdot \alpha_2^3 \cdot \alpha_3^1 \cdot \alpha_4^1 \cdot \alpha_5^1 &&= \alpha_1^{e_4^{-1}e_1} \cdot \alpha_2^{e_5^{-1}e_2} \cdot \alpha_3^{e_1^{-1}e_3} \cdot \alpha_4^{e_2^{-1}e_4} \cdot \alpha_5^{e_3^{-1}e_5},\\
a^{x^3} &= \alpha_1^1 \cdot \alpha_2^1 \cdot \alpha_3^1 \cdot \alpha_4^3 \cdot \alpha_5^2 &&= \alpha_1^{e_3^{-1}e_1} \cdot \alpha_2^{e_4^{-1}e_2} \cdot \alpha_3^{e_5^{-1}e_3} \cdot \alpha_4^{e_1^{-1}e_4} \cdot \alpha_5^{e_2^{-1}e_5},\\
a^{x^4} &= \alpha_1^2 \cdot \alpha_2^3 \cdot \alpha_3^2 \cdot \alpha_4^3 \cdot \alpha_5^1 &&= \alpha_1^{e_2^{-1}e_1} \cdot \alpha_2^{e_3^{-1}e_2} \cdot \alpha_3^{e_4^{-1}e_3} \cdot \alpha_4^{e_5^{-1}e_4} \cdot \alpha_5^{e_1^{-1}e_5},\\
a^{x^5} &= \alpha_1^1 \cdot \alpha_2^1 \cdot \alpha_3^1 \cdot \alpha_4^1 \cdot \alpha_5^1 &&= \alpha_1^{e_1^{-1}e_1} \cdot \alpha_2^{e_2^{-1}e_2} \cdot \alpha_3^{e_3^{-1}e_3} \cdot \alpha_4^{e_4^{-1}e_4} \cdot \alpha_5^{e_5^{-1}e_5}.
\end{aligned}$$

Hence we have confirmed the formula $a^{x^i} = \prod_{j=1}^{p} \alpha_{1\sigma^j}^{e_{1\sigma^{j-i}}^{-1} e_{1\sigma^j}}$ from the proof of the lemma. Consequently, a non-trivial relation between that a^{x^i} will give a non-trivial solution to $Ax \equiv 0 \pmod p$, where the matrix $A = (A_{\ell,j})$ has the form

$$A = \begin{pmatrix} e_5^{-1}e_1 & e_4^{-1}e_1 & e_3^{-1}e_1 & e_2^{-1}e_1 & e_1^{-1}e_1 \\ e_1^{-1}e_2 & e_5^{-1}e_2 & e_4^{-1}e_2 & e_3^{-1}e_2 & e_2^{-1}e_2 \\ e_2^{-1}e_3 & e_1^{-1}e_3 & e_5^{-1}e_3 & e_4^{-1}e_3 & e_3^{-1}e_3 \\ e_3^{-1}e_4 & e_2^{-1}e_4 & e_1^{-1}e_4 & e_5^{-1}e_4 & e_4^{-1}e_4 \\ e_4^{-1}e_5 & e_3^{-1}e_5 & e_2^{-1}e_5 & e_1^{-1}e_5 & e_5^{-1}e_5 \end{pmatrix}.$$

This has a non-trivial solution precisely when the matrix

$$\begin{aligned} B &= \begin{pmatrix} e_5^{-1} & e_4^{-1} & e_3^{-1} & e_2^{-1} & e_1^{-1} \\ e_1^{-1} & e_5^{-1} & e_4^{-1} & e_3^{-1} & e_2^{-1} \\ e_2^{-1} & e_1^{-1} & e_5^{-1} & e_4^{-1} & e_3^{-1} \\ e_3^{-1} & e_2^{-1} & e_1^{-1} & e_5^{-1} & e_4^{-1} \\ e_4^{-1} & e_3^{-1} & e_2^{-1} & e_1^{-1} & e_5^{-1} \end{pmatrix}. \\ &= \operatorname{circ}(e_5^{-1}, e_4^{-1}, e_3^{-1}, e_2^{-1}, e_1^{-1}) = \operatorname{circ}(1,2,1,2,1) \end{aligned}$$

has zero determinant (as the $e_i \in \mathbb{Z}_p$ are invertible). However, by Lemma 5.1.2 we have that

$$\det(B) \equiv e_5^{-1} + e_4^{-1} + e_3^{-1} + e_2^{-1} + e_1^{-1} \equiv 1+2+1+2+1 \equiv 2 \pmod 5.$$

Thus there are no non-trivial solutions to $Ax \equiv 0 \pmod 5$ and hence $\langle a^{x^j} | j = 0, \ldots, 4\rangle$ has rank 5 - as can easily be verified.

Proposition 5.2.2

By the work above we have that $[a, a^x] = 1$ and that $\langle a^{x^j} | j = 0, \ldots, 4\rangle$ is an elementary abelian 5-group of rank 5. Moreover

$$A_x = \begin{pmatrix} 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 5 & 0 & 0 & 0 & 0 \end{pmatrix} = 5 \cdot Y_{(1,2,3,4,5)} \qquad \text{and}$$

$$A_a^x = \begin{pmatrix} 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 3 & 2 \\ 2 & 0 & 0 & 0 & 3 \\ 3 & 2 & 0 & 0 & 0 \end{pmatrix} = \operatorname{circ}(0,3,2,0,0).$$

We define $(c_1, c_2, c_3, c_4) = (3, 2, 0, 0)$, $a_1 = \min\{i | c_i \neq 0\}$ and

$$a_i = \min\left\{ \ell \;\middle|\; \sum_{j=1}^{\ell} j \cdot c_j > \sum_{j=1}^{i-1} a_j \right\}$$

for $i = 2, \ldots, 5$. Thus $(a_1, a_2, a_3, a_4, a_5) = (1, 1, 1, 2, 2)$.

Finally, considering the representer polynomial of A_a^x we have that

$$f'_{A_a^x}(1) \equiv (3 + 4X)|_{X=1} \equiv 2 \equiv \sum_{i=1}^{5} a_i \not\equiv 0 \pmod 5$$

and hence $X = 1$ is a simple root of $f_{A_a^x}(X) \in \mathbb{Z}_5[X]$ as required.

We note that by considering x as a 5×5 array of the form

$$x = \begin{array}{l} (1,\ \ 6, 11, 16, 21) \\ (2,\ \ 9, 12, 19, 22) \\ (3,\ \ 7, 13, 17, 23) \\ (4, 10, 14, 20, 24) \\ (5,\ \ 8, 15, 18, 25) \end{array}, \tag{5.2.6}$$

the image of each entry in the first column of x under the action of a is in the same column, and one row below the original entry. Similarly, for columns $2, 3, 4$ and 5 the action of a corresponds to "descending" by $2, 1, 2$ and 1 rows respectively. Thus the image of 1 under $(ax)^5$ will be in the first column of x in (5.2.6), but will have descended $2 \equiv 1 + 2 + 1 + 2 + 1 \equiv \sum_{i=1}^{5} a_i \pmod 5$ rows.

5.3 An Equivalent Formulation

Our characterisation of $\langle a, x\rangle$ in terms of the matrices A_x and A_a^x given in Section 5.2 depended on the representer polynomial of a given circulant matrix. However, we may also consider the determinant of the circulant matrix. This characterisation relies on the following well known result regarding the determinant of a circulant matrix.

Theorem 5.3.1. *[KS12, Theorem 6] Let $C = \operatorname{circ}(c_0, c_1, \ldots, c_{n-1})$ be a circulant matrix with complex coefficients. Then*

$$\det(C) = \prod_{j=0}^{n-1} \left(\sum_{l=0}^{n-1} c_l \omega^{jl} \right),$$

where $\omega = \exp(2\pi i/n)$.

We now describe this approach using determinants.

Proposition 5.3.2. *Let $p \geqslant 3$ be a prime and let $c_1, \ldots, c_{p-1}$ be non-negative integers such that $\sum_{i=1}^{p-1} c_i = p$. Let $C = \operatorname{circ}(0, c_1, \ldots, c_{p-1})$, and let $f_C(X) = \sum_{i=1}^{p-1} c_i X^i \in \mathbb{Z}_p[X]$ be the representer polynomial of C. Then $|\det(C)|_p$ is divisible by p^2. Furthermore $|\det(C)|_p$ is divisible by p^3 if and only if $f_C(X) = 0$ or 1 is a root of $f_C(X)$ of multiplicity at least 2.*

Before proceeding to prove Proposition 5.3.2, we first give three general results that we will require in the proof.

Lemma 5.3.3. *Let $p \geqslant 3$ be a prime and $\omega = \exp(2\pi i/p)$ be a primitive p^{th} root of unity. Then $\mathbb{Q} \cap \mathbb{Z}[\omega] = \mathbb{Z}$. In particular, $1/p^i \notin \mathbb{Z}[\omega]$ for $i \geqslant 1$.*

Proof. We note that $\mathcal{S} = \{1, \omega, \ldots, \omega^{p-2}\}$ is a basis for both $\mathbb{Q}[\omega]$ over $\mathbb{Q}$ and $\mathbb{Z}[\omega]$ over $\mathbb{Z}$. Suppose that $\alpha \in (\mathbb{Q} \cap \mathbb{Z}[\omega])\backslash\mathbb{Z}$. Then α can be written as a $\mathbb{Z}$-linear combination of elements of $\mathcal{S}$, thus giving two distinct ways of writing α as a $\mathbb{Q}$-linear combination of elements of $\mathcal{S}$ in $\mathbb{Q}[\omega]$. This contradiction gives the result. □

Lemma 5.3.4. *Let $p \geqslant 3$ be a prime and let $\omega = \exp(2\pi i/p)$. Then*

(i) $\prod_{j=1}^{p-1}(1-\omega^{2j}) = p$; *and*

(ii) $\prod_{j=1}^{p-1}(\omega^j - \omega^{(p-1)j}) = p$.

Proof. Set $h(z) = z^p - 1 = \prod_{j=0}^{p-1}(z-\omega^{2j})$. Differentiating $h(z)$ with respect to z gives

$$pz^{p-1} = h'(z) = \prod_{j=1}^{p-1}(z-\omega^{2j}) + (z-1)\sum_{j=1}^{p-1}\prod_{\substack{i=1\\ i\neq j}}^{p-1}(z-\omega^{2i}).$$

It follows that $p = h'(1) = \prod_{j=1}^{p-1}(1-\omega^{2j})$ as required to prove (i).

Part (ii) then follows immediately since

$$\begin{aligned}\prod_{j=1}^{p-1}(\omega^j - \omega^{(p-1)j}) &= \prod_{j=1}^{p-1}\omega^j(1-\omega^{(p-2)j})\\ &= \omega^{p(p-1)/2}\prod_{j=1}^{p-1}(1-\omega^{(p-2)j})\\ &= \prod_{j=1}^{p-1}(1-\omega^{(p-2)j}) = \prod_{j=1}^{p-1}(1-\omega^{2j}) = p.\end{aligned}$$

□

Lemma 5.3.5. *Let $p \geqslant 3$ be a prime. Then*

$$1 + \sum_{i=2}^{p-1}\prod_{j=2}^{i}(1-j^{-1}) \equiv 0 \pmod{p}. \tag{5.3.7}$$

Proof. As the elements $(1-j^{-1})$ for $j = 2, \ldots, p-1$ are the non-zero, non-identity elements of $\mathbb{Z}_p$, it follows that $\prod_{j=2}^{p-1}(1-j^{-1}) \equiv p-1 \pmod{p}$. Thus (5.3.7) is equivalent to proving that $\sum_{i=2}^{p-2}\prod_{j=2}^{i}(1-j^{-1}) \equiv 0 \pmod{p}$. We will actually prove that

$$1 + \prod_{j=i+1}^{p-i}(1-j^{-1}) \equiv 0 \pmod{p} \tag{5.3.8}$$

for all $i = 2, \ldots, (p-1)/2$. This is sufficient to prove the result, since

$$\begin{aligned}\prod_{j=2}^{i}(1-j^{-1}) + \prod_{j=2}^{p-i}(1-j^{-1}) &= \prod_{j=2}^{i}(1-j^{-1})\left(1 + \prod_{j=i+1}^{p-i}(1-j^{-1})\right)\\ &\equiv 0 \pmod{p},\end{aligned}$$

from which the result follows immediately.

We prove that (5.3.8) holds by induction on $\ell = (p+1)/2 - i$. When $\ell = 1$, then $i = (p-1)/2$ and so (5.3.8) becomes $1 + (1 - ((p+1)/2)^{-1}) \equiv 0 \pmod p$, which clearly holds since $((p+1)/2)^{-1} \equiv 2 \pmod p$. Assume that (5.3.8) holds for $\ell = (p+1)/2 - k < (p+1)/2$. It follows that

$$\begin{aligned}
1 + \prod_{j=k}^{p-k+1} (1 - j^{-1}) &\equiv 1 + (1 - k^{-1})(1 - (p-k+1)^{-1}) \prod_{j=k+1}^{p-k} (1 - j^{-1}) \\
&\equiv 1 - (1 - k^{-1})(1 - (p-k+1)^{-1}) \\
&\equiv k^{-1} + (p-k+1)^{-1} - k^{-1}(p-k+1)^{-1} \\
&\equiv k^{-1}(p-k+1)^{-1}(p-k+1+k-1) \\
&\equiv 0 \pmod p.
\end{aligned}$$

Thus by induction (5.3.8) holds for all $i = 2, \ldots, (p-1)/2$, as required to complete the proof. □

We are now in a position to prove Proposition 5.3.2.

Proof of Proposition 5.3.2: If $c_i = p$ for some i, then $C = p \cdot \pi^i$. Thus the result clearly holds since $|\det(C)|_p = p^p$ so is divisible by p^3, whilst $f_C(X) = 0 \in \mathbb{Z}_p[X]$. Conversely, if $f_C(X) = 0 \in \mathbb{Z}_p[X]$, then some c_i must equal p, and hence $|\det(C)|_p = p^p$. Consequently, we may restrict to the case that $f_C(X) \neq 0$ and hence that $c_i < p$ for each $i = 1, \ldots, p-1$.

Consider the general form of the determinant of C and define

$$\begin{aligned}
q_j &= c_1\omega^j + c_2\omega^{2j} + \cdots + c_{p-2}\omega^{(p-2)j} - (c_1 + \cdots + c_{p-2})\omega^{(p-1)j} \\
&= c_1(\omega^j - \omega^{(p-1)j}) + c_2(\omega^{2j} - \omega^{(p-1)j}) + \cdots + c_{p-2}(\omega^{(p-2)j} - \omega^{(p-1)j}),
\end{aligned}$$

where $\omega = \exp(2\pi i/p)$. As $c_1 + \cdots + c_{p-1} = p$, applying Theorem 5.3.1 to $C = \mathrm{circ}(0, c_1, \ldots, c_{p-1})$ we have that

$$\begin{aligned}
\det(C) &= \prod_{j=0}^{p-1} \Big(c_1\omega^j + c_2\omega^{2j} + \cdots + c_{p-2}\omega^{(p-2)j} \\
&\qquad + (p - c_1 - \cdots - c_{p-2})\omega^{(p-1)j} \Big) \\
&= p \cdot \prod_{j=1}^{p-1} \Big(c_1\omega^j + c_2\omega^{2j} + \cdots + c_{p-2}\omega^{(p-2)j} \\
&\qquad + (p - c_1 - \cdots - c_{p-2})\omega^{(p-1)j} \Big) \\
&= p \cdot \prod_{j=1}^{p-1} (q_j + p \cdot w^{(p-1)j}).
\end{aligned}$$

We now extract the terms in this expansion with a factor of p, p^2 and at least p^3. Define

$$\alpha(C) = \sum_{i=1}^{p-1}\sum_{\substack{k=1\\k\neq i}}^{p-1} \omega^{-i-k} \prod_{\substack{j=1\\j\neq i,k}}^{p-1} (q_j + p\cdot\omega^{(p-1)j}),$$

$$\beta(C) = \sum_{i=1}^{p-1} \omega^{-i} \prod_{\substack{j=1\\j\neq i}}^{p-1} q_j, \qquad \text{and}$$

$$\gamma(C) = \prod_{j=1}^{p-1} q_j.$$

It follows that

$$\det(C) = p^3\cdot\alpha(C) + p^2\cdot\beta(C) + p\cdot\gamma(C) \tag{5.3.9}$$

We now consider $\beta(C)$ and $\gamma(C)$ in turn.

Claim 1: $\beta(C) = p\cdot\beta'(C)$ for some $\beta'(C)\in\mathbb{Z}[\omega]$.

Proof of Claim 1: Setting

$$\beta_j(C) = \frac{q_j}{w^j - w^{(p-1)j}},$$

we see that

$$\begin{aligned}\beta_j(C) = c_1 &+ c_2(1+\omega^{2j}+\omega^{4j}\cdots+\omega^{(p-1)j}+\omega^j)\\ &+ c_3(1+\omega^{2j}) + c_4(1+\omega^{2j}+\omega^{4j}\cdots+\omega^{(p-1)j}+\omega^j+\omega^{3j})\\ &+ c_5(1+\omega^{2j}+\omega^{4j}) + \cdots\cdots + c_{p-2}(1+\omega^{2j}+\omega^{4j}\cdots+\omega^{(p-3)j}).\end{aligned}$$

Consider the formulation of $\beta(C)$ as:

$$\begin{aligned}\beta(C) &= \sum_{i=1}^{p-1}\omega^{-i}\prod_{\substack{j=1\\j\neq i}}^{p-1}\Big(c_1(\omega^j-\omega^{(p-1)j}) + \cdots + c_{p-2}(\omega^{(p-2)j}-\omega^{(p-1)j})\Big)\\ &= \sum_{i=1}^{p-1}\omega^{-i}\prod_{\substack{j=1\\j\neq i}}^{p-1}(\omega^j-\omega^{(p-1)j})\beta_j(C)\\ &= \sum_{i=1}^{p-1}\omega^{-i}\prod_{\substack{j=1\\j\neq i}}^{p-1}(\omega^j-\omega^{(p-1)j})\prod_{\substack{j=1\\j\neq i}}^{p-1}\beta_j(C).\end{aligned} \tag{5.3.10}$$

If $k\equiv -(j+1)^{-1} \pmod p$, then $2k(j+1)+1\equiv -1 \pmod p$. Consequently

$$\delta_j := \omega + \omega^{2(j+1)+1} + \cdots + \omega^{2k(j+1)+1} + \cdots + \omega^{-1}$$

has $1-(j+1)^{-1} \pmod p$ terms. As a consequence, we have the relation

$$\begin{aligned}\omega^{p-j}-\omega^j &= (\omega^{p-j-1}-\omega^{j+1})(\omega+\omega^{2j+3}+\cdots+\omega^{2k(j+1)+1}+\cdots+\omega^{-1})\\ &= (\omega^{p-j-1}-\omega^{j+1})\delta_j\end{aligned} \tag{5.3.11}$$

for $j = 1, \ldots, p-2$.

Let $i \in \{1, \ldots, p-2\}$. By (5.3.11) we have that

$$\omega^{p-1} - \omega = \delta_1 \cdots \delta_{p-i-1}(\omega^i - \omega^{p-i}). \tag{5.3.12}$$

Applying (5.3.12) to (5.3.10) we obtained

$$\begin{aligned}
\beta(C) &= \omega \prod_{j=1}^{p-2}(\omega^j - \omega^{(p-1)j}) \prod_{j=1}^{p-2} \beta_j(C) + \sum_{i=1}^{p-2} \omega^{-i} \prod_{\substack{j=1\\ j\neq i}}^{p-1}(\omega^j - \omega^{(p-1)j}) \prod_{\substack{j=1\\ j\neq i}}^{p-1} \beta_j(C) \\
&= \omega \prod_{j=1}^{p-2}(\omega^j - \omega^{(p-1)j}) \prod_{j=1}^{p-2} \beta_j(C) + \sum_{i=1}^{p-2} w^{-i}\delta_1 \cdots \delta_{p-i-1} \prod_{j=1}^{p-2}(\omega^j - \omega^{(p-1)j}) \prod_{\substack{j=1\\ j\neq i}}^{p-1} \beta_j(C) \\
&= \Gamma(C) \prod_{j=1}^{p-2}(\omega^j - \omega^{(p-1)j}),
\end{aligned} \tag{5.3.13}$$

where

$$\Gamma(C) := \omega \prod_{j=1}^{p-2} \beta_j(C) + \sum_{i=1}^{p-2} \omega^{-i}\delta_1 \cdots \delta_{p-i-1} \prod_{\substack{j=1\\ j\neq i}}^{p-1} \beta_j(C).$$

Define $\Gamma(C)(X) \in \mathbb{Z}_p[X]$ to be the polynomial obtained by replacing every occurrence of ω in $\Gamma(C)$ by the indeterminant X. Similarly define $\beta_j(C)(X), \delta_j(X) \in \mathbb{Z}_p[X]$. Since $\beta_j(C)(1) = \beta_i(C)(1)$ for all $i, j = 1, \ldots, p-1$, we may denote this common value by ξ. Moreover, our observation above asserts that

$$\delta_j(1) \equiv 1 - (j+1)^{-1} \pmod{p}.$$

It follows that

$$\begin{aligned}
\Gamma(C)(1) &\equiv \xi^{p-2} + \sum_{i=1}^{p-2}(1-2^{-1}) \cdots (1-(p-i)^{-1})\xi^{p-2} \\
&\equiv \xi^{p-2}\left(1 + \sum_{i=1}^{p-2} \prod_{j=2}^{p-i}(1-j^{-1})\right) \\
&\equiv \xi^{p-2}\left(1 + \sum_{i=2}^{p-1} \prod_{j=2}^{i}(1-j^{-1})\right) \\
&\equiv 0 \pmod{p}
\end{aligned}$$

where the last equivalence arises from Lemma 5.3.5. Consequently, $(X-1)$ is a factor of $\Gamma(C)(X) \in \mathbb{Z}_p[X]$ and hence evaluating $\Gamma(C)(X)$ at $X = \omega$ yields

$$\Gamma(C) = (\omega - 1)F(\omega) + pG(\omega)$$

for some $F(\omega), G(\omega) \in \mathbb{Z}[\omega]$. Substituting into (5.3.13) gives

$$\beta(C) = \left(\prod_{j=1}^{p-2}(\omega^j - \omega^{(p-1)j})\right)((\omega-1)F(\omega) + pG(\omega)).$$

Finally, as

$$\omega - 1 = (\omega^{p-1} - \omega)(\omega^2 + \omega^4 + \cdots + \omega^{p-1})$$

we may appeal to Lemma 5.3.4(ii) to get

$$\begin{aligned}
\beta(C) &= \left(\prod_{j=1}^{p-1}(\omega^j - \omega^{(p-1)j})\right)(\omega^2 + \omega^4 + \cdots + \omega^{p-1})F(\omega) \\
&\quad + p\left(\prod_{j=1}^{p-2}(\omega^j - \omega^{(p-1)j})\right)G(\omega) \\
&= p(\omega^2 + \omega^4 + \cdots + \omega^{p-1})F(\omega) + p\left(\prod_{j=1}^{p-2}(\omega^j - \omega^{(p-1)j})\right)G(\omega) \\
&= p\beta'(C)
\end{aligned}$$

as required. •

Now we consider $\gamma(C)$.

Claim 2: $\gamma(C) = p \cdot \gamma'(C)$ for some $\gamma'(C) \in \mathbb{Z}$.

Proof of Claim 2: We may factorise $\gamma(C)$ as

$$\begin{aligned}
\gamma(C) &= \prod_{j=1}^{p-1}(\omega^j - \omega^{(p-1)j})\,(\,c_1 + c_2(1 + \omega^{2j} + \cdots + \omega^{(p-1)j} + \omega^j) + c_3(1 + \omega^{2j}) \\
&\quad + c_4(1 + \omega^{2j} + \cdots + \omega^{(p-1)j} + \omega^j + \omega^{3j}) + c_5(1 + \omega^{2j} + \omega^{4j}) + \cdots \\
&\quad \cdots + c_{p-2}(1 + \omega^{2j} + \cdots + \omega^{(p-3)j})\,) \\
&= \prod_{j=1}^{p-1}(\omega^j - \omega^{(p-1)j})\beta_j(C) \\
&= \prod_{j=1}^{p-1}(\omega^j - \omega^{(p-1)j})\prod_{j=1}^{p-1}\beta_j(C) \\
&= p \cdot \prod_{j=1}^{p-1}\beta_j(C)
\end{aligned} \tag{5.3.14}$$

where the last equality in (5.3.14) follows from Lemma 5.3.4(ii). If we consider $\prod_{j=1}^{p-1}\beta_j(C)$, then by taking s_i to be the coefficient of ω^{ij} in $\beta_j(C)$ we see that the s_i are non-negative integers and $s_0 = c_1 + \cdots + c_{p-2} = p - c_{p-1}$ is non-zero. Defining $S = \mathrm{circ}(s_0, s_1, \ldots, s_{p-1})$, we see that $\det(S) \in \mathbb{Z}$ and so

$$\prod_{j=1}^{p-1}\beta_j(C) = \frac{1}{s_0 + s_1 + \cdots + s_{p-1}}\det(S) \in \mathbb{Q} \cap \mathbb{Z}[\omega].$$

Appealing to Lemma 5.3.3, we conclude that $\prod_{j=1}^{p-1}\beta_j(C) \in \mathbb{Z}$ as required to prove Claim 2. •

It follows from (5.3.9) that

$$\det(C) = p^2\left(p\alpha(C) + p\beta'(C) + \gamma'(C)\right).$$

Since $p\alpha(C) + p\beta'(C) + \gamma'(C) = \det(C)/p^2 \in \mathbb{Q} \cap \mathbb{Z}[\omega] = \mathbb{Z}$, we conclude that $\det(C)$ is divisible by p^2. To complete the proof of the proposition, we make one final claim:

Claim 3: $\gamma'(C) = p \cdot \gamma''(C)$ for some $\gamma''(C) \in \mathbb{Z}[\omega]$ if and only if $X = 1$ is a root of $f_C(X) \in \mathbb{Z}_p[X]$ of multiplicity at least 2.

Proof of Claim 3: First note that $X = 1$ is a root of $f_C(X)$ of multiplicity at least 2 if and only if $X = 1$ is a root of the derivative $f'_C(X)$ of $f_C(X)$. This occurs precisely when $\sum_{i=1}^{p-1} i \cdot c_i \equiv 0 \pmod p$. Moreover

$$\begin{aligned}\sum_{i=1}^{p-1} i \cdot c_i &= \sum_{i=1}^{p-2} i \cdot c_i + (p-1)\left(p - \sum_{i=1}^{p-2} c_i\right) \\ &\equiv \sum_{i=1}^{p-2} (i+1) \cdot c_i \pmod p \\ &\equiv \sum_{i=1}^{(p-3)/2} \left(2i \cdot c_{2i-1} + (2i+1) \cdot c_{2i} \right) + (p-1) \cdot c_{p-2} \pmod p.\end{aligned}$$

Thus it suffices to prove that $\gamma'(C)$ has the desired factorisation precisely when

$$\sum_{i=1}^{(p-3)/2} \left(2i \cdot c_{2i-1} + (2i+1) \cdot c_{2i} \right) + (p-1) \cdot c_{p-2} \equiv 0 \pmod p.$$

In fact we will prove that

$$\gamma'(C) \equiv \left(\sum_{i=1}^{(p-3)/2} \left(i \cdot c_{2i-1} + \left(i + \frac{p+1}{2} \right) c_{2i} \right) + \frac{p-1}{2} \cdot c_{p-2} \right)^{p-1} \pmod p \tag{5.3.15}$$

We note that

$$\begin{aligned}\gamma'(C) &= \prod_{j=1}^{p-1} \big(c_1 + c_2(1 + \omega^{2j} + \cdots + \omega^{(p-1)j} + \omega^j) + c_3(1 + \omega^{2j}) \\ &\quad + c_4(1 + \omega^{2j} + \cdots + \omega^{(p-1)j} + \omega^j + \omega^{3j}) \\ &\quad + c_5(1 + \omega^{2j} + \omega^{4j}) + \cdots + c_{p-2}(1 + \omega^{2j} + \cdots + \omega^{(p-3)j}) \big) \\ &= \prod_{j=1}^{p-1} \Big((c_1 + c_3 + \cdots + c_{p-2} + c_2 + c_4 + \cdots + c_{p-3}) \\ &\quad + (c_3 + \cdots + c_{p-2} + c_2 + c_4 + \cdots + c_{p-3})\omega^{2j} \\ &\quad + (c_5 + \cdots + c_{p-2} + c_2 + c_4 + \cdots + c_{p-3})\omega^{4j} \\ &\quad + \cdots + (c_2 + c_4 + \cdots + c_{p-3})\omega^{(p-1)j} + (c_2 + c_4 + \cdots + c_{p-3})\omega^j \\ &\quad + (c_4 + c_6 + \cdots + c_{p-3})\omega^{3j} + \cdots + c_{p-3}\omega^{(p-4)j} \Big).\end{aligned} \tag{5.3.16}$$

Now let C' be the circulant matrix $C' = \mathrm{circ}(c'_0, \ldots, c'_{p-1})$ where c'_i is the coefficient of ω^{ij} in the j^{th} factor of (5.3.16) and set $\lambda := \sum_{i=0}^{p-1} c'_i$. Since $\gamma'(C) = \prod_{j=1}^{p-1} \sum_{i=0}^{p-1} c'_i \omega^{ij}$, we see that $\det(C') = \lambda \cdot \gamma'(C)$. An alternative formulation of λ is given by

$$\lambda = \left(\sum_{i=1}^{(p-3)/2} i \cdot c_{2i-1} + \left(i + \frac{p+1}{2} \right) c_{2i} \right) + \frac{p-1}{2} \cdot c_{p-2}. \tag{5.3.17}$$

There are now two possible cases to consider. First suppose that λ is divisible by p. Then an analogous argument to that used for the matrix C can be used to show that $\det(C')$ is divisible by p^2 (as the coefficient of $\omega^{(p-2)j}$ is zero, and so the matrix is of the required form). Since $\lambda < p^2$, it follows that $\gamma'(C)$ is divisible by p, and hence Claim 3 holds.

Now suppose that λ is not divisible by p. Then (5.3.15) becomes $\gamma'(C) \equiv 1 \pmod p$ by Fermat's Little Theorem. Combining our formulation of $\det(C')$ and Lemma 5.1.2 we have that $\lambda \equiv \det(C') \equiv \lambda \cdot \gamma'(C) \pmod p$, and hence as $\lambda \not\equiv 0 \pmod p$ it follows that $\gamma'(C) \equiv 1 \pmod p$ as required. This completes the proof of Claim 3. •

To complete the proof of Proposition 5.3.2 we recall that we have already reduced to the case that $f_C(X) \neq 0 \in \mathbb{Z}_p[X]$. Substituting the results of Claims 1 and 2 into (5.3.9) we obtain

$$\det(C) = p^3(\alpha(C) + \beta'(C)) + p^2\gamma'(C).$$

If $X = 1$ is a root of $f_C(X) \in \mathbb{Z}_p[X]$ of multiplicity at least 2, then by Claim 3 we obtain $\det(C) = p^3(\alpha(C)+\beta'(C)+\gamma''(C))$, and hence appealing to Lemma 5.3.3 we obtain $|\det(C)|_p \geqslant p^3$ as required. Conversely, if $X = 1$ is not a root of multiplicity at least 2 of $f_C(X)$, then Claim 3 asserts that there is no $\gamma''(C) \in \mathbb{Z}[\omega]$ such that $\gamma'(C) = p \cdot \gamma''(C)$. Thus $\det(C) = p^3(\alpha(C)+\beta'(C))+p^2\gamma'(C)$ and as $\gamma'(C) \in \mathbb{Z}$, appealing to Lemma 5.3.3 we obtain that $\det(C) = p^3m + p^2n$ where p and n are coprime. Thus $|\det(C)|_p = p^2$.

□

Theorem 5.0.1 now follows immediately from Propositions 5.2.2 and 5.3.2.

5.4 The $n = p^3$ Case

To consider the case that $G = \mathrm{Sym}(p^3)$, we follow a similar approach to that used in Section 5.2. We first develop the theoretical results in Subsections 5.4.1 and 5.4.2, before giving a worked example for the case that $p = 3$ in Subsection 5.4.3.

5.4.1 General Theory

In Section 5.3 we saw that the multiplicity of $X = 1$ as a root of the representer polynomial of given circulant matrices was important. We begin by noting that the multiplicity is preserved under cyclic shifts of our circulant matrix.

Lemma 5.4.1. *Let p be a prime and $C := \mathrm{circ}(c_0, c_1, \ldots, c_{p-1})$ be an integer circulant matrix. If $X = 1$ is a root of multiplicity i of $f_C(X) \in \mathbb{Z}_p[X]$, then it is a root of multiplicity i of $f_{\pi^j C}(X) \in \mathbb{Z}_p[X]$ for each $j = 0, \ldots, p-1$.*

Proof. Define $C_j = \pi^{-j}C$ for each $j = 0, \ldots, p-1$. We shall prove that $X = 1$ is a root of multiplicity i of the representer polynomial of each C_j. By assumption the result holds for $j = 0$ and hence assume it holds for $j = k$. We will prove the result holds for C_{k-1}. Indeed

$$\begin{aligned} f_{C_k}(X) &= c_k + c_{k+1}X + \cdots + c_{p-1}X^{p-k-1} + c_0X^{p-k} + \cdots + c_{k-1}X^{p-1} \\ &= (X-1)^i g(X) \end{aligned}$$

for some $g(X) \in \mathbb{Z}_p[X]$ with $g(1) \not\equiv 0 \pmod p$. Thus

$$\begin{aligned} f_{C_{k-1}}(X) &= c_{k-1} + c_k X + \cdots + c_{p-1}X^{p-k} + c_0 X^{p+1-k} + \cdots + c_{k-2}X^{p-1} \\ &= f_{C_k}(X) \cdot X + c_{k-1}(1 - X^p) \\ &= (X-1)^i g(X) \cdot X + c_{k-1}(1-X)^p. \end{aligned}$$

Hence $X = 1$ is a root of multiplicity i of $f_{C_{k-1}}(X)$, and thus by induction of $f_{C_j}(X)$ for each $j = 0, \ldots, p-1$. □

We may use Lemma 5.4.1 to consider quotients of W_p.

Lemma 5.4.2. *Let N be a non-trivial normal subgroup of W_p and define $Q := W_p/N$. Then* $\mathrm{Exp}\,(Q) \leqslant p$.

Proof. Denote the base group of W_p by Γ, so that $W_p = \Gamma \rtimes C_p$ where $C_p = \langle x \rangle$ - a cyclic group of order p. Assume for sake of contradiction that $\mathrm{Exp}\,(Q) = p^2$, and let $q \in Q$ satisfy $\mathrm{ord}(q) = p^2$. We consider elements of W_p to be ordered pairs of the form (γ, c) for $\gamma \in \Gamma$ and $c \in C_p$. Thus there exists $\gamma \in \Gamma$ such that $q = (\gamma, x^i)N$ for some $i = 1, \ldots, p-1$. Thus

$$q^p = (\gamma\gamma^{x^{-i}}\gamma^{x^{-2i}} \cdots \gamma^{x^{(1-p)i}}, 1)N = (\gamma\gamma^x\gamma^{x^2} \cdots \gamma^{x^{p-1}}, 1)N. \tag{5.4.18}$$

Moreover, $\gamma\gamma^x\gamma^{x^2} \cdots \gamma^{x^{p-1}} \notin N$ as $\mathrm{ord}(q) = p^2$.

If $N \nsubseteq \Gamma$, then there exists $\gamma_0 \in \Gamma$ such that $(\gamma_0, x^{-i}) \in N$. Consequently

$$q = (\gamma, x^i)(\gamma_0, x^{-i})N = (\gamma\gamma_0^{x^{-i}}, 1)N$$

contradicting the fact that q has order p^2. Thus $N \subseteq \Gamma$.

If $\Gamma = \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \times \cdots \times \langle \alpha_p \rangle$ where $\alpha_i^x = \alpha_{i-1}$ for $i = 2, \ldots, p$ and $\alpha_1^x = \alpha_p$, then from (5.4.18) we may deduce that

$$1 \neq \gamma\gamma^x\gamma^{x^2} \cdots \gamma^{x^{p-1}} = (\alpha_1\alpha_2 \cdots \alpha_p)^j \tag{5.4.19}$$

for some $j = 1, \ldots, p-1$. Moreover, for any $\delta \in \Gamma$ we have that

$$\delta\delta^x\delta^{x^2} \cdots \delta^{x^{p-1}} \in \langle \alpha_1\alpha_2 \cdots \alpha_p \rangle.$$

If there exists $\delta \in N$ such that $\delta\delta^x\delta^{x^2} \cdots \delta^{x^{p-1}} \neq 1$, then q^p is trivial in Q. Thus every $\delta \in N$ must satisfy $\delta\delta^x\delta^{x^2} \cdots \delta^{x^{p-1}} = 1$. Consequently, every $\delta \in N$ has the form

$$\delta = \alpha_1^{c_1}\alpha_2^{c_2} \cdots \alpha_p^{c_p} \tag{5.4.20}$$

with $c_1 + \cdots + c_p \equiv 0 \pmod p$.

Let $\delta \in N \backslash \{1\}$. Thus δ has the form (5.4.20). Set $C = \mathrm{circ}(c_1, c_2, \ldots, c_p)$ and let $f_C(X) \in \mathbb{Z}_p[X]$ be the corresponding representer polynomial. As $c_1 + \cdots + c_p \equiv 0 \pmod p$ we have that $X = 1$ is a root of $f_C(X)$. Assume that this root has multiplicity i. Thus $X = 1$ is a root of $f_C^{i-1}(X)$ but is not a root of $f_C^i(X)$. Hence set

$$\lambda := f_C^i(1) = c_{i+1} \cdot \frac{i!}{0!} + c_{i+2} \cdot \frac{(i+1)!}{1!} + \cdots + c_p \cdot \frac{(p-1)!}{(p-1-i)!} \in \mathbb{Z}_p^*.$$

Define d_j for $j = 1, \ldots, p$ by

$$d_{j+1} := \begin{cases} j!/(j-i)! & \text{if } j = i, \ldots, p-1; \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

so that $\lambda = \sum_{j=1}^{p} c_j d_j$.

Defining $C_j = \pi^{1-j} C$ for $j = 1, \ldots, p$ we have that $f^i_{C_1}(1) = f^i_C(1) = \lambda$. In fact $f^i_{C_k}(1) = \lambda$ for all $k = 1, \ldots p$. Indeed, by Lemma 5.4.1 we have that $f^{i-1}_{C_k}(1) = 0$ for each k. Moreover by iterating the calculations in the proof of the lemma we obtain

$$f_{C_k}(X) = f_{C_1}(X) \cdot X^{p-k+1} + (X-1)^p g_k(X)$$

for some $g_k(X) \in \mathbb{Z}_p[X]$. Consequently, $f^i_{C_k}(1) = f^i_{C_1}(1) = \lambda$. We conclude that

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_{p-1} & c_p \\ c_p & c_1 & \cdots & c_{p-2} & c_{p-1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_3 & c_4 & \cdots & c_1 & c_2 \\ c_2 & c_3 & \cdots & c_p & c_1 \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{p-1} \\ d_p \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \\ \lambda \end{bmatrix}.$$

As δ has the form given in (5.4.20) we see that

$$(\alpha_1 \alpha_2 \cdots \alpha_p)^\lambda = \prod_{i=1}^{p} (\delta^{x^{i-1}})^{d_i} \in N.$$

Thus by (5.4.18) and (5.4.19), $q^p = N$, contradicting the fact that $\mathrm{ord}(q) = p^2$. Thus the result holds. $\square$

A presentation of W_p was given by Drozd and Skuratovskii.

Theorem 5.4.3. *[DS08] The wreath product $C_p \wr C_p$ has a presentation given by*

$$C_p \wr C_p = \langle a, x \mid a^p = 1, x^p = 1, [a, a^{x^k}] = 1 \text{ for } 1 \leqslant k \leqslant (p-1)/2 \rangle.$$

Combining Corollary 5.1.4, Lemma 5.4.2 and Theorem 5.4.3 we obtain the following result.

Corollary 5.4.4. *Let G be a group and let $a, x \in G$ be elements of order p. Then $\langle a, x \rangle \cong W_p$ if and only if $ord(ax) = p^2$ and either*

(i) $[a, a^{x^i}] = 1$ for $i = 1, \ldots, (p-1)/2$; or

(ii) $[x, x^{a^i}] = 1$ for $i = 1, \ldots, (p-1)/2$.

Proof. Assume that $\langle a, x \rangle \cong W_p$. By Corollary 5.1.4 the desired commutator relations hold. Without loss assume that $[a, a^{x^i}] = 1$ for $i = 1, \ldots, (p-1)/2$. Thus a is in the base group of W_p. Moreover, the base group is defined by $\langle a^{x^i} \mid i = 0, \ldots, p-1 \rangle$ and hence $(ax)^p = a \cdot a^{x^{p-1}} \cdots a^x \neq 1$. Thus $\mathrm{ord}(ax) = p^2$.

Conversely, assume that the commutator relations hold. By Theorem 5.4.3 we have that $\langle a, x \rangle$ is isomorphic to a quotient of W_p. However, as $\mathrm{ord}(ax) = p^2$ this quotient must be W_p by Lemma 5.4.2. $\square$

To generalise the results of Sections 5.2 and 5.3 we begin by considering conjugate elements of full support in $\mathrm{Sym}(p^3)$. Indeed, let $G := \mathrm{Sym}(p^3)$, $a \in G$ be the standard p-element of G, $\mathcal{X} = a^G$ and let $x \in \mathcal{X}$. In such a situation the matrices A_x and A_a^x are both $p^2 \times p^2$ matrices. We make the following definition.

Definition 5.4.5. *Let $r > 1$ be an integer and let M be a $pr \times pr$ matrix. The* block sum matrix *of M, denoted $BS(M)$, is the $p \times p$ matrix given by*

$$(BS(M))_{i,j} := \sum_{v,w=0}^{r-1} M_{i+pv,j+pw}.$$

Given a $pr \times pr$ matrix M, we see that $BS(M)$ is the matrix obtained by partitioning M into $p \times p$ blocks and then forming the formal sum of these blocks. Of particular interest is the case when M can be partitioned into circulant blocks. In this case we see that $BS(M)$ is a circulant matrix.

We shall also consider a variant on the representer polynomial.

Definition 5.4.6. *Let M be a $pr \times pr$ matrix having $p \times p$ circulant blocks. We define the* reduced representer polynomial *of M to be the polynomial*

$$g_M(X) := g_0 + g_1 X + \cdots + g_{r-1} X^{r-1}$$

where

$$g_i := \sum_{j=0}^{r-1} \sum_{k=1}^{p} M_{jp+1,(j+i)p+k}$$

and $M_{jp+1,(j+i)p+k}$ is replaced by $M_{jp+1,(j+i-r)p+k}$ if $j + i \geqslant r$.

The reduced representer polynomial of the matrix M in some senses represents the matrix formed by replacing each $p \times p$ circulant block of M by its row/column sum.

5.4.2 The Results

Before giving a characterisation of which elements of $\mathcal{X}$ generate W_p with a, we make the following simple observation. If $\langle a^{x^i} \mid i = 0, \ldots, p-1 \rangle$ is an abelian group that is closed under conjugation by x, then

$$(ax)^p = a \cdot a^{x^{p-1}} \cdot a^{x^{p-2}} \cdots a^x.$$

In particular $(ax)^p$ is invariant under conjugation by both a and x and hence by $\langle a, x \rangle$.

We are now in a position to prove Theorem 5.0.2.

Proof of Theorem 5.0.2: Assume that $\langle a, x \rangle \cong W_p$ and denote the base group of W_p by Γ. By Corollary 5.1.4 we either have $\Gamma = \langle a^{x^i} | i = 0, \ldots, p-1 \rangle$ or $\Gamma = \langle x^{a^i} | i = 0, \ldots, p-1 \rangle$. Without loss of generality assume the former case holds. We will prove that the given conditions hold (an analogous argument may be used to show that in the latter case the conditions hold with the roles of a and x interchanged). By assumption the commutator relations hold and thus it remains to prove the conditions on A_x and A_a^x.

For distinct $i, j \in \{1, \dots, p^2\}$ such that $\mathrm{supp}(\alpha_i^x) \cap \mathrm{supp}(\alpha_j) \neq \varnothing$, we have that $|\mathrm{supp}(\alpha_i^x) \cap \mathrm{supp}(\alpha_j)| = 1$ or p. In the latter case the commutator relations assert that $\alpha_i^x \in \langle \alpha_j \rangle$ and this contributes an entry of p in the (i, j) position of A_x and zeros in all other entries of the i^{th} row and j^{th} column. In the former case, we see that there must be distinct $j_1 = j, j_2, \dots, j_p \in \{1, \dots, p^2\}$ such that $|\mathrm{supp}(\alpha_i^x) \cap \mathrm{supp}(\alpha_{j_\ell})| = 1$ for all $\ell = 1, \dots, p$. However, the commutator relations then imply that there exist distinct $i_1 = i, i_2, \dots, i_p \in \{1, \dots, p^2\}$ such that $\left|\mathrm{supp}(\alpha_{i_k}^x) \cap \mathrm{supp}(\alpha_{j_\ell})\right| = 1$ for all $k, \ell \in \{1, \dots, p\}$.

If every entry of A_x is equal to 0 or p, then we may decompose $a = a_1 \cdots a_p$ and $x = x_1 \cdots x_p$ with each pair (a_i, x_i) sitting inside a copy of $\mathrm{Sym}(p^2)$. Our assumptions on a and x together with the commutator relations assert that $\langle a_i, x_i \rangle$ is isomorphic to a proper quotient of W_p for each $i = 1, \dots, p$, and hence the same is true of $\langle a, x \rangle$. We conclude that there is a block of A_x in which every entry is equal to one and up to a suitable renumbering of the α_i, the matrix A_x has the required form.

Since A_x has the given form, it follows that $|\mathrm{supp}(\alpha_i) \cap \mathrm{supp}(\chi_j)| = 0$ or 1 for all $i, j \in \{1, \dots, p^2\}$. Thus we see that $w \in \mathrm{supp}(\chi_w)$ for $w \in \{1, \dots, p^2\}$. Setting $q := p^2 - p$, we consider x as

$$x = \begin{matrix} \left(1, 1 \cdot \chi_1, \dots, 1 \cdot \chi_1^{p-1}\right) & \cdots & \left(q+1, (q+1) \cdot \chi_{q+1}, \dots, (q+1) \cdot \chi_{q+1}^{p-1}\right) \\ \left(2, 2 \cdot \chi_2, \dots, 2 \cdot \chi_2^{p-1}\right) & \cdots & \left(q+2, (q+2) \cdot \chi_{q+2}, \dots, (q+2) \cdot \chi_{q+2}^{p-1}\right) \\ \vdots & & \vdots \\ \left(p, p \cdot \chi_p, \dots, p \cdot \chi_p^{p-1}\right) & \cdots & \left(p^2, (p^2) \cdot \chi_{p^2}, \dots, (p^2) \cdot \chi_{p^2}^{p-1}\right). \end{matrix} \tag{5.4.21}$$

In the subsequent work, we shall refer to the columns of x according to (5.4.21). By this, we mean that the first column of x is equal to $\{1, 2, \dots, p\}$, the second column is equal to $\{1 \cdot \chi_1, 2 \cdot \chi_2, \dots, p \cdot \chi_p\}$ and so on. We will also refer to the permutation σ_i corresponding to column i. This will be the permutation defined on column i by a cyclic permuting of the entries. Thus $\sigma_1 = (1, 2, \dots, p)$, $\sigma_2 = (1 \cdot \chi_1, 2 \cdot \chi_2, \dots, p \cdot \chi_p)$, and so on.

We consider the structure of A_a^x via the structure of A_x. First suppose that $\mathrm{supp}(\alpha_i^{x^\ell}) = \mathrm{supp}(\alpha_j)$ for some $i = 1, \dots, p$, $j = 1, \dots, p^2$ and $\ell = 1, \dots, p$. As noted above it follows that $\alpha_i^{x^\ell} \in \langle \alpha_j \rangle$, and hence the column of x in (5.4.21) corresponding to $\alpha_i^{x^\ell}$ will contribute a power of π, say π^r, to one diagonal block of A_a^x. Indeed, this is the only way that a diagonal block of A_a^x can be constructed. Moreover, as there is at least one block of A_x consisting of constant 1s, there exits some $q \in \{1, \dots, p\}$ such that none of the q^{th}, $(q+p)^{\text{th}}$, $\dots$, $(q + (p^2 - p))^{\text{th}}$ columns of x are equal to the support of some α_j. For each such set of columns, the commutator relations then ensure that the combination of the columns add circulant blocks to the matrix A_a^x corresponding to the way that these columns are mapped onto one another by a. It follows that the matrix A_a^x is a block matrix with $p \times p$ circulant blocks. Finally, as there exists at least one set of p columns as defined above, we may use these together with the commutator relations to see that the diagonal blocks of A_a^x must be equal.

It remains to prove that the block sum matrix or the reduced representer polynomial of A_a^x has the desired form. To do this we consider $y := (ax)^p$. The commutator relations ensure that the columns of x in (5.4.21) are permuted under the action of a and hence the i^{th} column of x in (5.4.21) is mapped to the $(pr+i)^{\text{th}}$ column of x under y for some value of r. First assume that each column of x is invariant under y. It follows from the commutator relations that $y = \sigma_1^{i_1}\sigma_2^{i_2}\cdots\sigma_{p^2}^{i_{p^2}}$ for some $i_1, \dots, i_{p^2} \in \{0, \dots, p-1\}$. However, as y is invariant under conjugation by $\langle a, x\rangle$ we may conclude that the i_j are all equal.

We now construct polynomials $f_j(X) \in \mathbb{Z}_p[X]$ for $j = 1, \dots, p$ recursively. To do this initially set $f_j(X) = 0$ - the zero polynomial. Now consider $((j-1)p+1)\cdot ax$. Redefine $f_j(X)$ to be $f_j(X) := f_j(X) + X^r$ where r is the number of rows descended in (5.4.21) to go from the entry $((j-1)p+1)$ to the entry $((j-1)p+1)\cdot ax$. Repeat this with $((j-1)p+1)$ replaced by $((j-1)p+1)\cdot (ax)^k$ for $k = 1, \dots, p-1$. Thus $f_j(X)$ encodes the circulant nature of the action of a on x in the columns of (5.4.21) corresponding to the orbit of the $((j-1)p+1)^{\text{th}}$ column under the action of $\langle ax\rangle$. We see that the contribution of these columns to $BS(A_a^x)$ is precisely the $p\times p$ circulant matrix for which $f_j(X)$ is the representer polynomial. Moreover, a similar analysis to that used in Section 5.2 shows that $X = 1$ is a simple root of $f_j(X) \in \mathbb{Z}_p[X]$ precisely when $i_{(j-1)p+1} \not\equiv 0 \pmod p$ since

$$f_j'(1) \equiv \begin{cases} 0 & \text{if } i_{(j-1)p+1} \equiv 0 \pmod p; \\ i_{(j-1)p+1}^{-1} & \text{otherwise.} \end{cases} \tag{5.4.22}$$

Considering the representer polynomial of the block sum matrix $BS(A_a^x)$ we see that

$$f_{BS(A_a^x)}(X) = \sum_{j=1}^{p} f_j(X).$$

Since the i_j are all equal, we see that $f'_{BS(A_a^x)}(1) \equiv 0 \pmod p$ and $X = 1$ is a root of multiplicity at least two of the representer polynomial of $BS(A_a^x)$.

To obtain the final conclusion we note that as the i_j are all equal, we have that $f_1'(1) = f_2'(1) = \cdots = f_p'(1)$. Combining this with (5.4.22) we see that $y = 1$ precisely when $f_i'(1) = pk$ for some $k \in \mathbb{N}\backslash\{0\}$ and for all $i = 1, \dots, p$. This occurs precisely when

$$f'_{BS(A_a^x)}(1) = \sum_{j=1}^{p} f_j'(1) = kp^2.$$

Thus $X = 1$ is a root of $f'_{BS(A_a^x)}(X) \in \mathbb{Z}_{p^2}[X]$ precisely when $y = 1$. Hence as $\langle a, x\rangle \cong W_p$ we conclude that $X = 1$ is not a root of $f'_{BS(A_a^x)}(X) \in \mathbb{Z}_{p^2}[X]$. Hence if the columns of x in (5.4.21) are invariant under the action of y, then case $(c1)$ holds.

Now assume that the columns of x are not fixed by y. We shall prove that case $(c2)$ must be true. Define block j of x to be the set of columns $jp+1, \dots, jp+p$ of x in (5.4.21) for $j = 0, \dots, p-1$ and consider the matrix A_a^x. Each entry of A_a^x corresponds to the image of a column of x under a. There are two possibilities; either the column i is left invariant under the action of a, or it is mapped onto the column $pr+i$ for some $1 \leqslant r \leqslant p-1$. This corresponds to the column being mapped from block j to block $j+r \pmod p$. The former case occurs precisely when a power of

π is added to a diagonal block of A_a^x, whilst in the latter case, a power of π is added to an off diagonal block of A_a^x which is r blocks to the right of the leading diagonal. We see that the partial representer polynomial $g_{A_a^x}(X)$ of A_a^x is of the form

$$g_{A_a^x}(X) = g_0 + g_1 X^1 + \cdots + g_{p-1} X^{p-1},$$

where for each $r = 0, \ldots, p-1$ the coefficient g_r is the number of columns i of x in (5.4.21) that are mapped to the column $i + pr$. Equivalently,

$$g_r = \sum_{j=0}^{p-1} \left| \left\{ i \;\middle|\; \begin{array}{c} \text{column } i \text{ is in block } j \text{ and is mapped to a column in} \\ \text{block } j + r \pmod p \text{ under the action of } a \end{array} \right\} \right|.$$

Consider the action of y on the column i of x. This corresponds to applying ax to the column p times. It follows that the action of y is determined by the p successive actions of a. Since each column of x has an orbit of size p^2 under the action of ax, we see that each of the actions of a on the columns of x occurs p times in the action of y on the columns of x. Thus to obtain the number of blocks that column i has passed through in reaching its image under the action of y, we need to sum the number of blocks passed through for each successive action of a. It follows that the total number of blocks passed through by all columns of x under the action of y is

$$p \cdot \sum_{r=0}^{p-1} r \cdot g_r = p \cdot g'_{A_a^x}(1). \tag{5.4.23}$$

Since y is invariant under the action of x, we see that if column i in block j is mapped to column $i + pr$ in block $j + r \pmod p$ under the action of y, then every column of block j will be mapped to the corresponding column in block $j + r \pmod p$ under the action of y. Hence y simply permutes the blocks of x.

Suppose that block j of x passes through $p \cdot k_j + \ell_j$ blocks under the action of y for some $k_j, \ell_j \in \{0, \ldots, p-1\}$. Since y is invariant under the action of a, we deduce that the k_j are all equal to some common value say k. It follows that the total number of blocks passed through by all columns of x under the action of y is

$$\sum_{j=0}^{p-1} p \cdot (p \cdot k + \ell_j) = p \left(p^2 k + \sum_{j=0}^{p-1} \ell_j \right). \tag{5.4.24}$$

Combining (5.4.23) and (5.4.24) we obtain

$$g'_{A_a^x}(1) \equiv \sum_{j=0}^{p-1} \ell_j \pmod{p^2}.$$

Finally, we note that $\sum_{j=0}^{p-1} \ell_j \equiv 0 \pmod{p^2}$ precisely when $\ell_j = 0$ for each $j = 0, \ldots, p-1$. Hence $g'_{A_a^x}(1) \equiv 0 \pmod{p^2}$ precisely when the columns of x in (5.4.21) are fixed by y. Since by assumption the columns of x are permuted by y, we conclude that case $(c2)$ holds.

Conversely assume that the given conditions hold (again the argument for when a and x are interchanged is analogous). By the above we see that $(ax)^p \neq 1$. Combining this with the commutator relations and Corollary 5.4.4 we obtain that $\langle a, x \rangle \cong W_p$, as required. $\square$

As in Section 5.2 we note that the conditions given in Theorem 5.0.2 are all required. As before this is evident in the proof of the theorem for the conditions on the matrices A_x and A_a^x. To see that the commutator relations are all necessary, we note that if $G := \mathrm{Sym}(125)$ and a is the standard 5-element of G, then we may take

$$\begin{aligned} x =& (1,26,51,76,101)(2,27,56,79,106)(3,28,61,77,111)(4,29,66,80,116) \\ & (5,30,71,78,121)(6,41,54,91,117)(7,42,59,94,122)(8,43,64,92,102) \\ & (9,44,69,95,107)(10,45,74,93,112)(11,49,72,83,110)(12,50,52,81,115) \\ & (13,46,57,84,120)(14,47,62,82,125)(15,48,67,85,105)(16,36,68,90,118) \\ & (17,37,73,88,123)(18,38,53,86,103)(19,39,58,89,108)(20,40,63,87,113) \\ & (21,32,70,100,119)(22,33,75,98,124)(23,34,55,96,104)(24,35,60,99,109) \\ & (25,31,65,97,114). \end{aligned}$$

The reader may check that A_x and A_a^x have the required form and $[a, a^x] = 1$. However, $[a, a^{x^2}] \neq 1$.

5.4.3 A Worked Example

As in Section 5.2 we follow the technical steps of the proof of Theorem 5.0.2 in the case of a couple of specific examples. Indeed, consider $G := \mathrm{Sym}(27)$ and the following G-conjugates of the standard 3-element a of G,

$$x_1 = \begin{array}{l} (1,10,20)(4,14,22)(7,18,27) \\ (2,13,23)(5,17,25)(8,12,21) \\ (3,16,26)(6,11,19)(9,15,24) \end{array} \tag{5.4.25}$$

and

$$x_2 = \begin{array}{l} (1,10,22)(4,14,19)(7,12,25) \\ (2,13,23)(5,17,20)(8,15,26) \\ (3,16,24)(6,11,21)(9,18,27) \end{array} . \tag{5.4.26}$$

It is easily seen that $\langle a, x_1 \rangle \cong \langle a, x_2 \rangle \cong W_3$. We shall show that for the pair (a, x_1) conditions (i), (ii) and $(iii)(c1)$ of Theorem 5.0.2 hold, whilst for the pair (a, x_2) conditions (i), (ii) and $(iii)(c2)$ hold.

$\boldsymbol{(a, x_1)}$ **:**

As in the proof of Theorem 5.0.2 we consider the columns of x_1 in (5.4.25) and note that columns

2, 5 and 8 ensure that the matrix A_{x_1} has a block of constant ones. Indeed

$$A_{x_1} = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

whilst

$$A_a^{x_1} = \left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

Consider the representation of x_1 in (5.4.25) and the element

$$\begin{aligned} y_1 := (ax_1)^3 =& (1,2,3)(4,5,6)(7,8,9)(10,13,16)(11,14,17) \\ & (12,15,18)(19,22,25)(20,23,26)(21,24,27). \end{aligned}$$

Denoting the permutation corresponding to the i^{th} column of x_1 by σ_i we see that

$$y_1 = \sigma_1^{i_1} \cdot \sigma_2^{i_2} \cdot \sigma_3^{i_3} \cdot \sigma_4^{i_4} \cdot \sigma_5^{i_5} \cdot \sigma_6^{i_6} \cdot \sigma_7^{i_7} \cdot \sigma_8^{i_8} \cdot \sigma_9^{i_9},$$

where $i_1 = i_2 = \cdots = i_9 = 1$.

We now define the polynomials $f_j(X) \in \mathbb{Z}_3[X]$ as described in the proof of the theorem. First set $f_1(X) := 0 \in \mathbb{Z}_3[X]$ and consider the orbit of 1 under ax_1. Since $1^{ax_1} = 13$ and 13 is in the second row of x_1 in (5.4.25), we have "descended" by 1 row. Hence we redefine $f_1(X) := f_1(X) + X^1 = X^1$. As $13^{ax_1} = 22$ and 22 is in the first row of x_1 we have cyclically descended a further 2 rows, and so we set $f_1(X) := f_1(X) + X^2 = X + X^2$. Finally, $22^{ax_1} = 2$ and 2 is in the second row of x_1. Thus we have descended by a further row, and our final polynomial is

$$f_1(X) := f_1(X) + X^1 = 2X + X^2 \in \mathbb{Z}_3[X].$$

By considering the images of the first column of x_1 under the action of $(ax_1)^1$, $(ax_1)^2$ and $(ax_1)^3$

we see that we have accounted for an entry of 1 in each of the circled entries of

$$A_a^{x_1} = \left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right). \tag{5.4.27}$$

Hence the contribution to the block sum matrix $BS(A_a^{x_1})$ from these columns is

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} = \mathrm{circ}(0, 2, 1).$$

This is the matrix for which $f_1(X)$ is the representer polynomial.

Similarly $f_2(X) = 2X + X^2 \in \mathbb{Z}_3[X]$, which accounts for additional entries of 1 in (5.4.27). This gives

$$A_a^{x_1} = \left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right). \tag{5.4.28}$$

Finally, $f_3(X) = 2X + X^2 \in \mathbb{Z}_3[X]$ which contributes a 1 to the remaining entries of (5.4.28). We note that in general, it is not the case that the $f_i(X) \in \mathbb{Z}_p[X]$ are all equal.

The final step of the proof of Theorem 5.0.2 considers the representer polynomials $f_1(X), f_2(X), f_3(X) \in \mathbb{Z}_3[X]$ and $f_{BS(A_a^x)}(X) \in \mathbb{Z}_9[X]$. We see that $f_j'(1) = 1 = i_j^{-1} \in \mathbb{Z}_3$ for all j, as given in (5.4.22). Moreover, considering $f_j'(X) \in \mathbb{Z}_9(X)$ we have that $f_j'(1) = 4$ for all j. Consequently

$$f'_{BS(A_a^{x_1})}(1) = \sum_{j=1}^{3} f_j'(1) = 3 \neq 0 \in \mathbb{Z}_9$$

and case $(c1)$ holds.

$(\boldsymbol{a}, \boldsymbol{x_2})$ **:**

As with x_1, we see that columns 2, 5 and 8 of x_2 in (5.4.26) ensure that A_{x_2} has the desired form, namely

$$A_{x_2} = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

whilst

$$A_a^{x_2} = \left(\begin{array}{ccc|ccc|ccc} 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \end{array}\right).$$

In this case we see that

$$\begin{aligned} y_2 := (ax_2)^3 =& (1,5,8)(2,6,9)(3,4,7)(10,17,15)(11,18,13) \\ & (12,16,14)(19,25,24)(20,26,22)(21,27,23) \end{aligned}$$

and so clearly y_2 does not fix the columns of x_2 in (5.4.26). We thus consider the blocks of columns of x_2 in (5.4.26), where block 0 is given by the first three columns, block 1 by the middle three columns and block 2 by the final three columns. Denoting the columns of x_2 by $c_1, \ldots, c_9$ we see that the action of a on the columns of x_2 is given by $c_2 \mapsto c_5$, $c_5 \mapsto c_8$, $c_8 \mapsto c_2$ and $c_i \mapsto c_i$ for $i = 1, 3, 4, 6, 7, 9$. It follows that there are six columns whose image under a stays in the same block, meaning that $g_0 = 6$. The remaining three columns are mapped from block i to block $i+1$ (mod 3) for some $i = 0, 1, 2$. Hence $g_1 = 3$ and $g_2 = 0$.

Considering the reduced representer polynomial of $A_a^{x_2}$ we have that

$$g_{A_a^{x_2}}(X) = \sum_{j=0}^{2} g_i X^i = 6 + 3X.$$

Moreover we see that the columns of x_2 in (5.4.26) are permuted under the action of ax_2 in the following cyclic manner

$$c_1 \mapsto c_2 \mapsto c_6 \mapsto c_4 \mapsto c_5 \mapsto c_9 \mapsto c_7 \mapsto c_8 \mapsto c_3 \mapsto c_1.$$

It follows that column i is mapped under y_2 to column $i+3$ (or $i-6$ if $i+3>9$) and in doing so passes through a single block of x_2. Thus using the notation of (5.4.24) we have that $k_j = 0$ and $\ell_j = 1$ for all $j = 0,1,2$.

Consequently the total number of blocks passed through by all columns of x_2 under the action of y_2 is

$$9 = 3 \cdot g'_{A_a^{x_2}}(1)$$

and

$$g'_{A_a^{x_2}}(1) \equiv 3 \equiv \sum_{j=0}^{2} \ell_j \pmod 9.$$

In particular $g'_{A_a^{x_2}}(1) \not\equiv 0 \pmod 9$ and case $(c2)$ holds.

5.5 The General Case

We now consider the most general setting that $G := \mathrm{Sym}(n)$ for some $n \geqslant p^2$ and let

$$a = (1,2,\ldots,p)(p+1,p+2,\ldots,2p)\cdots(p(r-1)+1,p(r-1)+2,\ldots,pr) \in G$$

for some $r \geqslant p$. As previously we set $\mathcal{X} := a^G$ and consider $x \in \mathcal{X}$. Denote the p-cycles forming a (respectively x) by $\alpha_1,\ldots,\alpha_r$ (respectively $\chi_1,\ldots,\chi_r$). We note that if $\langle a,x\rangle \cong W_p$, then $|\mathrm{supp}(\alpha_i) \cap \mathrm{supp}(\chi_j)| = 0, 1$ or p for each $i,j \in \{1,\ldots,r\}$.

Before considering the most general situation, we look at the case that $\mathrm{supp}(a) = \mathrm{supp}(x)$ and prove Theorem 5.0.3.

Proof of Theorem 5.0.3: Assume that $\langle a,x\rangle \cong W_p$. Without loss of generality assume that a is in the base group of W_p. We will show that the given conditions hold true (if x is in the base group, then an analogous argument may be used to show that the conditions hold with the roles of a and x interchanged). The commutator relations follow immediately.

First consider the case that $\mathrm{supp}(\alpha_i) = \mathrm{supp}(\chi_j)$ for some $i,j \in \{1,\ldots,r\}$. This results in matrix entries $(A_x)_{i,i} = (A_a^x)_{j,j} = p$. By a suitable renumbering of the α_i and χ_j, we may assume that such a situation arises for pairs (α_i,χ_i) with $i = 1,\ldots,m$ for some m. This gives the diagonal block D_1 of A_x and E_1 of A_a^x. It follows that for all other α_i we have $\mathrm{supp}(\alpha_i^x) \cap \mathrm{supp}(\alpha_i) = \varnothing$.

If $\alpha_i^{x^\ell} \in \langle \alpha_{i_\ell}\rangle$ for $\ell = 1,\ldots,p$ and for some i_ℓ, then we may consider these α_{i_l} and the χ_j that share their common support as lying inside a copy of $\mathrm{Sym}(p^2)$. Thus we may apply the results of Section 5.2 to obtain the general structures of D_2 and E_2. In all other cases, we use a similar approach to find a subset $I \subseteq \{1,\ldots,r\}$ of size p^2 such that $\prod_{i\in I}\alpha_i$ and $\prod_{i\in I}\chi_i$ are conjugate elements of full support inside a copy of $\mathrm{Sym}(p^3)$. The results of Section 5.4 then give the structures of D_3 and E_3. The final condition on one of the blocks of E_2 or E_3 then follows from the fact that ax has order p^2 and the results of Sections 5.2 and 5.4.

Conversely, assume that the conditions in Theorem 5.0.3 hold (again the argument for when a and x are interchanged is analogous). By Corollary 5.4.4 it remains to check that ax has order p^2.

Suppose that D_1 is an $m_1 \times m_1$ matrix, D_2 is a $pm_2 \times pm_2$ matrix and D_3 is a $p^2m_3 \times p^2m_3$ matrix for some m_1, m_2 and m_3. Define $a_1 = \alpha_1 \cdots \alpha_{m_1}$,

$$\begin{aligned} a_{2,i} &= \alpha_{m_1+p(i-1)+1} \cdots \alpha_{m_1+p(i-1)+p} && \text{for } i = 1, \ldots, m_2, \text{ and} \\ a_{3,j} &= \alpha_{m_1+pm_2+p^2(j-1)+1} \cdots \alpha_{m_1+pm_2+p^2(j-1)+p^2} && \text{for } j = 1, \ldots, m_3. \end{aligned}$$

Define $x_1, x_{2,1}, \ldots, x_{2,m_2}, x_{3,1}, \ldots, x_{3,m_3}$ analogously. We see that the pair (a_1, x_1) corresponds to the blocks D_1 and E_1 of A_x and A_a^x, each pair $(a_{2,i}, x_{2,i})$ corresponds to a block of D_2 and its associated block in E_2, whilst every pair $(a_{3,j}, x_{3,j})$ corresponds to blocks of D_3 and E_3. Moreover, since for each pair (a_*, x_*) we have $\mathrm{supp}(a_*) = \mathrm{supp}(x_*)$ and $[a_*, a_*^{x_*^i}] = 1$, it suffices to check that for one such pair, the element $a_*x_* \in \mathrm{Sym}(\mathrm{supp}(a_*))$ has order p^2. However, this follows immediately from condition (iv) and the results of Sections 5.2 and 5.4. Thus $\langle a, x\rangle \cong W_p$. $\square$

In the general setting, we note that by the preceding arguments we must have that $|\mathrm{supp}(a) \cap \mathrm{supp}(x)|$ is divisible by p. Thus the above result will still hold if $n = pr + s$ for some $s < p$. However this is not the case in general, since if we no longer require $\mathrm{supp}(a) = \mathrm{supp}(x)$, the condition that a and x are G-conjugate becomes weaker. Indeed, if we consider $G = \mathrm{Sym}(p^2)$, we have that W_p is embedded into G using the generators $a = (1, 2, \ldots, p)$ and

$$x = (1, p+1, \ldots, p(p-1)+1)(2, p+2, \ldots, p(p-1)+2) \cdots (p, 2p, \ldots, p^2).$$

However, we may also use x and y to generate W_p, where

$$\begin{aligned} y = (1, 2, \ldots, p)&(p+1, p+2, \ldots, 2p) \cdots \\ &\cdots (p(p-2)+1, p(p-2)+2, \ldots, p(p-1)). \end{aligned}$$

If we now relax our assumption that $\mathrm{supp}(a) = \mathrm{supp}(x)$ and move into $G := \mathrm{Sym}(p(p+1))$, we see that taking $x \in G$ as above and

$$\overline{y} = y \cdot (p^2+1, p^2+2, \ldots, p(p+1)),$$

we have that $\langle x, \overline{y}\rangle \cong W_p$. However, it is clear that the generation of W_p results from the elements $x, y \in \mathrm{Sym}(p^2) \hookrightarrow G$. It follows that although x and $\overline{y}$ are G-conjugate, this conjugation is in some sense artificial. Consequently, producing a theorem such as Theorem 5.0.3 would not be realistic in the most general setting.

Chapter 6

Cuspidal Characters of Finite Groups

Let G be a finite group of Lie type. In Chapters 2 and 3 we saw how the building, Δ, of G is a rich structure which admits the construction of modules for G over finite fields from modules defined on smaller subgroups known as parabolic subgroups. We then mirrored the structure of a building in the form of minimal parabolic systems, with appropriate groups being known as parabolic subgroups. Each parabolic subgroup, P, contained in Δ admits an amiable decomposition as $P = U \cdot L$, with unipotent radical U and Levi complement L. For an arbitrary finite group, we saw how the p-core of a subgroup can in some senses replicate the role of the unipotent radical. However, in general there will be no complement to the p-core of a parabolic subgroup. The subsequent simplicial complex of parabolic subgroups then permitted the construction of modules over finite fields in a comparable manner to the motivating case of groups of Lie type.

Stepping away from the building site of construction work, we may return to the familiar setting of characters of finite groups defined over the complex numbers. Although a great knowledge of such characters already exists, there is a natural desire to gain a deeper understanding of them. One way to gain such insight is to classify the known characters of a given group, G, dependant on whether they satisfy certain pre-described properties. An example of such a property for a group of Lie type is that of being a cuspidal character. This relies on the *truncation* of a character to each parabolic subgroup P of G, which in turn depends on the unipotent radical of P.

In the case that G is a sporadic simple group, the complex characters of G are described in full in [CCN$^+$09]. Using the p-core of each parabolic subgroup of G, we may consider a notion of cuspidal characters in this more general setting. Since such a notion depends on the p-core, we obtain p-cuspidal characters for each prime divisor p of G. These p-cuspidal characters are the primary focus of this chapter.

This chapter is arranged as follows. We begin in Section 6.1 by formally defining an X-parabolic system for a group G with respect to a given subgroup X. Such systems are a natural generalisation of the p-minimal parabolic systems introduced in Chapter 2. This is followed by a brief exposition of cuspidal characters of groups of Lie type. These motivate the subsequent definition of cuspidal characters of finite groups with respect to X-parabolic systems, whose properties are then investigated.

The subsequent four sections provide a full survey of the p-cuspidal characters of the sporadic simple groups, with 2-, 3- and 5-cuspidal characters being considered in Sections 6.2, 6.3 and 6.4 respectively. The remaining p-cuspidal characters for $p > 5$ are studied in Section 6.5. With the exception of the baby monster, $\mathbb{B}$, for the prime $p = 2$, we fully determine the p-cuspidal characters of each sporadic group in these sections. When $G = \mathbb{B}$ and $p = 2$ we make the assumption that Co_2 has a unique 22-dimensional representation over $GF(2)$. With this assumption, our calculations leave us with two potential 2-cuspidal characters, but at the time of writing we have been unable to ascertain for definite whether or not they are indeed 2-cuspidal. For ease of reference, a summary of the 2-minimal parabolic systems for each of the sporadic simple groups together with their associated 2-cuspidal characters is given in Tables 6.1 and 6.2. A complete list of p-cuspidal characters for the sporadic groups in the case that $p > 2$ is then given in Table 6.3. In each table, for a given group G the character χ_i will equal the corresponding character of G given in the character table in [CCN+09].

The chapter concludes in Section 6.6 with a brief discussion on possible geometries arising from the p-cuspidal characters summarised in Tables 6.1-6.3.

6.1 Parabolic Systems and Cuspidal Characters

6.1.1 Parabolic Systems

Let G be a finite group and let X be a subgroup of G. Let I be an index set with $|I| = n \geqslant 0$.

Definition 6.1.1. *An* X-parabolic system, $\mathfrak{X}$, *of* G *of rank* n *is a set of pairs of subgroups of* G, (P_J, Q_J), *indexed by subsets* J *of* I *such that*

(i) for each $J \subseteq I$, $X \leqslant P_J$, $Q_J \trianglelefteq P_J$;

(ii) for $K \subseteq J \subseteq I$, $Q_J \leqslant Q_K$;

(iii) $P_I = G$ *and* $Q_I = 1$; *and*

(iv) $X = P_\varnothing$.

We shall write $\mathfrak{X} = \{(P_J, Q_J) | J \subseteq I\}$ and note that by part (ii) of Definition 6.1.1 that all Q_J are subgroups of $Q_\varnothing$ and that $Q_\varnothing \trianglelefteq P_\varnothing = X$. We allow the possibility that $(P_J, Q_J) = (P_K, Q_K)$ with $J \neq K$, but this will not occur in most of the cases that follow.

For ease of notation, if our index set is $I = \{1, 2, \ldots, n\}$ for some $n \geqslant 1$, then given a subset $\{i_1, i_2, \ldots, i_r\} \subseteq I$ with $i_j < i_{j+1}$ for all j, we denote the subgroups $P_{\{i_1,i_2,\ldots,i_r\}}$ and $Q_{\{i_1,i_2,\ldots,i_r\}}$ by $P_{i_1 i_2 \cdots i_r}$ and $Q_{i_1 i_2 \cdots i_r}$ respectively.

Example 6.1.2. *Let* $G = \langle r, s \mid r^4 = s^2 = (rs)^2 = 1\rangle$ *be the dihedral group of order* 8, $X = \langle r \rangle$ *and let* $I = \{1, 2\}$. *Then defining*

$$\begin{aligned} &P_{12} = G, \quad P_1 = G, \quad P_2 = G, \quad P_\varnothing = X; \quad \textit{and} \\ &Q_{12} = 1, \quad Q_1 = \langle r^2 \rangle, \quad Q_2 = X, \quad Q_\varnothing = X, \end{aligned}$$

we have that $\mathfrak{X} = \{(P_J, Q_J) | J \subseteq I\}$ *is an* X*-parabolic system of* G *of rank* 2. •

Family	Group	2-Minimal Parabolic System	Rank	Geometric/ Non-Geometric	2-Cuspidal Characters (character degrees)
Mathieu Groups	M_{11}	$\{P_1 \sim 2_-^{1+2}.\mathrm{Sym}(3), P_2 \sim 3^2.SD_{16}\}$	2	Geometric	None
		$\{P_1 \sim 2_-^{1+2}.\mathrm{Sym}(3), P_3 \sim \mathrm{Alt}(6).2\}$	2	Geometric	None
		$\{P_2 \sim 3^2.SD_{16}, P_3 \sim \mathrm{Alt}(6).2\}$	2	Geometric	χ_3 (10), χ_4 (10)
	M_{12}	$\{P_1 \sim 4^2 2.\mathrm{Sym}(3), P_2 \sim 2_+^{1+4}.\mathrm{Sym}(3)\}$	2	Geometric	None
	M_{22}	$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_2 \sim 2^4.\mathrm{Sym}(5)\}$	2	Geometric	χ_3 (45), χ_4 (45)
	M_{23}	$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_3 \sim 2^{4+2}.\mathrm{Sym}(3), P_7 \sim 2^4.\mathrm{Sym}(5)\}$	3	Geometric	χ_3 (45), χ_4 (45)
		$\{P_3 \sim 2^{4+2}.\mathrm{Sym}(3), P_4 \sim 2^{4+2}.\mathrm{Sym}(3), P_7 \sim 2^4.\mathrm{Sym}(5)\}$	3	Geometric	χ_3 (45), χ_4 (45)
		$\{P_2 \sim 2^{4+2}.\mathrm{Sym}(3), P_3 \sim 2^{4+2}.\mathrm{Sym}(3), P_7 \sim 2^4.\mathrm{Sym}(5)\}$	3	Geometric	χ_3 (45), χ_4 (45)
		$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_6 \sim 2^4.\mathrm{Sym}(5), P_7 \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	χ_3 (45), χ_4 (45)
		$\{P_2 \sim 2^{4+2}.\mathrm{Sym}(3), P_6 \sim 2^4.\mathrm{Sym}(5), P_7 \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	χ_3 (45), χ_4 (45)
		$\{P_3 \sim 2^{4+2}.\mathrm{Sym}(3), P_6 \sim 2^4.\mathrm{Sym}(5), P_7 \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	χ_3 (45), χ_4 (45)
		$\{P_4 \sim 2^{4+2}.\mathrm{Sym}(3), P_6 \sim 2^4.\mathrm{Sym}(5), P_7 \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	χ_3 (45), χ_4 (45)
	M_{24}	$\{P_1 \sim 2^{6+3}.\mathrm{Sym}(3), P_2 \sim 2^{6+3}.\mathrm{Sym}(3), P_3 \sim 2^{6+3}.\mathrm{Sym}(3)\}$	3	Geometric	χ_3 (45), χ_4 (45)
Leech Lattice and Conway Groups	HS	$\{P_1 \sim 4.2^4.\mathrm{Sym}(5), P_2 \sim 4^3 2^2.\mathrm{Sym}(3)\}$	2	Geometric	None
	J_2	$\{P_1 \sim 2^{2+4}.3\,\mathrm{Sym}(3), P_2 \sim 2_+^{1+4}.L_2(4)\}$	2	Geometric	None
	Co_1	$\{P_1 \sim [2^{20}].\mathrm{Sym}(3), P_2 \sim [2^{20}].\mathrm{Sym}(3), P_3 \sim [2^{20}].\mathrm{Sym}(3), P_4 \sim [2^{20}].\mathrm{Sym}(3)\}$	4	Geometric	χ_2 (276), χ_8 (37674), χ_{11} (94875)
	Co_2	$\{P_1 \sim [2^{15}].\mathrm{Sym}(5), P_2 \sim [2^{17}].\mathrm{Sym}(3), P_3 \sim [2^{17}].\mathrm{Sym}(3)\}$	3	Geometric	χ_3 (253), χ_{10} (9625), χ_{11} (9625), χ_{12} (10395), χ_{13} (10395), χ_{16} (31625), χ_{31} (239085), χ_{32} (239085)
	Co_3	$\{P_1 \sim 2^{4+4+1}.\mathrm{Sym}(3), P_2 \sim 2^{4+4+1}.\mathrm{Sym}(3), P_3 \sim 2^{4+4+1}.\mathrm{Sym}(3)\}$	3	Geometric	None
	McL	$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_5 \sim 2^4.\mathrm{Sym}(5), P_5^\sigma \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_5 \sim 2^4.\mathrm{Sym}(5), P_5^\sigma \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_2 \sim 2^{4+2}.\mathrm{Sym}(3), P_5 \sim 2^4.\mathrm{Sym}(5), P_5^\sigma \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_2^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_5 \sim 2^4.\mathrm{Sym}(5), P_5^\sigma \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_3 \sim 2^{4+2}.\mathrm{Sym}(3), P_5 \sim 2^4.\mathrm{Sym}(5), P_5^\sigma \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_2^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_5^\sigma \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_2 \sim 2^{4+2}.\mathrm{Sym}(3), P_5 \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_3 \sim 2^{4+2}.\mathrm{Sym}(3), P_5^\sigma \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_3 \sim 2^{4+2}.\mathrm{Sym}(3), P_5 \sim 2^4.\mathrm{Sym}(5)\}$	3	Non-Geometric	None
		$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_2 \sim 2^{4+2}.\mathrm{Sym}(3), P_5^\sigma \sim 2^4.\mathrm{Sym}(5)\}$	3	Geometric	None
		$\{P_1^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_2^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_5 \sim 2^4.\mathrm{Sym}(5)\}$	3	Geometric	None
		$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_1^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_5 \sim 2^4.\mathrm{Sym}(5)\}$	3	Geometric	None
		$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_1^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_5^\sigma \sim 2^4.\mathrm{Sym}(5)\}$	3	Geometric	None
		$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_1^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_2 \sim 2^{4+2}.\mathrm{Sym}(3)\}$	3	Geometric	None
		$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_1^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_2^\sigma \sim 2^{4+2}.\mathrm{Sym}(3)\}$	3	Geometric	None
		$\{P_1 \sim 2^{4+2}.\mathrm{Sym}(3), P_1^\sigma \sim 2^{4+2}.\mathrm{Sym}(3), P_3 \sim 2^{4+2}.\mathrm{Sym}(3), P_4 \sim 2^{4+2}.\mathrm{Sym}(3)\}$	4	Geometric	None
	Suz	$\{P_1 \sim 2^{4+6+1}.L_2(4), P_2 \sim 2^{4+6+2}.(3 \times \mathrm{Sym}(3)), P_3 \sim 2^{6+4+2}.(3 \times \mathrm{Sym}(3))\}$	3	Geometric	None

Table 6.1: The 2-minimal parabolic systems and their associated 2-cuspidal characters for the Mathieu, Leech Lattice and Conway sporadic simple groups.

Family	Group	2-Minimal Parabolic System	Rank	Geometric/ Non-Geometric	2-Cuspidal Characters (character degrees)
Monster Group and Subgroups	He	$\{P_1 \sim 2^{6+3}.\,\mathrm{Sym}(3), P_2 \sim 2^{6+3}.\,\mathrm{Sym}(3), P_4 \sim 2^{6+3}.\,\mathrm{Sym}(3)\}$	3	Geometric	None
		$\{P_1 \sim 2^{6+3}.\,\mathrm{Sym}(3), P_3 \sim 2^{6+3}.\,\mathrm{Sym}(3), P_4 \sim 2^{6+3}.\,\mathrm{Sym}(3)\}$	3	Geometric	None
	HN	$\{P_1 \sim 2_+^{1+8}.\,\mathrm{Alt}(5) \wr \mathbb{Z}_2, P_2 \sim 2^{2+3+6+2}3\,\mathrm{Sym}(3)\}$	2	Geometric	None
	Th	$\{P_1 \sim 2^{1+8}.\,\mathrm{Alt}(9), P_2 \sim 2^{5+6+2+1}.\,\mathrm{Sym}(3)\}$	2	Geometric	χ_2 (248)
	Fi_{22}	$\{P_1 \sim [2^{16}].\,\mathrm{Sym}(3), P_2 \sim [2^{16}].\,\mathrm{Sym}(3), P_3 \sim [2^{14}].\,\mathrm{Sym}(5)\}$	3	Geometric	None
	Fi_{23}	$\{P_1 \sim [2^{17}].\,\mathrm{Sym}(3), P_3 \sim [2^{17}].\,\mathrm{Sym}(3), P_5 \sim [2^{17}].\,\mathrm{Sym}(3), P_8 \sim [2^{15}].\,\mathrm{Sym}(5)\}$	4	Geometric	None
		$\{P_1 \sim [2^{17}].\,\mathrm{Sym}(3), P_4 \sim [2^{17}].\,\mathrm{Sym}(3), P_5 \sim [2^{17}].\,\mathrm{Sym}(3), P_8 \sim [2^{15}].\,\mathrm{Sym}(5)\}$	4	Geometric	None
		$\{P_1 \sim [2^{17}].\,\mathrm{Sym}(3), P_2 \sim [2^{17}].\,\mathrm{Sym}(3), P_5 \sim [2^{17}].\,\mathrm{Sym}(3), P_8 \sim [2^{15}].\,\mathrm{Sym}(5)\}$	4	Non-Geometric	None
		$\{P_1 \sim [2^{17}].\,\mathrm{Sym}(3), P_7 \sim [2^{15}].\,\mathrm{Sym}(5), P_8 \sim [2^{15}].\,\mathrm{Sym}(5)\}$	3	Non-Geometric	None
	Fi'_{24}	$\{P_1 \sim [2^{20}].L_2(2), P_2 \sim [2^{20}].L_2(2), P_3 \sim [2^{20}].L_2(2), P_4 \sim [2^{20}].L_2(2)\}$	4	Geometric	None
	$\mathbb{B}$	$\{P_1 \sim [2^{40}].\,\mathrm{Sym}(3), P_2 \sim [2^{40}].\,\mathrm{Sym}(3), P_3 \sim [2^{40}].\,\mathrm{Sym}(3), P_5 \sim [2^{38}].\,\mathrm{Sym}(5)\}$	4	Geometric	Possibly χ_{10} (1407126890), χ_{11} (3214743741)
	$\mathbb{M}$	$\left\{ \begin{array}{c} P_1 \sim [2^{45}].L_2(2), P_2 \sim [2^{45}].L_2(2), P_3 \sim [2^{45}].L_2(2) \\ P_4 \sim [2^{45}].L_2(2), P_5 \sim [2^{45}].L_2(2) \end{array} \right\}$	5	Geometric	None
Pariahs	J_1	$\{J_1\}$	1	Geometric	None
	$O'N$	$\{P_1 \sim 4^3 2^2.\,\mathrm{Sym}(3), P_2 \sim 4.L_3(4)2\}$	2	Geometric	None
	J_3	$\{P_1 \sim 2^{2+4}.(3 \times \mathrm{Sym}(3)), P_2 \sim 2_-^{1+4}.L_2(4)\}$	2	Geometric	None
	Ru	$\{P_1 \sim 2^{5+6}.\,\mathrm{Sym}(5), P_2 \sim 2^{5+6+2}.\,\mathrm{Sym}(3)\}$	2	Geometric	χ_2 (378), χ_3 (378)
	J_4	$\{P_1 \sim [2^{20}].\,\mathrm{Sym}(3), P_2 \sim [2^{20}].\,\mathrm{Sym}(3), P_3 \sim [2^{18}].\,\mathrm{Sym}(5)\}$	3	Geometric	χ_2 (1333), χ_3 (1333)
	Ly	$\{P_1 \sim [2^7].\,\mathrm{Sym}(3), P_2 \sim [2^5].\,\mathrm{Sym}(5)\}$	2	Geometric	None
		$\{P_1 \sim [2^7].\,\mathrm{Sym}(3), P_3 \sim 2.\,\mathrm{Sym}(9)\}$	2	Geometric	None

Table 6.2: The 2-minimal parabolic systems and their associated 2-cuspidal characters for the Monster group, its subgroups and the pariah sporadic simple groups.

Family	Group	Prime, p	p-Minimal Parabolic System	Rank	Geometric/ Non-Geometric	p-Cuspidal Characters (character degrees)
Mathieu Groups	M_{11}	3	$\{M_{11}\}$	1	Geometric	χ_6 (16), χ_7 (16)
		11	$\{M_{11}\}$	1	Geometric	χ_2 (10), χ_3 (10), χ_4 (10)
	M_{12}	3	$\{P_1 \sim 3^2.GL_2(3), P_2 \sim 3^2.GL_2(3)\}$	2	Geometric	χ_4 (16), χ_5 (16)
	M_{23}	23	$\{M_{23}\}$	1	Geometric	χ_2 (22)
Leech Lattice and Conway Groups	J_2	5	$\{J_2\}$	1	Geometric	χ_6 (36)
Monster Group and Subgroups	He	7	$\{He\}$	1	Geometric	χ_2 (51), χ_3 (51)
	HN	5	$\{P_1 \sim 5^{1+4}.(2^{1+4}.5.4), P_2 \sim 5^{2+1+2}.4.\,\mathrm{Alt}(5)\}$	2	Geometric	χ_4 (760)
	Th	3	$\{P_1 \sim 3^{(1+2)+4+2}.GL_2(3), P_2 \sim 3^{(2+3)+4}.GL_2(3)\}$	2	Geometric	χ_2 (248)
		5	$\{Th\}$	1	Geometric	χ_2 (248)
	Fi_{22}	3	$\{P_1 \sim [3^8].2.PGL_2(3), P_2 \sim [3^8].2.PGL_2(3), P_3 \sim [3^8].2.PGL_2(3)\}$	3	Geometric	χ_2 (78)
Pariahs	Ly	5	$\{P_1 \sim 5_+^{1+4}.4.PGL_2(5), P_2 \sim 5^{3+2}.4.PGL_2(5), P_3 \sim 5_+^{1+4}.4.PGL_2(5)\}$	3	Geometric	χ_2 (2480) χ_3 (2480)

Table 6.3: The p-cuspidal characters and associated p-minimal parabolic systems of the sporadic simple groups in the case that $p > 2$.

Given an X-parabolic system $\mathfrak{X} = \{(P_J, Q_J)|J \subseteq I\}$ of G and $J \subseteq I$, we set $\overline{P}_J := P_J/Q_J$. Furthermore, for any subgroup $Q_J \leqslant Y \leqslant P_J$, we use the standard bar notation $\overline{Y} := Y/Q_J$. We may use $\mathfrak{X}$ to form an $\overline{X}$-parabolic system, $\overline{\mathfrak{X}}_J$, of rank $|J|$ for $\overline{P}_J$ given by

$$\overline{\mathfrak{X}}_J = \{(\overline{P}_J \cap \overline{P}_K, \overline{Q}_K)|(P_K, Q_K) \in \mathfrak{X}, K \subseteq J\}.$$

For each prime p, taking $S \in Syl_p(G)$ and $B := N_G(S)$, we may define a B-parabolic system for G. Indeed, we recall from Chapter 2 that a subgroup P of G is called *p-minimal (with respect to B)* if $B < P$ and B is contained in a unique maximal subgroup of P. We note that our definition of p-minimality is slightly more general than that given in [RS84], where the subgroup P was also required to have non-trivial p-core, $O_p(P)$. Defining

$$\mathscr{M}(G,B) = \{P|P \text{ is a } p\text{-minimal subgroup of } G \text{ (with respect to } B)\},$$

then a set

$$\mathscr{M}_0 = \{P_i|P_i \in \mathscr{M}(G,B), i \in I\}$$

is called a *minimal parabolic system of characteristic p for G* or a *p-minimal parabolic system of G*, if $G = \langle P_i|i \in I\rangle$ and $G \neq \langle P_j|j \in I\backslash\{i\}\rangle$ for any $i \in I$. The *rank* of $\mathscr{M}_0$ is $|I|$. We call $\mathscr{M}_0$ a *geometric* p-minimal parabolic system if for all $J, K \subseteq I$ we have $P_{J\cap K} = P_J \cap P_K$. Otherwise $\mathscr{M}_0$ is called *non-geometric*.

The promised B-parabolic system, $\mathfrak{X} = \{(P_J, Q_J)|J \subseteq I\}$, is given by defining

$$P_J = \begin{cases} \langle P_j|j \in J\rangle & \text{if } \varnothing \neq J \subseteq I; \text{ and} \\ B & \text{if } J = \varnothing. \end{cases}$$

and $Q_J = O_p(P_J)$ for all $J \subseteq I$. It follows that every group G has a p-minimal parabolic system, and hence a B-parabolic system for each prime p (although this is not the case if the additional condition on non-trivial p-cores is imposed). The minimal parabolic systems for the sporadic groups (in the case that S is non-cyclic) are described in [RS84].

6.1.2 Cuspidal Characters

Given a finite simple group, G, of Lie type, having a Borel subgroup X, let P_J be a standard parabolic subgroup of G and let U_J be the associated unipotent radical. For an irreducible character χ of G, we recall from [Car93] that the truncation, $T_{P_J/U_J}(\chi)$, of χ is the character of P_J defined by

$$(T_{P_J/U_J}(\chi))(p) = \frac{1}{|U_J|}\sum_{u \in U_J} \chi(up)$$

for $p \in P_J$. The character χ is then called *cuspidal* if for all standard parabolic subgroups $P_J \neq G$, the truncation $T_{P_J/U_J}(\chi)$ is zero. Equivalently, a character χ is cuspidal if $(\chi|_{U_J}, 1|_{U_J}) = 0$ for all $U_J \neq X$ (see [Car93, Proposition 9.1.1]).

Cuspidal characters play an important role in determining the irreducible characters of finite simple groups of Lie type, and an exposition of them may be found in [Car93, Chapter 9]. It would be of interest to form an analogous definition of cuspidal characters for the sporadic simple groups, and see whether any interesting properties arise. To do this, we note that for a finite simple group, G, of Lie type as described above, taking Q_J to be the unipotent radical of P_J we obtain an X-parabolic system of G given by $\mathfrak{X} = \{(P_J, Q_J) | J \subseteq I\}$. We may use this observation to define a form of cuspidal characters in a more general setting. Indeed, let G be an arbitrary finite group and denote by $\mathrm{Irr}(G)$ the set of all irreducible complex characters of G.

Definition 6.1.3. *Let $\mathfrak{X}$ be an X-parabolic system of G where $X \leqslant G$, and let $\chi \in \mathrm{Irr}(G)$. Then χ is called* $\mathfrak{X}$-cuspidal *if for all $(P_J, Q_J) \in \mathfrak{X}$ with $Q_J \neq 1$ we have*

$$\sum_{g \in Q_J} \chi(g) = 0. \tag{6.1.1}$$

The condition (6.1.1) will be known as the *cuspidal condition on Q_J* and is equivalent to $(\chi_{Q_J}, 1_{Q_J}) = 0$.[1] Clearly, when the index set $I = \varnothing$, we have $G = P_\varnothing = X$ and $Q_\varnothing = 1$, and hence every irreducible character is vacuously $\mathfrak{X}$-cuspidal. When $\mathfrak{X}$ is a B-parabolic system associated to a p-minimal parabolic system of G, then any $\mathfrak{X}$-cuspidal character will also be called a *p-cuspidal character* of G.

We recall the notion of the intertwining number of two modules.

Definition 6.1.4. *[Kar94] Let F be a field of characteristic 0, with algebraic closure $\overline{F}$, let G be a finite group and let V and W be $\overline{F}G$-modules. The* intertwining number*, denoted $i(V, W)$, is defined by*

$$i(V, W) := \dim_F \mathrm{Hom}_{\overline{F}G}(V, W)$$

The intertwining number of modules will be of importance due to its connection with the inner product of the associated characters.

Theorem 6.1.5. *[Kar94, Chapter 3, Theorem 1.1] Let F be an arbitrary field of characteristic 0 and let λ and μ be arbitrary characters of G afforded by $\overline{F}G$-modules V and W respectively. Then*

$$(\lambda, \mu) = i(V, W).$$

We may use Theorem 6.1.5 to prove an analogue of Proposition 9.1.3 of [Car93], the proof of which is almost identical to that used in Carter's Proposition.

Theorem 6.1.6. *Let $\mathfrak{X}$ be an X-parabolic system of G and $\chi \in \mathrm{Irr}(G)$. Then there exists $(P_J, Q_J) \in \mathfrak{X}$ and an $\overline{\mathfrak{X}}_J$-cuspidal character ψ of $\overline{P_J} = P_J/Q_J$ such that $(\chi, \psi^G) \neq 0$.*

Proof. Let $\mathcal{S} = \{J \subseteq I | (\chi_{Q_J}, 1_{Q_J}) \neq 0\}$. Note that $\mathcal{S} \neq \varnothing$ as $Q_I = 1$. Let J be a minimal element of $\mathcal{S}$ and let V be an irreducible $\mathbb{C}G$-module that affords χ. Define

$$V' = \{v \in V | v \cdot u = v \text{ for all } u \in Q_J\}.$$

[1] In an abuse of terminology, we will also sometimes refer to the cuspidal relation holding for P_J when (6.1.1) occurs.

By Theorem 6.1.5, as $(\chi_{Q_J}, 1_{Q_J}) \neq 0$, there exists a non-zero $\mathbb{C}Q_J$-homomorphism from the trivial $\mathbb{C}Q_J$-module to V, and hence V' is non-empty.

Clearly V' is a linear subspace of V, and given $g \in P_J$ and $u \in Q_J$ we have that

$$(vg)u = vgug^{-1}g = vg,$$

as $Q_J \trianglelefteq P_J$. Hence V' is a $\mathbb{C}P_J$-module.

Consider V' as a $\mathbb{C}\overline{P}_J$-module, having associated character $\phi = \sum_i \phi_i$ (with the ϕ_i irreducible $\mathbb{C}\overline{P}_J$-characters). So V' affords $\phi_{P_J} = \sum_i (\phi_i)_{P_J}$ and V affords the character χ_{P_J}. Now V' is a $\mathbb{C}P_J$-submodule of V, hence each $(\phi_i)_{P_J}$ is a component of χ_{P_J}. Consequently

$$((\phi_i)^G, \chi) = ((\phi_i)_{P_J}, \chi_{P_J}) \neq 0,$$

and χ is a component of $(\phi_i)^G$. Thus it remains to prove that ϕ_i is a cuspidal character.

If ϕ_i is not cuspidal, then $((\phi_i)_{Q_K}, 1_{Q_K}) \neq 0$ for some $K \subsetneq J$. It follows that $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}Q_K}(1, V) \neq 0$ and hence $(\chi_{Q_K}, 1_{Q_K}) \neq 0$. Hence $K \in \mathcal{S}$, contradicting the minimality of J. Thus the result holds true. $\square$

We illustrate this behaviour with an example.

Example 6.1.7. $G = \mathrm{Alt}(7)$ *has a rank* 2 *characteristic* 2 *minimal parabolic system* $\{P_1, P_2\}$ $(\subseteq \mathscr{M}(G, B))$ *with* $B \cong \mathrm{Dih}(8)$, $P_1 \cong \mathrm{Dih}(8) : C_3$ *and* $P_2 \cong \mathrm{Sym}(4)$. *Take* $B = \langle (1,2)(3,4), (1,3)(5,6) \rangle$, $P_1 = \langle B, (5,6,7) \rangle$ *and* $P_2 = \langle B, (1,2,5)(3,4,6) \rangle$. *Let* $\mathfrak{X}$ *be the* B*-parabolic system given by* $\{(P_J, Q_J) | J \subseteq \{1,2\}\}$ *where* $P_\varnothing = B$. *So* $Q_\varnothing = B$, $Q_1 = \langle (1,2)(3,4), (1,3)(2,4) \rangle$, $Q_2 = \langle (1,3)(5,6), (2,4)(5,6) \rangle$ *and* $Q_{\{1,2\}} = 1$. *It follows using [CCN+ 09] that for* $i = 1, 2$ *we have*

$$\sum_{g \in Q_i} \chi(g) = \chi(1) + 3\chi(2A) \neq 0$$

for any $\chi \in \mathrm{Irr}(G)$. *Thus* G *has no* $\mathfrak{X}$*-cuspidal characters.*

For $\overline{P_i} = P_i/Q_i \cong \mathrm{Sym}(3)$ *we have that* $\overline{Q_\varnothing} \cong C_2$. *It follows that there is one* $\overline{\mathfrak{X}}_i$*-cuspidal character, namely the sign character. We denote this character by* ϕ_i *and also think of it as a* P_i *character. Using the notation from [CCN+ 09], calculations show that the constituent characters of* ϕ_1^G *are* $\chi_3, \chi_4, \chi_7, \chi_9$. *Meanwhile the constituent characters of* ϕ_2^G *are* $\chi_3, \chi_4, \chi_5, \chi_7, \chi_8, \chi_9$.

For $\overline{P_\varnothing} = 1$ *we note that trivial character will be* $\overline{\mathfrak{X}}_\varnothing$*-cuspidal, and it lifts to the trivial character* 1_B. *We have that the constituent characters of* 1_B^G *are* $\chi_1, \chi_2, \chi_5, \chi_6, \chi_7, \chi_8, \chi_9$. *Thus we observe - not very surprisingly - that Proposition 9.1.5 of [Car93] does not extend to our more general situation, since* $(\phi_1^G, \phi_2^G) = 5$, *whilst* $\phi_1^G \neq \phi_2^G$. •

Although by definition, to determine if a character is cuspidal we must check the cuspidal condition for every subgroup in the X-parabolic system, we shall shortly see that this is not actually necessary. First we give the analogue of [Car93, Proposition 9.1.2].

Proposition 6.1.8. *Let* G *be a group,* $X \leqslant G$ *and* $\chi \in \mathrm{Irr}(G)$. *If* $\mathfrak{X}$ *is an* X*-parabolic system of* G *of rank* n *having underlying indexing set* I, *then the following are equivalent:*

(i) χ *is a* $\mathfrak{X}$*-cuspidal character of* G.

(ii) $(\chi_{Q_J}, 1_{Q_J}) = 0$ *for all* $J \subseteq I$ *such that* $Q_J \neq 1$.

(iii) $(\chi, 1_{Q_J}^G) = 0$ *for all* $J \subseteq I$ *such that* $Q_J \neq 1$.

(iv) $\sum_{q \in Q_J} \chi(qg) = 0$ *for all* $J \subseteq I$ *such that* $Q_J \neq 1$ *and all* $g \in G$.

(v) $\sum_{q \in Q_J} \chi(gq) = 0$ *for all* $J \subseteq I$ *such that* $Q_J \neq 1$ *and all* $g \in G$.

Proof. *(i)* $\Rightarrow$ *(ii)*. Assume that χ is a $\mathfrak{X}$-cuspidal character of G. Thus for each pair of subgroups (P_J, Q_J) for $J \subseteq I$ we have that either $Q_J = 1$ or

$$\sum_{q \in Q_J} \chi(q) = 0.$$

In particular

$$\sum_{q \in Q_J} \chi(q) 1_{Q_J}(q) = 0$$

and hence $(\chi_{Q_J}, 1_{Q_J}) = 0$ for all $J \subseteq I$ such that $Q_J \neq 1$.

(ii) $\Rightarrow$ *(iv)* Let Q_J be such that $Q_J \neq 1$ (if no such Q_J exists, the result is vacuously true). Let ρ be an irreducible representation corresponding to χ, let ρ' be an irreducible constituent of $\rho|_{Q_J}$ and let d denote the degree of ρ'. The module corresponding to ρ' has basis $\{e_1, \ldots, e_d\}$ and hence we may define coefficient functions ρ'_{ij} for $i, j = 1, \ldots, d$ by

$$e_i g = \sum_{j=1}^{d} \rho'_{ij}(g) e_j.$$

By the orthogonality relations for the coefficient functions (as given in [Car93, Section 6.1]), it follows that

$$(\rho'_{ij}, (1_{Q_J})_{11}) = 0$$

for all $i, j = 1, \ldots, d$, as 1_{Q_J} is not an irreducible constituent of χ_{Q_J}. Thus

$$\sum_{q \in Q_J} \rho'_{ij}(q) = 0$$

for all i, j and hence

$$\sum_{q \in Q_J} \rho'(q) = 0$$

for all i, j. Since this holds for all irreducible components ρ' of χ_{Q_J} we deduce that

$$\sum_{q \in Q_J} \rho(q) = 0. \tag{6.1.2}$$

Now let $g \in G$ be given. Multiplying (6.1.2) on the right by $\rho(g)$ gives

$$\sum_{q \in Q_J} \rho(qg) = \left(\sum_{q \in Q_J} \rho(q) \right) \rho(g) = 0.$$

Consequently, taking traces we obtain

$$\sum_{q \in Q_J} \chi(qg) = 0.$$

$(iv) \Rightarrow (i)$ Taking $g = 1$ we see that χ is a $\mathfrak{X}$-cuspidal character of G.

$(ii) \Rightarrow (v) \Rightarrow (i)$ This follows analogously by multiplying on the left by $\rho(g)$ in (6.1.2).

$(ii) \Leftrightarrow (iii)$ This follows by Frobenius reciprocity. □

Proposition 6.1.8 infers that we only have to check that the cuspidal condition holds for certain "maximal" subgroups of a parabolic system to ascertain that a character is cuspidal.

Corollary 6.1.9. *Let G be a group, $X \leqslant G$, $I = \{1, \ldots, n\}$ and let $\mathfrak{X} = \{(P_J, Q_J) | J \subseteq I\}$ be an X-parabolic system of G. Define*

$$\mathfrak{Y} := \{J \subseteq I | Q_J \neq 1 \text{ and if } J \subsetneq K \subseteq I, \text{ then } Q_K = 1\}.$$

Then $\chi \in \mathrm{Irr}(G)$ is cuspidal precisely when

$$\sum_{q \in Q_J} \chi(q) = 0$$

for all Q_J such that $J \in \mathfrak{Y}$.

Proof. The condition is clearly necessary. To see that it is sufficient, let $J' \subseteq I$ be such that $Q_{J'} \neq 1$. We shall show that

$$\sum_{q \in Q_{J'}} \chi(q) = 0.$$

Since $Q_{J'} \neq 1$, we see that $J' \subsetneq I$ and there exists some $J \in \mathfrak{Y}$ such that $J' \subseteq J$. Consequently $Q_J \subseteq Q_{J'}$. By assumption

$$\sum_{q \in Q_J} \chi(q) = 0,$$

and so $(\chi|_{Q_J}, 1_{Q_J}) = 0$. The proof of Proposition 6.1.8 asserts that

$$\sum_{q \in Q_J} \chi(qg) = 0$$

for all $g \in G$.

Let T denote a right transversal of Q_J in $Q_{J'}$. Then

$$\sum_{q \in Q_{J'}} \chi(q) = \sum_{t \in T} \left(\sum_{q \in Q_J} \chi(qt) \right) = 0$$

as required. □

The final result that we will use in classifying the p-cuspidal characters of the sporadic simple groups concerns irreducible characters of odd degree.

Lemma 6.1.10. *Let G be a finite group and p an odd prime such that $|G| = p^a m$ for some $a \geqslant 1$ with $(p, m) = 1$. Assume that G has a p-minimal parabolic system containing a parabolic subgroup with non-trivial p-core. If for each G-conjugacy class, C, of elements of order p^b for $b \leqslant a$ and all $g \in C$ we have that*

$$\langle g \rangle \cap \{g \in G \mid ord(g) = p^b\} \subseteq C, \tag{6.1.3}$$

then every p-cuspidal character of G has even degree.

Proof. Assume that condition (6.1.3) holds for all non-trivial powers of p. Then a non-trivial p-core, Q, of a parabolic subgroup will intersect every conjugacy class of p-elements in a set of even order. Thus if the degree of $\chi \in \mathrm{Irr}(G)$ is odd, then the same is true of

$$\sum_{g \in Q} \chi(g),$$

and hence χ is not a p-cuspidal character of G. □

6.2 2-Cuspidal Characters

We now work systematically through the sporadic simple groups, determining for each group G and each 2-minimal parabolic system of G, which characters $\chi \in \mathrm{Irr}(G)$ are 2-cuspidal. A summary of our results is given in Tables 6.1 and 6.2. Throughout, the notation $\chi_i \in \mathrm{Irr}(G)$ is the same as that used in [CCN+09]. We shall also use the standard notation from [CCN+09] for the conjugacy classes of G.

6.2.1 The Mathieu Groups

M_{11}

There are three 2-minimal parabolic subgroups of M_{11}, namely

$$P_1 \sim 2_-^{1+2}.\mathrm{Sym}(3), \quad P_2 \sim 3^2.SD_{16}, \quad \text{and} \quad P_3 \sim \mathrm{Alt}(6).2,$$

(where SD_{16} is the semidihedral group of order 16) and these give rise to three 2-minimal parabolic systems, each of rank 2. Since $O_2(P_2) = O_2(P_3) = 1$, we must consider a Sylow 2-subgroup of M_{11}. Such a subgroup will intersect the M_{11}-conjugacy classes $1A$, $2A$, $4A$, $8A$ and $8B$ in 1, 5, 6, 2 and 2 elements respectively. It follows that the cuspidal relation for a Sylow 2-subgroup holds for $\chi_3, \chi_4 \in \mathrm{Irr}(M_{11})$ (both of degree 10). Consequently, χ_3 and χ_4 are 2-cuspidal characters of the minimal parabolic system $\{P_2, P_3\}$. Finally, as $O_2(P_1)$ contains 1, 1 and 6 elements from the classes $1A$, $2A$ and $4A$ respectively and

$$\chi_i(1A) + \chi_i(2A) + 6 \cdot \chi_i(4A) = 8$$

for $i = 3, 4$, we see that the minimal parabolic systems containing P_1 admit no 2-cuspidal characters.

M_{12}

There are no 2-cuspidal characters for the unique 2-minimal parabolic system of M_{12} given by

$$\{P_1 \sim 4^2 2.\,\mathrm{Sym}(3), P_2 \sim 2_+^{1+4}.\,\mathrm{Sym}(3)\}.$$

To see this, we observe that $O_2(P_1)$ intersects the M_{12}-conjugacy classes $1A$, $2A$, $2B$, $4A$ and $4B$ in 1, 4, 15, 6 and 6 elements respectively, whilst $O_2(P_2)$ intersects these classes in 1, 12, 7, 6 and 6 elements respectively. Consequently the only character satisfying the cuspidal relation for $O_2(P_1)$ is χ_{13} (of degree 120). However

$$\chi_{13}(1A) + 12 \cdot \chi_{13}(2A) + 7 \cdot \chi_{13}(2B) + 6 \cdot \chi_{13}(4A) + 6 \cdot \chi_{13}(4B) = 64.$$

M_{22}

There is a unique 2-minimal parabolic system for M_{22}, namely

$$\{P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), P_2 \sim 2^4.\,\mathrm{Sym}(5)\}.$$

The 2-cores $O_2(P_1)$ and $O_2(P_2)$ intersect the M_{22}-conjugacy classes $1A$, $2A$, $4A$ and $4B$ in 1, 27, 12, 24 and 1, 15, 0 and 0 elements respectively. The only elements $\chi \in \mathrm{Irr}(M_{22})$ satisfying

$$\chi(1A) + 27 \cdot \chi(2A) + 12 \cdot \chi(4A) + 24 \cdot \chi(4B) = \chi(1A) + 15 \cdot \chi(2A) = 0$$

are the two characters of degree 45, χ_3 and χ_4.

M_{23}

The group M_{23} has seven conjugacy classes of 2-minimal parabolic subgroups, six of which feature in 2-minimal parabolic systems of M_{23}. Using the notation of [RS84], these subgroups are

$$\begin{array}{lll} P_1 \sim 2^{4+2}.\,\mathrm{Sym}(3), & P_2 \sim 2^{4+2}.\,\mathrm{Sym}(3), & P_3 \sim 2^{4+2}.\,\mathrm{Sym}(3) \\ P_4 \sim 2^{4+2}.\,\mathrm{Sym}(3), & P_6 \sim 2^4.\,\mathrm{Sym}(5), & P_7 \sim 2^4.\,\mathrm{Sym}(5). \end{array}$$

The 2-minimal parabolic systems are given by $\{P_1, P_3, P_7\}$, $\{P_3, P_4, P_7\}$, $\{P_2, P_3, P_7\}$, $\{P_1, P_6, P_7\}$, $\{P_2, P_6, P_7\}$, $\{P_3, P_6, P_7\}$ and $\{P_4, P_6, P_7\}$.

Considering the maximal 2-parabolic subgroups of these systems, we see that the maximal 2-parabolic subgroups involving P_6 and P_7 have trivial 2-cores. Thus we need to check sub-maximal parabolics in order to apply Corollary 6.1.9. The 2-cores $O_2(P_i)$ for $i = 1, 2, 3, 4$ intersect the M_{23}-classes $1A$, $2A$ and $4A$ in 1, 27 and 36 elements respectively. The remaining sub-maximal parabolics, namely P_6, P_7, P_{13} and P_{34}, have rank 4 elementary abelian 2-groups for their 2-cores. Thus the non-trivial elements of their 2-cores lie in the M_{23}-class $2A$. For each sub-maximal parabolic, the only irreducible M_{23}-characters satisfying the cuspidal relation are χ_3 and χ_4. Hence for each of the 2-minimal parabolic systems, the characters χ_3 and χ_4 of degree 45 are 2-cuspidal characters.

Parabolic Subgroup	2-core	Order of intersection with M_{24}-class				
		$1A$	$2A$	$2B$	$4A$	$4B$
P_{12}	2^6	1	45	18	0	0
P_{13}	2^{6+2}	1	57	54	72	72
P_{23}	2^{4+3}	1	29	42	56	0

Table 6.4: The 2-cores of maximal parabolic subgroups of M_{24}.

M_{24}

The Mathieu group M_{24} has a unique 2-minimal parabolic system given by

$$\{P_1 \sim 2^{6+3}.\,\mathrm{Sym}(3), P_2 \sim 2^{6+3}.\,\mathrm{Sym}(3), P_3 \sim 2^{6+3}.\,\mathrm{Sym}(3)\}.$$

The maximal parabolic subgroups P_{12}, P_{13} and P_{23} all have non-trivial 2-cores, and their intersections with the M_{24}-conjugacy classes are summarised in Table 6.4. It follows that the characters χ_3, χ_4, χ_{12}, χ_{13}, χ_{15} and χ_{16} satisfy the cuspidal relation for P_{12}, as do χ_3, χ_4, χ_5, χ_6, χ_{12}, χ_{13}, χ_{15} and χ_{16} for P_{13} and χ_3, χ_4, χ_5, χ_6 and χ_8 for P_{23}. We conclude that χ_3 and χ_4 - both of degree 45 - are the only 2-cuspidal characters of M_{24}.

6.2.2 The Leech Lattice and Conway Groups

HS

The Higman-Sims group has a unique 2-minimal parabolic system of the form

$$\{P_1 \sim 4.2^4.\,\mathrm{Sym}(5), P_2 \sim 4^3 2^2.\,\mathrm{Sym}(3)\}.$$

Considering $O_2(P_1)$, we see that it intersects the HS-conjugacy classes $1A$, $2A$, $2B$, $4A$, $4B$ and $4C$ in 1, 31, 0, 2, 30 and 0 elements respectively. Consequently, there are no 2-cuspidal characters of HS, as the cuspidal relation does not hold for P_1.

J_2

There is a unique 2-minimal parabolic system of J_2 given by

$$\{P_1 \sim 2^{2+4}.3\,\mathrm{Sym}(3), P_2 \sim 2_+^{1+4}.L_2(4)\}.$$

The intersections of $O_2(P_1)$ with the J_2-classes $1A$, $2A$, $2B$ and $4A$ have orders 1, 3, 24 and 36 respectively. Consequently, the cuspidal relation on P_1 holds for the irreducible characters χ_4, χ_5, χ_{14} and χ_{15}. Meanwhile, the 2-core $O_2(P_2)$ intersects the given J_2-classes in 1, 11, 0 and 20 elements respectively, meaning that the cuspidal relation holds on P_2 for the characters χ_8, χ_9 and χ_{18}. We conclude that J_2 admits no 2-cuspidal characters.

Co_1

The largest Conway group, Co_1, admits a unique 2-minimal parabolic system, having rank 4. Its minimal parabolic subgroups are given by $P_i \sim [2^{20}].\,\mathrm{Sym}(3)$ for $i = 1, \ldots, 4$ and the corresponding 2-maximal parabolic subgroups have the form $P_{123} \sim 2^{2+12+3}.(\mathrm{Sym}(3) \times L_3(2))$,

Conjugacy Class	1	$2A$	$2B$	$2C$	$4A$	$4B$	$4C$	$4D$	$4E$	$4F$
P_{123}	1	1095	1344	4984	336	22512	18816	38976	0	43008
P_{124}	1	1095	576	6264	720	13680	17280	25920	0	0
P_{134}	1	759	0	1288	0	0	0	0	0	0
P_{234}	1	551	0	2520	240	15120	896	13440	0	0

Table 6.5: The intersections of the 2-cores of the maximal 2-parabolic subgroups of Co_1 with the Co_1-conjugacy classes.

Parabolic Subgroup	Corresponding Characters (character degrees)
P_{123}	χ_2 (276), χ_4 (1771), χ_5 (8855), χ_8 (37674), χ_{11} (94875), χ_{13} (345345), χ_{15} (483000), χ_{21} (1434510), χ_{23} (1771000), χ_{27} (2464749), χ_{28} (2464749)
P_{124}	χ_2 (276), χ_8 (37674), χ_{11} (94875), χ_{21} (1434510), χ_{27} (2464749), χ_{28} (2464749)
P_{134}	χ_2 (276), χ_4 (1771), χ_5 (8855), χ_8 (37674), χ_{11} (94875), χ_{13} (345345), χ_{15} (483000), χ_{21} (1434510), χ_{23} (1771000), χ_{27} (2464749), χ_{28} (2464749)
P_{234}	χ_2 (276), χ_4 (1771), χ_8 (37674), χ_9 (44275), χ_{11} (94875), χ_{13} (345345)

Table 6.6: The elements of $\mathrm{Irr}(Co_1)$ satisfying the cuspidal relation for each maximal 2-parabolic subgroup of Co_1.

$P_{124} \sim 2^{4+12}.(\mathrm{Sym}(3) \times 3Sp_4(2))$, $P_{134} \sim 2^{11}.M_{24}$ and $P_{234} \sim 2^{1+8+6}.L_4(2)$. The orders of the intersections of the 2-cores of the maximal parabolic subgroups with the Co_1-conjugacy classes are given in Table 6.5. A summary of the elements of $\mathrm{Irr}(Co_1)$ which satisfy the cuspidal relation for each of the maximal parabolics is given in Table 6.6. We conclude that Co_1 admits three 2-cuspidal characters, namely χ_2, χ_8 and χ_{11}.

Co_2

The group Co_2 has a 2-minimal parabolic system of the form $\{P_1, P_2, P_3\}$, where $P_1 \sim [2^{15}].\mathrm{Sym}(5)$ and $P_i \sim [2^{17}].\mathrm{Sym}(3)$ for $i = 2,3$. This system has maximal parabolic subgroups $P_{12} \sim 2^{4+10}.(\mathrm{Sym}(3) \times \mathrm{Sym}(5))$, $P_{13} \sim 2^{10}.M_{22}2$ and $P_{23} \sim 2^{1+8+6}.L_3(2)$. The orders of the intersections of the 2-cores of these maximal parabolics with the relevant Co_2-conjugacy classes are given in Table 6.7. It follows that the cuspidal relation holds on P_{12} for the characters χ_2, χ_3, χ_{10}, χ_{11}, χ_{12}, χ_{13}, χ_{16}, χ_{21}, χ_{22}, χ_{23}, χ_{31}, χ_{32} and χ_{37}, it holds on P_{13} for χ_3, χ_5, χ_9, χ_{10}, χ_{11}, χ_{12}, χ_{13}, χ_{16}, χ_{25}, χ_{31} and χ_{32}, and the cuspidal relation holds on P_{23} for the irreducible characters χ_2, χ_3, χ_5, χ_7, χ_8, χ_9, χ_{10}, χ_{11}, χ_{12}, χ_{13}, χ_{16}, χ_{20}, χ_{21}, χ_{22}, χ_{23}, χ_{25}, χ_{31}, χ_{32}, χ_{36}, χ_{37} and χ_{40}. Consequently, Co_2 admits eight 2-cuspidal characters namely χ_3 (of degree 253), χ_{10} (9625), χ_{11} (9625), χ_{12} (10395), χ_{13} (10395), χ_{16} (31625), χ_{31} (239085) and χ_{32} (239085).

2-core	Order of intersection with Co_2-conjugacy class											
	$1A$	$2A$	$2B$	$2C$	$4A$	$4B$	$4C$	$4D$	$4E$	$4F$	$4G$	$8A$
$O_2(P_{12})$	1	125	490	2328	240	1440	2400	1680	1920	5760	0	0
$O_2(P_{13})$	1	77	330	616	0	0	0	0	0	0	0	0
$O_2(P_{23})$	1	141	634	1848	240	3808	2464	336	8064	8064	0	7168

Table 6.7: The 2-cores of the maximal parabolic subgroups of Co_2.

Co_3

There is a unique 2-minimal parabolic system of Co_3 given by

$$\{P_1 \sim 2^{4+4+1}.\,\mathrm{Sym}(3), P_2 \sim 2^{4+4+1}.\,\mathrm{Sym}(3), P_3 \sim 2^{4+4+1}.\,\mathrm{Sym}(3)\},$$

which has maximal parabolic subgroups $P_{12} \sim 2^{2+6}.3(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$, $P_{13} \sim 4^3 2.L_3(2)$ and $P_{23} \sim 4.2^4.Sp_4(2)$. We have that $O_2(P_{23})$ intersects the Co_3-conjugacy classes $1A$, $2A$, $2B$, $4A$ and $4B$ in 1, 31, 0, 2 and 30 elements respectively, and hence the cuspidal condition does not hold for P_{23}. Thus Co_3 admits no 2-cuspidal characters.

McL

The McLaughlin group has a multitude of 2-minimal parabolic systems comprising of the 2-minimal parabolic subgroups P_i and P_i^σ for $i = 1, \dots, 5$, where σ is the non-trivial outer automorphism of McL (as seen in Example 2.5.6). Here $P_3^\sigma = P_3$, $P_4^\sigma = P_4$, $P_i \sim 2^{4+2}.\,\mathrm{Sym}(3)$ for $i = 1, 2, 3, 4$ and $P_5 \sim 2^4.\,\mathrm{Sym}(5)$. These subgroups give rise to the minimal parabolic systems

$$\begin{array}{llll}
\{P_1, P_5, P_5^\sigma\}, & \{P_1^\sigma, P_5, P_5^\sigma\}, & \{P_2, P_5, P_5^\sigma\}, & \{P_2^\sigma, P_5, P_5^\sigma\}, \\
\{P_3, P_5, P_5^\sigma\}, & \{P_1, P_2^\sigma, P_5^\sigma\}, & \{P_1^\sigma, P_2, P_5\}, & \{P_1, P_3, P_5^\sigma\}, \\
\{P_1^\sigma, P_3, P_5\}, & \{P_1, P_2, P_5^\sigma\}, & \{P_1^\sigma, P_2^\sigma, P_5\}, & \{P_1, P_1^\sigma, P_5\}, \\
\{P_1, P_1^\sigma, P_5^\sigma\}, & \{P_1, P_1^\sigma, P_2\}, & \{P_1, P_1^\sigma, P_2^\sigma\}, & \{P_1, P_1^\sigma, P_3, P_4\}.
\end{array}$$

Since the 2-cores of P_5 and P_5^σ are elementary abelian of rank 4, we see that any minimal parabolic system containing either of these minimal parabolics will not admit a 2-cuspidal character. Conversely, any 2-minimal parabolic system not containing these subgroups will contain the parabolic subgroup $P_{11^\sigma} := \langle P_1, P_1^\sigma \rangle$. Since $O_2(P_{11^\sigma})$ intersects the McL-conjugacy classes $1A$, $2A$ and $4A$ in 1, 19 and 12 elements respectively, we see that the cuspidal relation does not hold for P_{11^σ} and hence none of the 2-minimal parabolic systems of McL admit a 2-cuspidal character.

Suz

The group Suz has a unique 2-minimal parabolic system, which has rank 3. Its minimal parabolic subgroups satisfy $P_1 \sim 2^{4+6+1}.L_2(4)$, $P_2 \sim 2^{4+6+2}.(3 \times L_2(2))$ and $P_3 \sim 2^{6+4+2}.(3 \times L_2(2))$. The maximal parabolic subgroups are given by $P_{12} \sim 2^{1+6}.U_4(2)$, $P_{13} \sim 2^{2+8}.(\mathrm{Sym}(3) \times L_2(4))$ and $P_{23} \sim 2^{4+6}3.Sp_4(2)'$. The 2-core $O_2(P_{12})$ intersects the Suz-conjugacy classes $1A$, $2A$ and $4A$ in 1, 55 and 72 elements respectively. We deduce that there are no 2-cuspidal characters of Suz.

6.2.3 The Monster Group and its Subgroups

He

There are four 2-minimal parabolic subgroups of He given by $P_1 \cong P_4 \sim 2^{6+3}.\,\mathrm{Sym}(3)$ and $P_2 \cong P_3 \sim 2^{6+3}.\,\mathrm{Sym}(3)$. These give rise to the 2-minimal parabolic systems $\{P_1, P_2, P_4\}$ and $\{P_1, P_3, P_4\}$. Considering the maximal parabolic subgroups P_{14} and $P_{13} \cong P_{24}$ we see that $O_2(P_{14})$ intersects the He-conjugacy classes $1A$, $2A$, $2B$, $4A$, $4B$ and $4C$ in 1, 18, 45, 0, 0 and 0 elements respectively, whilst $O_2(P_{13})$ intersects the respective conjugacy classes in 1, 42, 29, 0, 56 and 0 elements. It follows that the cuspidal relation for P_{14} holds for the characters $\chi_7, \chi_8 \in \mathrm{Irr}(He)$, whilst for $P_{13} \cong P_{24}$ the cuspidal relation holds for $\chi_4, \chi_5 \in \mathrm{Irr}(He)$. Since each 2-minimal parabolic system contains P_{14} and either P_{13} or P_{24}, we conclude that there are no 2-cuspidal characters of He.

HN

There is a unique 2-minimal parabolic system of HN given by

$$\{P_1 \sim 2_+^{1+8}.\,\mathrm{Alt}(5) \wr \mathbb{Z}_2, P_2 \sim 2^{2+3+6+2}3\,\mathrm{Sym}(3)\}.$$

We consider the minimal parabolic subgroup P_1, whose character table is given in [Har76, TABLE IV], and we adopt the notation given in [Har76] for the P_1-conjugacy classes. We have that each P_1-class is either contained in, or is disjoint from $O_2(P_1)$. It follows that

$$O_2(P_1) = 1_1 \cup 2_1 \cup 2_2 \cup 2_3 \cup 4_1.$$

Considering the centralizer orders of 1_1, 2_1, 2_2, 2_3 and 4_1 in P_1 and the orders of the centralizers of 2-elements in HN, we see that the P_1-classes 2_1 and 2_2 are contained in the HN-class $2B$, the P_1-class 2_3 lies in either HN-class $2A$ or $2B$, and the P_1-class 4_1 is contained in the HN-class $4A$. It follows that $|O_2(P_1) \cap 2A| = 0$ or 120, $|O_2(P_1) \cap 2B| = 151$ or 271 and $|O_2(P_1) \cap 4A| = 240$. It follows that the cuspidal relation on P_1 does not hold for any $\chi \in \mathrm{Irr}(HN)$, and hence there are no 2-cuspidal characters of HN.

Th

The 2-minimal parabolic system

$$\{P_1 \sim 2^{1+8}.\,\mathrm{Alt}(9), P_2 \sim 2^{5+6+2+1}.\,\mathrm{Sym}(3)\}$$

of Th is unique. Considering fusion within the maximal subgroup $2^5.L_5(2) > P_2$ we find that $O_2(P_2)$ intersects the Th-conjugacy classes $1A$, $2A$, $4A$, $4B$, $8A$ and $8B$ in 1, 687, 656, 7104, 4864 and 3072 elements respectively. It follows that the cuspidal relation holds on P_2 for $\chi_2, \chi_6 \in \mathrm{Irr}(Th)$. Considering the normal subgroups of a Sylow 2-subgroup of $2^5.L_5(2)$ having order 2^9 and exponent 4, we see that for each such subgroup the only element of $\mathrm{Irr}(Th)$ for which the cuspidal relation holds is χ_2 of degree 248. Thus χ_2 is the unique 2-cuspidal character of Th.

Fi_{22}

There is a unique 2-minimal parabolic system of Fi_{22} given by $\{P_1, P_2, P_3\}$ where $P_i \sim [2^{16}].\,\mathrm{Sym}(3)$ for $i = 1, 2$ and $P_3 \sim [2^{14}].\,\mathrm{Sym}(5)$. This system has maximal parabolic subgroups $P_{12} \sim 2^{9+4+2}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$, $P_{13} \sim 2^{2+8}.U_4(2)2$ and $P_{23} \sim 2^{10}.M_{22}$. The 2-core $O_2(P_{13})$ intersects the Fi_{22}-conjugacy classes $1A$, $2A$, $2B$, $2C$, $4A$, $4B$, $4C$, $4D$ and $4E$ in 1, 2, 271, 270, 480, 0, 0, 0 and 0 elements respectively. We see that the cuspidal relation does not hold on P_{13} and hence Fi_{22} admits no 2-cuspidal characters.

Fi_{23}

The group Fi_{23} has eight 2-minimal parabolic subgroups, seven of which feature in 2-minimal parabolic systems. Using the notation of [RS84] these have the form $P_i \sim [2^{17}].\operatorname{Sym}(3)$ for $i = 1, \ldots, 5$ and $P_i \sim [2^{15}].\operatorname{Sym}(5)$ for $i = 7, 8$. These give rise to the geometric 2-minimal parabolic systems $\{P_1, P_3, P_5, P_8\}$ and $\{P_1, P_4, P_5, P_8\}$ and the non-geometric systems $\{P_1, P_2, P_5, P_8\}$ and $\{P_1, P_7, P_8\}$. The maximal parabolic subgroups of these systems are

$$\begin{array}{cc}
P_{125} \sim 2^{10+4}\operatorname{Sym}(3) \times \operatorname{Alt}(7), & P_{128} = P_{138} = P_{148} \sim 2^2 \times 2^{1+8}(3 \times U_4(2))2, \\
P_{135} \sim [2^{14}].(\operatorname{Sym}(3) \times L_3(2)), & P_{145} \sim [2^{14}].(\operatorname{Sym}(3) \times L_3(2)), \\
P_{158} \sim 2Fi_{22}, & P_{17} \sim [2^{14}].(\operatorname{Sym}(3) \times \operatorname{Sym}(5)), \\
P_{18} \sim [2^{11}].U_4(2)2, & P_{258} = P_{358} = P_{458} \sim 2^{11}M_{23}, \\
P_{78} \sim 2^{11}.M_{21}2. &
\end{array}$$

It is easy to check that the cuspidal relation does not hold for P_{158}, and hence the three 2-minimal parabolic systems of rank 4 do not admit any 2-cuspidal characters.

Finally, we consider the maximal parabolic subgroup $P_{18} < P_{158}$. We see that the 2-core $O_2(P_{18})$ intersects the Fi_{23}-conjugacy classes $1A$, $2A$, $2B$, $2C$, $4A$, $4B$, $4C$ and $4D$ in 1, 3, 273, 811, 0, 960, 0 and 0 elements respectively. Consequently, the cuspidal relation does not hold for P_{18}, and hence there are no 2-cuspidal characters of Fi_{23}.

Fi'_{24}

There is a unique 2-minimal parabolic system of Fi'_{24}, which has rank 4. The maximal parabolic subgroups are $P_a \sim 2^{1+12}.(3U_4(3) \cdot 2)$, $P_b \sim 2^{3+12+2}.(\operatorname{Sym}(3) \times Sp_4(2)')$, $P_c \sim 2^{8+6+3}.(L_3(2) \times \operatorname{Sym}(3))$ and $P_d \sim 2^{11}.M_{24}$. Since $O_2(P_d)$ is elementary abelian, we consider the minimum value that each $\chi \in \operatorname{Irr}(Fi'_{24})$ takes on elements of order 2. We immediately deduce that the only possible 2-cuspidal character of Fi'_{24} is χ_2 of degree 8671. For χ_2 to be 2-cuspidal, we would require an integer solution to

$$\chi_2(1A) + j \cdot \chi_2(2A) + (2^{11} - j - 1) \cdot \chi_2(2B) = 0.$$

Since no such solution exists, we conclude that there are no 2-cuspidal characters of Fi'_{24}.

$\mathbb{B}$

The baby monster has five conjugacy classes of 2-minimal parabolic subgroups having representatives $P_i \sim [2^{40}].\operatorname{Sym}(3)$ for $i = 1, \ldots, 4$ and $P_5 \sim [2^{38}].\operatorname{Sym}(5)$. These give rise to a unique 2-minimal parabolic system $\{P_1, P_2, P_3, P_5\}$. The maximal parabolic subgroups of this system are given by $P_{123} \sim 2^{9+16+6+4}.L_4(2)$, $P_{125} \sim 2^{3+32}(L_3(2) \times \operatorname{Sym}(5))$, $P_{135} \sim 2^{2+10+20}(\operatorname{Sym}(3) \times M_{22}2)$ and $P_{235} \sim 2_+^{1+22}.Co_2$. All of these maximal parabolic subgroups are 2-radical. Indeed, from [Yos05] we observe that all 2-parabolic subgroups generated by $P_1, \ldots, P_5$ are 2-radical with the exception of P_3, P_4 and P_{34}. The reader can find further information regarding the structure of the 2-radical parabolic subgroups in [Yos05].

Element, m_i	$\lvert m_i^G \rvert$	$\mathrm{Stab}_{Co_2}(m_i)$	$\lvert\mathrm{Stab}_{Co_2}(m_i)\rvert$
m_1	$2,049,300$	$(2^4 \times 2^{1+4}).\,\mathrm{Sym}(8)$	$20,643,840 = 2^{16}.3^2.5.7$
m_2	$1,619,200$	$U_4(3) : Dih(8)$	$26,127,360 = 2^{10}.3^6.5.7$
m_3	$476,928$	$HS : 2$	$88,704,000 = 2^{10}.3^2.5^3.7.11$
m_4	$46,575$	$2^{10} : M_{22} : 2$	$908,328,960 = 2^{18}.3^2.5.7.11$
m_5	$2,300$	$U_6(2) : 2$	$18,393,661,440 = 2^{16}.3^6.5.7.11$

Table 6.8: The orbits of the 22-dimensional $GF(2)Co_2$-module, M.

Conjugacy class, $*$	$\lvert C_{\mathbb{B}}(*)\rvert$
$2A$	$2^{38}.3^9.5^2.7^2.11.13.17.19$
$2B$	$2^{41}.3^6.5^3.7.11.23$
$2C$	$2^{27}.3^6.5^2.7^2.13.17$
$2D$	$2^{38}.3^5.5^2.7$
$4A$	$2^{32}.3^2.5^3.7.11$
$4B$	$2^{32}.3^6.5.7$
$4C$	$2^{30}.3^3.7$

Conjugacy class, $*$	$\lvert C_{\mathbb{B}}(*)\rvert$
$4D$	$2^{26}.3^4.5.7$
$4E$	$2^{25}.3^4.5.7$
$4F$	$2^{29}.3^2.5$
$4G$	$2^{27}.3^2.5$
$4H$	$2^{25}.3^2$
$4I$	$2^{14}.3^3.5^2.13$
$4J$	$2^{23}.3.5$

Table 6.9: The orders of centralizers of elements of order 2 and 4 in $\mathbb{B}$.

Noting that $P_{235} \cong C_{\mathbb{B}}(2B)$, we focus on the 2-core $Q := O_2(P_{235}) \cong 2_+^{1+22}$ and we shall assume that Co_2 admits a unique class of 22-dimensional modules over $GF(2)$, namely that given in [ABL$^+$], having representative M. Consider the orbits of M under Co_2. There are five such orbits, represented by elements $m_1, \ldots, m_5$. Details of orbits and stabilizers of the m_i under Co_2 are given in Table 6.8, whilst the orders of the centralizers of elements of order 2 and 4 in $\mathbb{B}$ are given in Table 6.9. As an aside, we note that the stabilizers of m_2, m_3, m_4 and m_5 are all maximal in Co_2.

Let z denote the central involution of $Z(Q)$. Thus each m_i gives rise to a subgroup of Q of order 4. Hence as each element of $\mathbb{B}$ of order 4 is conjugate to its cube, we see that each m_i either gives rise to two Co_2-orbits of involutions or two Co_2-orbits of elements of order 4, each having size as given in Table 6.8. Since Q contains $4,196,351$ elements of order 2 and $4,192,256$ elements of order 4, we conclude that m_1, m_4 and m_5 correspond to orbits of involutions, whilst m_2 and m_3 correspond to orbits of elements of order 4. Comparing the stabilizer orders in Table 6.8 with the centralizer orders in Table 6.9 we conclude that both orbits arising from m_2 contain elements of the $\mathbb{B}$-class $4B$, whilst both orbits arising from m_3 contain elements of the $\mathbb{B}$-class $4A$.

Let n_i denote a non-central involution in the subgroup of Q that is the pre-image of m_i in M for each $i = 1, 4, 5$. Since $Q/Z(Q)$ is elementary abelian and $|Z(Q)| = 2$ we see that $[Q : C_Q(\langle z, n_i\rangle)] = 2$, and hence n_i and $z \cdot n_i$ are P_{235}-conjugate.Thus the P_{235}-orbits of n_1, n_4 and n_5 have respective orders 4098600, 93150 and 4600. Since $|\mathrm{Stab}_{Co_2}(m_4)| = 2^{18}.3^2.5.7.11$, we have that $|\mathrm{Stab}_{P_{235}}(n_4)| = 2^{40}.3^2.5.7.11$, and hence elements of the orbit of n_4 lie in the $\mathbb{B}$-class $2B$. This leaves the elements n_1 and n_5 for which $|\mathrm{Stab}_{P_{235}}(n_1)| = 2^{38}.3^2.5.7$ and $|\mathrm{Stab}_{P_{235}}(n_5)| = 2^{38}.3^6.5.7.11$.

Let x be an element of the $\mathbb{B}$-class $2A$. By the $\mathbb{ATLAS}$ we have that there are five orbits of $x^{\mathbb{B}}$ under the action of $C_{\mathbb{B}}(x) \sim 2.{}^2E_6(2) : 2$, as summarised in Table 6.10. Let $x, y \in 2A_{\mathbb{B}}$ be such that $xy = z \in 2B$. Thus $\langle x, y\rangle \leqslant Q$. From Table 6.10 we see that $[C_{\mathbb{B}}(x) : C_{\mathbb{B}}(x) \cap C_{\mathbb{B}}(xy)] = 3,968,055$

Element, y_i	$y_1 = x$	y_2	y_3	y_4	y_5
$y_i^{C_{\mathbb{B}}(x)}$	1	$3,968,055$	$23,113,728$	$2,370,830,336$	$11,174,042,880$
$\mathbb{B}$-class of xy_i	$1A$	$2B$	$2C$	$3A$	$4B$

Table 6.10: The orbits of elements of the $\mathbb{B}$-class $2A$ under the action of $C_{\mathbb{B}}(x)$ for $x \in 2A$.

and hence

$$|C_{\mathbb{B}}(x) \cap Q| = |C_{\mathbb{B}}(x) \cap C_{\mathbb{B}}(xy)| = 77,148,607,752,437,760 = 2^{38}.3^6.5.7.11.$$

It follows that n_5 is an element of the $\mathbb{B}$-class $2A$, and that the orbit of n_5 under P_{235} accounts for all elements of class $2A$ in Q.

Finally, consider the P_{235}-orbit of n_1. Consideration of the centralizer order and the preceding paragraph allows us to deduce that n_1 is in $2B_{\mathbb{B}}$ or $2D_{\mathbb{B}}$. If $n_1 \in 2B_{\mathbb{B}}$, then then there would exist an element of $2B_{\mathbb{B}}$ admitting a P_{235}-orbit of size 4098600. However, the sizes of such orbits are given in [RW04, Theorem 1] and none have this order. Hence $n_1 \in 2D_{\mathbb{B}}$. We conclude that $Q = O_2(P_{235})$ contains 1, 4600, 93151, 4098600, 953856 and 3238400 elements of the $\mathbb{B}$-classes $1A$, $2A$, $2B$, $2D$, $4A$ and $4B$ respectively, and thus the cuspidal relation on P_{235} holds for the characters χ_{10} and χ_{11} of respective degrees $1,407,126,890$ and $3,214,743,741$.

At the time of writing, we have been unable to ascertain if the cuspidal relations for P_{123}, P_{125} or P_{135} hold for either χ_{10} or χ_{11}, and hence whether χ_{10} and χ_{11} are 2-cuspidal characters of $\mathbb{B}$.

$\mathbb{M}$

The monster group has a unique 2-minimal parabolic system, $\{P_1, P_2, P_3, P_4, P_5\}$, where $P_i \sim [2^{45}].L_2(2)$ for $i = 1, \ldots, 5$. The maximal parabolic subgroups are given by $P_{1234} \sim 2^{5+5+16+10}.L_5(2)$, $P_{1235} \sim 2^{4+1+2+8+8+12+4}.(L_4(2) \times \mathrm{Sym}(3))$, $P_{1245} \sim 2^{3+36}.(L_3(2) \times 3Sp_4(2))$, $P_{1345} \sim 2^{2+11+22}.(\mathrm{Sym}(3) \times M_{24})$ and $P_{2345} \sim 2^{1+24}.Co_1$.

We observe that there is no $\chi \in \mathrm{Irr}(\mathbb{M})$ that satisfies the cuspidal relation for $O_2(P_{2345})$. Indeed, let z be an involution of $\mathbb{M}$ in class $2B$ and let Λ be the leach lattice as defined in [Asc94]. Moreover, let Λ_i be the set of all vectors in Λ of *type* i defined as

$$\Lambda_i := \{v \in \Lambda | (v, v)/16 = i\}.$$

Then calculations in [Asc94] show that

$$\begin{aligned}
|\Lambda_2| &= 196,560 &&= 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13, \\
|\Lambda_3| &= 2^{12}(2^{12} - 1) &&= 2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13, \text{ and} \\
|\Lambda_4| &= 398,034,000 &&= 2^4 \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13.
\end{aligned}$$

Let $G := \cdot 0$ - the automorphism group of Λ, and let $\widetilde{G} := G/\langle \varepsilon_X \rangle$ (where ε_X is the scalar map defined on Λ by -1). Thus $\widetilde{G}$ is equal to Co_1. Since ε_X acts trivially on $\widetilde{\Lambda} := \Lambda/2\Lambda$, and G acts transitively on the set Λ_2 ([Asc94, Lemma 22.12(1)]) and on each of the sets Λ_3, Λ_4 ([Asc94, Lemma 22.14(1)]), it follows that $\widetilde{G}$ acts transitively on the sets $\widetilde{\Lambda_2}$, $\widetilde{\Lambda_3}$ and $\widetilde{\Lambda_4}$ (where $\widetilde{\Lambda_i}$ is the image of Λ_i in $\widetilde{\Lambda}$).

Without loss, we may assume that z is the central involution of the extra-special group $O_2(P_{2345})$ and we define $\overline{P_{2345}} := P_{2345}/\langle z\rangle$. By considering the action of G on $\widetilde{\Lambda}$ we have that

$$\left|\mathrm{Stab}_G(\widetilde{\lambda_2})\right| = 2^{17}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 23,$$
$$\left|\mathrm{Stab}_G(\widetilde{\lambda_3})\right| = 2^{9}\cdot 3^7\cdot 5^3\cdot 7\cdot 11\cdot 23,\ \text{and}$$
$$\left|\mathrm{Stab}_G(\widetilde{\lambda_4})\right| = 2^{17}\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$$

(where $\lambda_i \in \Lambda_i$). It follows that

$$\begin{aligned}\left|\mathrm{Stab}_{\overline{P_{2345}}}(\widetilde{\lambda_2})\right| &= 2^{41}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 23,\\ \left|\mathrm{Stab}_{\overline{P_{2345}}}(\widetilde{\lambda_3})\right| &= 2^{33}\cdot 3^7\cdot 5^3\cdot 7\cdot 11\cdot 23,\ \text{and}\\ \left|\mathrm{Stab}_{\overline{P_{2345}}}(\widetilde{\lambda_4})\right| &= 2^{41}\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23.\end{aligned} \tag{6.2.4}$$

The question remains, how does the element λ_i lift to the extra-special group $2^{1+24} = O_2(P_{2345})$? Once this is established, we may then use the centralizer orders

$$\begin{aligned}|C_{\mathbb{M}}(2A)| &= 2^{42}\cdot 3^{13}\cdot 5^6\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47,\\ |C_{\mathbb{M}}(2B)| &= 2^{46}\cdot 3^{9}\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 23,\\ |C_{\mathbb{M}}(4A)| &= 2^{34}\cdot 3^{7}\cdot 5^3\cdot 7\cdot 11\cdot 23,\\ |C_{\mathbb{M}}(4B)| &= 2^{27}\cdot 3^{6}\cdot 5^2\cdot 7^2\cdot 13\cdot 17,\\ |C_{\mathbb{M}}(4C)| &= 2^{34}\cdot 3^{4}\cdot 5^7,\ \text{and}\\ |C_{\mathbb{M}}(4D)| &= 2^{27}\cdot 3^{3}\cdot 5^2\cdot 7\cdot 13\end{aligned} \tag{6.2.5}$$

to determine the fusion within $O_2(P_{2345})$. Indeed, we have that $C_{\overline{P_{2345}}}(\widetilde{\lambda_2}) \sim 2^{24}.Co_2$, $C_{\overline{P_{2345}}}(\widetilde{\lambda_3}) \sim 2^{24}.Co_3$ and $C_{\overline{P_{2345}}}(\widetilde{\lambda_4}) \sim 2^{24}.(2^{11} : M_{24})$.

There are two possible ways in which a $\widetilde{\lambda_i}$ can lift into 2^{1+24}, namely to an abelian subgroup of order 4 of the form $\langle \lambda_i, z\rangle$ having exponent 2 or 4. The former case occurs when $\mathrm{ord}(\lambda_i) = 2$, whilst the latter case occurs when $\mathrm{ord}(\lambda_i) = 4$ and hence $\lambda_i^2 = z$. Since $(4A_{\mathbb{M}})^3 = 4A_{\mathbb{M}}$, $(4B_{\mathbb{M}})^3 = 4B_{\mathbb{M}}$, $(4C_{\mathbb{M}})^3 = 4C_{\mathbb{M}}$ and $(4D_{\mathbb{M}})^3 = 4D_{\mathbb{M}}$, we may use the centralizer and stabilizer orders from (6.2.5) and (6.2.4) to see that the only possible elements of order 4 in P_{2345} must lie in the $\mathbb{M}$-conjugacy class $4A$. Since the exponent of 2^{1+24} is 4, we conclude that the elements of the orbit $\widetilde{\lambda_3}^{\widetilde{G}}$ lift to cyclic groups of order 4 containing 1, z and two elements from the $\mathbb{M}$-class $4A$. This means that

$$|O_2(P_{2345}) \cap 4A| = 2\cdot\left|\widetilde{\Lambda_3}\right| = |\Lambda_3| = 2^{12}(2^{12}-1) = 2^{12}\cdot 3^2\cdot 5\cdot 7\cdot 13 = 16,773,120.$$

Next we consider the lifts of $\widetilde{\lambda_2}$ and $\widetilde{\lambda_4}$. We see that these must lift to the elementary abelian subgroups $\langle \lambda_2, z\rangle$ and $\langle \lambda_4, z\rangle$ respectively. Since there are only two $\mathbb{M}$-classes of involutions, we have that $(\lambda_2 z)^{g_2} = \lambda_2$ and $(\lambda_4 z)^{g_4} = \lambda_4$ for some $g_2, g_4 \in \mathbb{M}$. To determine which element lifts to class $2A$ and which element lifts to $2B$, we note that by [MS02, Lemma 4.4] for $x \neq z$ an involution of P_{2345}, either $C_{P_{2345}}(x) \sim 2^{1+23}.Co_2$ (if x is not 2-central) or $C_{P_{2345}}(x) \sim 2^{1+23}.(2^{11} : M_{24})$ if x is 2-central. Here

$$\begin{aligned}\left|2^{1+23}.Co_2\right| &= 2^{42}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 23\ \text{and}\\ \left|2^{1+23}.(2^{11} : M_{24})\right| &= 2^{45}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23.\end{aligned} \tag{6.2.6}$$

Combining (6.2.6) with (6.2.4) and the fact that the 2-central elements of P_{2345} lie in $2B$, we have that $\widetilde{\lambda_2}$ lifts to an elementary abelian subgroup generated by z and an involution of the $\mathbb{M}$-class $2A$, whilst $\widetilde{\lambda_4}$ lifts to a subgroup generated by z and an element of $2B$. Since z also lies in $2B$, we conclude that

$$\begin{aligned} |O_2(P_{2345}) \cap 2A| &= 2 \cdot \left|\widetilde{\Lambda_2}\right| &&= |\Lambda_2| &&= 196,560, \\ |O_2(P_{2345}) \cap 2B| &= 2 \cdot \left|\widetilde{\Lambda_4}\right| + 1 &&= |\Lambda_4|/24 + 1 &&= 16,584,751 \quad \text{and} \\ |O_2(P_{2345}) \cap 4A| &= 2 \cdot \left|\widetilde{\Lambda_3}\right| &&= |\Lambda_3| &&= 16,773,120. \end{aligned}$$

As there is no $\chi \in \mathrm{Irr}(\mathbb{M})$ satisfying

$$\chi(1A) + 196560 \cdot \chi(2A) + 16584751 \cdot \chi(2B) + 16773120 \cdot \chi(4A) = 0,$$

it follows that there are no 2-cuspidal characters of $\mathbb{M}$.

6.2.4 The Pariahs

J_1

The normalizer of a Sylow 2-subgroup of J_1 is maximal. Thus as the cuspidal relation does not hold for such a Sylow subgroup, J_1 has no 2-cuspidal characters.

$O'N$

The group $O'N$ admits a unique 2-minimal parabolic system of the form

$$\{P_1 \sim 4^3 2^2.\,\mathrm{Sym}(3), P_2 \sim 4.L_3(4)2\}.$$

The generators of $O_2(P_2)$ are elements of the $O'N$-conjugacy class $4A$. Thus we see that the cuspidal relation does not hold on P_2 for any $\chi \in \mathrm{Irr}(O'N)$. Hence $O'N$ admits no 2-cuspidal characters.

J_3

There is a unique 2-minimal parabolic system of J_3, given by

$$\{P_1 \sim 2^{2+4}.(3 \times \mathrm{Sym}(3)), P_2 \sim 2_-^{1+4}.L_2(4)\}.$$

Considering the 2-core $O_2(P_2)$, it contains 1, 11 and 20 elements from the J_3-conjugacy classes $1A$, $2A$ and $4A$ respectively. It follows that J_3 does not admit any 2-cuspidal characters.

Ru

There are three 2-minimal parabolic subgroups of Ru given by $P_1 \sim 2^{5+6}.\,\mathrm{Sym}(5)$ and $P_i \sim 2^{5+6+2}.\,\mathrm{Sym}(3)$ for $i = 2, 3$. Since $P_3 \leqslant P_1$, we obtain a unique 2-minimal parabolic system, namely $\{P_1, P_2\}$. Considering the 2-cores of P_1 and P_2, we see that $O_2(P_1)$ intersects the Ru-conjugacy classes $1A$, $2A$, $2B$, $4A$, $4B$, $4C$, $4D$, $8A$, $8B$ and $8C$ in 1, 271, 0, 512, 64, 240, 960, 0, 0 and 0 elements respectively, whilst $O_2(P_2)$ intersects the given Ru-classes in respectively 1, 367, 192, 608, 448, 1296, 1440, 1536, 768 and 1536 elements. It follows that the cuspidal relation holds on P_1 for $\chi_2, \chi_3 \in \mathrm{Irr}(Ru)$ and on P_2 for $\chi_2, \chi_3, \chi_4 \in \mathrm{Irr}(Ru)$. We conclude that the two characters χ_2 and χ_3 of degree 378 are the only 2-cuspidal characters of Ru.

J_4-conjugacy class, C	$1A$	$2A$	$2B$	$4A$	$4B$	$4C$	$8A$	$8B$	$8C$
$\lvert O_2(P_{12}) \cap C\rvert$	1	3579	4868	22848	100800	72704	0	57344	0
$\lvert O_2(P_{13}) \cap C\rvert$	1	3067	5892	21312	54720	46080	0	0	0
$\lvert O_2(P_{23}) \cap C\rvert$	1	1387	2772	4032	0	0	0	0	0

Table 6.11: The intersections of the 2-cores of the maximal parabolic subgroups of J_4 with the J_4-conjugacy classes.

J_4

There is a unique 2-minimal parabolic system of J_4 given by $\{P_1, P_2, P_3\}$ where $P_i \sim [2^{20}].\operatorname{Sym}(3)$ for $i = 1, 2$ and $P_3 \sim [2^{18}].\operatorname{Sym}(5)$. The maximal 2-parabolic subgroups of this system are given by $P_{12} \sim [2^{18}].L_3(2)$, $P_{13} \sim [2^{17}].(\operatorname{Sym}(3) \times \operatorname{Sym}(5))$ and $P_{23} \sim 2^{1+12}3M_{22}2$. By considering centralizer orders, powering up classes and conjugation of representatives of certain P_{12}-, P_{13}- and P_{23}-conjugacy classes by random elements in J_4, we may determine the fusion of $O_2(P_{12})$-, $O_2(P_{13})$- and $O_2(P_{23})$-classes in J_4. We detail the orders of the intersections of the 2-cores of the maximal parabolic subgroups with the J_4-conjugacy classes in Table 6.11. Consequently, the cuspidal relation holds on P_{12} for the characters χ_2, χ_3, χ_4 and χ_5, it holds on P_{13} for χ_2, χ_3, χ_4, χ_5, χ_6, χ_7, χ_9, χ_{10}, χ_{12} and χ_{13}, whilst the cuspidal relation holds on P_{23} for the irreducible characters χ_2 and χ_3. We conclude that J_4 admits two 2-cuspidal characters, χ_2 and χ_3, both of degree 1333.

Ly

There are six 2-minimal parabolic subgroups of Ly, three of which feature in the two 2-minimal parabolic systems of Ly. These are $P_1 \sim [2^7].\operatorname{Sym}(3)$, $P_2 \sim [2^5].\operatorname{Sym}(5)$, $P_3 \sim 2.\operatorname{Sym}(9)$ and they give rise to the systems $\{P_1, P_2\}$ and $\{P_1, P_3\}$. Since $|O_2(P_1)| = 2^7$, $|O_2(P_2)| = 2^5$ and $|O_2(P_3)| = 2$, it is easy to see that the cuspidal relation does not hold for any of the 2-minimal parabolic subgroups of Ly, and hence there are no 2-cuspidal characters of Ly.

6.3 3-Cuspidal Characters

We now describe the 3-cuspidal characters for each of the sporadic groups.

6.3.1 The Mathieu Groups

M_{11}

The normalizer of a Sylow 3-subgroup of M_{11} is the maximal subgroup $M_9 : 2$ of M_{11}. Consequently, we see that M_{11} admits two 3-cuspidal characters, χ_6 and χ_7, both of degree 16.

M_{12}

The group M_{12} has a unique 3-minimal parabolic system

$$\{P_1 \sim 3^2.GL_2(3), P_2 \sim 3^2.GL_2(3)\}.$$

The non-trivial elements of the 3-cores $O_3(P_1)$ and $O_3(P_2)$ lie in the M_{12}-conjugacy class $3A$. It follows that the 3-cuspidal characters of M_{12} are χ_4 and χ_5, both of which have degree 16.

M_{22}

There is a unique 3-minimal parabolic system of M_{22}, namely

$$\{P_1 \cong M_{10}, P_2 \cong L_3(4)\}.$$

Since $O_3(P_i) = 1$ for $i = 1, 2$, an element $\chi \in \mathrm{Irr}(M_{22})$ will be 3-cuspidal if and only if the cuspidal relation holds for a Sylow 3-subgroup. Since this is never the case, M_{22} has no 3-cuspidal characters.

M_{23}

The 3-cores of the two 3-minimal parabolic subgroups comprising the unique 3-minimal parabolic system

$$\{P_1 \cong M_{11}, P_2 \sim L_3(4) : 2_2\}$$

of M_{23} are both trivial. Since the cuspidal relation does not hold for a Sylow 3-subgroup, we conclude that M_{23} admits no 3-cuspidal characters.

M_{24}

There are three 3-minimal parabolic subgroups of M_{24} given by $P_1 \sim 3.\,\mathrm{Sym}(6)$, $P_2 \sim 2^6 3_+^{1+2} 2_+^{1+2}$ and $P_3 \sim M_{12} : 2$. These give rise to two 3-minimal parabolic systems of M_{24}, namely $\{P_1, P_3\}$ and $\{P_2, P_3\}$. It is easy to observe that there is no $\chi \in \mathrm{Irr}(M_{24})$ satisfying the cuspidal relation for a Sylow 3-subgroup of M_{24}. It follows that neither 3-minimal parabolic system of M_{24} admits a 3-cuspidal character.

6.3.2 The Leech Lattice and Conway Groups

HS

As a Sylow 3-subgroup of HS has order 9, it is easy to see that the cuspidal relation will not hold for such a subgroup, and hence HS admits no 3-cuspidal characters.

J_2

There is a unique 3-minimal parabolic system of J_2 given by

$$\{P_1 \sim 3.\,\mathrm{Alt}(6)2, P_2 \cong U_3(3)\}.$$

The non-trivial elements of $O_3(P_1)$ are contained in the J_2-conjugacy class $3A$. We immediately see that J_2 has no 3-cuspidal characters.

Co_1

There is a solitary 3-minimal parabolic system of Co_1, which has rank 3. Its maximal parabolic subgroups are $P_{12} \sim 3^{1+4}.Sp_4(3) \cdot 2$, $P_{13} \sim 3^{3+4}.GL_2(3)^2$ and $P_{23} \sim 3^6.2M_{12}$.

The Co_1-fusion of the 3-core of P_{23} is described in [Cur80]. We see that $O_3(P_{23})$ intersects the Co_1-conjugacy classes $1A$, $3A$, $3B$, $3C$ and $3D$ in 1, 24, 264, 440 and 0 elements respectively. It follows that the cuspidal relation does not hold for P_{23} and hence Co_1 admits no 3-cuspidal characters.

Co_2

The unique 3-minimal parabolic system of Co_2 has the form

$$\{P_1 \sim 3^{1+4}_{+}.2^{1+4}_{-}\,\mathrm{Sym}(5), P_2 \sim 3^4.L_2(9)\,\mathrm{Dih}(8)\}.$$

Since $O_3(P_2)$ intersects the respective Co_2-conjugacy classes $1A$, $3A$ and $3B$ in 1, 20 and 60 elements, we see that there are no 3-cuspidal characters of Co_2.

Co_3

There are two 3-minimal parabolic subgroups of Co_3 and they form the unique 3-minimal parabolic system

$$\{P_1 \sim 3^{1+4}_{+}.4\,\mathrm{Sym}(6), P_2 \sim 3^5.(M_{11} \times 2)\}.$$

The 3-cores $O_3(P_1)$ and $O_3(P_2)$ intersect the Co_3-classes $1A$, $3A$, $3B$ and $3C$ in 1, 2, 240 and 0 and 1, 110, 132 and 0 elements respectively. It follows that the cuspidal relation holds on P_1 for $\chi_6, \chi_7 \in \mathrm{Irr}(Co_3)$ and on P_2 for $\chi_{10}, \chi_{11} \in \mathrm{Irr}(Co_3)$. Consequently, there are no 3-cuspidal characters of Co_3.

McL

The McLaughlin group has a unique 3-minimal parabolic system given by

$$\{P_1 \sim 3^4.M_{10}, P_2 \sim 3^{1+4}_{+}.2\,\mathrm{Sym}(5)\}.$$

The 3-core $O_3(P_1)$ intersects the McL-conjugacy classes $1A$, $3A$ and $3B$ in 1, 20 and 60 elements respectively. It follows that no $\chi \in \mathrm{Irr}(McL)$ satisfies the cuspidal relation for P_1 and hence McL has no 3-cuspidal characters.

Suz

The unique 3-minimal parabolic system

$$\{P_1 \sim 3^5.M_{11}, P_2 \sim 3^{2+4}.2(\mathrm{Alt}(4) \times 2^2)2\}$$

of Suz does not admit any 3-cuspidal characters. To see this, we note that the 3-core $O_3(P_1)$ intersects the Suz-classes $1A$, $3A$, $3B$ and $3C$ in 1, 22, 220 and 0 elements respectively, meaning that the cuspidal relation does not hold on P_1 for any $\chi \in \mathrm{Irr}(Suz)$.

6.3.3 The Monster Group and its Subgroups

He

Since a Sylow 3-subgroup of He has order 27 and exponent 3, we easily observe that there are no 3-cuspidal characters of He.

HN

The unique 3-minimal parabolic system of HN is

$$\{P_1 \sim 3_+^{1+4}.2\,\mathrm{Sym}(5), P_2 \sim 3^4.2(\mathrm{Alt}(4) \times \mathrm{Alt}(4))4\}.$$

The character table of the subgroup $M = 3^{1+4}.SL_2(5) \leqslant HN$ is given as [Har76, TABLE II]. Since $M \leqslant P_1$, it follows that $O_3(M) = O_3(P_1)$. Moreover, every M-conjugacy class is either contained in, or disjoint from $O_3(M)$. Using the notation from [Har76], we have that

$$O_3(M) = 1 \cup 3_1 \cup 3_1^2 \cup 3_2 \cup 3_3.$$

Considering centralizer orders in M and HN, we see that 3_1 and 3_1^2 are contained in the HN-class $3B$, whilst 3_2 and 3_3 lie in either $3A$ or $3B$. It follows that $|O_3(P_1) \cap 3A| = |O_3(M) \cap 3A| = 0, 120$ or 240 and $|O_3(P_1) \cap 3B| = |O_3(M) \cap 3B| = 2, 122$ or 242. For each of these possibilities, we see that the cuspidal relation would not hold for P_1, and hence there are no 3-cuspidal characters of HN.

Th

The Thompson group has a single 3-minimal parabolic system. It is of rank 2 and has the form

$$\{P_1 \sim 3^{(1+2)+4+2}.GL_2(3), P_2 \sim 3^{(2+3)+4}.GL_2(3)\}.$$

The 3-cores $O_3(P_1)$ and $O_3(P_2)$ intersect the respective Th-conjugacy classes $1A$, $3A$, $3B$, $3C$, $9A$, $9B$ and $9C$ in 1, 270, 2186, 4104, 2106, 5184 and 5832 elements and 1, 756, 2672, 4590, 648, 5184 and 5832 elements respectively. It follows that for both P_1 and P_2 there is a unique element of $\mathrm{Irr}(Th)$ satisfying the cuspidal relation, namely χ_2 of degree 248. We conclude that χ_2 is the unique 3-cuspidal character of Th.

Fi_{22}

There is a unique 3-minimal parabolic system of Fi_{22}, which has rank 3 and minimal parabolic subgroups of the form $P_i \sim [3^8].2.PGL_2(3)$ for $i = 1, 2, 3$. The maximal parabolic subgroups of this system are

$$P_{12} \cong P_{13} \sim 3^{4+2}.L_3(3) \quad \text{and} \quad P_{23} \sim 3_+^{1+6}.2^2.SL_2(3).\,\mathrm{Sym}(4).$$

The maximal parabolics P_{12} and P_{13} are maximal subgroups of the maximal subgroups of Fi_{22} which are isomorphic to $O_7(3)$, and hence their 3-cores can be easily computed. Meanwhile,

Conjugacy Class, C	$1A$	$3A$	$3B$	$3C$	$3D$
$\lvert O_3(P_{12}) \cap C\rvert$	1	0	260	234	234
$\lvert O_3(P_{13}) \cap C\rvert$	1	0	260	234	234
$\lvert O_3(P_{23}) \cap C\rvert$	1	72	386	576	1152

Table 6.12: The orders of the intersections of the Fi_{22}-conjugacy classes with the 3-cores of maximal 3-parabolic subgroups.

by [Asc97, (39.6)], the 3-core $O_3(P_{23})$ is isomorphic to the Fitting subgroup of the normalizer in Fi_{22} of an element of the Fi_{22}-class $3B$. The orders of the intersections of the 3-cores of these maximal parabolics with the Fi_{22}-conjugacy classes is summarised in Table 6.12.

We see that the cuspidal relation holds on P_{12} and P_{13} for the characters $\chi_2, \chi_5 \in \mathrm{Irr}(Fi_{22})$, and it holds on P_{23} for χ_2. Consequently, χ_2 (of degree 78) is the unique 3-cuspidal character of Fi_{22}.

Fi_{23}

The group Fi_{23} has a unique 3-minimal parabolic system given by

$$\{P_1 \sim [3^{12}].2^2PGL_2(3), P_2 \sim [3^{12}].2^2PGL_2(3), P_3 \sim [3^9].2L_2(3)^3 2\,\mathrm{Sym}(3)\}.$$

The corresponding 3-maximal parabolic subgroups are

$$P_{12} \sim 3^{3+7}.GL_3(3), \quad P_{13} \sim 3_+^{1+8} : 2_-^{1+6} : 3_+^{1+2} : 2\,\mathrm{Sym}(4) \quad \text{and} \quad P_{23} \sim D_4(3)\,\mathrm{Sym}(3).$$

Using the information on Fi_{23}-fusion within $O_3(P_{13})$ given in [Wil87, Table 2], and the fact that the non-trivial elements of $Z(3_+^{1+8})$ lie in the Fi_{23}-class $3B$, we see that $O_3(P_{13})$ intersects the Fi_{23}-classes $3A$, $3B$, $3C$ and $3D$ in 864, 1538, 3456 and 13824 elements respectively. (We note that the above fusion can also be calculated within P_{23}. Indeed, there are nine P_{23}-classes of elements of order 3, say $3a, \ldots, 3i$, having centralizer orders in P_{23} of 408146688, 37791360, 37791360, 12737088, 2834352, 944784, 314928, 78732 and 17496 respectively. It follows that the P_{23}-classes satisfy the following inclusions; $3c, 3d \subseteq 3A$, $3a, 3e \subseteq 3B$, $3b, 3g \subseteq 3C$ and $3f, 3h, 3i \subseteq 3D$). Consequently, the cuspidal relation does not hold on P_{13} for any $\chi \in \mathrm{Irr}(Fi_{23})$, and Fi_{23} admits no 3-cuspidal characters.

Fi_{24}'

As with Fi_{23} we see that there is a unique 3-minimal parabolic system of Fi_{24}', namely

$$\{P_1 \sim [3^{15}].2^2PGL_2(3), P_2 \sim [3^{15}].2^2PGL_2(3), P_3 \sim [3^{15}].2\,\mathrm{Sym}(5)\},$$

having maximal parabolics $P_{12} \sim 3^{3+7+3}.L_3(3)2$, $P_{13} \sim 3^{2+4+8}.(SL_2(3) \times \mathrm{Alt}(5))2$ and $P_{23} \sim 3_+^{1+10}.U_5(2)2$.

Consider the extra-special 3-core, $O_3(P_{23}) \cong 3_+^{1+10}$. Since there is a unique irreducible 10-dimensional $GF(3)U_5(2)$-module, M, we can explicitly determine the sizes of the orbits of elements of this module. These are summarised in Table 6.13. We see that there are at most four classes of non-central elements of 3_+^{1+10}.

Element, m	$\mathrm{Stab}_{U_5(2)}(m)$	$\lvert\mathrm{Stab}_{U_5(2)}(m)\rvert$	$\lvert m^{U_5(2)}\rvert$
m_1	$3^4.\,\mathrm{Alt}(5)$	$4,860$	$2,816$
m_2	$3.\,\mathrm{Alt}(6)$	$1,080$	$12,672$
m_3	$2^{1+6}.3^{1+2}.3$	$10,368$	$1,320$
m_4	$3^3.\,\mathrm{Alt}(4)$	324	$42,240$

Table 6.13: The non-trivial orbits of the unique irreducible 10-dimensional $GF(3)U_5(2)$-module.

Let $z \in Z(3_+^{1+10})\backslash\{1\}$ and let $x \in 3_+^{1+10}\backslash Z(3_+^{1+10})$. Thus x represents a non-zero vector in M. Then x, x^2, zx, zx^2, z^2x and z^2x^2 are all Fi_{24}'-conjugate. Indeed, from the $\mathbb{ATLAS}$ we have that for any 3-element $w \in Fi_{24}'$, the elements w and w^2 are Fi_{24}'-conjugate. Suppose that $g \in Fi_{24}'$ is such that $x^g = x^2$. Then $(zx)^g = zx^2$ as z is central. Thus zx and zx^2 lie in a common Fi_{24}'-class, and are joined by $(zx)^2 = z^2x^2$ and $(zx^2)^2 = z^2x$. Finally, as zx and z^2x are Fi_{24}'-conjugate, there exists $h \in Fi_{24}'$ satisfying $z^2x = (zx)^h = zx^h$ and hence $zx = x^h$. We conclude that the orbits of m_1, m_2, m_3 and m_4 give rise to orbits of $3_+^{1+10}\backslash Z(3_+^{1+10})$ of respective sizes 8448, 38016, 3960 and 126720.

Label the orbit of 3_+^{1+10} arising from m_i by M_i for $i = 1, \ldots, 4$. Considering the orders of stabilizers given in Table 6.13 together with the centralizer orders of elements of order 3 in Fi_{24}' given in the $\mathbb{ATLAS}$, we deduce that $M_1, M_2, M_3 \subseteq 3A \cup 3B \cup 3C$, whilst elements in M_4 could form a subset of $3A$, $3B$, $3C$, $3D$ or $3E$. Since $N(3B) \cong P_{23}$, it follows that the non-trivial central elements of 3_+^{1+10} lie in the Fi_{24}'-class $3B$.

The menagerie of information obtained above results in 135 different possibilities for the fusion of 3-elements within $O_3(P_{23}) \cong 3_+^{1+10}$. Feeding each possibility into MAGMA and allowing it to roam over all 108 complex characters of Fi_{24}', we see that the cuspidal relation never holds for P_{23}, and hence Fi_{24}' admits no 3-cuspidal characters.

Aside 6.3.1. *We note that the Fi_{24}'-fusion within the 3-core $O_3(P_{23})$ has previously been studied by Wilson. Indeed, in [Wil87, Section 2.2] Wilson calculates that $O_3(P_{23})$ contains* 3960, 8450, 38016 *and* 126720 *elements from the respective Fi_{24}'-classes $3A$, $3B$, $3C$ and $3D$. However, these calculations are based heavily on an unpublished preprint, and we have been unable to verify them.*

$\mathbb{B}$

There is a unique 3-minimal parabolic system of $\mathbb{B}$, which has rank 3. The maximal parabolic subgroups of this system are $P_{12} \sim 3^{3+7}.GL_3(3)$, $P_{13} \sim 3^{2+3+6}.GL_2(3)^2$ and $P_{23} \sim 3^{1+8}.2^{1+6}PSp_4(3){\cdot}2$. Considering the minimum value that each $\chi \in \mathrm{Irr}(\mathbb{B})$ takes on 3-elements, we see that the only possible 3-cuspidal character of $\mathbb{B}$ is χ_2 of degree 4371. However, as $\mathbb{B}$ satisfies the condition of Lemma 6.1.10, we see that χ_2 cannot be 3-cuspidal, and hence $\mathbb{B}$ admits no 3-cuspidal characters.

$\mathbb{M}$

The monster group, $\mathbb{M}$, has a unique 3-minimal parabolic system, which has rank 3. Its maximal parabolic subgroups are given by

$$P_{12} \sim 3^{3+8+6}2^4.L_3(3),\; P_{13} \sim 3^{2+5+10}.(GL_2(3) \times M_{11}) \text{ and } P_{23} \sim 3_+^{1+12}.2\,\mathrm{Suz}\,2.$$

By considering the 3-core $O_3(P_{23})$, we see that it has exponent 3. Moreover, appealing to [IM97, Lemma 1.5] we see that

$$|O_3(P_{23}) \cap 3A| = 196,560 \qquad \text{and} \qquad |O_3(P_{23}) \cap 3B| = 1,397,762.$$

It immediately follows that $\mathbb{M}$ admits no 3-cuspidal characters.

6.3.4 The Pariahs

J_1

The cuspidal relation does not hold for a Sylow 3-subgroup of J_1. Hence there are no 3-cuspidal characters for the unique 3-minimal parabolic system, $\{J_1\}$, of J_1.

$O'N$

A Sylow 3-subgroup of $O'N$ has order 81. Moreover, since there is a unique $O'N$-conjugacy class of non-trivial 3-elements, we see that the cuspidal relation does not hold for a Sylow 3-subgroup of $O'N$. Consequently, $O'N$ admits no 3-cuspidal characters.

J_3

The normalizer of a Sylow 3-subgroup of J_3 is maximal. Moreover, such a Sylow subgroup intersects the J_3-conjugacy classes $1A$, $3A$, $3B$, $9A$, $9B$ and $9C$ in 1, 18, 8, 72, 72 and 72 elements respectively. Checking the cuspidal relation for each $\chi \in \mathrm{Irr}(J_3)$ for a Sylow 3-subgroup, we see that there are no 3-cuspidal characters of J_3.

Ru

Since a Sylow 3-subgroup of Ru has order 27 and there is a unique Ru-conjugacy class of non-trivial 3-elements, it is easy to see that Ru admits no 3-cuspidal characters.

J_4

The non-trivial elements of a Sylow 3-subgroup of J_4 lie in the J_4-class $3A$. Since such a subgroup has order 27, it is easy to see that the cuspidal relation does not hold for a Sylow 3-subgroup of J_4. Hence, J_4 has no 3-cuspidal characters.

Ly

There is a unique 3-minimal parabolic system of Ly given by

$$\{P_1 \sim 3^{2+4}.8\,\mathrm{Sym}(5), P_2 \sim 3^5.(M_{11} \times 2)\}.$$

Considering the minimum value that each $\chi \in \mathrm{Irr}(Ly)$ takes on elements of order 3, we see that the only possible candidates for 3-cuspidal characters are χ_7 and χ_8 of degree 120064. However, these characters take strictly negative values on all 3-elements, and hence the cuspidal relation cannot hold for them for both $O_3(P_2)$ and a Sylow 3-subgroup of Ly. We conclude that there are no 3-cuspidal characters of Ly.

6.4 5-Cuspidal Characters

The groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, Suz, He, Fi_{22}, Fi_{23}, Fi'_{24}, J_1, $O'N$, J_3, Ru and J_4 have Sylow 5-subgroups of exponent 5 and of order at most 5^3. Thus, considering the minimal values that an irreducible character takes on elements of order 5 together with the character degree, we see that none of these groups admit a 5-cuspidal character. We now consider the remaining eleven sporadic groups in turn.

HS

A Sylow 5-subgroup of HS intersects the HS-conjugacy classes $1A$, $5A$, $5B$ and $5C$ in 1, 4, 40 and 80 elements respectively. It follows immediately that HS has no 5-cuspidal characters.

J_2

The normalizer in J_2 of a Sylow 5-subgroup, S, is maximal and S intersects each of the J_2-conjugacy classes $5A$, $5B$, $5C$ and $5D$ in 6 elements. From this we deduce that J_2 has a unique 5-cuspidal character given by χ_6 of degree 36.

Co_1

There is a unique 5-minimal parabolic system of Co_1 given by

$$\{P_1 \sim 5^3.(4 \times \mathrm{Alt}(5)) \cdot 2, P_2 \sim 5^{1+2}.GL_2(5)\}.$$

Considering the minimum value that each $\chi \in \mathrm{Irr}(Co_1)$ takes on elements of order 5, and the order of the 5-cores of the minimal parabolic subgroups, we see that there are no 5-cuspidal characters of Co_1.

Co_2, Co_3, McL

If $G \in \{Co_2, Co_3, McL\}$, then a Sylow 5-subgroup of G intersects the G-conjugacy classes $1A$, $5A$ and $5B$ in 1, 4 and 120 elements respectively. It follows that the cuspidal relation does not hold for a Sylow 5-subgroup for any $\chi \in \mathrm{Irr}(G)$ and hence G has no 5-cuspidal characters.

HN

There is a unique 5-minimal parabolic system of HN given by

$$\{P_1 \sim 5^{1+4}.(2^{1+4}.5.4), P_2 \sim 5^{2+1+2}.4.\mathrm{Alt}(5)\}.$$

The character table of P_1 is given as [Har76, Table III], whilst a partial character table of P_2 - featuring the conjugacy classes of elements of order 2 and classes contained in $O_5(P_2)$ - is given in

Class	1_1	2_1	2_2	5_1	5_2	5_3	5_7	5_8	5_9	5_{10}	5_{13}
Size	1	625	3750	24	50	50	600	600	600	600	600
Order	1	2	2	5	5	5	5	5	5	5	5
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	1	1	1	1	1	1	1
χ_3	2	-2	0	2	2	2	2	2	2	2	2
χ_4	2	-2	0	2	2	2	2	2	2	2	2
χ_5	2	-2	0	2	2	2	2	2	2	2	2
χ_6	2	-2	0	2	2	2	2	2	2	2	2
χ_7	3	3	1	3	3	3	3	3	3	3	3
χ_8	3	3	1	3	3	3	3	3	3	3	3
χ_9	3	3	-1	3	3	3	3	3	3	3	3
χ_{10}	3	3	-1	3	3	3	3	3	3	3	3
χ_{11}	4	4	0	4	4	4	4	4	4	4	4
χ_{12}	4	4	0	4	4	4	4	4	4	4	4
χ_{13}	4	-4	0	4	4	4	4	4	4	4	4
χ_{14}	4	-4	0	4	4	4	4	4	4	4	4
χ_{15}	5	5	1	5	5	5	5	5	5	5	5
χ_{16}	5	5	-1	5	5	5	5	5	5	5	5
χ_{17}	6	-6	0	6	6	6	6	6	6	6	6
χ_{18}	6	-6	0	6	6	6	6	6	6	6	6
χ_{19}	10	2	0	10	$5 \cdot w^3 + 5 \cdot w^2$	$-5 \cdot w^3 - 5 \cdot w^2 - 5$	0	0	0	0	0
χ_{20}	10	2	0	10	$-5 \cdot w^3 - 5 \cdot w^2 - 5$	$5 \cdot w^3 + 5 \cdot w^2$	0	0	0	0	0
χ_{21}	20	-4	0	20	$10 \cdot w^3 + 10 \cdot w^2$	$-10 \cdot w^3 - 10 \cdot w^2 - 10$	0	0	0	0	0
χ_{22}	20	-4	0	20	$-10 \cdot w^3 - 10 \cdot w^2 - 10$	$10 \cdot w^3 + 10 \cdot w^2$	0	0	0	0	0
χ_{23}	20	-4	0	20	$-10 \cdot w^3 - 10 \cdot w^2 - 10$	$10 \cdot w^3 + 10 \cdot w^2$	0	0	0	0	0
χ_{24}	20	-4	0	20	$10 \cdot w^3 + 10 \cdot w^2$	$-10 \cdot w^3 - 10 \cdot w^2 - 10$	0	0	0	0	0
χ_{25}	24	0	4	24	24	24	-1	-1	-1	-1	-1
χ_{26}	24	0	-4	24	24	24	-1	-1	-1	-1	-1
χ_{27}	30	6	0	30	$-15 \cdot w^3 - 15 \cdot w^2 - 15$	$15 \cdot w^3 + 15 \cdot w^2$	0	0	0	0	0
χ_{28}	30	6	0	30	$-15 \cdot w^3 - 15 \cdot w^2 - 15$	$15 \cdot w^3 + 15 \cdot w^2$	0	0	0	0	0
χ_{29}	30	6	0	30	$15 \cdot w^3 + 15 \cdot w^2$	$-15 \cdot w^3 - 15 \cdot w^2 - 15$	0	0	0	0	0
χ_{30}	30	6	0	30	$15 \cdot w^3 + 15 \cdot w^2$	$-15 \cdot w^3 - 15 \cdot w^2 - 15$	0	0	0	0	0
χ_{31}	40	-8	0	40	$-20 \cdot w^3 - 20 \cdot w^2 - 20$	$20 \cdot w^3 + 20 \cdot w^2$	0	0	0	0	0
χ_{32}	40	8	0	40	$20 \cdot w^3 + 20 \cdot w^2$	$-20 \cdot w^3 - 20 \cdot w^2 - 20$	0	0	0	0	0
χ_{33}	40	-8	0	40	$20 \cdot w^3 + 20 \cdot w^2$	$-20 \cdot w^3 - 20 \cdot w^2 - 20$	0	0	0	0	0
χ_{34}	40	8	0	40	$-20 \cdot w^3 - 20 \cdot w^2 - 20$	$20 \cdot w^3 + 20 \cdot w^2$	0	0	0	0	0
χ_{35}	48	0	0	48	48	48	-2	-2	-2	-2	-2
χ_{36}	48	0	0	48	48	48	-2	-2	-2	-2	-2
χ_{37}	50	10	0	50	$25 \cdot w^3 + 25 \cdot w^2$	$-25 \cdot w^3 - 25 \cdot w^2 - 25$	0	0	0	0	0
χ_{38}	50	10	0	50	$-25 \cdot w^3 - 25 \cdot w^2 - 25$	$25 \cdot w^3 + 25 \cdot w^2$	0	0	0	0	0
χ_{39}	60	-12	0	60	$30 \cdot w^3 + 30 \cdot w^2$	$-30 \cdot w^3 - 30 \cdot w^2 - 30$	0	0	0	0	0
χ_{40}	60	-12	0	60	$-30 \cdot w^3 - 30 \cdot w^2 - 30$	$30 \cdot w^3 + 30 \cdot w^2$	0	0	0	0	0
χ_{41}	120	0	-4	-5	0	0	-5	0	5	-5	5
χ_{42}	120	0	4	-5	0	0	-5	0	5	-5	5
χ_{43}	120	0	-4	-5	0	0	0	5	$-5 \cdot w^3 - 5 \cdot w^2 - 5$	0	$5 \cdot w^3 + 5 \cdot w^2$
χ_{44}	120	0	4	-5	0	0	$5 \cdot w^3 + 5 \cdot w^2 + 5$	-5	0	$-5 \cdot w^3 - 5 \cdot w^2$	0
χ_{45}	120	0	4	-5	0	0	0	5	$5 \cdot w^3 + 5 \cdot w^2$	0	$-5 \cdot w^3 - 5 \cdot w^2 - 5$
χ_{46}	120	0	-4	-5	0	0	$-5 \cdot w^3 - 5 \cdot w^2$	-5	0	$5 \cdot w^3 + 5 \cdot w^2 + 5$	0
χ_{47}	120	0	-4	-5	0	0	$5 \cdot w^3 + 5 \cdot w^2 + 5$	-5	0	$-5 \cdot w^3 - 5 \cdot w^2$	0
χ_{48}	120	0	4	-5	0	0	0	5	$-5 \cdot w^3 - 5 \cdot w^2 - 5$	0	$5 \cdot w^3 + 5 \cdot w^2$
χ_{49}	120	0	-4	-5	0	0	0	5	$5 \cdot w^3 + 5 \cdot w^2$	0	$-5 \cdot w^3 - 5 \cdot w^2 - 5$
χ_{50}	120	0	4	-5	0	0	$-5 \cdot w^3 - 5 \cdot w^2$	-5	0	$5 \cdot w^3 + 5 \cdot w^2 + 5$	0

Table 6.14: A partial character table of the 5-minimal parabolic subgroup $P_2 \sim 5^{2+1+2}.4.\mathrm{Alt}(5)$ of HN (where $w = \exp(2\pi i/5)$).

Table 6.14. By considering the restriction of $\chi_2 \in \mathrm{Irr}(HN)$ to P_1 and P_2, we may calculate the HN-fusion within $O_5(P_1)$ and $O_5(P_2)$. We see that $O_5(P_1)$ intersects the HN-classes $1A$, $5A$, $5B$, $5C$, $5D$ and $5E$ in 1, 400, 324, 800, 800 and 800 elements respectively, whilst $O_5(P_2)$ intersects the given classes in 1, 0, 624, 650, 650, and 1200 elements respectively. Consequently, the cuspidal relation holds on P_1 for $\chi_4 \in \mathrm{Irr}(HN)$ and on P_2 for $\chi_4, \chi_5 \in \mathrm{Irr}(HN)$. We conclude that χ_4 (of degree 760) is the unique 5-cuspidal character of HN.

Th

The normalizer of a Sylow 5-subgroup of Th is a maximal subgroup. Thus a character $\chi \in \mathrm{Irr}(Th)$ will be 5-cuspidal precisely when the cuspidal condition holds for the Sylow subgroup. Since Th has a unique conjugacy class of elements of order 5, we observe that there is a unique 5-cuspidal character of Th, namely χ_2 of degree 248.

$\mathbb{B}$

There is a unique 5-minimal parabolic system of $\mathbb{B}$ given by

$$\{P_1 \sim 5^{1+4}.2^{1+4}\,\mathrm{Sym}(5)2, P_2 \sim 5^{2+1+2}.GL_2(5)\}.$$

Considering the minimum value that each $\chi \in \mathrm{Irr}(\mathbb{B})$ takes on $\mathbb{B}$-classes of 5-elements, we see that the only possible 5-cuspidal character of $\mathbb{B}$ is χ_2. However, $\deg(\chi_2) = 4371$, and $\mathbb{B}$ satisfies the conditions of Lemma 6.1.10. Thus there are no 5-cuspidal characters of $\mathbb{B}$.

$\mathbb{M}$

The monster has a unique 5-minimal parabolic system, namely

$$\{P_1 \sim 5^{1+6}.2(J_2 \times 2)2, P_2 \sim 5^{2+2+4}\,\mathrm{Sym}(3).GL_2(5)\}.$$

Using a similar approach to that used for the baby monster, with the 5-core $O_5(P_1)$, we deduce that there are no 5-cuspidal characters of $\mathbb{M}$.

Ly

From [RS84] we see that there is a unique 5-minimal parabolic system of Ly, having rank 3. Its minimal parabolic subgroups are $P_1 \sim 5_+^{1+4}.4PGL_2(5)$, $P_2 \sim 5^{3+2}.4PGL_2(5)$ and $P_3 \sim 5_+^{1+4}.4PGL_2(5)$. The maximal 5-parabolic subgroups are given by

$$P_{12} \sim 5^3.SL_3(5), \quad P_{13} \sim 5_+^{1+4}.2\,\mathrm{Alt}(6)4 \quad \text{and} \quad P_{23} \cong G_2(5).$$

By considering the elementary abelian subgroup $O_5(P_{12}) \cong 5^3$, we see that the only possible 5-cuspidal characters of Ly are χ_2 and χ_3 (both of degree 2480), and that for these characters to be 5-cuspidal, we must have that the non-trivial elements of $O_5(P_{12})$ are contained in the Ly-conjugacy class $5A$. Defining S to be our given Sylow 5-subgroup of Ly, we see that S has a unique normal elementary abelian 5-subgroup of order 5^3. Considering this within the maximal subgroup

$G_2(5) \leqslant Ly$, we see that the non-trivial elements of $O_5(P_{12})$ are indeed contained in the Ly-class $5A$. It remains to check the cuspidal relation for χ_2 and χ_3 for the extra-special 5-core $O_5(P_{13})$ of order 5^5.

By constructing $O_5(P_{13})$ within both P_{13} and P_{23} we may deduce that it intersects the Ly-conjugacy classes $1A$, $5A$ and $5B$ in 1, 724 and 2400 elements respectively. It follows that the cuspidal relation holds on $O_5(P_{13})$ for both χ_2 and χ_3, and hence we see that χ_2 and χ_3 are 5-cuspidal characters of Ly.

6.5 p-Cuspidal Characters ($p > 5$)

In the case that $p > 5$, most sporadic groups with order divisible by p have a cyclic Sylow p-subgroup of order p. The exceptions are $(Co_1, p = 7)$, $(He, p = 7)$, $(Th, p = 7)$, $(Fi'_{24}, p = 7)$, $(\mathbb{B}, p = 7)$, $(\mathbb{M}, p = 7, 11, 13)$, $(O'N, p = 7)$ and $(J_4, p = 11)$.

The normalizer of a Sylow 7-subgroup, S, of He is maximal in He, and hence there is a unique 7-minimal parabolic system given by $\{He\}$. Consequently, an element $\chi \in \mathrm{Irr}(He)$ will be 7-cuspidal precisely when the cuspidal condition holds for S. We have that S contains 1, 42, 42, 132, 63 and 63 elements from the He-classes $1A$, $7A$, $7B$, $7C$, $7D$ and $7E$ respectively. It follows that the 7-cuspidal characters of He are χ_2 and χ_3 of degree 51.

In all other cases, since the sporadic group in question contains no elements of order p^a for $a > 1$, we may consider the minimum value that each irreducible character takes on elements of order p, to conclude that there are no p-cuspidal characters.

We conclude by considering the p-cuspidal characters arising from sporadic groups having a cyclic Sylow p-subgroup of order p. Since such a subgroup will necessarily be the p-core of its normalizer, it is easy to see that the cuspidal relation must hold for the Sylow subgroup. Moreover, a character will be cuspidal precisely when this is true. This gives an additional four cuspidal characters for the sporadic groups; the 10-dimensional characters $\chi_2, \chi_3, \chi_4 \in \mathrm{Irr}(M_{11})$ are 11-cuspidal and the 22-dimensional character $\chi_2 \in \mathrm{Irr}(M_{23})$ is 23-cuspidal.

6.6 Possible Connections with Geometries

In Section 6.1 we saw how geometries could be used to classify characters. Indeed, the definition of p-cuspidal characters is entirely dependant on a given p-minimal parabolic geometry. Having classified the p-cuspidal characters of the sporadic groups[2], we may now turn the situation on its head and ask the following question.

Question 6.6.1. *Let G be a group, p a prime divisor of $|G|$ and suppose that χ is a p-cuspidal character of G. Are there any "nice" geometries arising from χ?*

For the tuples (G, p) given by $(M_{11}, 11)$ and $(M_{23}, 23)$, it is unlikely that the p-cuspidal characters of G will give rise to any interesting geometries. Indeed, in both cases a Sylow p-subgroup of G

[2] with the exception of the 2-cuspidal characters of $\mathbb{B}$

is cyclic of order p, and the resulting p-cuspidal characters have degree $p-1$. Similarly, when (G, p) is $(M_{11}, 3)$ or $(Th, 5)$, there is a unique class of elements of order p, and a Sylow p-subgroup has exponent p. Hence it is improbable that the resulting p-cuspidal characters lead to interesting geometries. Finally, as the maximal 3-parabolic subgroups of M_{12} are both normalizers of elementary abelian subgroups of rank 2 generated by elements of the M_{12}-class $3A$, we shall not examine these 3-cuspidal characters any further. Thus we consider the remaining p-cuspidal characters given in Tables 6.1-6.3, with the caveat that - at the time of writing - our subsequent comments are purely speculative and serve more as a survey of literature that may be related to the remaining p-cuspidal characters.

In the early 1990s, Margolin looked at a geometry for the Mathieu group M_{24} [Mar93]. Margolin's interest stemmed from the two 1333-dimensional irreducible $GF(2)J_4$-representations. Since $2^{11} : M_{24}$ is a maximal subgroup of J_4, Margolin considered the restriction of these representations to $2^{11} : M_{24}$, namely as a faithful 1288-dimensional representation and a 45-dimensional representation having kernel 2^{11}. Hence Margolin sought to find a simple explanation for this 45-dimensional representation and this resulted in the construction of a geometry. We note that both 1333-dimensional J_4-characters are 2-cuspidal, as are both of the resulting irreducible M_{24}-characters of degree 45, along with their irreducible restrictions to M_{22} and M_{23}.

Considering the 3-cuspidal character of Th, we note that Th is a subgroup of the exceptional group $E_8(q)$ when $q = 3$ (see for example [Wil09]). The lowest character degree of $E_8(3)$ is 248. Thus it would be worth considering whether this subgroup inclusion is related to the fact that this character of lowest degree is 3-cuspidal, and if so, is there an amiable geometry arising from this character?

The existence of the geometry constructed by Margolin and the subgroup inclusion of Th in $E_8(3)$ may be completely unrelated to the existence of the aforementioned p-cuspidal characters. Indeed, there do not appear to exist publications on geometries related to the other p-cuspidal characters described in Sections 6.2-6.5. However, it would be worth investigating these two situations further. Moreover, it would be worth considering the p-cuspidal characters from Tables 6.1-6.3 from a general geometric viewpoint, and thus looking at answering Question 6.6.1. However, such consideration is far beyond the scope of this thesis.

Chapter 7

Conclusions and Future Work

We conclude the thesis by briefly summarising our findings and suggesting prospective areas of further study for the interested reader. For ease of reading, we break this down under the four main chapter headings.

7.1 Homology of Presheaves of Abelian Groups

In Chapter 3, our main contribution to the subject area was the calculation of the zero-homology groups of universal panel-irreducible presheaves of the symmetric group $\mathrm{Sym}(6)$ and the the Mathieu groups M_{11} and M_{22} together with calculating the irreducible quotients of the zero-homology groups of such presheaves for M_{12}, M_{23} and M_{24}. These presheaves were all defined over the finite field, $GF(2)$. Further work would allow for the undetermined homology groups to be explicitly constructed. However, describing their structure would be a complex affair.

The main aim in calculating these homology groups/quotients of these homology groups was to try to ascertain the answer to Question 3.9.1, namely for a given group G, a 2-minimal parabolic system $\mathcal{S} := \{P_i \mid i = 1, \ldots, n\}$ for G such that $P_i \sim O_2(P_i).\,\mathrm{Sym}(3)$ for all $i = 1, \ldots, n$ and an irreducible $GF(2)G$-module V, does there exist a universal panel-irreducible presheaf, λ, for G defined on $\mathcal{S}$ over $GF(2)$ such that V is a quotient of $H_0(\lambda)$? We saw in Section 3.8 that the groups $\mathrm{Sym}(6)$, M_{12} and M_{24} satisfied the given hypothesis and in each case and for every 2-minimal parabolic system $\mathcal{S}$ of the given group, every irreducible module over $GF(2)$ was indeed a quotient of the zero-homology group of some universal panel-irreducible presheaf defined on $\mathcal{S}$ over $GF(2)$. Future research into this area would be of interest, to try and establish the answer to Question 3.9.1.

Further areas that we did not consider in Chapter 3, but that are large areas of potential future research include presheaves defined over fields other than $GF(2)$, and presheaves defined on the order complex of simplicial complexes other than those defined by minimal parabolic systems. It is likely that such areas would be of great interest and may well shed further light on the presheaves defined on 2-minimal parabolic systems over $GF(2)$ that we have examined.

A final area of possible research in this area would be to try and develop the theoretical-side of the subject further. During our work, we have focused predominantly on the computational-side of things. However, we believe that further theoretical results, either expanding the results of

Section 3.6 or taking a completely new approach to the subject, would be beneficial in developing the field of study further.

7.2 π-Product Graphs in Symmetric Groups

The main result of Chapter 4 was Theorem 4.0.1. This concerned the case that $G := \mathrm{Sym}(n)$, $t = (1, 2) \cdots (2m-1, 2m) \in G$ and X is the G-conjugacy class of t. The result fully classified when the $\{4\}$-product graph $\mathcal{P}_{\{4\}}(G, X)$ was connected, and in the case that it was connected, the diameter was shown to equal 2. The result followed by a combinatorial analysis of the possible connected components of related x-graphs, and was constructive in nature. Indeed, given an element $x \in X$ such that the order of tx does not equal 4, then by following the proof of Theorem 4.0.1 the reader may construct an element $y \in X$ such that $d(t, y) = d(y, x) = 1$.

The graph $\mathcal{P}_{\{4\}}(G, X)$ was chosen as two elements $x, y \in X$ are neighbours in $\mathcal{P}_{\{4\}}(G, X)$ precisely when $\langle x, y\rangle \cong \mathrm{Dih}(8)$. A natural generalisation of this is to consider when the graph $\mathcal{P}_{\{2^a\}}(G, X)$ is connected for $a > 2$, where $x, y \in X$ are neighbouring vertices precisely when $\langle x, y\rangle \cong \mathrm{Dih}(2^{a+1})$. The connectivity of such graphs was determined in Theorem 4.0.2, where crude upper and lower bounds were also obtained for the diameter in the case of connected graphs. Explicit values of the diameters of the connected graphs $\mathcal{P}_{\{8\}}(G, X)$ were obtained in Theorem 4.0.3 in the case that the support of t has order 8. However, we did not investigate the diameters of the connected graphs from Theorem 4.0.2 any further in the case that $a > 3$. This is an area that would be worth contemplating in the future, and a more general approach to that used in proving Theorem 4.0.1 might be worth considering in such future work.

The final two results of Chapter 4, namely Theorems 4.0.4 and 4.0.5 were in a similar vein to Theorem 4.0.2. However, all three of these theorems concerned π-product graphs where $\pi = \{\ell\}$ and ℓ is closely related to $|\mathrm{supp}(t)|$. We did not consider the situation when such a relation did not hold. Since such a greater generality is in a similar vein to the greater generality of Theorem 4.0.1, it may be worth looking at the possibility of generalising Theorems 4.0.2, 4.0.4 and 4.0.5 in the future.

7.3 Conjugate p-elements of Full Support that Generate the Wreath Product $C_p \wr C_p$

Motivated by the graph-theoretic approach of Chapter 4, we considered possible generalisations of the x-graph in Chapter 5 resulting in matrices A_x^a and A_a^x being defined in Definition 5.1.1. These matrices encapsulated some of the information that a generalised x-graph would contain, whilst losing other information. However, they proved to have adequate versatility for us to determine when two conjugate elements of order p having full support in a given symmetric group generate the wreath product $C_p \wr C_p$ of two cyclic groups of order p. In the case of x-graphs, the work of Bates, Bundy, Perkins and Rowley [BBPR03b] illustrated that x-graphs may be applied to obtain a number of different results. The matrices A_a^x and A_x^a lack the majority of this versatility and were defined specifically for our chosen line of research. However, it may be worth considering if they have further applications. It is unlikely that the results of Chapter 5 can in themselves be extended.

7.4 Cuspidal Characters of Finite Groups

The final chapter of the thesis considered cuspidal characters of finite groups. The idea was to consider an analogue to cuspidal characters of groups of Lie type for an arbitrary finite group. Many of the results presented in [Car93] proved to have analogues in the finite setting and this enabled p-cuspidal characters of finite groups to be considered. Moreover, we showed in Corollary 6.1.9 that it suffices to check the cuspidal relation for a certain subset of parabolic subgroups.

The main body of Chapter 6 was devoted to considering the p-cuspidal characters of each sporadic simple group. All p-cuspidal characters were determined with the exception of the 2-cuspidal characters of the baby monster, $\mathbb{B}$. This is obviously an area that would warrant further work, as would consideration of possible geometries associated to the other p-cuspidal characters of the sporadic groups - a topic briefly mentioned in Section 6.6.

The approach used throughout Chapter 6 involved p-minimal parabolic systems of groups. As we saw, these were specific cases of X-parabolic systems as defined in Definition 6.1.1. We did not investigate these systems in greater generality. Further work could involve considering such systems and contemplating if other forms of cuspidal characters could be defined for finite groups using other X-parabolic systems.

Appendix A

Vertex Term Calculations for Universal Panel-Irreducible Presheaves of M_{23} and M_{24} over $GF(2)$

In Chapter 3 we calculated the irreducible quotients of the zero-homology groups of the universal panel-irreducible presheaves defined on the 2-minimal parabolic systems of M_{23} and M_{24} over $GF(2)$. These minimal parabolic systems all have rank 3, and thus the associated simplicial complexes have chamber, panel and vertex terms. In this appendix, we give a full examination of the vertex terms of these presheaves.

A.1 Vertex Term Calculations for Universal Panel-Irreducible Presheaves of M_{23} over $GF(2)$

As seen in Chapter 3, there are seven 2-minimal parabolic systems of M_{23}, each of rank 3. We denote these by $\mathcal{S}_i := \{P_i, P_6, P_7\}$ for $i = 1, 2, 3, 4$, $\mathcal{S}_5 := \{P_1, P_3, P_7\}$, $\mathcal{S}_6 := \{P_2, P_3, P_7\}$ and $\mathcal{S}_7 := \{P_3, P_4, P_7\}$. Here we are adopting the notation of [RS84, p77], and so $P_i \sim 2^{4+2}.\,\mathrm{Sym}(3)$ for $i = 1, 2, 3, 4$ and $P_j \sim 2^4.\,\mathrm{Sym}(5)$ for $j = 6, 7$. The diagrams of the geometries arising from $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_4$ are given in Figure A.1, whilst the diagrams arising from $\mathcal{S}_3$, $\mathcal{S}_5$, $\mathcal{S}_6$ and $\mathcal{S}_7$ are given in Figures A.2, A.3, A.4 and A.5 respectively.

The minimal parabolic subgroups P_i each admit two classes of irreducible $GF(2)P_i$-modules, for $i = 1, \ldots, 4$, denoted by 1_{P_i} and 2_{P_i}. For $j = 6, 7$ there are three classes of irreducible $GF(2)P_j$-modules, namely 1_{P_j}, 4_{P_j} and $\overline{4}_{P_j}$. Continuing the notation of Chapter 3, we denote the presheaf defined on the minimal parabolic system $\{P_i, P_j, P_k\}$ having a $*_{dim}$-dimensional irreducible $GF(2)P_*$-module at panels of type P_* by $\lambda^{(ijk)}_{i_{dim}, j_{dim}, k_{dim}}$ (using the bar notation to differentiate between classes of 4-dimensional irreducibles in the case of P_6 and P_7). The vertex terms of $\lambda^{(ijk)}_{i_{dim}, j_{dim}, k_{dim}}$ are themselves zero-homology groups of presheaves defined on rank two 2-minimal parabolic systems of P_{ij}, P_{ik} and P_{jk}. We denote by $\lambda^{(ij)}_{i_{dim}, j_{dim}}$ the universal panel-irreducible presheaf of P_{ij} having a 1-dimensional irreducible module at its chambers and satisfying

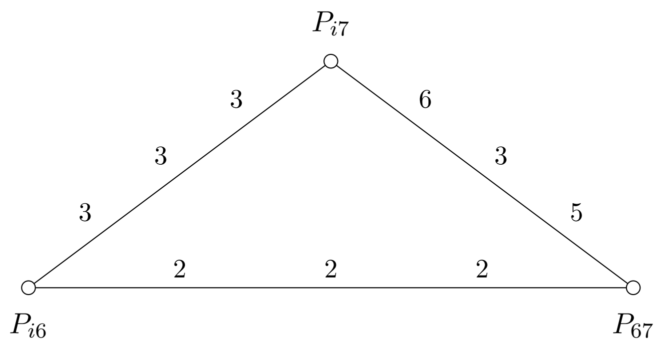


Figure A.1: The diagram of the geometries arising from the 2-minimal parabolic systems $\mathcal{S}_i$ of M_{23} for $i = 1, 2, 4$.

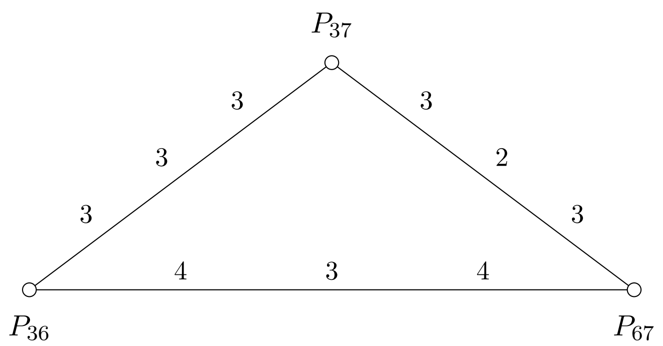


Figure A.2: The diagram of the geometry arising from the 2-minimal parabolic system $\mathcal{S}_3$ of M_{23}.

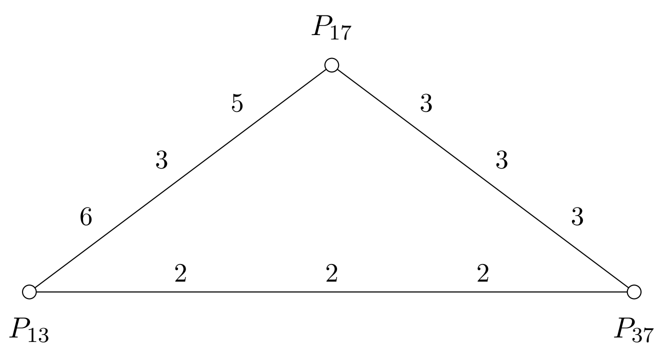


Figure A.3: The diagram of the geometry arising from the 2-minimal parabolic system $\mathcal{S}_5$ of M_{23}.

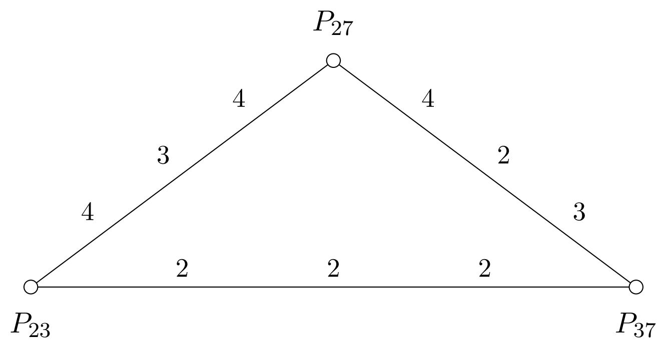


Figure A.4: The diagram of the geometry arising from the 2-minimal parabolic system $\mathcal{S}_6$ of M_{23}.

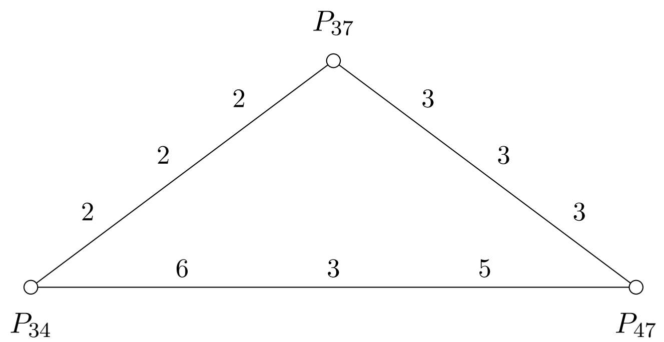


Figure A.5: The diagram of the geometry arising from the 2-minimal parabolic system $\mathcal{S}_7$ of M_{23}.

Parabolic Subgroup, P	Shape of P	Irreducible $GF(2)P$-Modules
P_{13}	$2^4.L_3(2)$	$1_{P_{13}}$, $3_{P_{13}}$, $\overline{3}_{P_{13}}$, $8_{P_{13}}$
P_{16}	$2^4.\operatorname{Alt}(7)$	$1_{P_{16}}$, $4_{P_{16}}$, $\overline{4}_{P_{16}}$, $6_{P_{16}}$, $14_{P_{16}}$, $20_{P_{16}}$
P_{17}	$2^4.(3 \times \operatorname{Alt}(5)).2$	$1_{P_{17}}$, $2_{P_{17}}$, $4^a_{P_{17}}$, $4^b_{P_{17}}$, $4^c_{P_{17}}$, $4^d_{P_{17}}$, $8_{P_{17}}$
P_{23}	$2^4.\operatorname{Alt}(7)$	$1_{P_{23}}$, $4_{P_{23}}$, $\overline{4}_{P_{23}}$, $6_{P_{23}}$, $14_{P_{23}}$, $20_{P_{23}}$
P_{26}	$2^4.\operatorname{Alt}(7)$	$1_{P_{26}}$, $4_{P_{26}}$, $\overline{4}_{P_{26}}$, $6_{P_{26}}$, $14_{P_{26}}$, $20_{P_{26}}$
P_{27}	$2^4.(3 \times \operatorname{Alt}(5)).2$	$1_{P_{27}}$, $2_{P_{27}}$, $4^a_{P_{27}}$, $4^b_{P_{27}}$, $4^c_{P_{27}}$, $4^d_{P_{27}}$, $8_{P_{27}}$
P_{34}	$2^4.L_3(2)$	$1_{P_{34}}$, $3_{P_{34}}$, $\overline{3}_{P_{34}}$, $8_{P_{34}}$
P_{36}	$2^4.\operatorname{Alt}(7)$	$1_{P_{36}}$, $4_{P_{36}}$, $\overline{4}_{P_{36}}$, $6_{P_{36}}$, $14_{P_{36}}$, $20_{P_{36}}$
P_{37}	M_{22}	$1_{P_{37}}$, $10_{P_{37}}$, $\overline{10}_{P_{37}}$, $34_{P_{37}}$, $98_{P_{37}}$, $140_{P_{37}}$
P_{46}	$2^4.\operatorname{Alt}(7)$	$1_{P_{46}}$, $4_{P_{46}}$, $\overline{4}_{P_{46}}$, $6_{P_{46}}$, $14_{P_{46}}$, $20_{P_{46}}$
P_{47}	$2^4.(3 \times \operatorname{Alt}(5)).2$	$1_{P_{47}}$, $2_{P_{47}}$, $4^a_{P_{47}}$, $4^b_{P_{47}}$, $4^c_{P_{47}}$, $4^d_{P_{47}}$, $8_{P_{47}}$
P_{67}	$M_{21}.2$	$1_{P_{67}}$, $9_{P_{67}}$, $\overline{9}_{P_{67}}$, $16_{P_{67}}$, $64_{P_{67}}$

Table A.1: The irreducible modules over $GF(2)$ of the 2-parabolic subgroups of M_{23} of rank 2.

$(\lambda^{(ij)}_{i_{dim},j_{dim}})_{P_i} := (\lambda^{(ijk)}_{i_{dim},j_{dim},k_{dim}})_{P_i}$ and $(\lambda^{(ij)}_{i_{dim},j_{dim}})_{P_j} := (\lambda^{(ijk)}_{i_{dim},j_{dim},k_{dim}})_{P_j}$.[1] Similarly for P_{ik} and P_{jk}. Consequently $(\lambda^{(ijk)}_{i_{dim},j_{dim},k_{dim}})_{P_{ij}} := H_0(\lambda^{(ij)}_{i_{dim},j_{dim}})$. It follows that we must consider the universal panel-irreducible presheaves of P_{13}, P_{16}, P_{17}, P_{23}, P_{26}, P_{27}, P_{34}, P_{36}, P_{37}, P_{46}, P_{47} and P_{67}. A summary of the irreducible modules over $GF(2)$ admitted by each of these parabolic subgroups is given in Table A.1. We note that $P_{16} = P_{23} = P_{26} = P_{36} = P_{46}$ and $P_{17} = P_{27} = P_{47}$. However, we will subsequently be considering presheaves for each group P_{ij} defined on the minimal parabolic system $\{P_i, P_j\}$. Thus, we have listed each of these labellings of the groups as separate entries in Table A.1.

There are four universal panel-irreducible presheaves for each parabolic subgroup not containing P_6 or P_7, six universal panel-irreducible presheaves for each parabolic subgroup containing either P_6 or P_7, and nine universal panel-irreducible presheaves for P_{67}. To calculate the zero-homology groups of each of these presheaves, we use the methods described in Chapter 3, namely considering possible quotients of induced modules, straightforward geometric spanning arguments and looking at which irreducible modules admit certain presheaves.[2] In the case of presheaves defined on P_{23}, we also use Ronan's Duality Theorem to obtain the dimension of $H_0(\lambda^{(23)}_{2,2})$. With the dimension determined, the exact structure of the homology group may then be obtained by considering the possible quotients of induced modules. Details of the universal panel-irreducible presheaves of the parabolic subgroups P_{13}, P_{16}, P_{17}, P_{23}, P_{26}, P_{27}, P_{34}, P_{36}, P_{37}, P_{46}, P_{47} and P_{67} are given in Tables A.2-A.13. Using these tables and the observation above that $(\lambda^{(ijk)}_{i_{dim},j_{dim},k_{dim}})_{P_{ij}} := H_0(\lambda^{(ij)}_{i_{dim},j_{dim}})$, the reader can easily determine the vertex terms of the universal panel-irreducible presheaves of M_{23}.

[1] We note that - due to the construction of the universal presheaf - this definition is independent of the values of k and k_{dim}.

[2] As $P_{37} \cong M_{22}$, a full discussion of the zero-homology groups of presheaves for P_{37} is given in Chapter 3. In that discussion P_3 and P_7 are labelled as P_1 and P_2 respectively.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$	Notes about λ
$\lambda_{1,1}^{(13)}$	1	1	$\left(\lambda_{1,1}^{(13)}\right)^* \cong \lambda_{2,2}^{(13)}$
$\lambda_{1,2}^{(13)}$	3	3	$\left(\lambda_{1,2}^{(13)}\right)^* \cong \lambda_{2,1}^{(13)}$
$\lambda_{2,1}^{(13)}$	$\overline{3}$	$\overline{3}$	$\left(\lambda_{2,1}^{(13)}\right)^* \cong \lambda_{1,2}^{(13)}$
$\lambda_{2,2}^{(13)}$	8	8	$\left(\lambda_{2,2}^{(13)}\right)^* \cong \lambda_{1,1}^{(13)}$

Table A.2: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_1, P_3\}$ of the parabolic subgroup P_{13} of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$	Notes about $H_0(\lambda)$
$\lambda_{1,1}^{(16)}$	1	1	
$\lambda_{1,4}^{(16)}$	6	$6/\left(\frac{4\oplus\overline{4}}{6}\right)$	
$\lambda_{1,\overline{4}}^{(16)}$	$\overline{4}$	$\overline{4}$	
$\lambda_{2,1}^{(16)}$	6	6	
$\lambda_{2,4}^{(16)}$	14	$14/(14\oplus 1/20/1)$	The submodule $1/20/1$ is uniserial.
$\lambda_{2,\overline{4}}^{(16)}$	4, 20	$4\oplus 20$	

Table A.3: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_1, P_6\}$ of the parabolic subgroup P_{16} of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$
$\lambda_{1,1}^{(17)}$	1	1
$\lambda_{1,4}^{(17)}$	4^a	4^a
$\lambda_{1,\overline{4}}^{(17)}$	4^b	4^b
$\lambda_{2,1}^{(17)}$	2	2
$\lambda_{2,4}^{(17)}$	8	8
$\lambda_{2,\overline{4}}^{(17)}$	4^c, 4^d	$4^c\oplus 4^d$

Table A.4: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_1, P_7\}$ of the parabolic subgroup P_{17} of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$	Notes about λ
$\lambda_{1,1}^{(23)}$	1	1	$\left(\lambda_{1,1}^{(23)}\right)^* \cong \lambda_{2,2}^{(23)}$
$\lambda_{1,2}^{(23)}$	6, 14	$6\oplus 14$	$\left(\lambda_{1,2}^{(23)}\right)^* \cong \lambda_{2,1}^{(23)}$
$\lambda_{2,1}^{(23)}$	4, $\overline{4}$	$4/6\oplus\overline{4}/6$	$\left(\lambda_{2,1}^{(23)}\right)^* \cong \lambda_{1,2}^{(23)}$
$\lambda_{2,2}^{(23)}$	14, 20	$20/1/14/1/20\oplus(14/1\oplus 20/1)/14$	$\left(\lambda_{2,2}^{(23)}\right)^* \cong \lambda_{1,1}^{(23)}$

Table A.5: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_2, P_3\}$ of the parabolic subgroup P_{23} of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$	Notes about $H_0(\lambda)$
$\lambda_{1,1}^{(26)}$	1	1	
$\lambda_{1,4}^{(26)}$	6	$6/\left(\frac{4\oplus\overline{4}}{6}\right)$	
$\lambda_{1,\overline{4}}^{(26)}$	None	0	
$\lambda_{2,1}^{(26)}$	6	6	
$\lambda_{2,4}^{(26)}$	14	$14/(14\oplus 1/20/1)$	The submodule 1/20/1 is uniserial.
$\lambda_{2,\overline{4}}^{(26)}$	4, $\overline{4}$, 20	$4\oplus\overline{4}\oplus 20/1$	

Table A.6: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_2, P_6\}$ of the parabolic subgroup P_{26} of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$
$\lambda_{1,1}^{(27)}$	1	1
$\lambda_{1,4}^{(27)}$	4^a	4^a
$\lambda_{1,\overline{4}}^{(27)}$	4^c	4^c
$\lambda_{2,1}^{(27)}$	2	2
$\lambda_{2,4}^{(27)}$	8	8
$\lambda_{2,\overline{4}}^{(27)}$	4^b, 4^d	$4^b\oplus 4^d$

Table A.7: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_2, P_7\}$ of the parabolic subgroup P_{27} of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$	Notes about λ
$\lambda_{1,1}^{(34)}$	1	1	$\left(\lambda_{1,1}^{(34)}\right)^* \cong \lambda_{2,2}^{(34)}$
$\lambda_{1,2}^{(34)}$	$\overline{3}$	$\overline{3}$	$\left(\lambda_{1,2}^{(34)}\right)^* \cong \lambda_{2,1}^{(34)}$
$\lambda_{2,1}^{(34)}$	3	3	$\left(\lambda_{2,1}^{(34)}\right)^* \cong \lambda_{1,2}^{(34)}$
$\lambda_{2,2}^{(34)}$	8	8	$\left(\lambda_{2,2}^{(34)}\right)^* \cong \lambda_{1,1}^{(34)}$

Table A.8: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_3, P_4\}$ of the parabolic subgroup P_{34} of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$
$\lambda_{1,1}^{(36)}$	1	1
$\lambda_{1,4}^{(36)}$	None	0
$\lambda_{1,\overline{4}}^{(36)}$	4, $\overline{4}$	$4\oplus\overline{4}$
$\lambda_{2,1}^{(36)}$	None	0
$\lambda_{2,4}^{(36)}$	6, 14	$6\oplus 14/1$
$\lambda_{2,\overline{4}}^{(36)}$	20	20/1

Table A.9: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_3, P_6\}$ of the parabolic subgroup P_{36} of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$
$\lambda_{1,1}^{(37)}$	1	1
$\lambda_{1,4}^{(37)}$	10	$10/\overline{10}/1$
$\lambda_{1,\overline{4}}^{(37)}$	None	0
$\lambda_{2,1}^{(37)}$	$\overline{10}$	$\overline{10}/1$
$\lambda_{2,4}^{(37)}$	98	$98/1/\left(\frac{34}{1/10\oplus\overline{10}/1}\bigoplus 10/\overline{10}/1\right)/34/1$
$\lambda_{2,\overline{4}}^{(37)}$	140	$140/(1\oplus 1/34/1/10/34)$

Table A.10: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_3, P_7\}$ of the parabolic subgroup P_{37} of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$	Notes about $H_0(\lambda)$
$\lambda_{1,1}^{(46)}$	1	1	
$\lambda_{1,4}^{(46)}$	6	$6/\left(\frac{4\oplus\overline{4}}{6}\right)$	
$\lambda_{1,\overline{4}}^{(46)}$	$\overline{4}$	$\overline{4}$	
$\lambda_{2,1}^{(46)}$	6	6	
$\lambda_{2,4}^{(46)}$	14	$14/(14\oplus 1/20/1)$	The submodule 1/20/1 is uniserial.
$\lambda_{2,\overline{4}}^{(46)}$	4, 20	$4\oplus 20$	

Table A.11: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_4, P_6\}$ of the parabolic subgroup P_{46} of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$
$\lambda_{1,1}^{(47)}$	1	1
$\lambda_{1,4}^{(47)}$	4^a	4^a
$\lambda_{1,\overline{4}}^{(47)}$	4^b	4^b
$\lambda_{2,1}^{(47)}$	2	2
$\lambda_{2,4}^{(47)}$	8	8
$\lambda_{2,\overline{4}}^{(47)}$	4^c, 4^d	$4^c\oplus 4^d$

Table A.12: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_4, P_7\}$ of the parabolic subgroup P_{47} of M_{23}.

Presheaf, λ	Irreducible quotients of $H_0(\lambda)$	$H_0(\lambda)$	Notes about $H_0(\lambda)$
$\lambda^{(67)}_{1,1}$	1	1	
$\lambda^{(67)}_{1,4}$	9	9/1/1	The module 9/1/1 is uniserial.
$\lambda^{(67)}_{1,\overline{4}}$	None	0	
$\lambda^{(67)}_{4,1}$	$\overline{9}$	$\overline{9}$/1/1	The module $\overline{9}$/1/1 is uniserial.
$\lambda^{(67)}_{4,4}$	64	64	
$\lambda^{(67)}_{4,\overline{4}}$	None	0	
$\lambda^{(67)}_{\overline{4},1}$	None	0	
$\lambda^{(67)}_{\overline{4},4}$	None	0	
$\lambda^{(67)}_{\overline{4},\overline{4}}$	16	16	

Table A.13: The universal panel-irreducible presheaves defined on the 2-minimal parabolic system $\{P_6, P_7\}$ of the parabolic subgroup P_{67} of M_{23}.

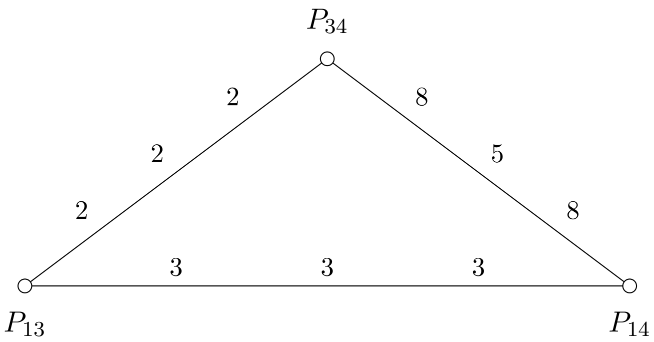


Figure A.6: The diagram of the geometry arising from the unique 2-minimal parabolic system of M_{24}.

A.2 Vertex Term Calculations for Universal Panel-Irreducible Presheaves of M_{24} over $GF(2)$

There is a unique 2-minimal parabolic system of M_{24} given by $\mathcal{S} := \{P_1, P_3, P_4\}$, where each P_i has shape $2^{6+3}.\mathrm{Sym}(3)$. The maximal parabolic subgroups are given by $P_{13} \sim 2^6.3\,\mathrm{Sym}(6)$, $P_{14} \sim 2^{6+2}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$ and $P_{34} \sim 2^{4+3}.L_3(2)$. Calculations show that the incidence graph of a residue of flag of cotype $\{1, 3\}$ of the corresponding geometry has 1-diameter and 3-diameter both equal to 8 and girth equal to 10. The corresponding graph of a residue of flag of cotype $\{1, 4\}$ has 1- and 4-diameter equal to 2 and girth equal to 4, whilst that of a residue of flag of cotype $\{3, 4\}$ has 3- and 4-diameter equal to 3 and girth equal to 6. The diagram of the geometry arising from $\mathcal{S}$ is given in Figure A.6.

Each minimal parabolic subgroup P_i admits two classes of irreducible $GF(2)P_i$-modules, 1_{P_i} and 2_{P_i}. Meanwhile, the maximal parabolic subgroups P_{13}, P_{14} and P_{34} admit seven, four and four classes of irreducible modules over $GF(2)$ respectively, given by $1_{P_{13}}$, $4_{P_{13}}$, $\overline{4}_{P_{13}}$, $6_{P_{13}}$, $\overline{6}_{P_{13}}$, $16_{P_{13}}$

Presheaf, $\lambda_{i,j,k}$	Vertex Terms		
	$(\lambda_{i,j,k})_{P_{13}}$	$(\lambda_{i,j,k})_{P_{14}}$	$(\lambda_{i,j,k})_{P_{34}}$
$\lambda_{1,1,1}$	1	1	1
$\lambda_{1,1,2}$	1	$\overline{2}$	3
$\lambda_{1,2,1}$	$\overline{4}/1 \oplus \overline{6}$	1	$\overline{3}$
$\lambda_{1,2,2}$	$\overline{4}/1 \oplus \overline{6}$	$\overline{2}$	8
$\lambda_{2,1,1}$	$4/1 \oplus 6$	2	1
$\lambda_{2,1,2}$	$4/1 \oplus 6$	4	3
$\lambda_{2,2,1}$	$16 \oplus \frac{18}{6 \oplus \overline{6}}$	2	$\overline{3}$
$\lambda_{2,2,2}$	$16 \oplus \frac{18}{6 \oplus \overline{6}}$	4	8

Table A.14: The vertex terms of the universal panel-irreducible presheaves for M_{24} over $GF(2)$.

and $18_{P_{13}}$, $1_{P_{14}}$, $2_{P_{14}}$, $\overline{2}_{P_{14}}$ and $4_{P_{14}}$ and $1_{P_{34}}$, $3_{P_{34}}$, $\overline{3}_{P_{34}}$ and $8_{P_{34}}$.

We proceed by defining $\lambda^{(ij)}_{i_a,j_b}$ to be the universal panel-irreducible presheaf defined on the maximal parabolic subgroup P_{ij} having a 1-dimensional chamber term, and a- and b-dimensional terms at panels of type P_i and P_j respectively. Thus $H_0(\lambda^{(ij)}_{i_1,j_1}) \cong 1_{P_{ij}}$ for each $(i,j) \in \{(1,3),(1,4),(3,4)\}$. Since $[P_i : B] = 3$ for $i = 1,3,4$, we note that the duality relations $\left(\lambda^{(ij)}_{i_1,j_1}\right)^* \cong \lambda^{(ij)}_{i_2,j_2}$ and $\left(\lambda^{(ij)}_{i_1,j_2}\right)^* \cong \lambda^{(ij)}_{i_2,j_1}$ hold true. In particular, combining Ronan's Duality Theorem with the Euler characteristic of $\lambda^{(ij)}_{i_2,j_2}$ we see that

$$\dim(H_0(\lambda^{(ij)}_{i_2,j_2})) = \chi(\lambda^{(ij)}_{i_2,j_2}) + 1 = [P_{ij} : P_i] + 1. \tag{A.2.1}$$

Presheaves defined on P_{13}

The irreducible $GF(2)P_{13}$-modules $1_{P_{13}}$, $4_{P_{13}}$, $\overline{4}_{P_{13}}$, $6_{P_{13}}$, $\overline{6}_{P_{13}}$, $16_{P_{13}}$ and $18_{P_{13}}$ each admit a unique universal panel-irreducible presheaf, these respectively being $\lambda^{(13)}_{1_1,3_1}$, $\lambda^{(13)}_{1_2,3_1}$, $\lambda^{(13)}_{1_1,3_2}$, $\lambda^{(13)}_{1_2,3_1}$, $\lambda^{(13)}_{1_1,3_2}$, $\lambda^{(13)}_{1_2,3_2}$ and $\lambda^{(13)}_{1_2,3_2}$. By considering possible quotients of the induced modules $(1_{P_1})^{P_{13}}_{P_1}$, $(2_{P_1})^{P_{13}}_{P_1}$, $(1_{P_3})^{P_{13}}_{P_3}$ and $(2_{P_3})^{P_{13}}_{P_3}$ in conjunction with (A.2.1) we see that

$$H_0(\lambda^{(13)}_{1_1,3_2}) \cong \overline{4}/1 \oplus \overline{6}, \quad H_0(\lambda^{(13)}_{1_2,3_1}) \cong 4/1 \oplus 6 \quad \text{and} \quad H_0(\lambda^{(13)}_{1_2,3_2}) \cong 16 \oplus \frac{18}{6 \oplus \overline{6}}.$$

Presheaves defined on P_{14} and P_{34}

Since $[P_{14} : P_1] = [P_{14} : P_4] = 3$ and $[P_{34} : P_3] = [P_{34} : P_4] = 7$, the homology groups $H_0(\lambda^{(14)}_{1_a,4_b})$ and $H_0(\lambda^{(34)}_{3_a,4_b})$ for $a, b \in \{1,2\}$ may be directly calculated and all turn out to be irreducible. We have that $H_0(\lambda^{(14)}_{1_1,4_2}) \cong \overline{2}_{P_{14}}$, $H_0(\lambda^{(14)}_{1_2,4_1}) \cong 2_{P_{14}}$, $H_0(\lambda^{(14)}_{1_2,4_2}) \cong 4_{P_{14}}$, $H_0(\lambda^{(34)}_{3_1,4_2}) \cong 3_{P_{34}}$, $H_0(\lambda^{(34)}_{3_2,4_1}) \cong \overline{3}_{P_{34}}$ and $H_0(\lambda^{(34)}_{3_2,4_2}) \cong 8_{P_{34}}$.

Presheaves defined on M_{24}

As in Chapter 3, we denote the presheaf for M_{24} having a 1-dimensional chamber term and i-, j- and k-dimensional panel terms at panels of type P_1, P_3 and P_4 by $\lambda_{i,j,k}$. Utilising the calculations above, we detail the vertex terms of these presheaves in Table A.14.

Appendix B

x-Graphs Related to π-Product Graphs in Symmetric Groups

We now present the various x-graphs that were omitted for the sake of brevity in Chapter 4.

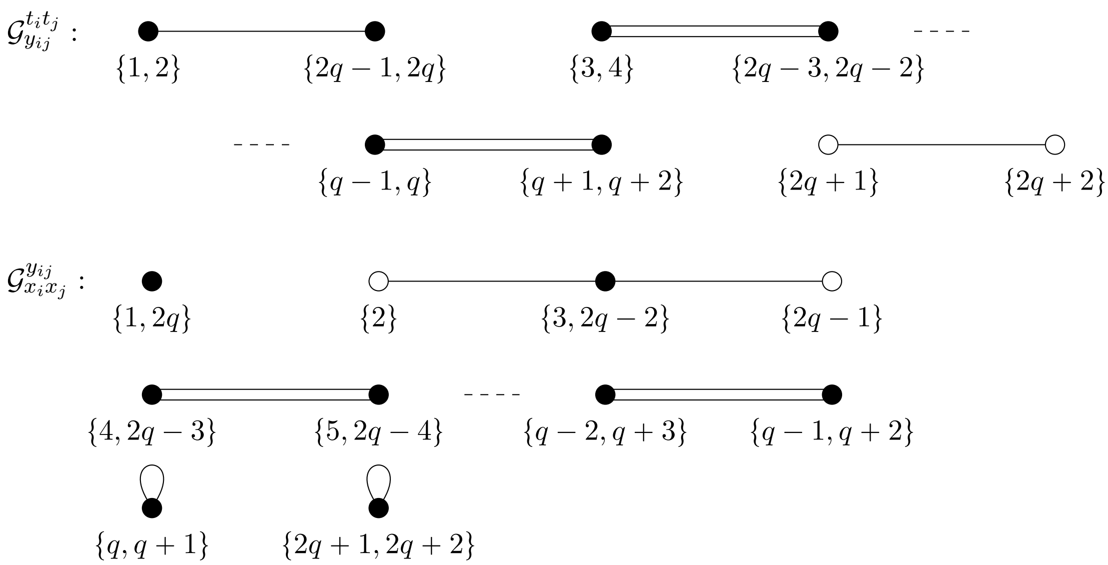


Figure B.1: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(i) in the case that $r = 0$ and q is even.

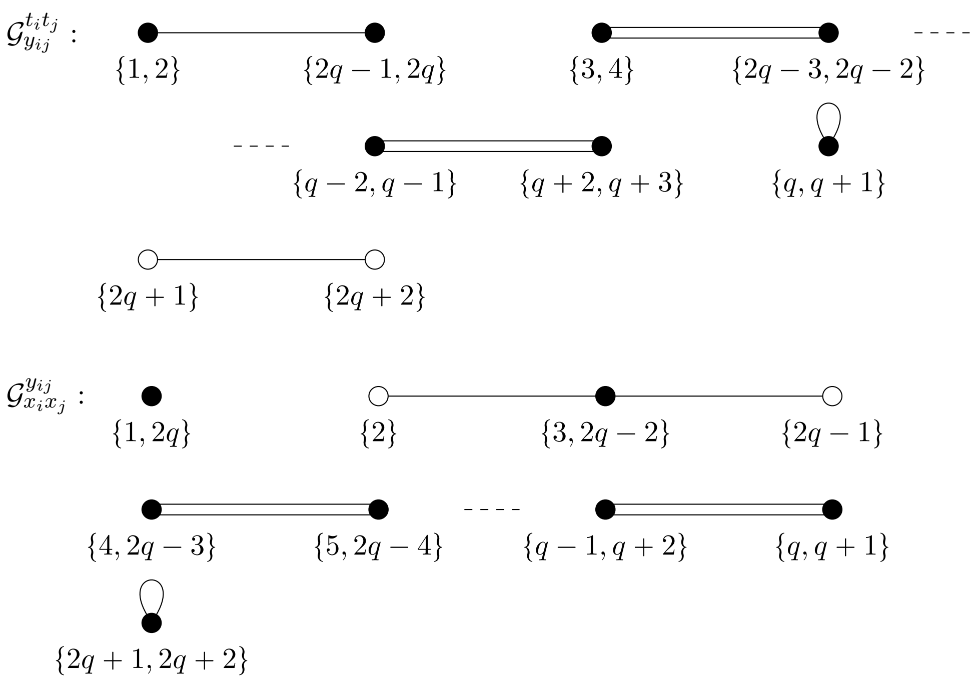


Figure B.2: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(i) in the case that $r = 0$ and q is odd.

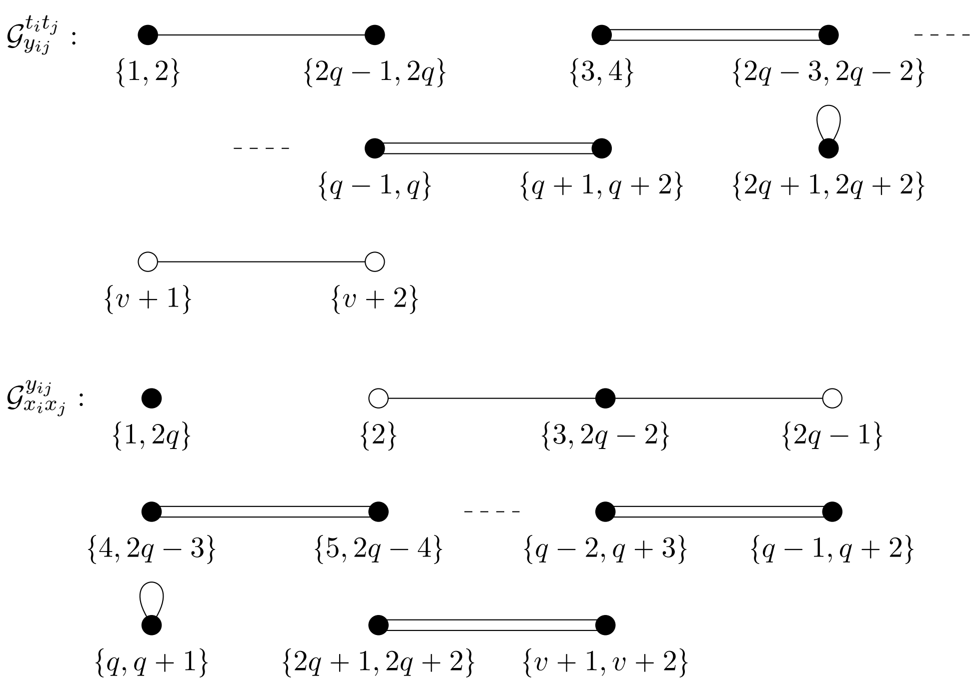


Figure B.3: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(i) in the case that $r = 1$ and q is even.

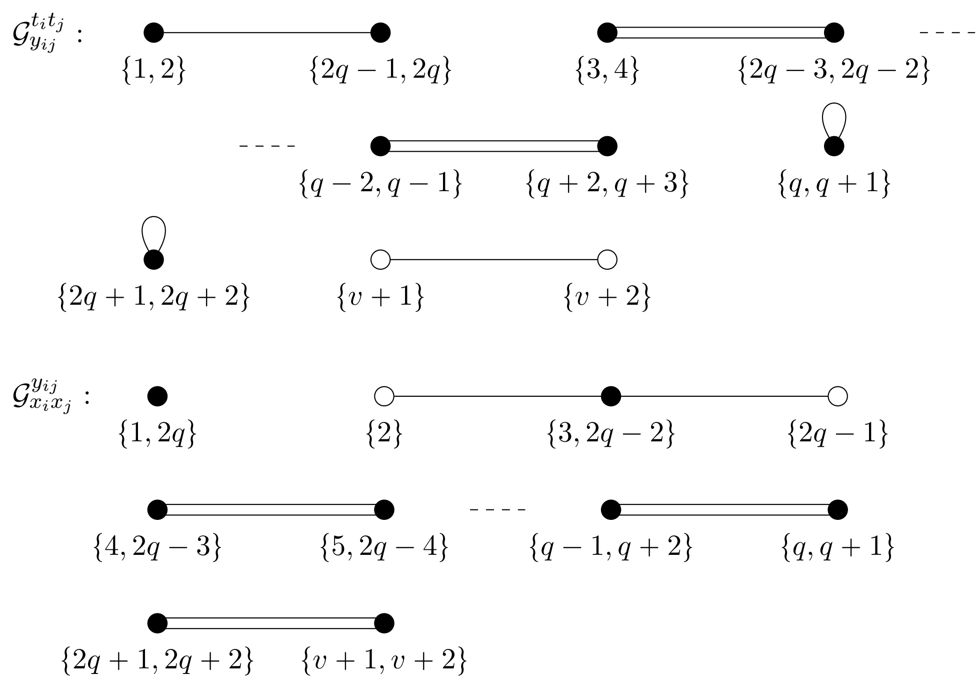


Figure B.4: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(i) in the case that $r = 1$ and q is odd.

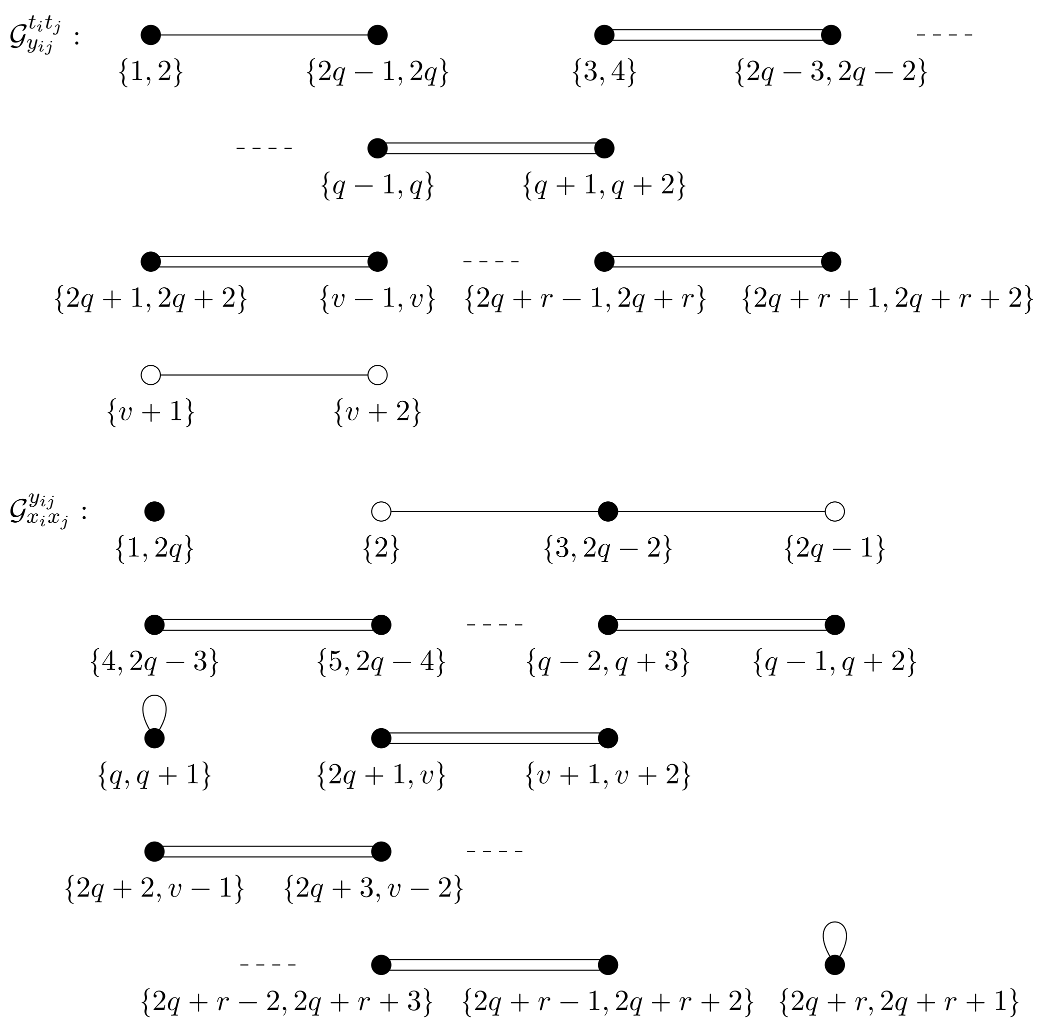


Figure B.5: The x-graphs $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ from Lemma 4.2.6(i) in the case that $r>1$ and q,r are both even.

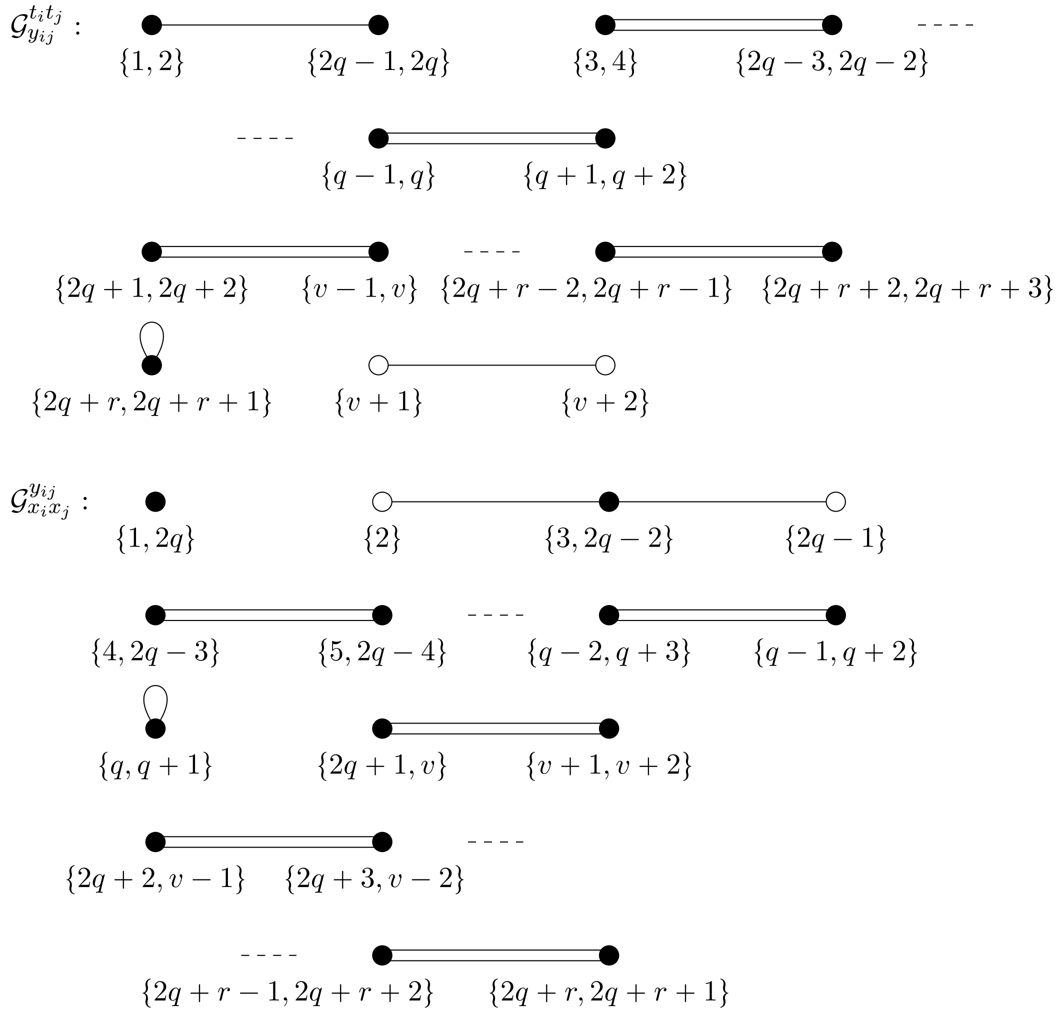


Figure B.6: The x-graphs $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ from Lemma 4.2.6(i) in the case that $r > 1$ and q is even and r is odd.

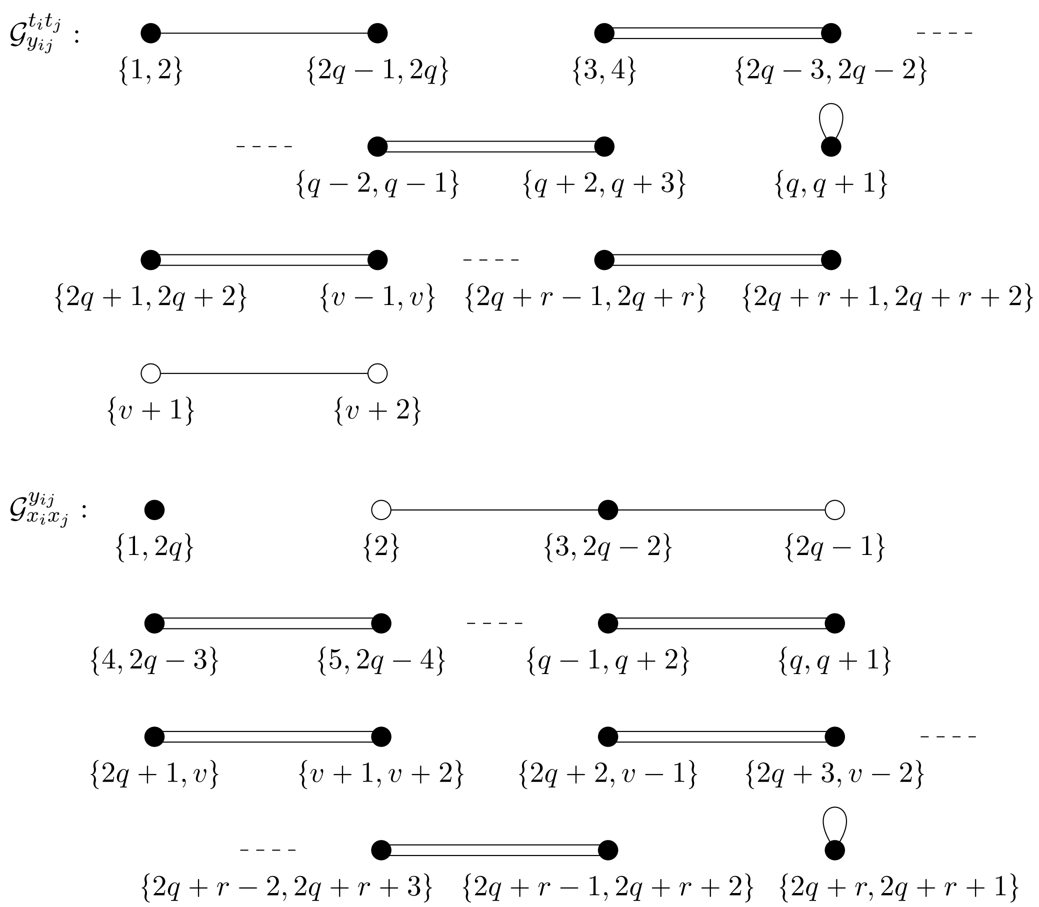


Figure B.7: The x-graphs $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ from Lemma 4.2.6(i) in the case that $r > 1$ and q is odd and r is even.

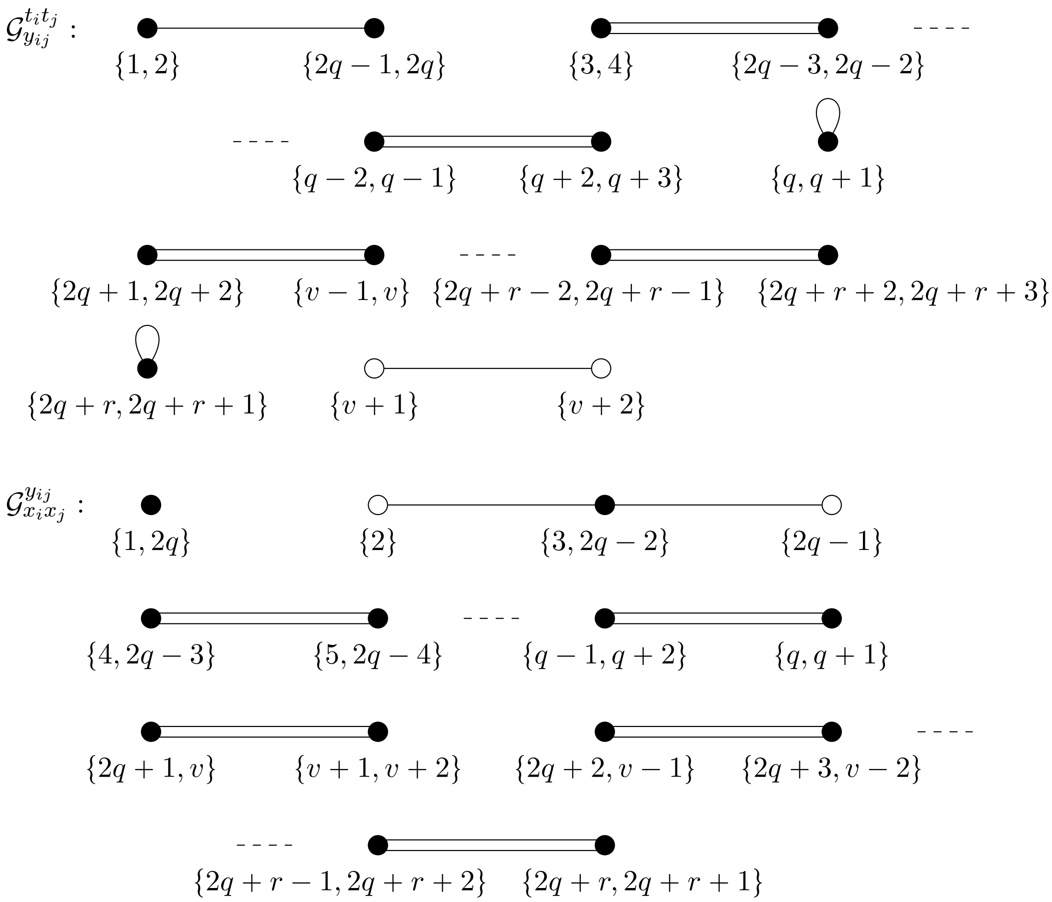


Figure B.8: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(i) in the case that $r > 1$ and q, r are both odd.

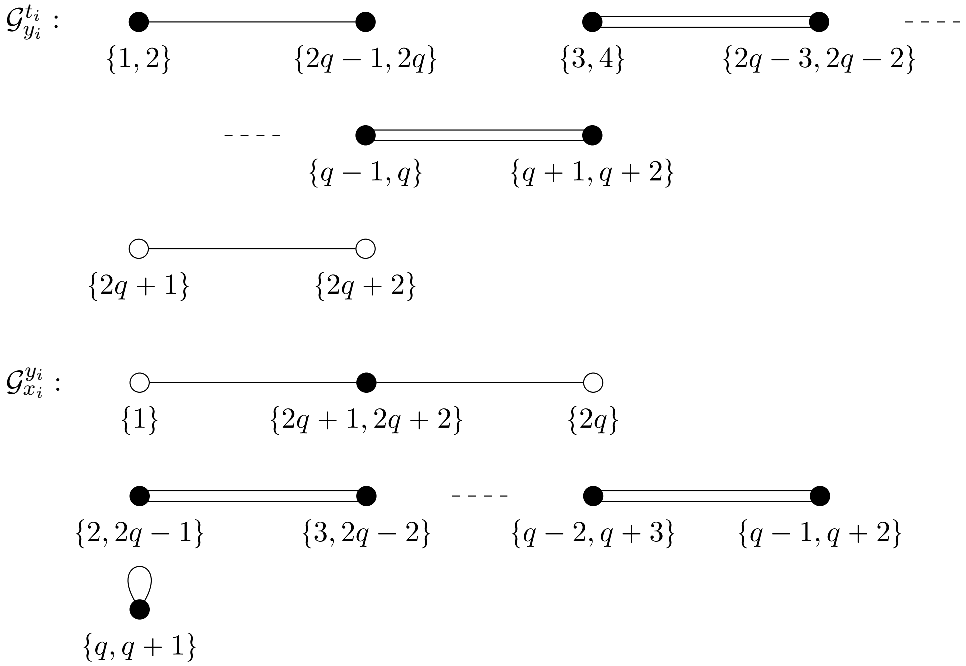


Figure B.9: The x-graphs $\mathcal{G}_{y_i}^{t_i}$ and $\mathcal{G}_{x_i}^{y_i}$ from Lemma 4.2.6(ii) in the case that q is even.

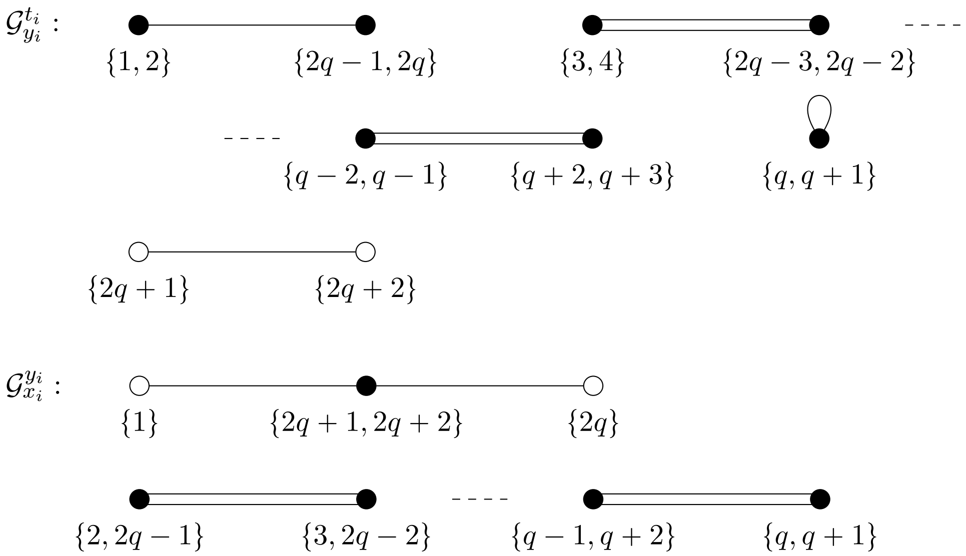


Figure B.10: The x-graphs $\mathcal{G}_{y_i}^{t_i}$ and $\mathcal{G}_{x_i}^{y_i}$ from Lemma 4.2.6(ii) in the case that q is odd.

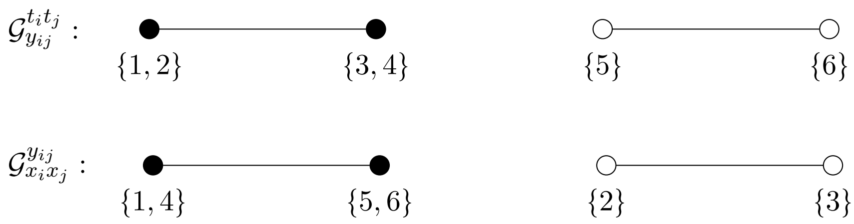


Figure B.11: The x-graphs $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ from Lemma 4.2.6(iii) in the case that $q = 2$.

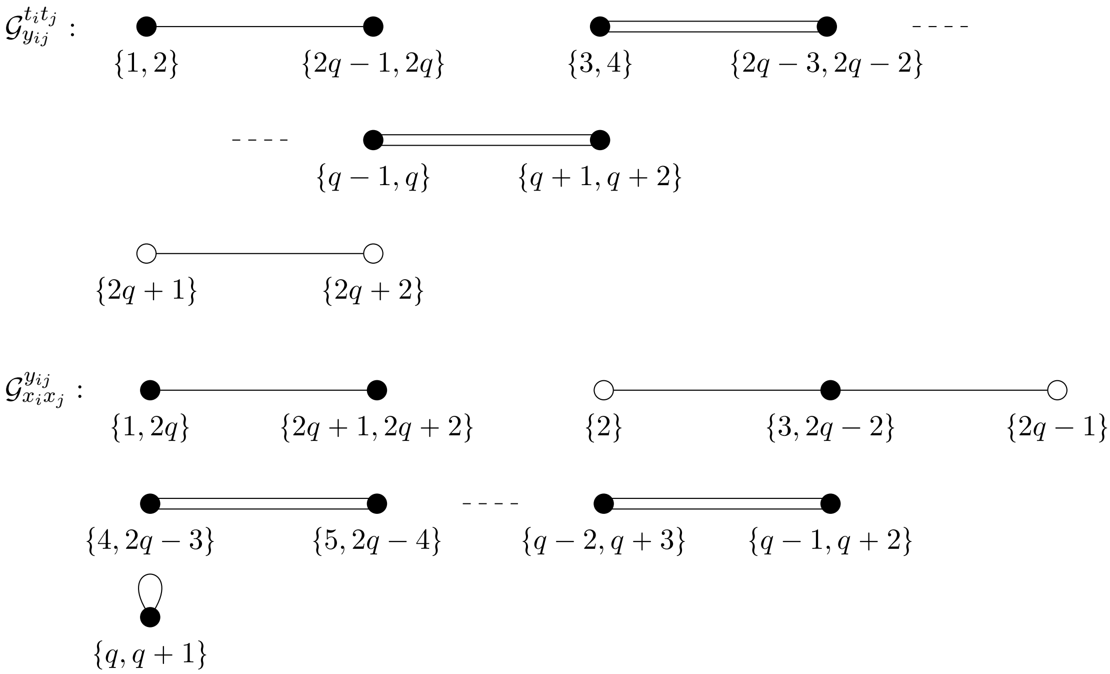


Figure B.12: The x-graphs $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ from Lemma 4.2.6(iii) in the case that $q \geqslant 3$ is even.

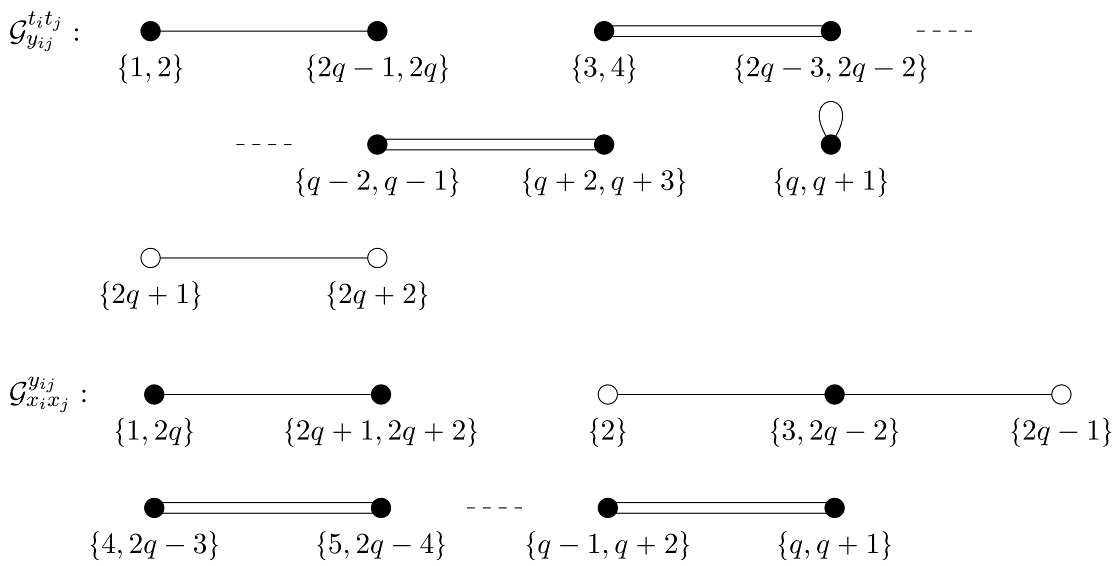


Figure B.13: The x-graphs $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ from Lemma 4.2.6(iii) in the case that $q \geqslant 3$ is odd.

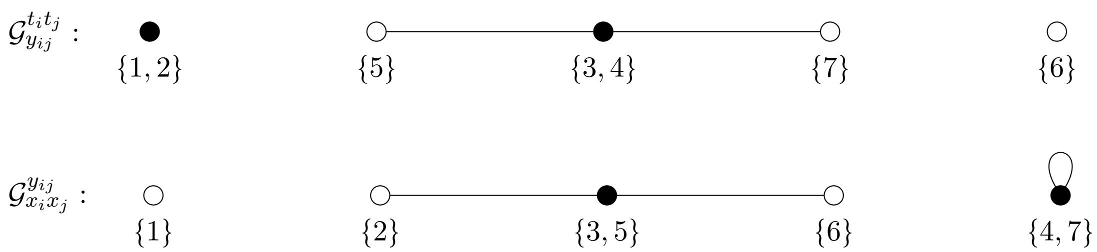


Figure B.14: The x-graphs $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ from Lemma 4.2.6(iv) in the case that $q = r = 1$.

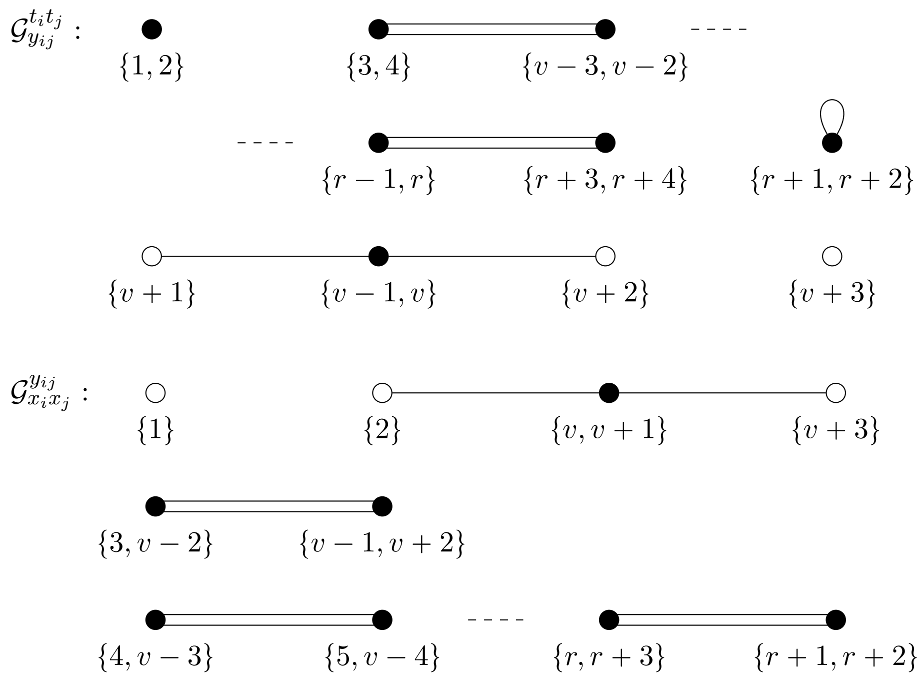


Figure B.15: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(iv) in the case that $q = 1$ and $r > 1$ is even.

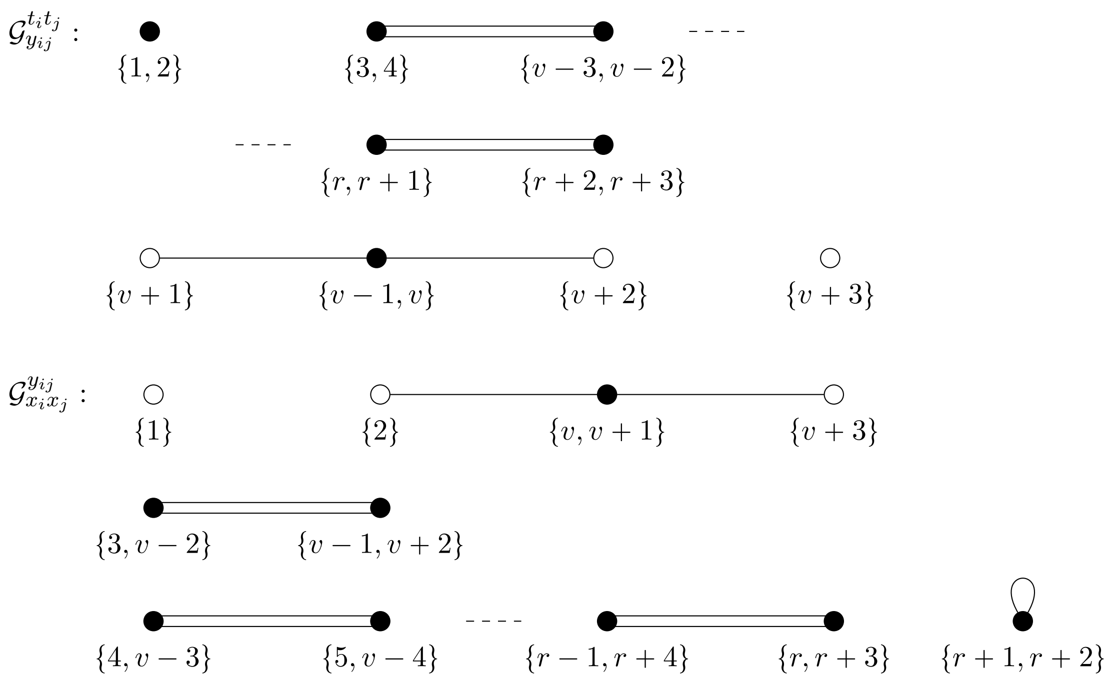


Figure B.16: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(iv) in the case that $q = 1$ and $r > 1$ is odd.

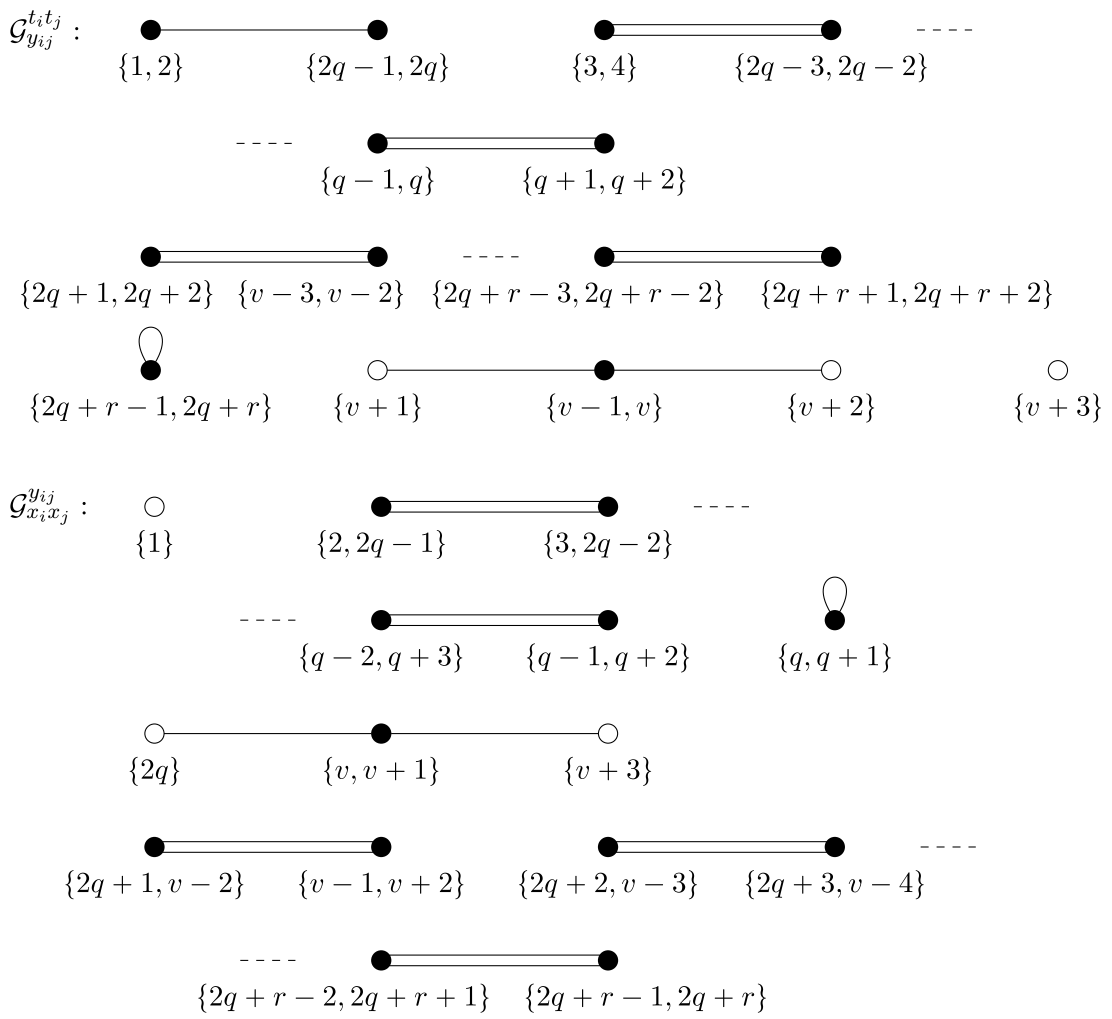


Figure B.17: The x-graphs $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ from Lemma 4.2.6(iv) in the case that $q>1$ and q,r are both even.

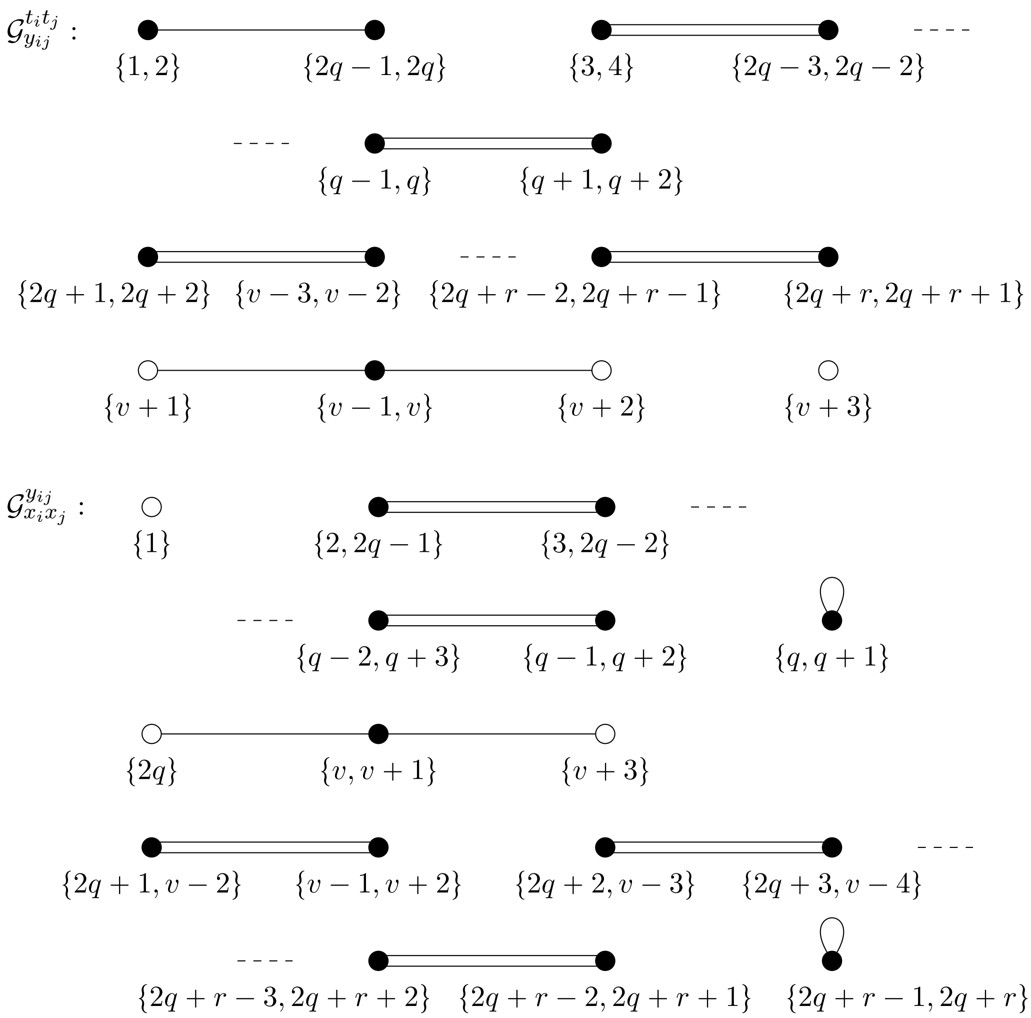


Figure B.18: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(iv) in the case that $q > 1$ is even and r is odd.

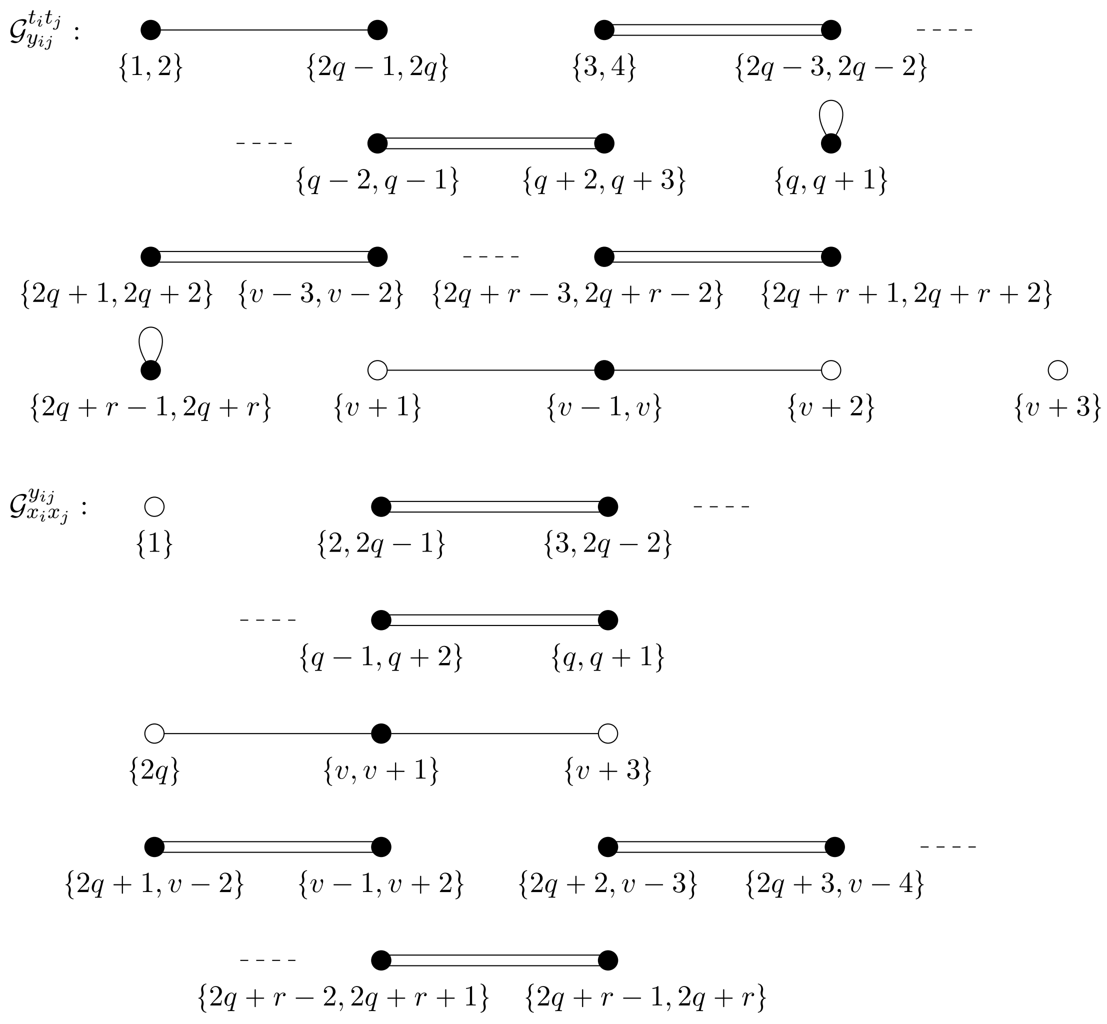


Figure B.19: The x-graphs $\mathcal{G}^{t_it_j}_{y_{ij}}$ and $\mathcal{G}^{y_{ij}}_{x_ix_j}$ from Lemma 4.2.6(iv) in the case that $q>1$ is odd and r is even.

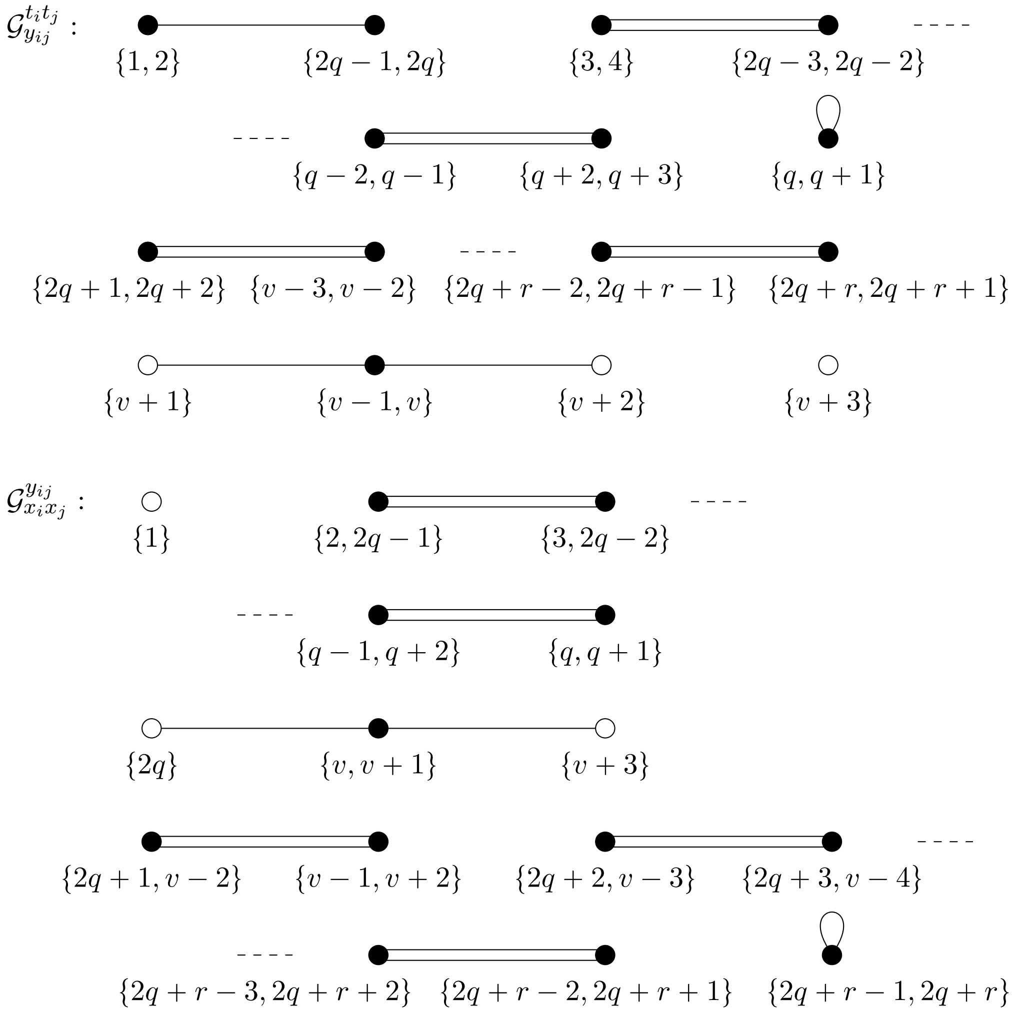


Figure B.20: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(iv) in the case that $q > 1$ and q, r are both odd.

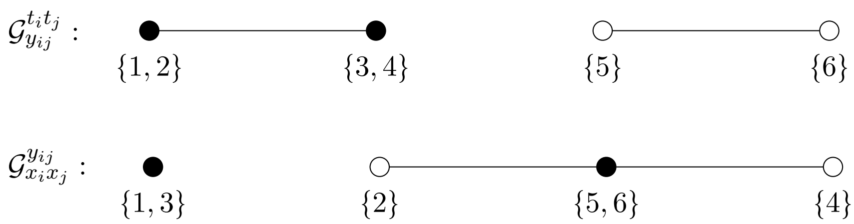


Figure B.21: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(v) in the case that $q = r = 1$.

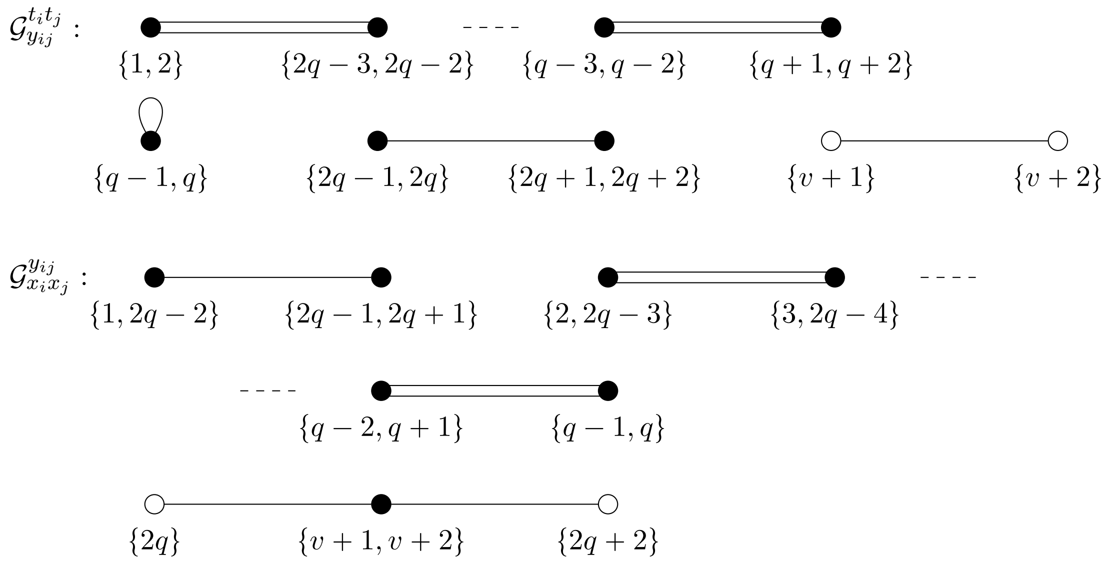


Figure B.22: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(v) in the case that $q > 1$ is even and $r = 1$.

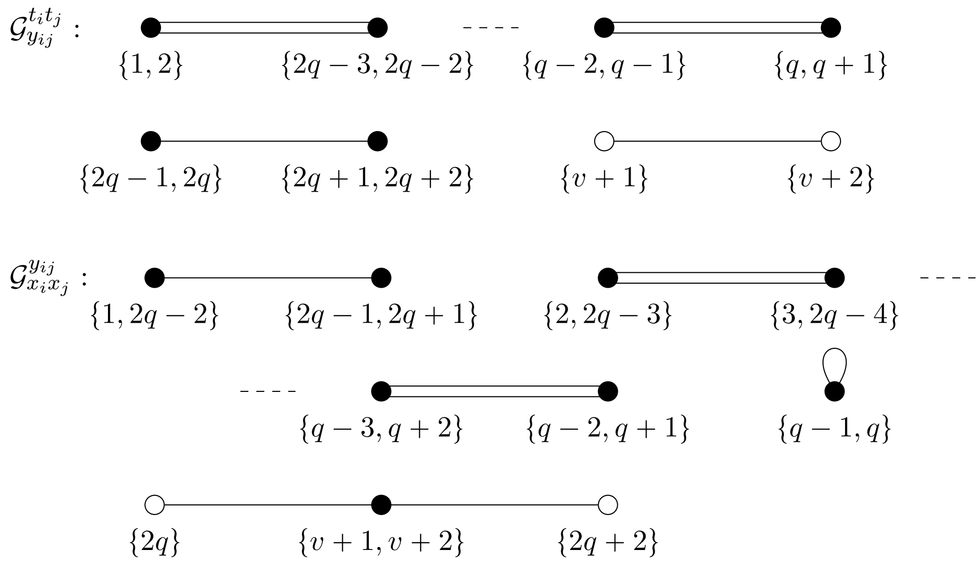


Figure B.23: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(v) in the case that $q > 1$ is odd and $r = 1$.

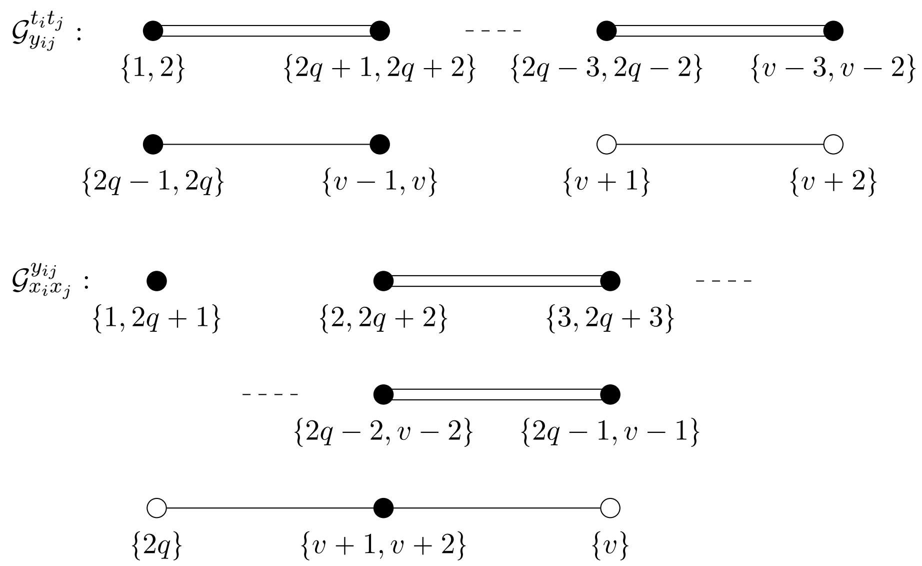


Figure B.24: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(v) in the case that $q = r > 1$.

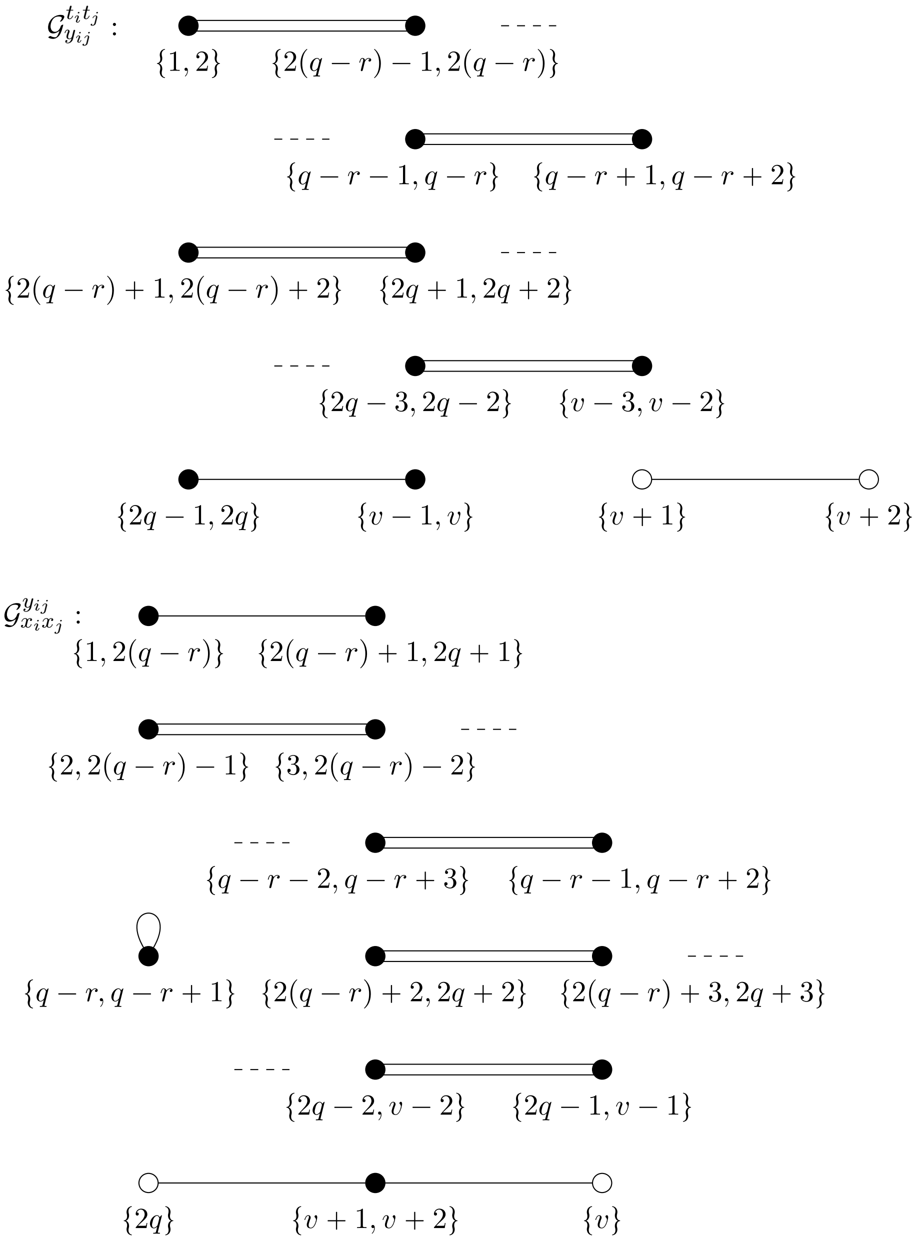


Figure B.25: The x-graphs $\mathcal{G}_{y_{ij}}^{t_i t_j}$ and $\mathcal{G}_{x_i x_j}^{y_{ij}}$ from Lemma 4.2.6(v) in the case that $q > r > 1$ and q and r have the same parity.

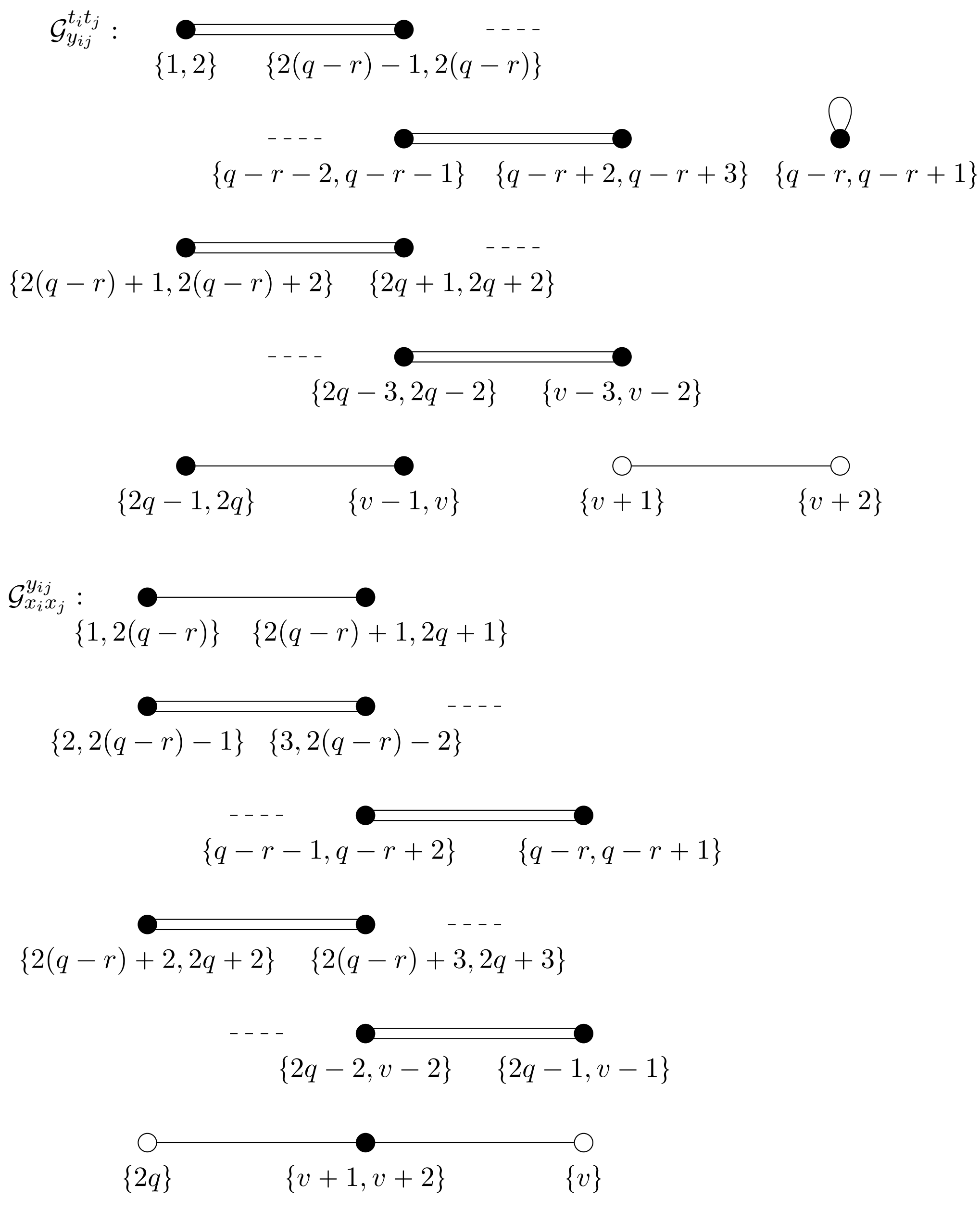


Figure B.26: The x-graphs $\mathcal{G}_{y_{ij}}^{t_it_j}$ and $\mathcal{G}_{x_ix_j}^{y_{ij}}$ from Lemma 4.2.6(v) in the case that $q > r > 1$ and q and r have different parities.

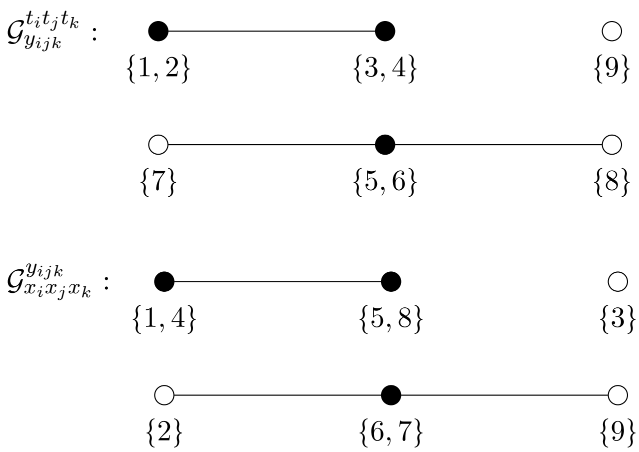


Figure B.27: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that $q = r = s = 1$.

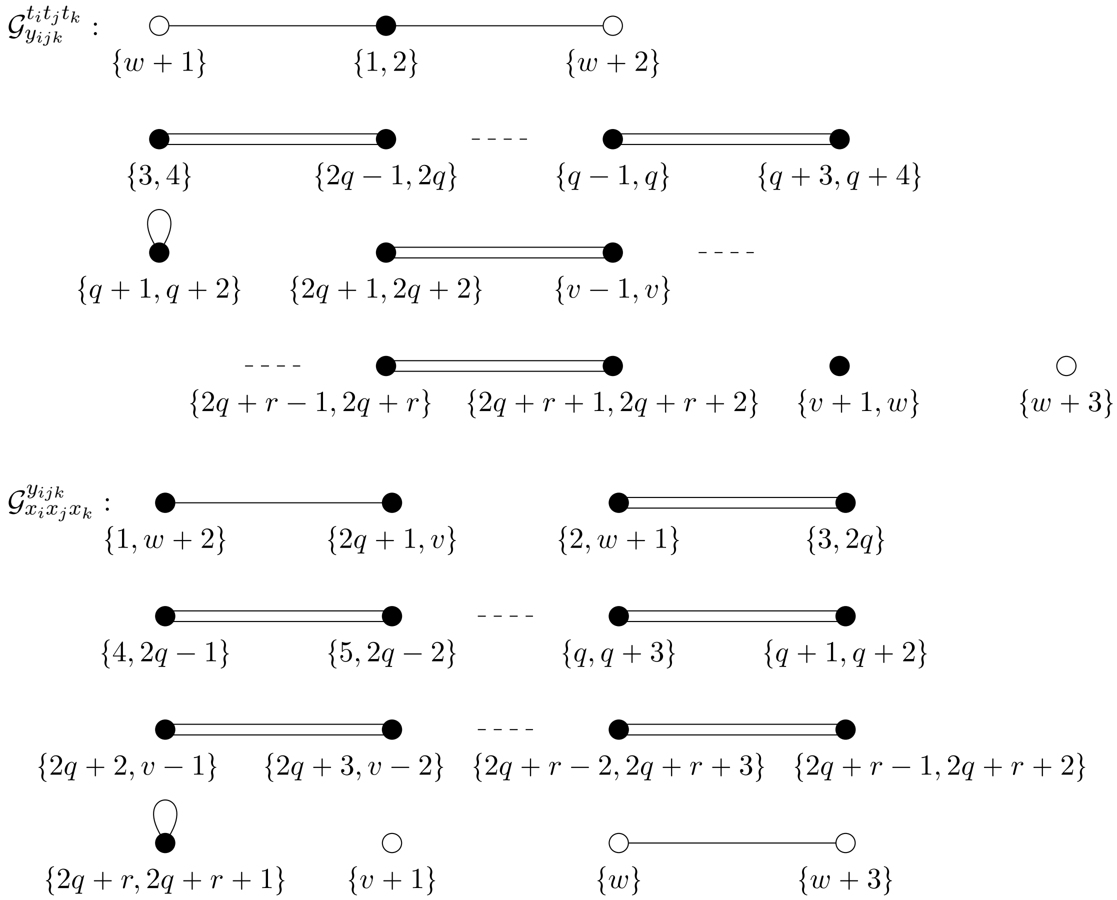


Figure B.28: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that q, r are both even and $s = 1$.

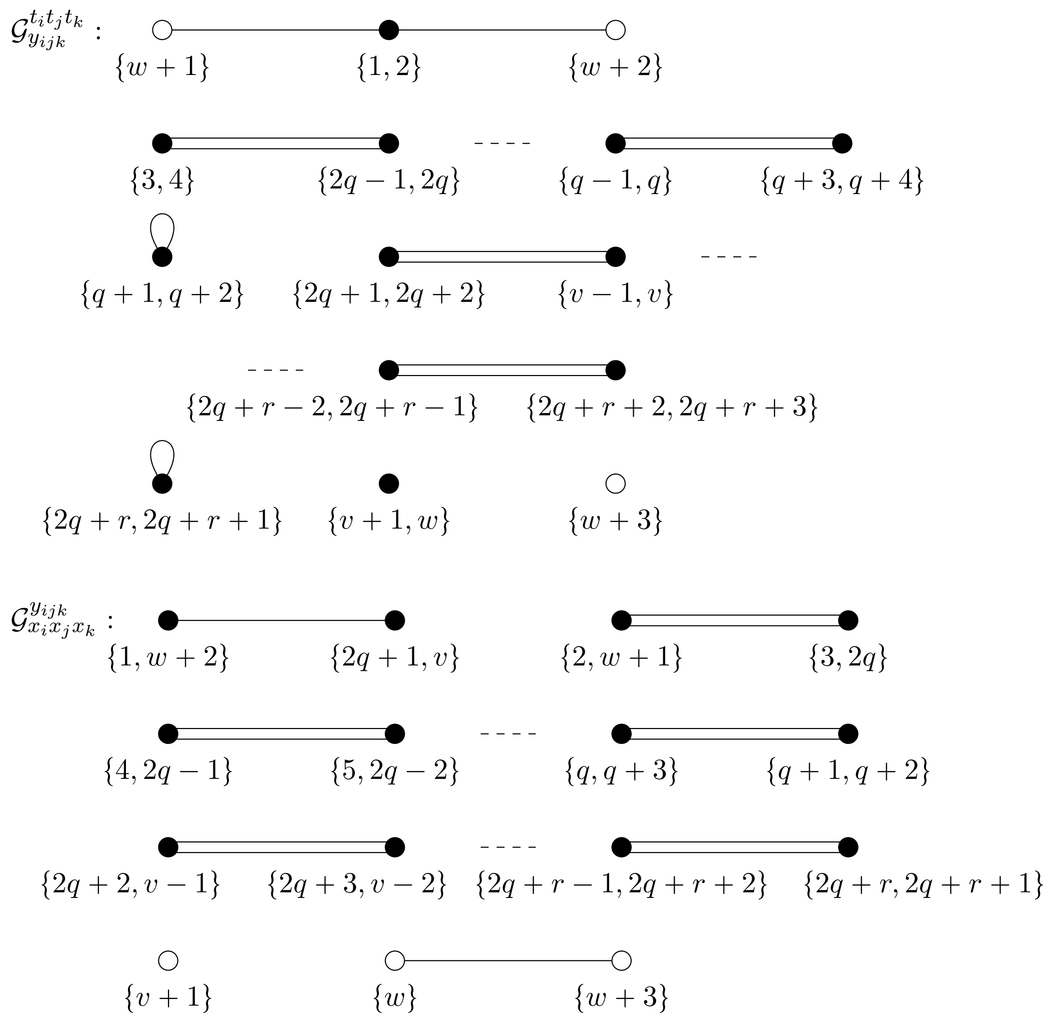


Figure B.29: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that q is even, r is odd and $s = 1$.

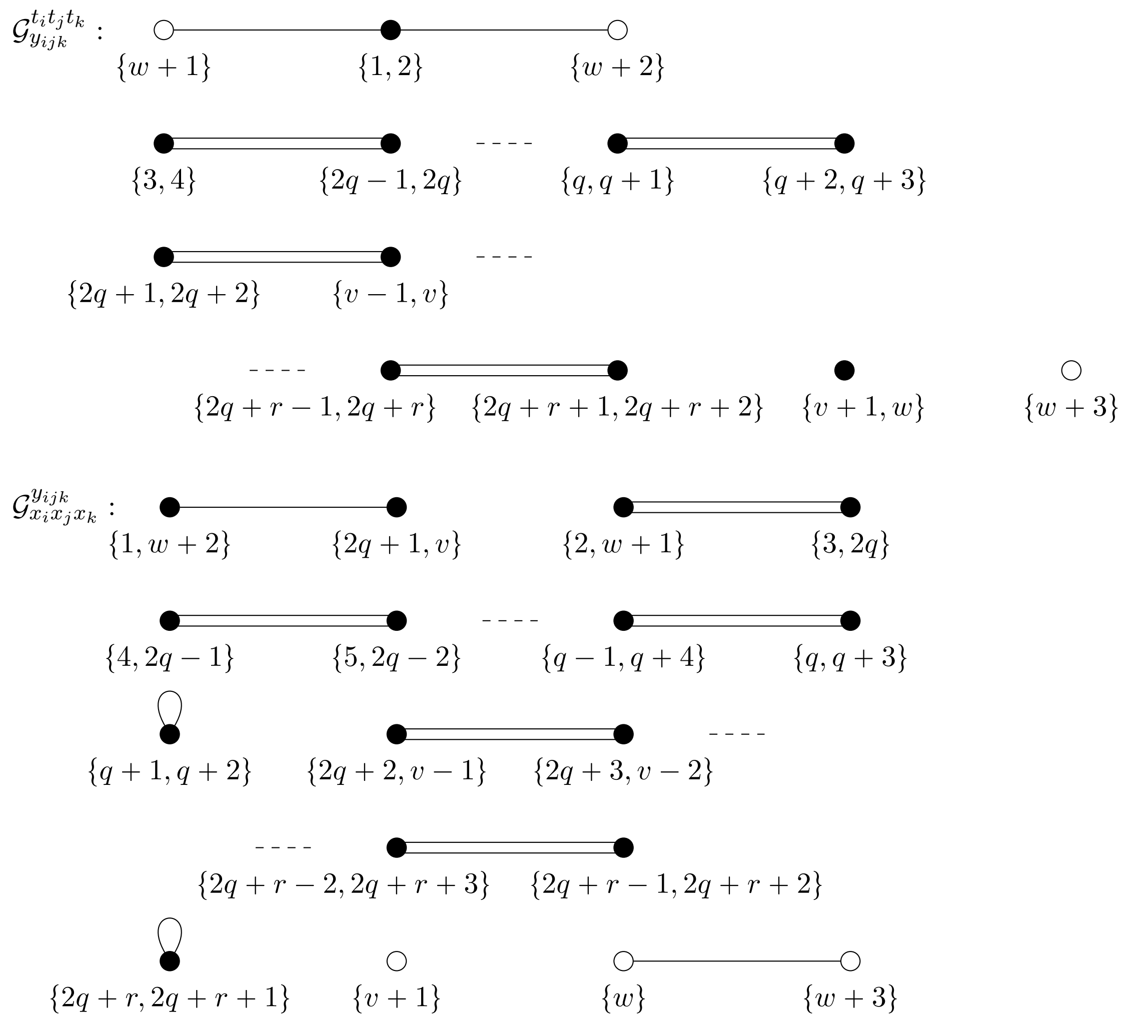


Figure B.30: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that q is odd, r is even and $s=1$.

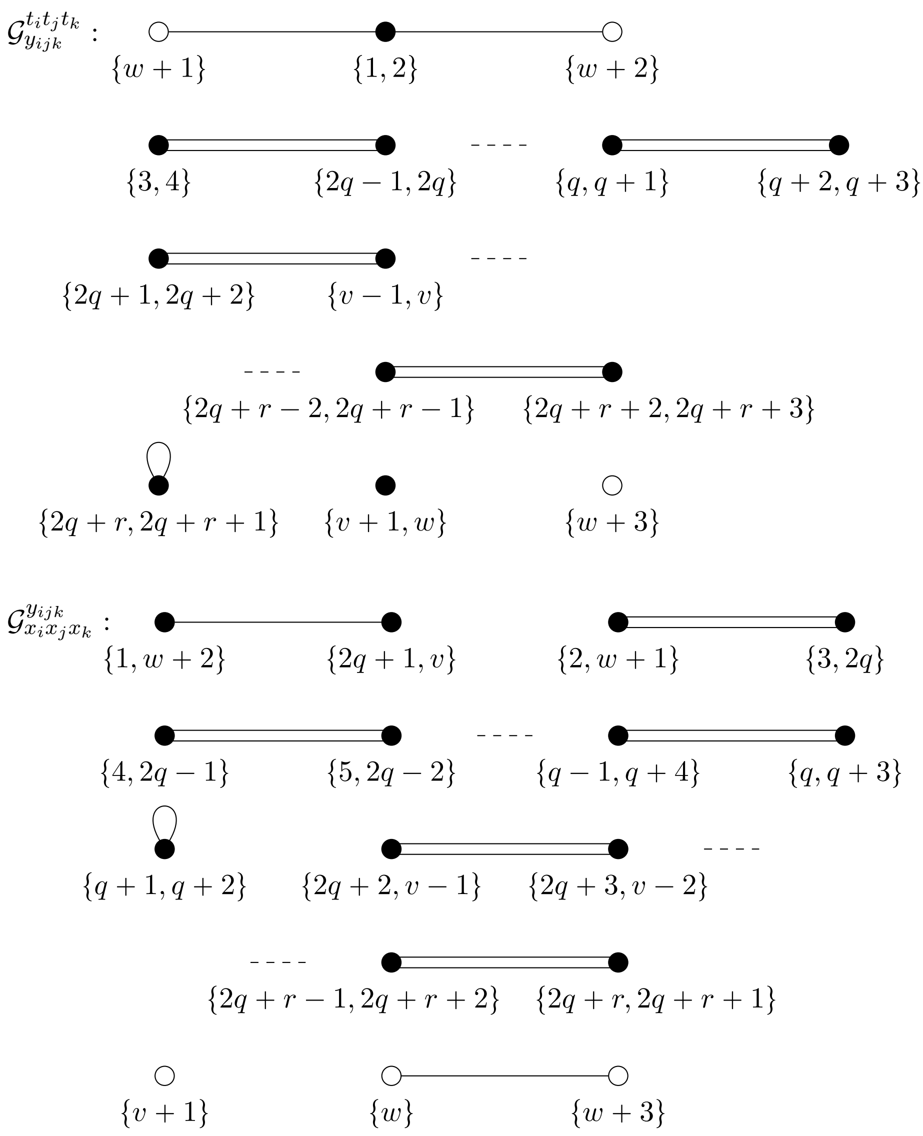


Figure B.31: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that q, r are both odd and $s = 1$.

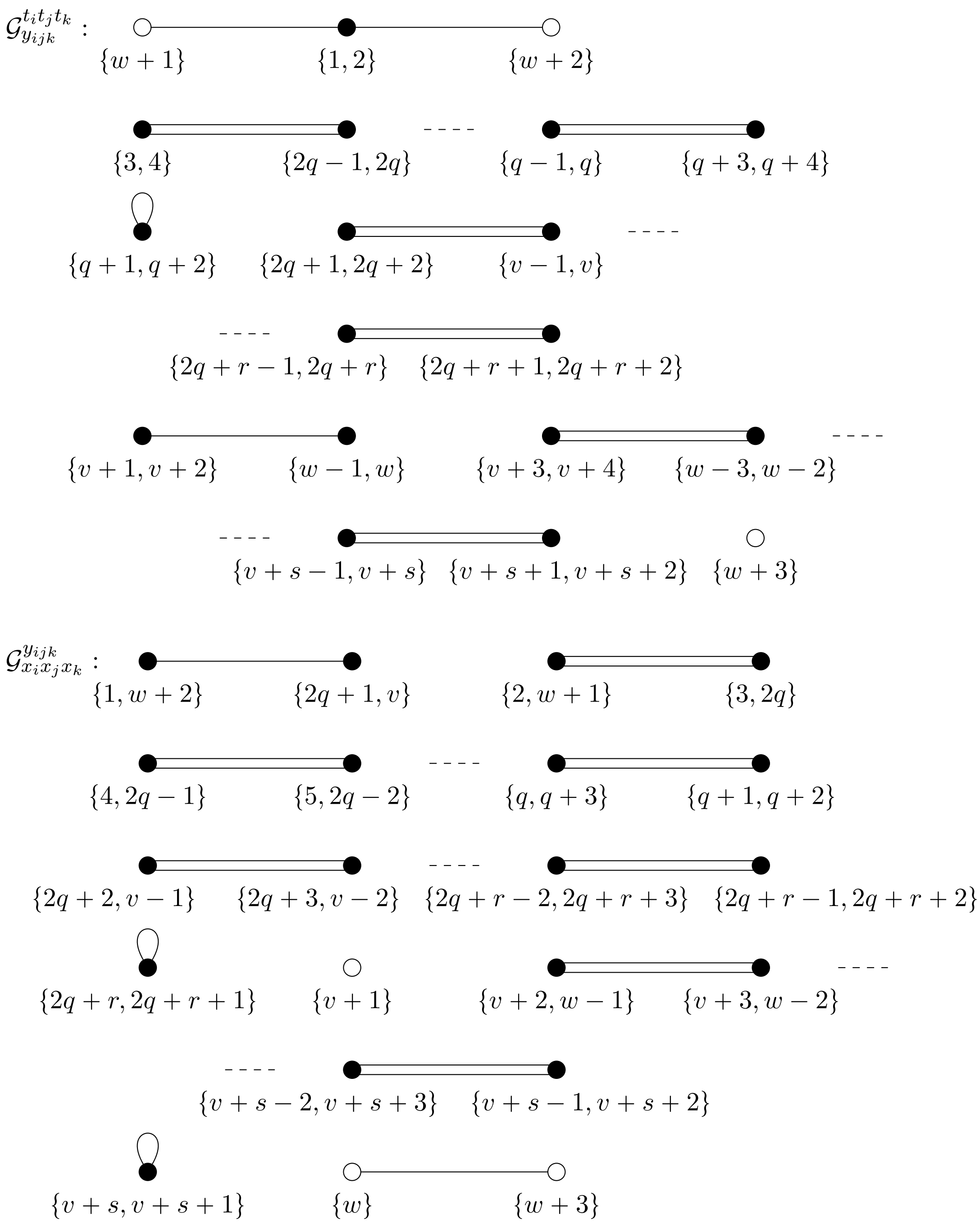


Figure B.32: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_i t_j t_k}$ and $\mathcal{G}_{x_i x_j x_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that $q \geqslant r \geqslant s > 1$ are all even.

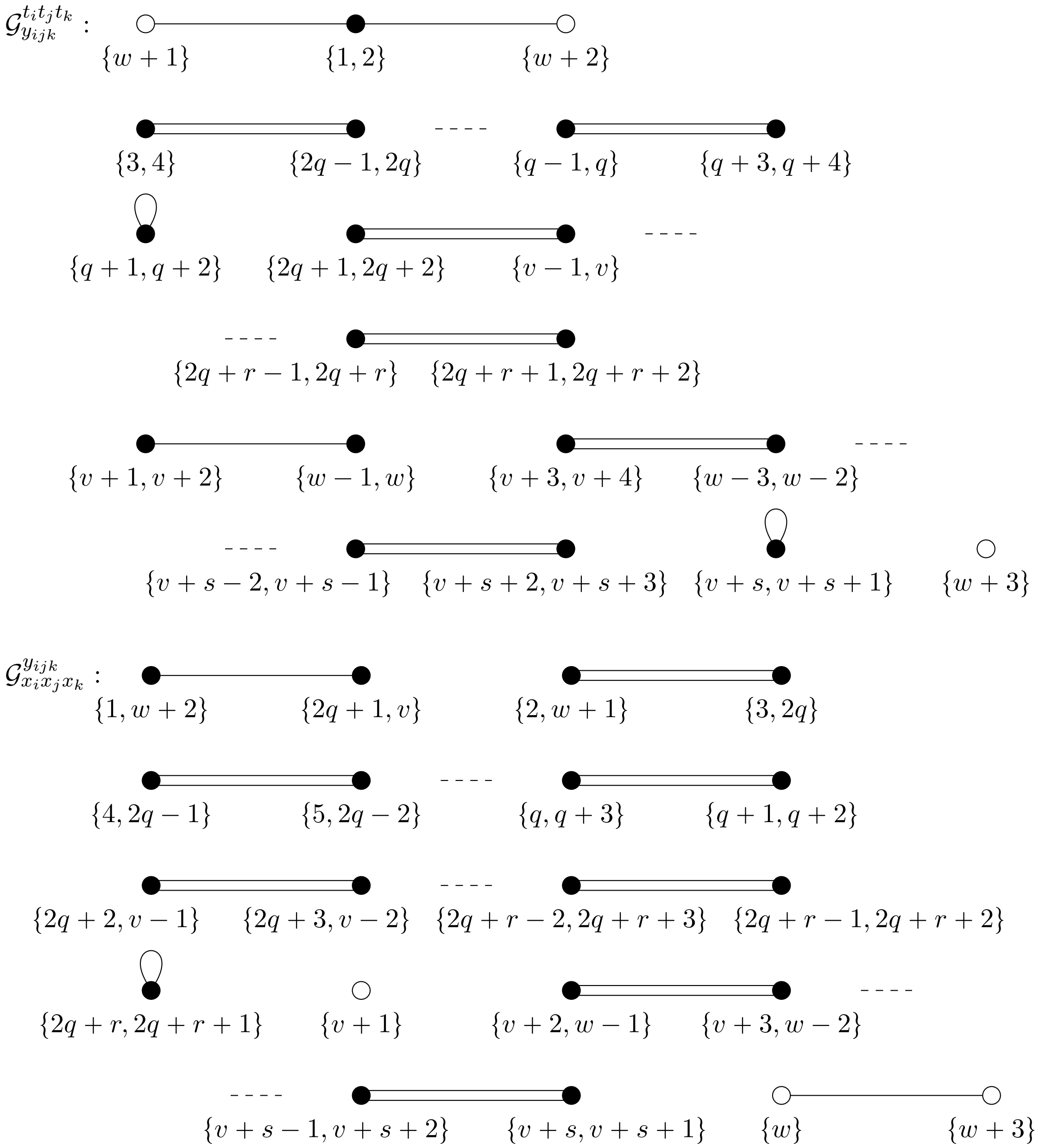


Figure B.33: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that $q \geqslant r \geqslant s > 1$ with q, r even and s odd.

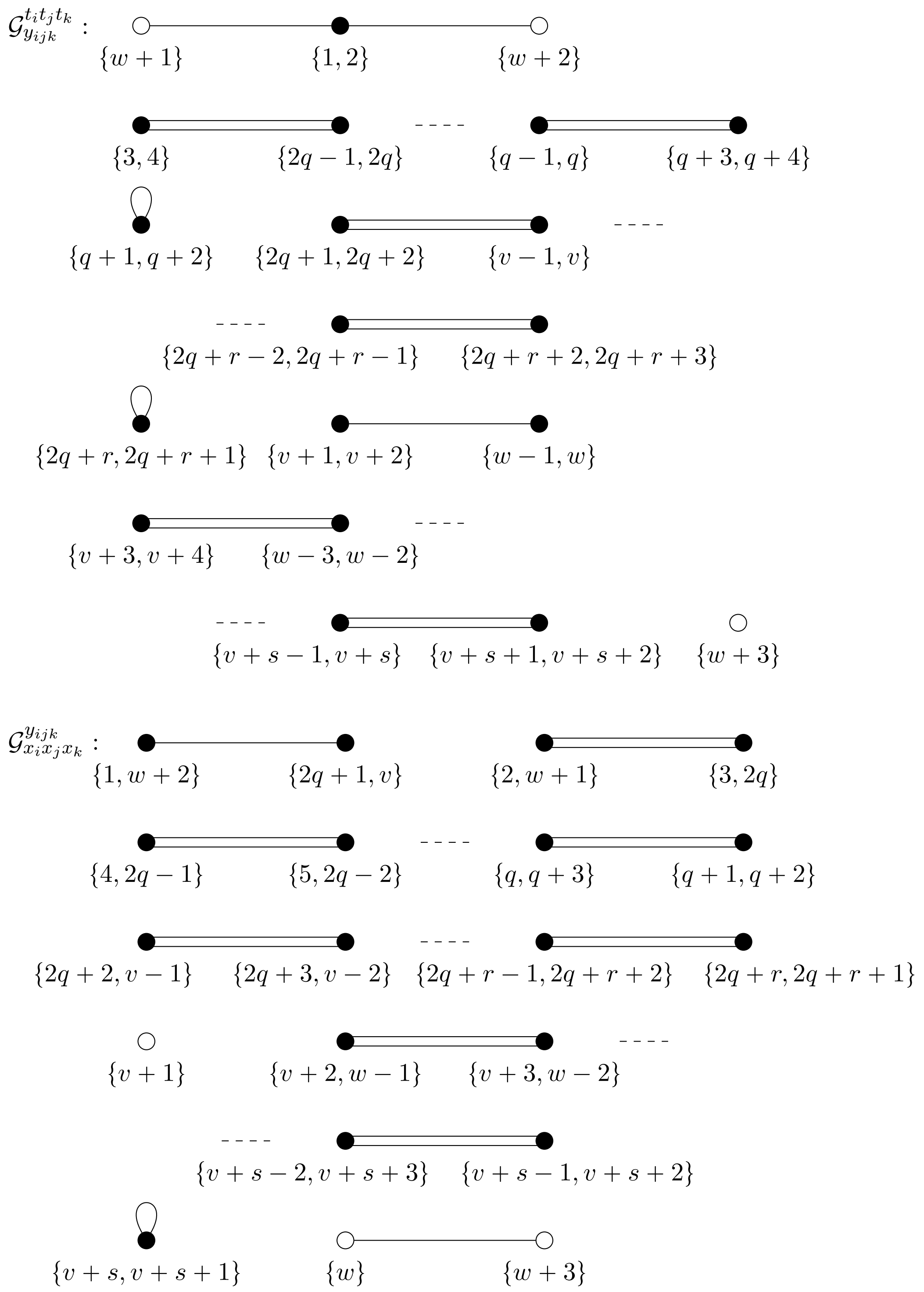


Figure B.34: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that $q \geqslant r \geqslant s > 1$ with q, s both even and r odd.

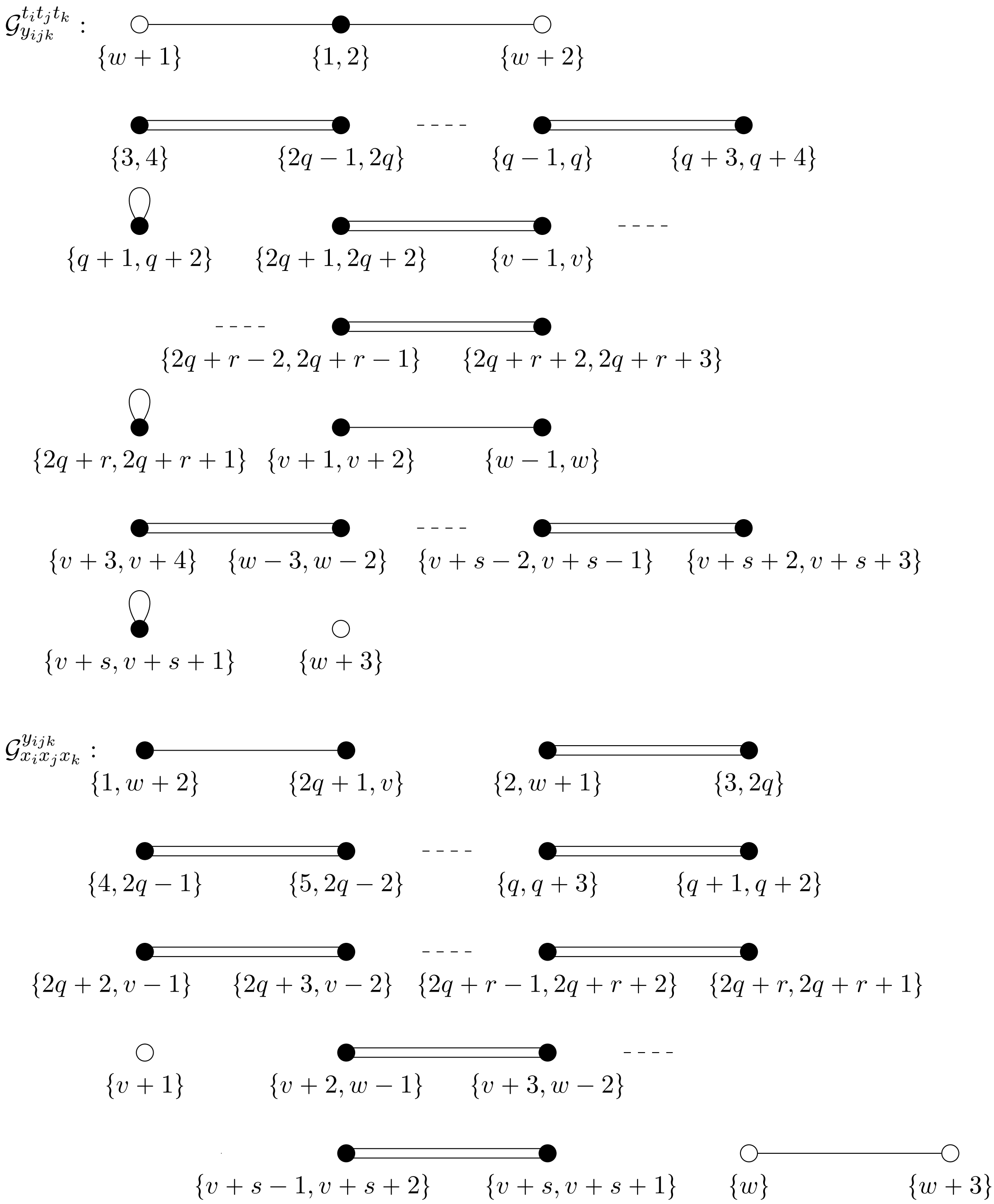


Figure B.35: The x-graphs $\mathcal{G}^{t_it_jt_k}_{y_{ijk}}$ and $\mathcal{G}^{y_{ijk}}_{x_ix_jx_k}$ from Lemma 4.2.7(i) in the case that $q \geqslant r \geqslant s > 1$ with q even and r, s both odd.

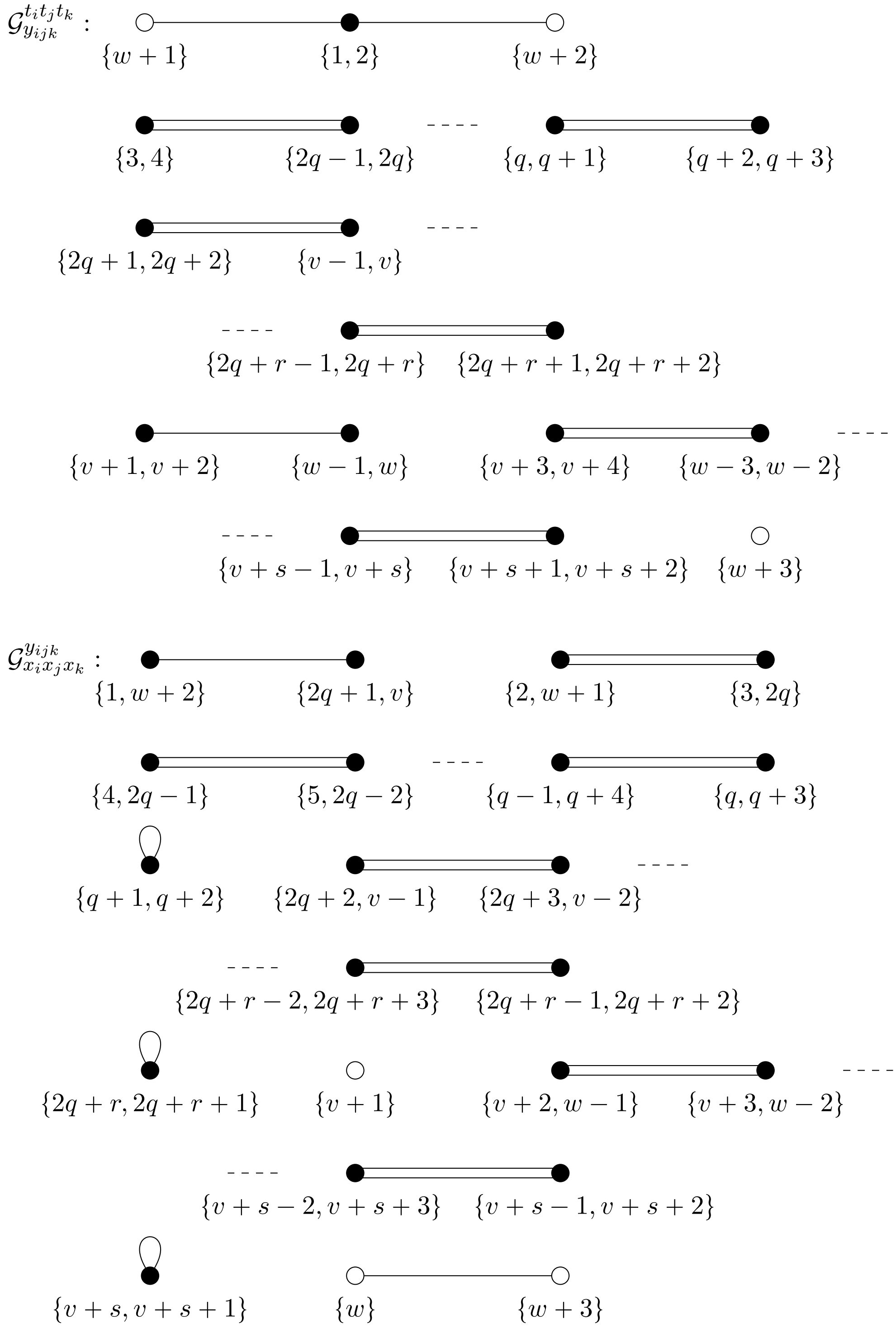


Figure B.36: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that $q \geqslant r \geqslant s > 1$ with q odd and r, s both even.

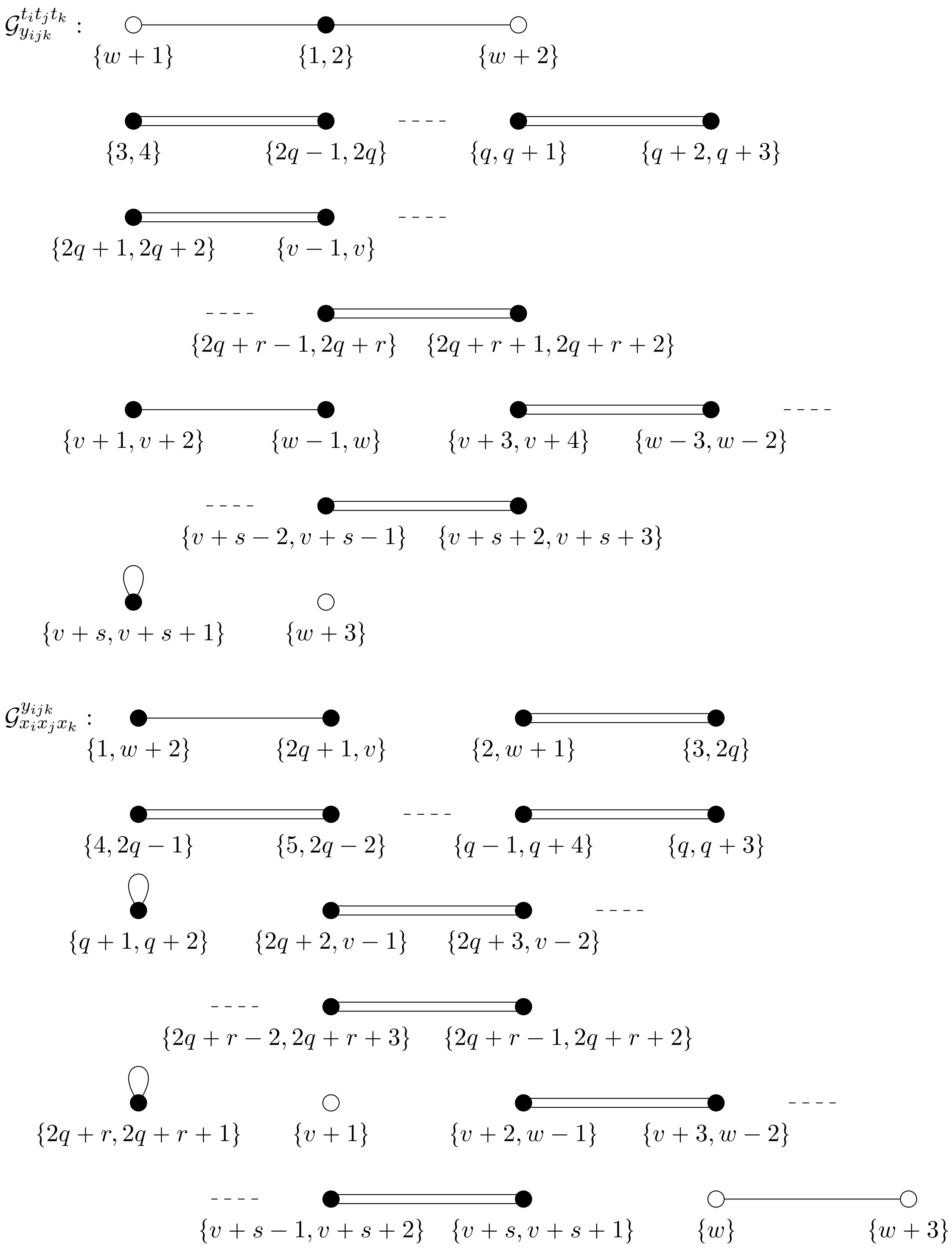


Figure B.37: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_i t_j t_k}$ and $\mathcal{G}_{x_i x_j x_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that $q \geqslant r \geqslant s > 1$ with q, s both odd and r even.

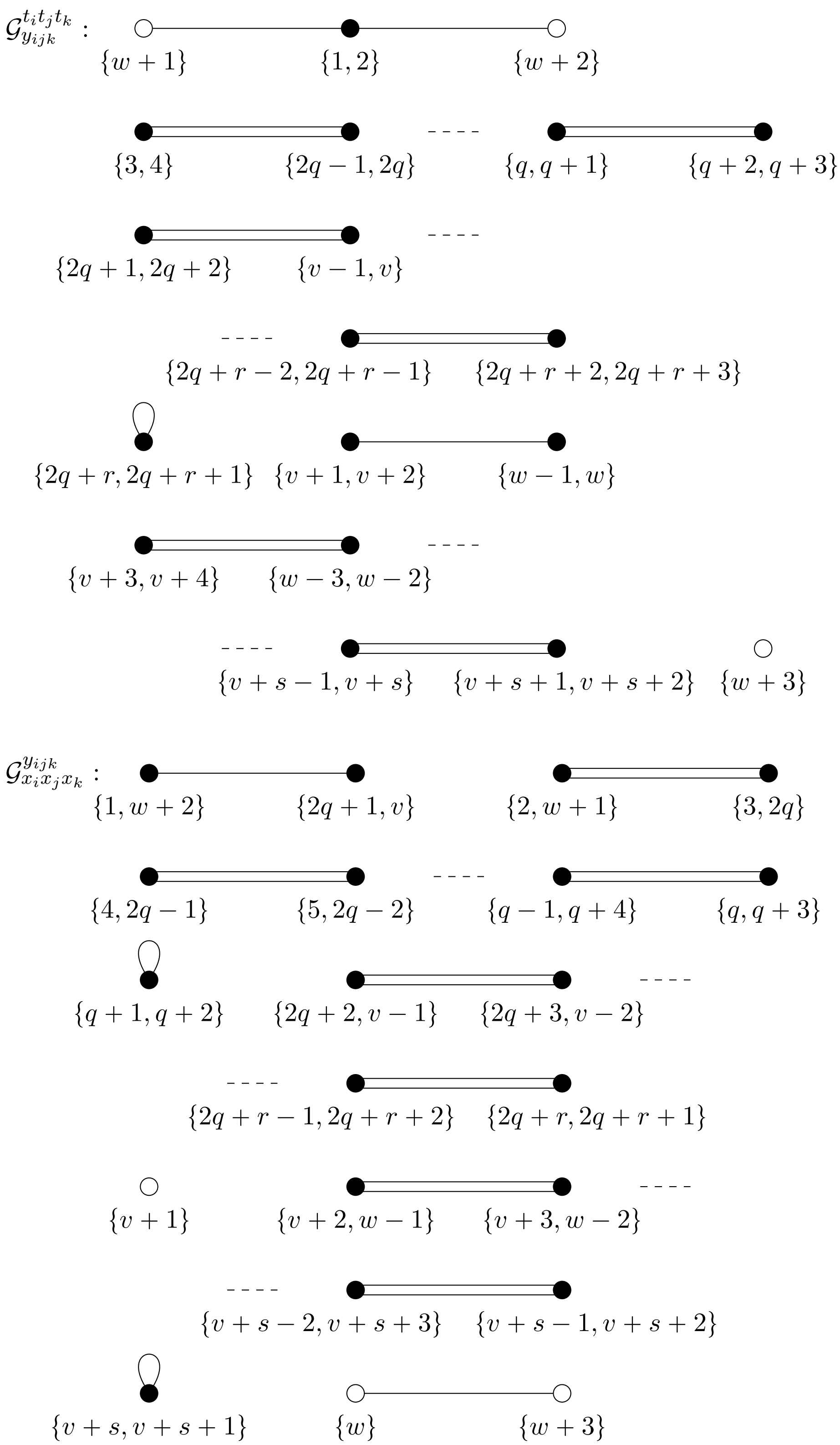


Figure B.38: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that $q \geqslant r \geqslant s > 1$ with q, r both odd and s even.

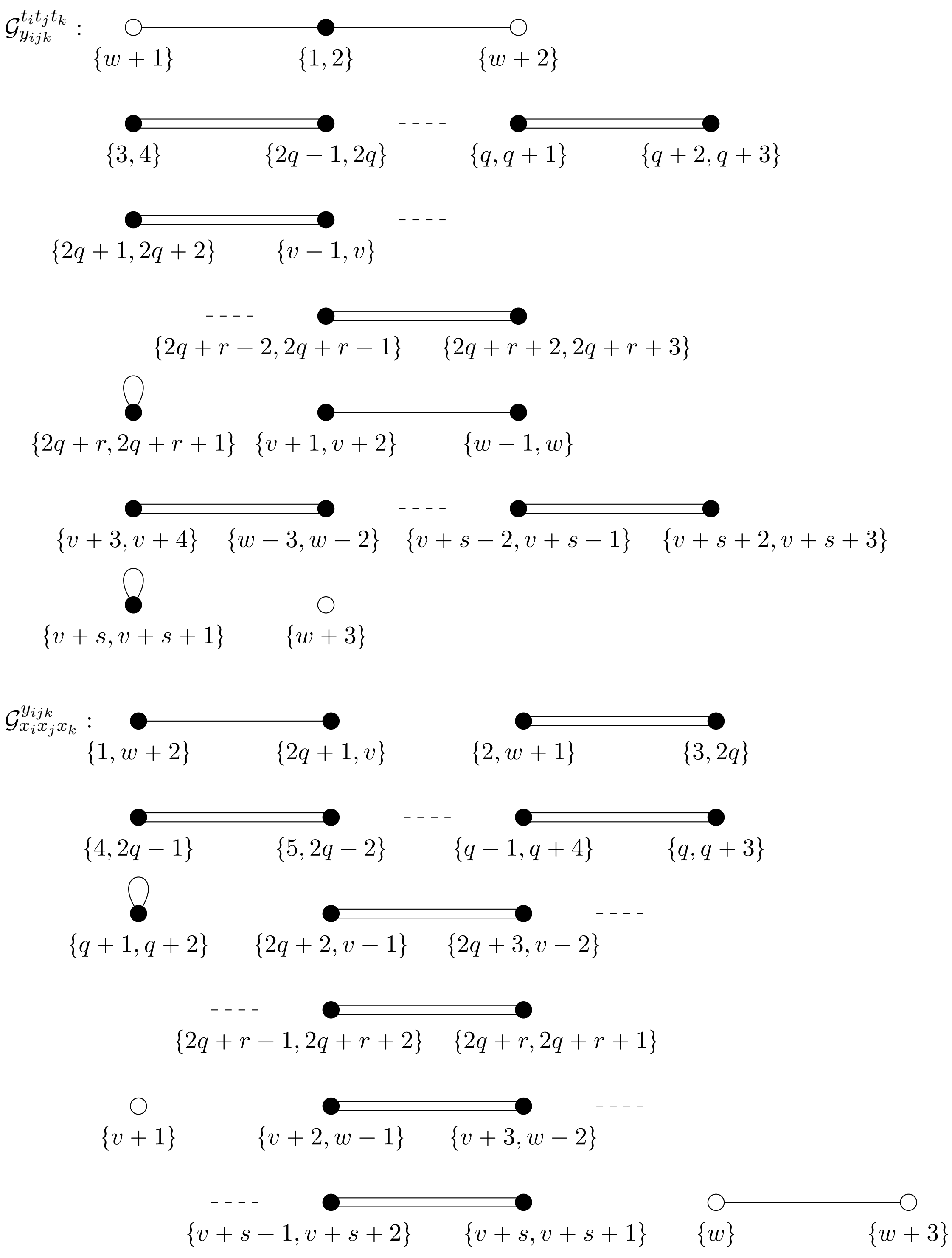


Figure B.39: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(i) in the case that $q \geqslant r \geqslant s > 1$ all odd.

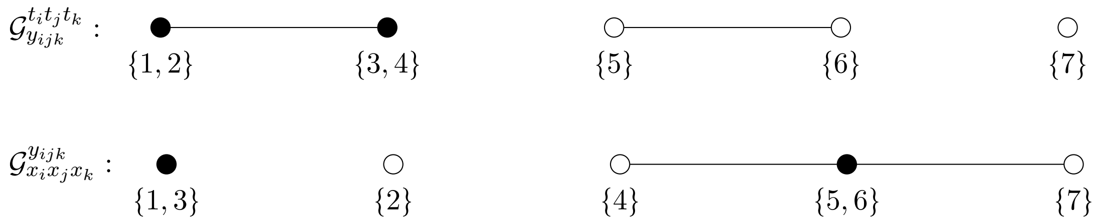


Figure B.40: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q = r = 1$.

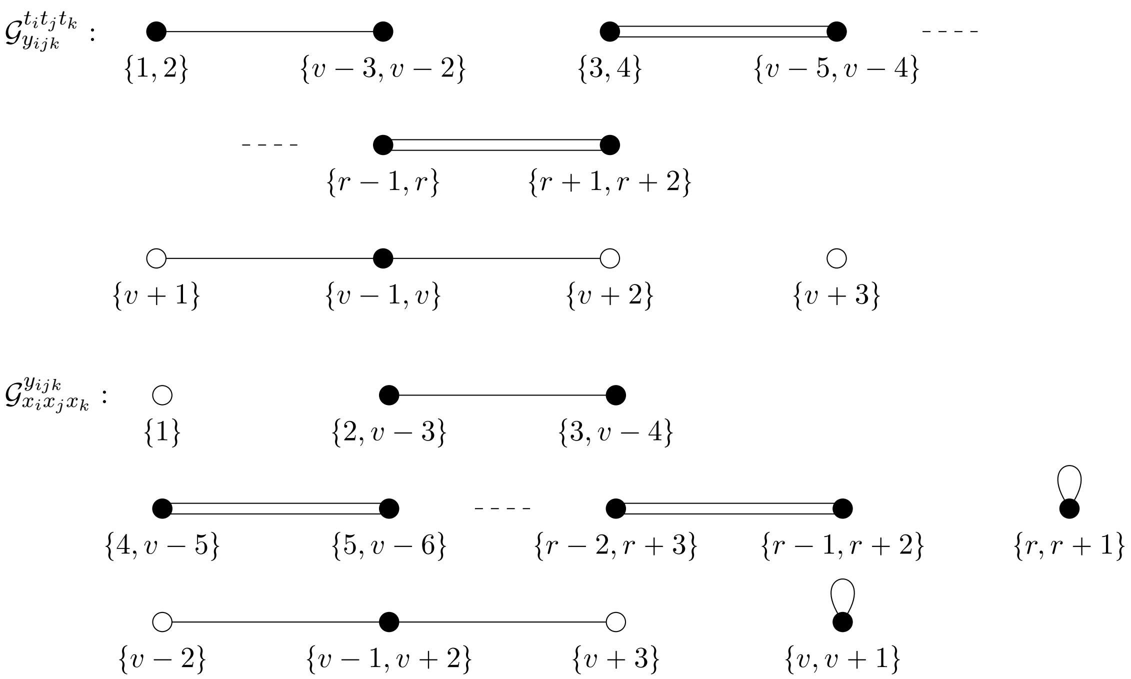


Figure B.41: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q = 1$ and $r > 1$ is even.

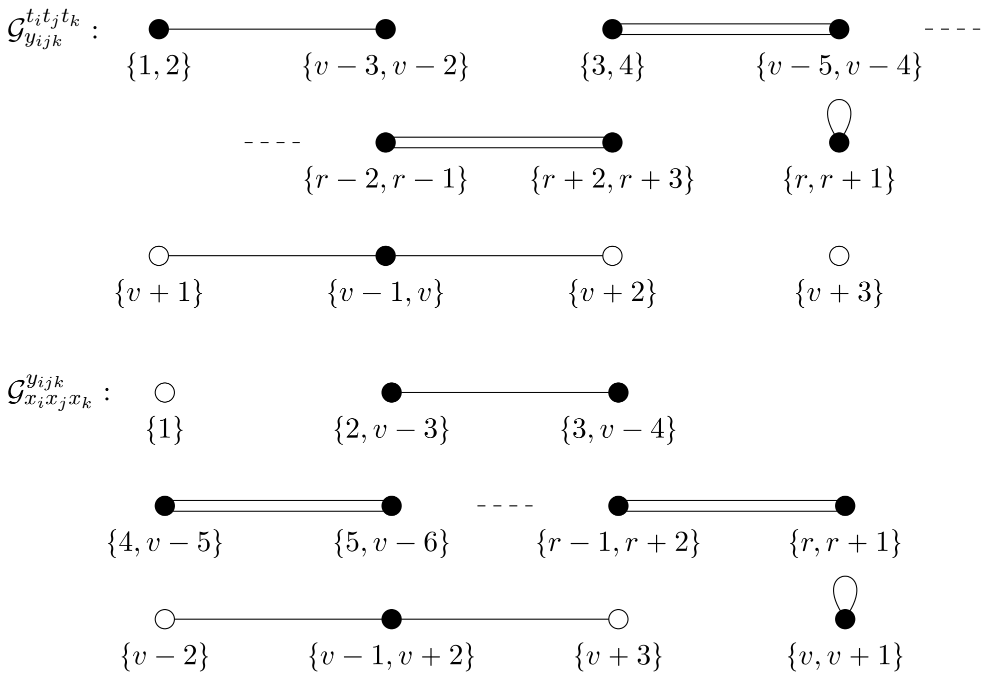


Figure B.42: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q=1$ and $r>1$ is odd.

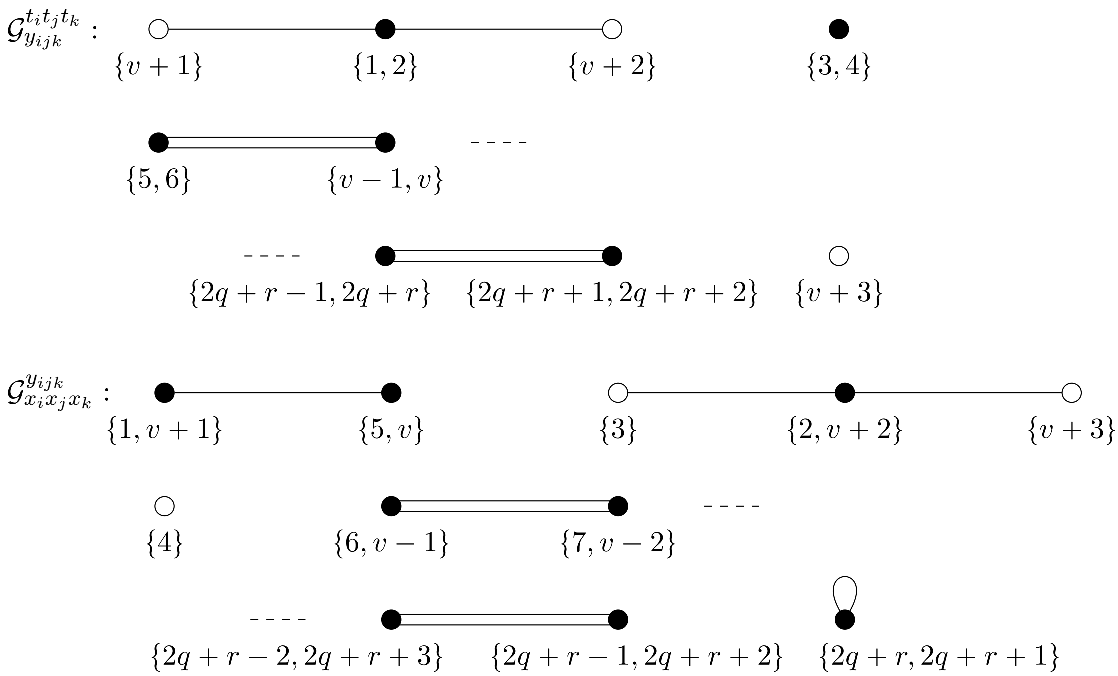


Figure B.43: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q=2$ and $r>1$ is even.

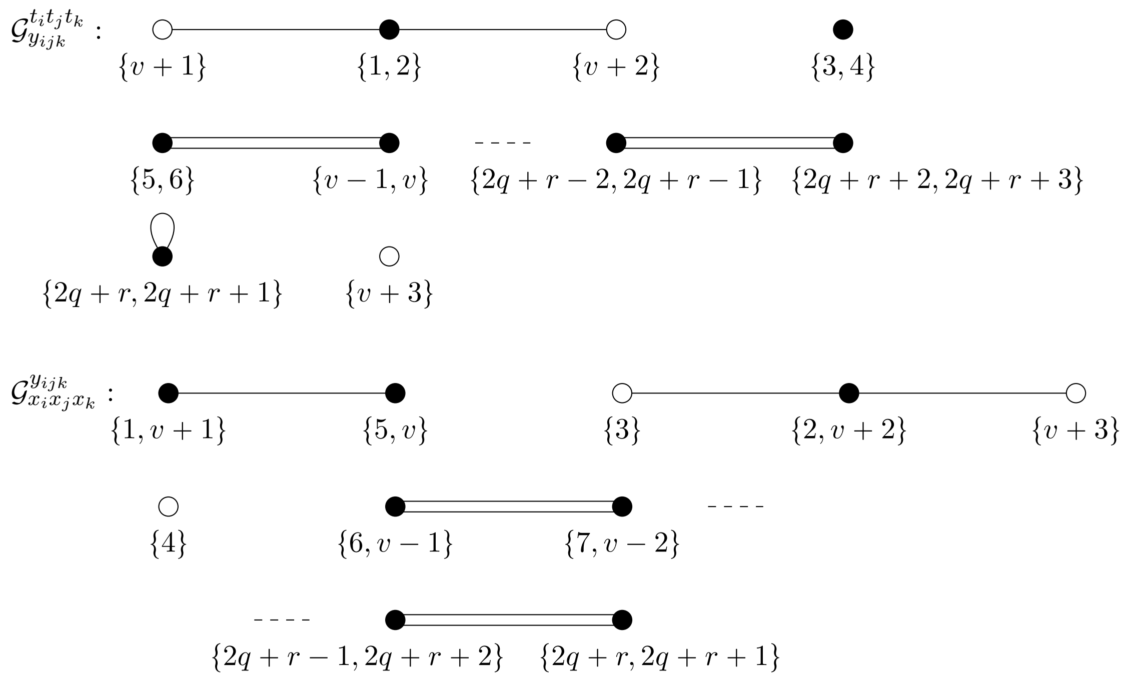


Figure B.44: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q=2$ and $r>1$ is odd.

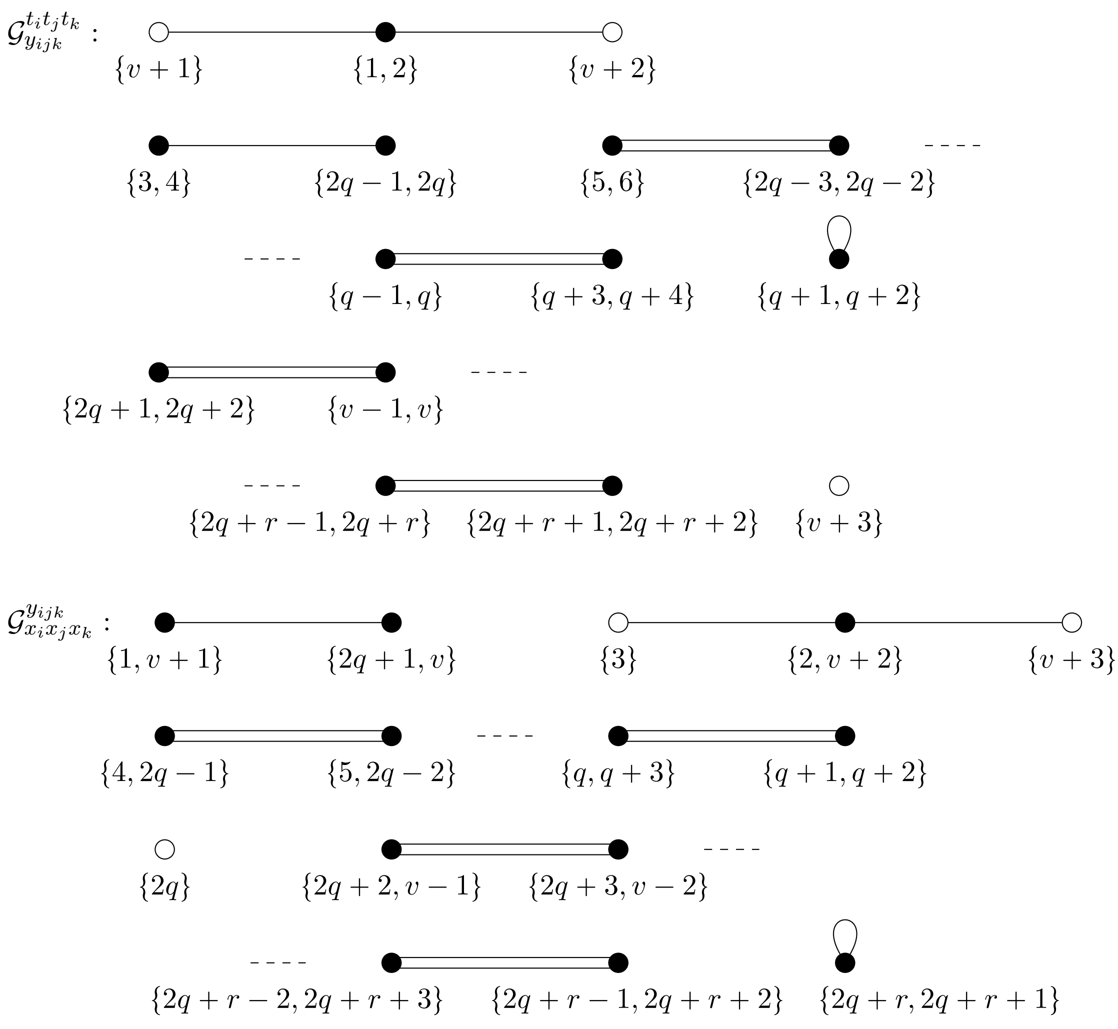


Figure B.45: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q > 2$, $r > 1$ and q and r are both even.

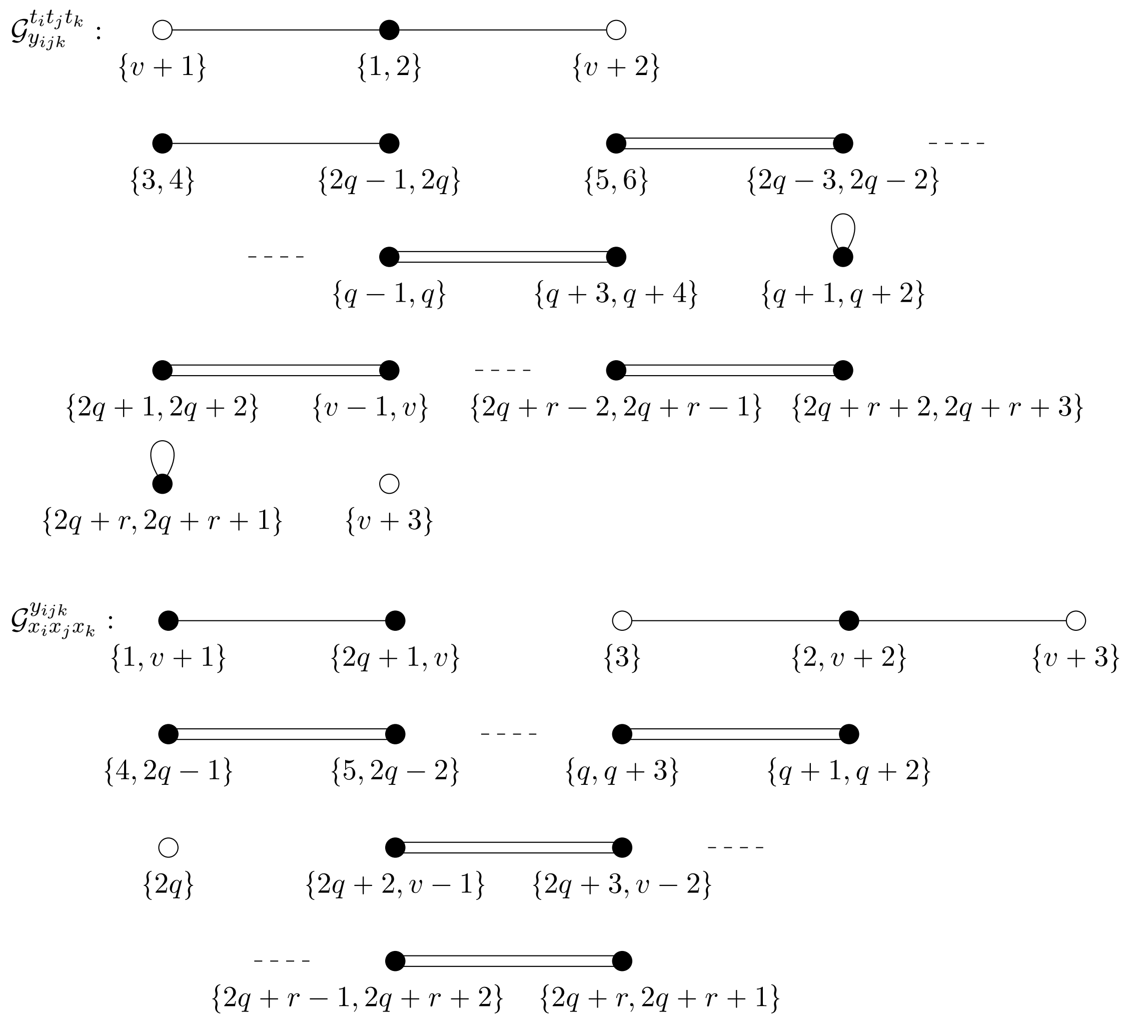


Figure B.46: The x-graphs $\mathcal{G}^{t_i t_j t_k}_{y_{ijk}}$ and $\mathcal{G}^{y_{ijk}}_{x_i x_j x_k}$ from Lemma 4.2.7(ii) in the case that $q > 2$ is even and $r > 1$ is odd.

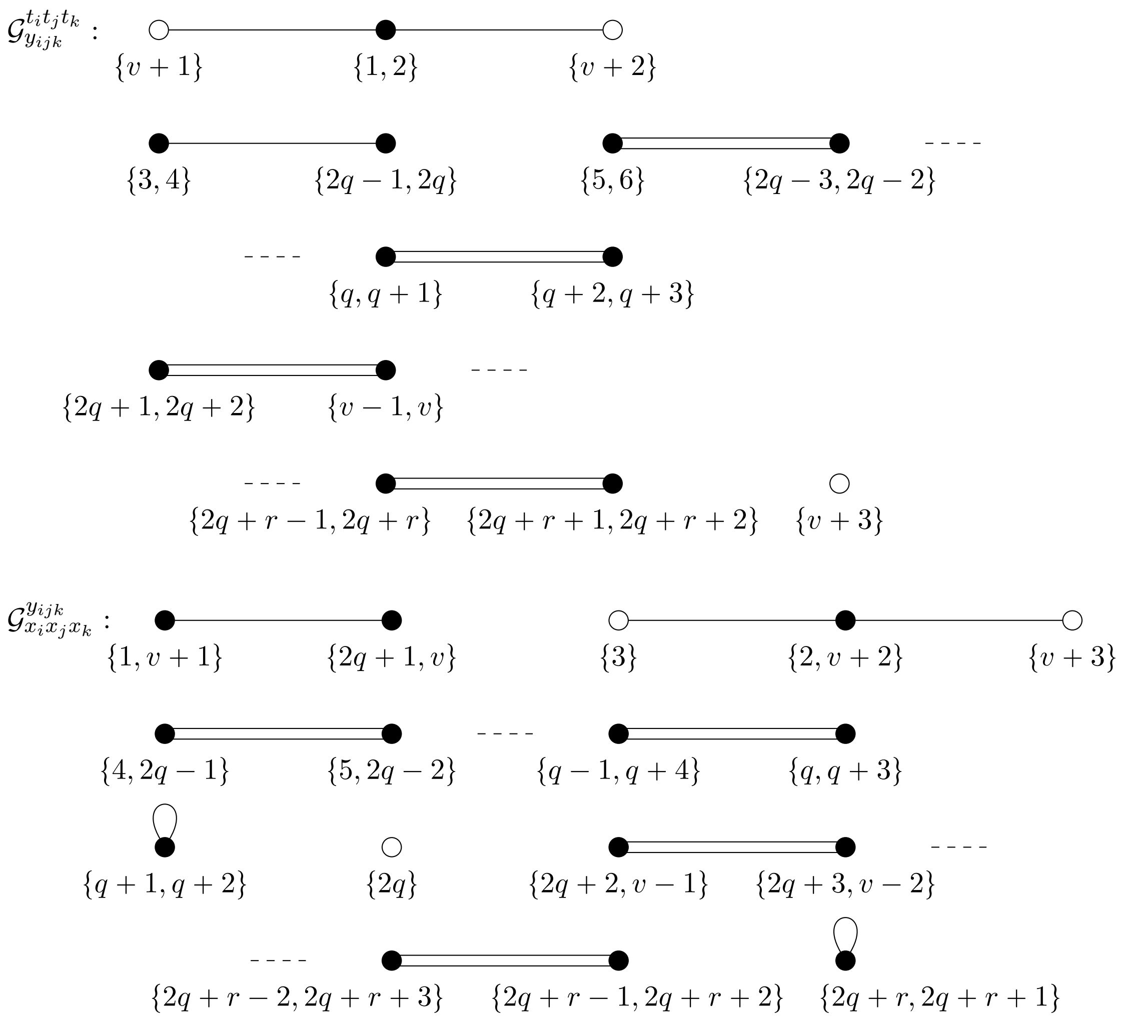


Figure B.47: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_it_jt_k}$ and $\mathcal{G}_{x_ix_jx_k}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q > 2$ is odd and $r > 1$ is even.

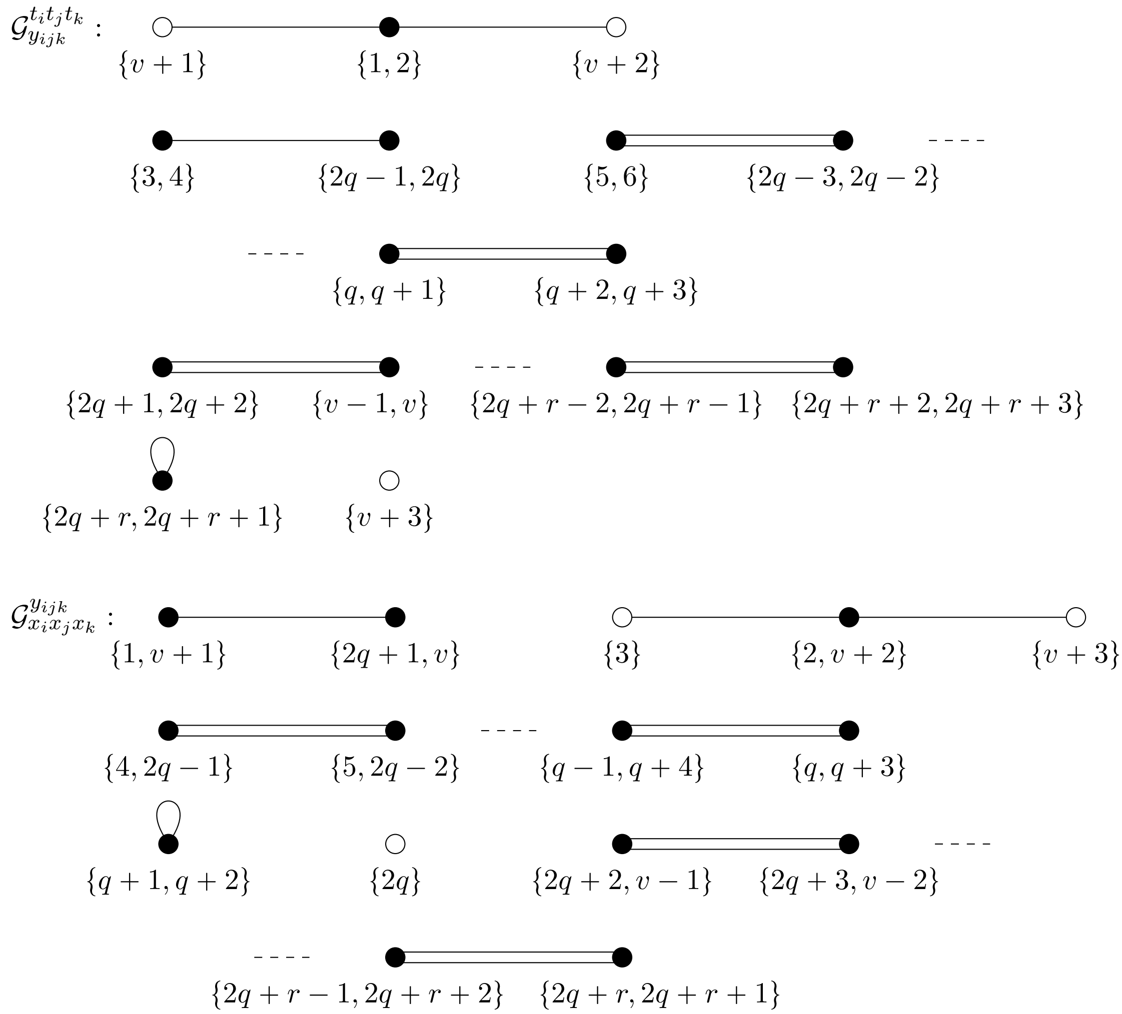


Figure B.48: The x-graphs $\mathcal{G}_{y_{ijk}}^{t_i t_j t_k}$ and $\mathcal{G}_{x_i x_j x_k}^{y_{ijk}}$ from Lemma 4.2.7(ii) in the case that $q > 2$, $r > 1$ and q and r are both odd.

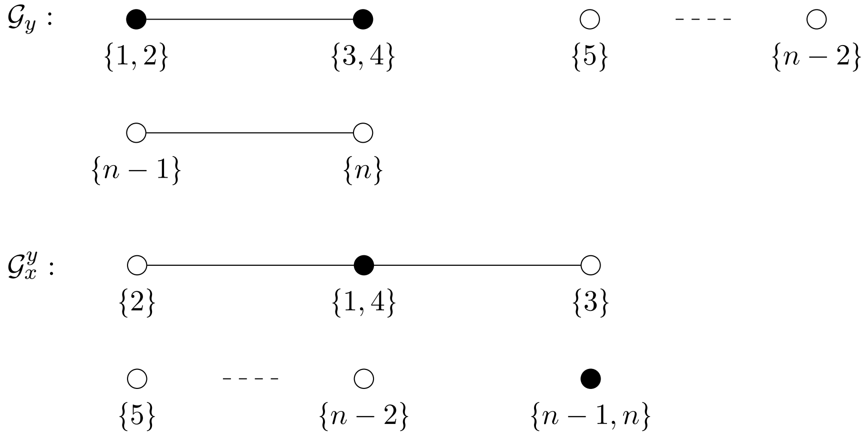


Figure B.49: The x-graphs $\mathcal{G}_y$ and $\mathcal{G}_x^y$ from Lemma 4.2.8 in the case that $\mathcal{G}_x$ contains a cycle of two black vertices and $m = 2$.

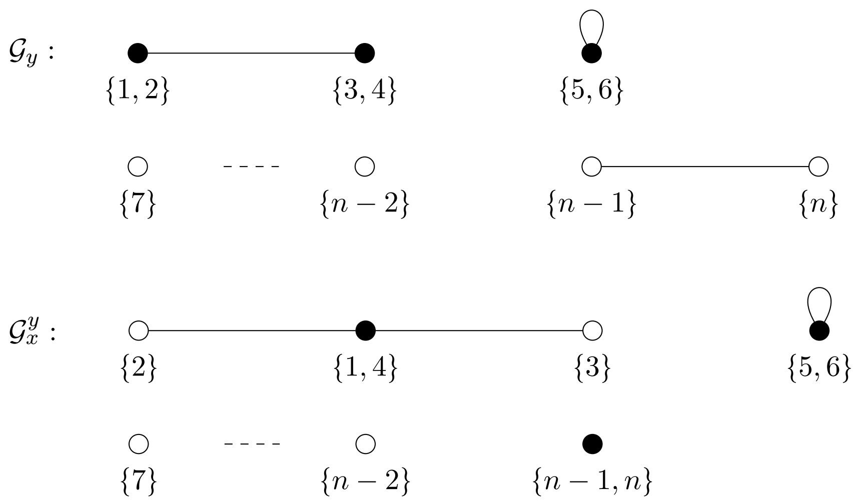


Figure B.50: The x-graphs $\mathcal{G}_y$ and $\mathcal{G}_x^y$ from Lemma 4.2.8 in the case that $\mathcal{G}_x$ contains a cycle of two black vertices and $m = 3$.

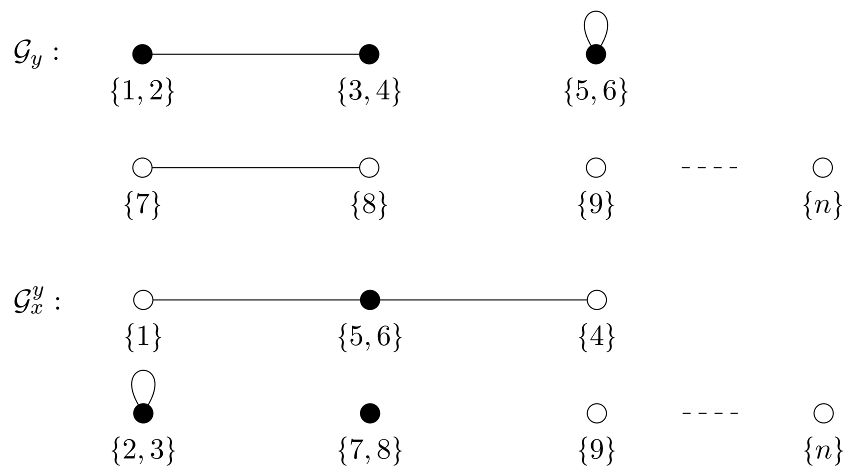


Figure B.51: The x-graphs $\mathcal{G}_y$ and $\mathcal{G}_x^y$ from Lemma 4.2.8 in the case that $\mathcal{G}_x$ contains a cycle of three black vertices and $m = 3$.

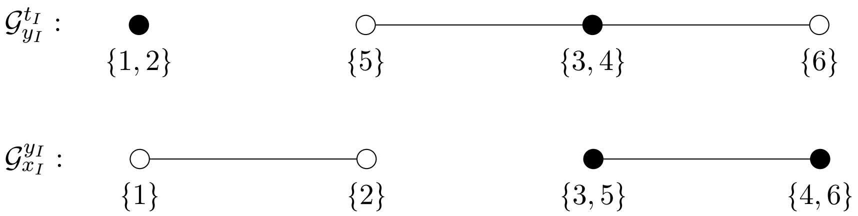


Figure B.52: The x-graphs $\mathcal{G}_{y_I}^{t_I}$ and $\mathcal{G}_{x_I}^{y_I}$ from (4.2.1) when $u = 1$.

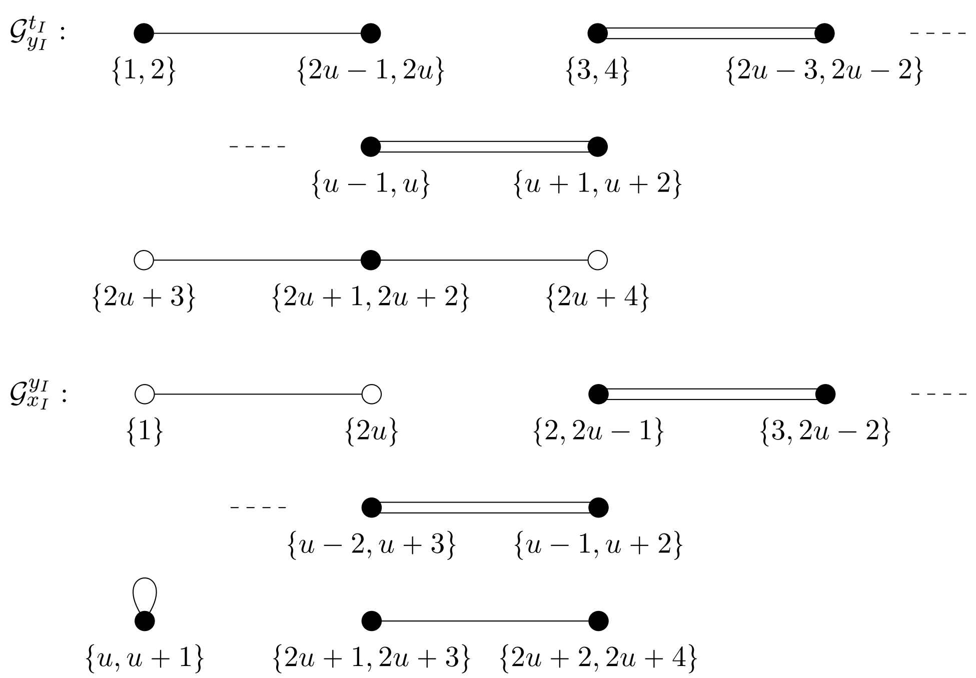


Figure B.53: The x-graphs $\mathcal{G}_{y_I}^{t_I}$ and $\mathcal{G}_{x_I}^{y_I}$ from (4.2.1) when $u > 1$ is even.

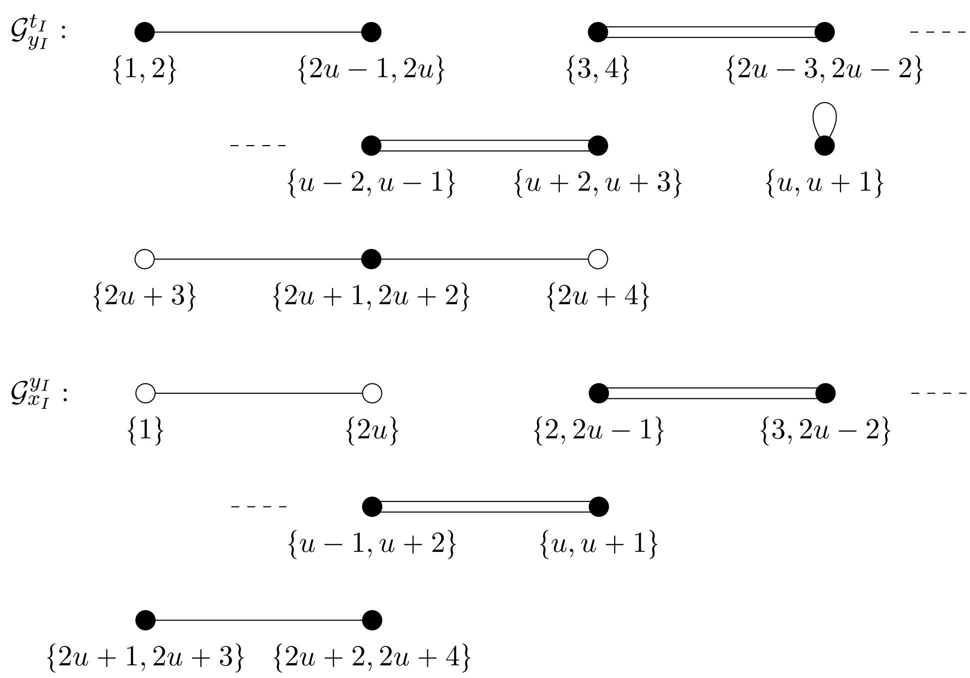


Figure B.54: The x-graphs $\mathcal{G}_{y_I}^{t_I}$ and $\mathcal{G}_{x_I}^{y_I}$ from (4.2.1) when $u>1$ is odd.

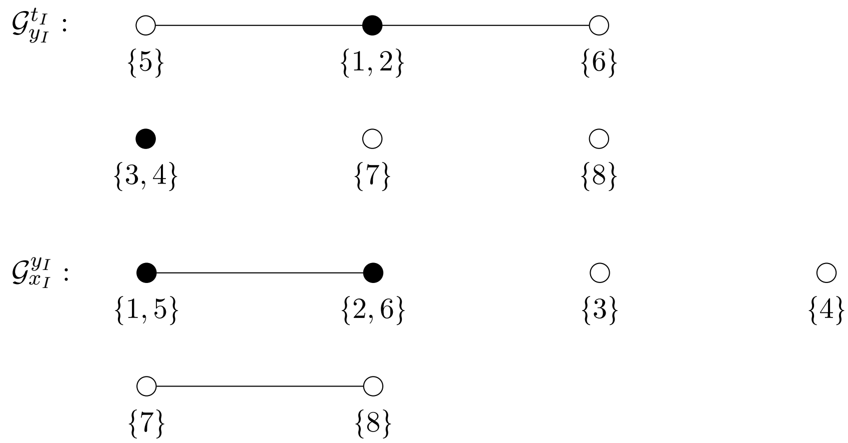


Figure B.55: The x-graphs $\mathcal{G}^{t_I}_{y_I}$ and $\mathcal{G}^{y_I}_{x_I}$ from (4.2.2).

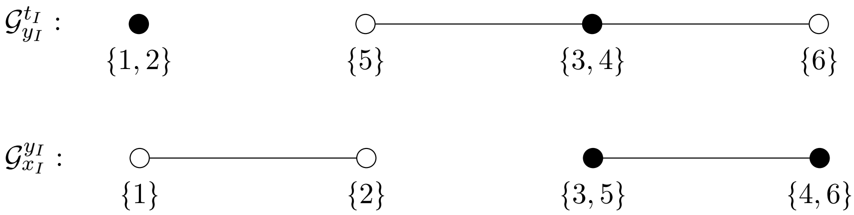


Figure B.56: The x-graphs $\mathcal{G}^{t_I}_{y_I}$ and $\mathcal{G}^{y_I}_{x_I}$ from (4.2.3) with $u = 1$.

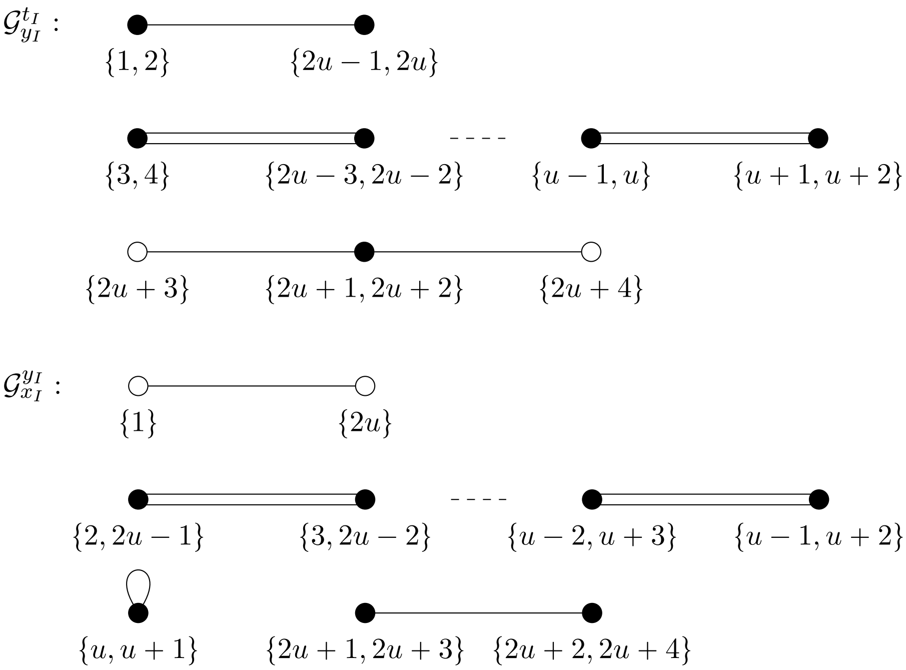


Figure B.57: The x-graphs $\mathcal{G}^{t_I}_{y_I}$ and $\mathcal{G}^{y_I}_{x_I}$ from (4.2.3) with $u > 1$ even.

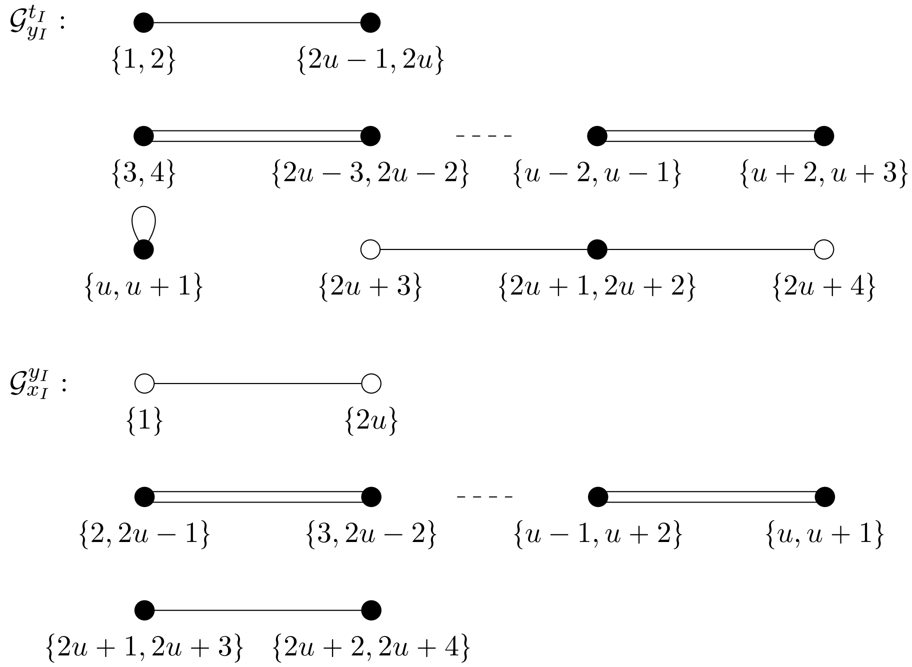


Figure B.58: The x-graphs $\mathcal{G}_{y_I}^{t_I}$ and $\mathcal{G}_{x_I}^{y_I}$ from (4.2.3) with $u > 1$ odd.

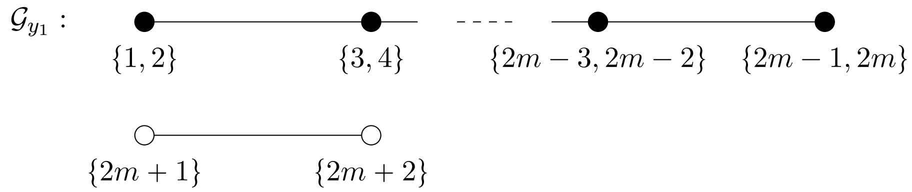


Figure B.59: The x-graph $\mathcal{G}_{y_1}$ from (4.3.5).

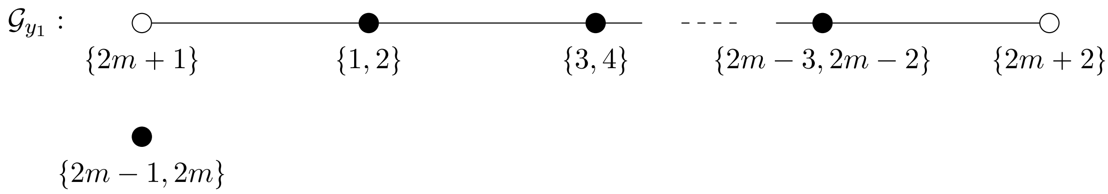


Figure B.60: The x-graph $\mathcal{G}_{y_1}$ from (4.3.6).

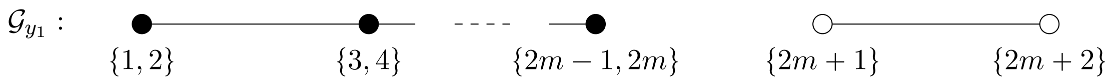


Figure B.61: The x-graph $\mathcal{G}_{y_1}$ from (4.3.7).

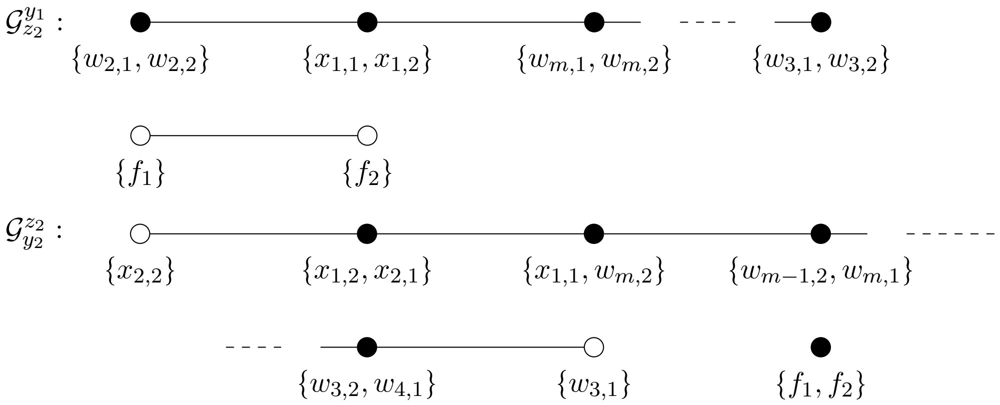


Figure B.62: The x-graphs $\mathcal{G}_{z_2}^{y_1}$ and $\mathcal{G}_{y_2}^{z_2}$ from (4.3.11) and (4.3.12) respectively.

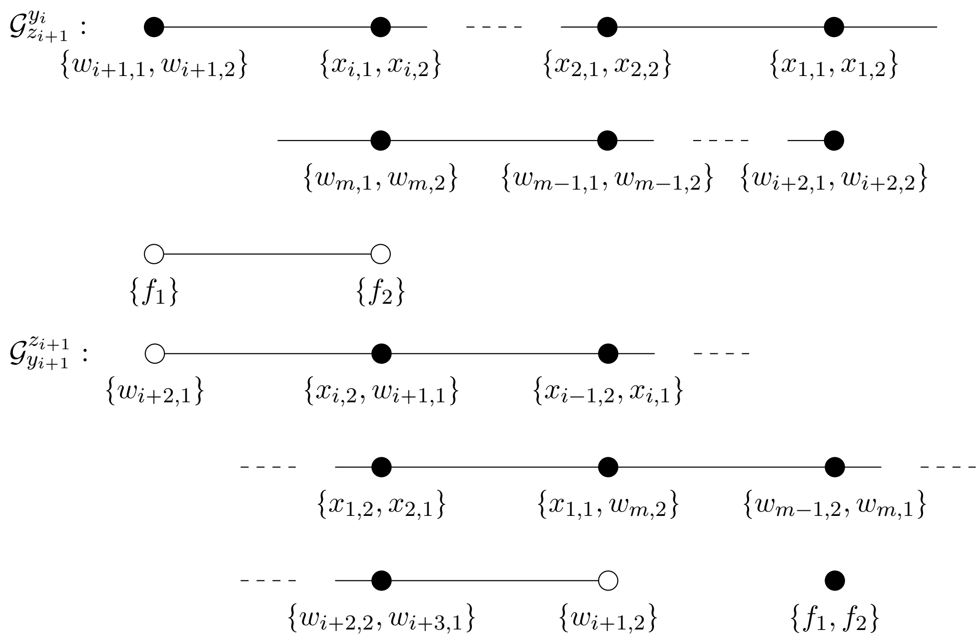


Figure B.63: The x-graphs $\mathcal{G}_{z_{i+1}}^{y_i}$ and $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ from (4.3.10) and (4.3.12) respectively.

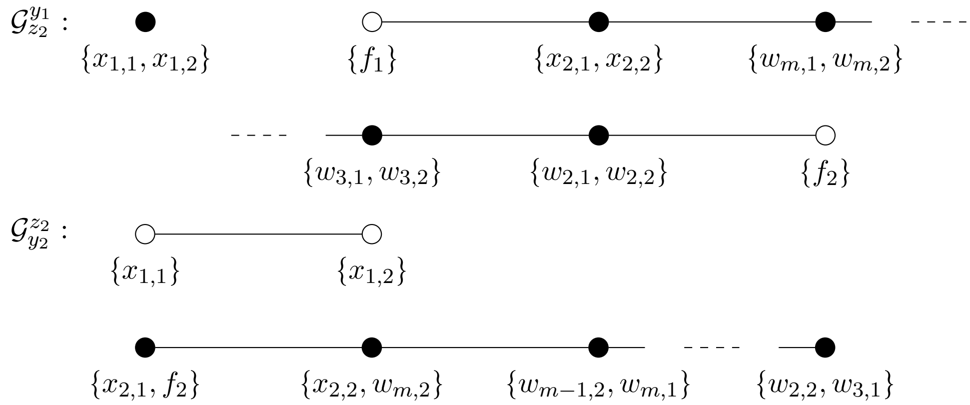


Figure B.64: The x-graphs $\mathcal{G}_{z_2}^{y_1}$ and $\mathcal{G}_{y_2}^{z_2}$ from (4.3.13) and (4.3.14) respectively.

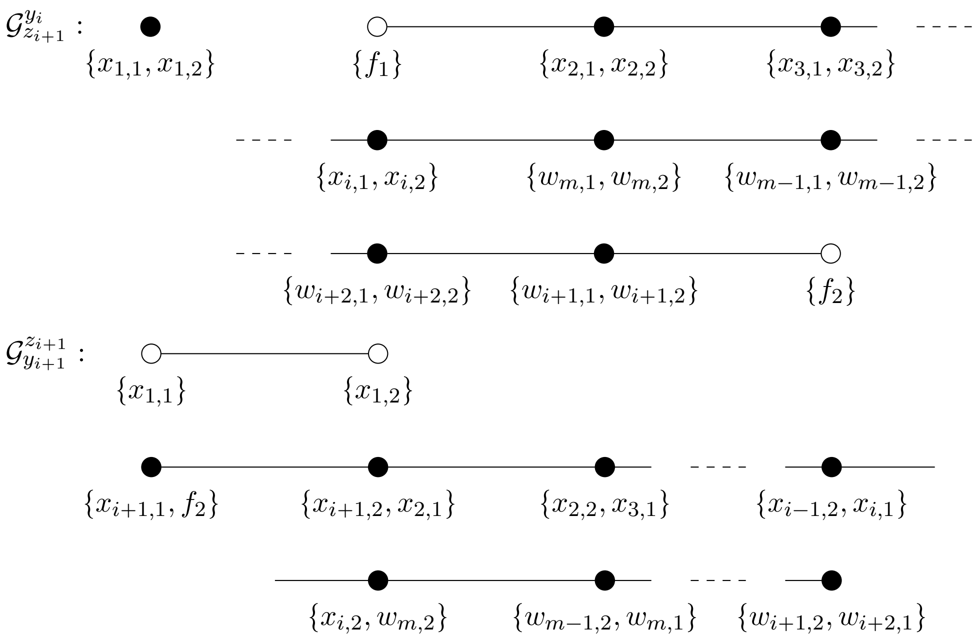


Figure B.65: The x-graphs $\mathcal{G}_{z_{i+1}}^{y_i}$ and $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ from (4.3.13) and (4.3.14) respectively.

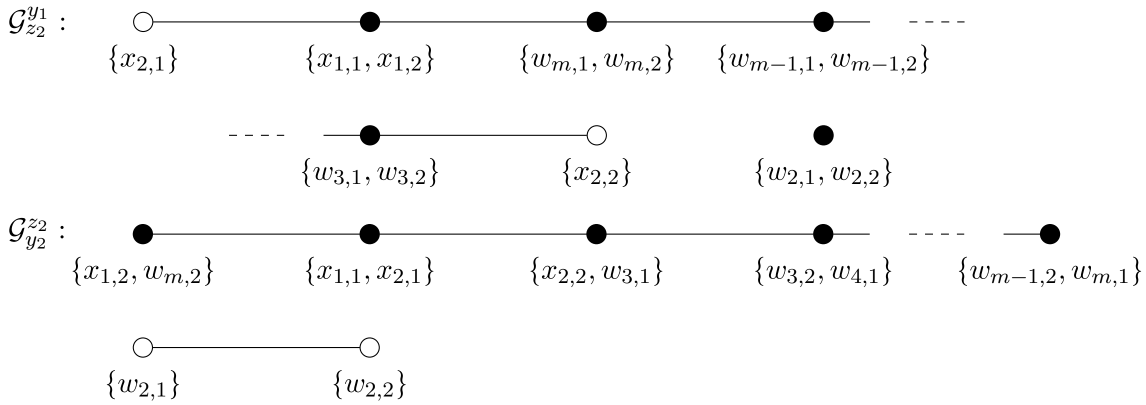


Figure B.66: The x-graphs $\mathcal{G}_{z_2}^{y_1}$ and $\mathcal{G}_{y_2}^{z_2}$ from (4.3.15) and (4.3.16) respectively.

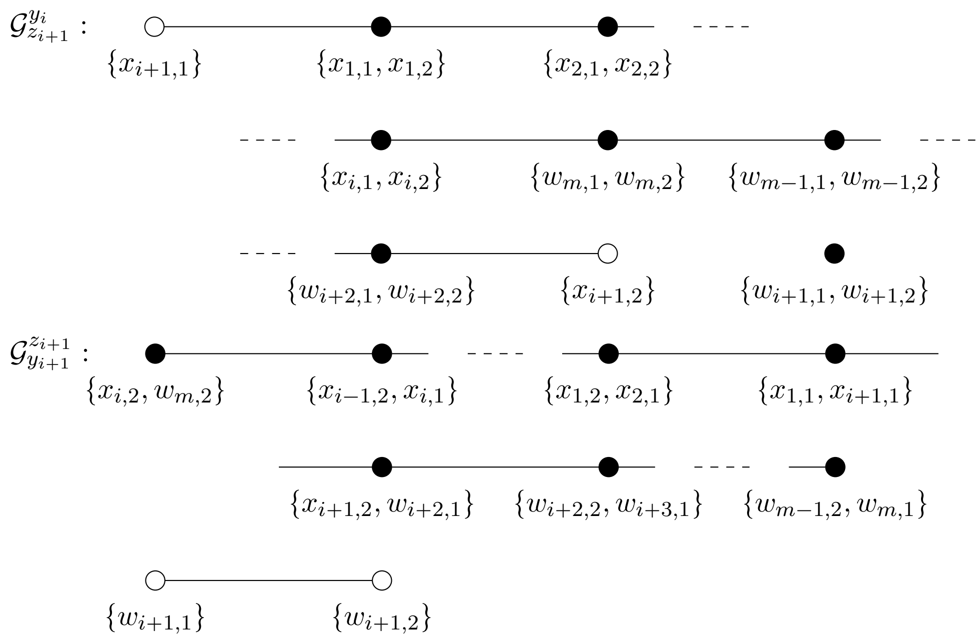


Figure B.67: The x-graphs $\mathcal{G}_{z_{i+1}}^{y_i}$ and $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ from (4.3.15) and (4.3.16) respectively.

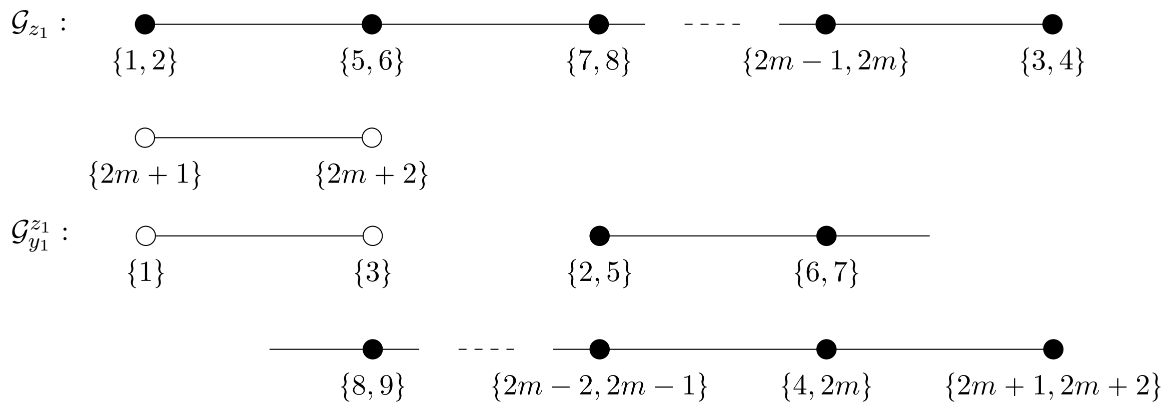


Figure B.68: The x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ from (4.3.17).

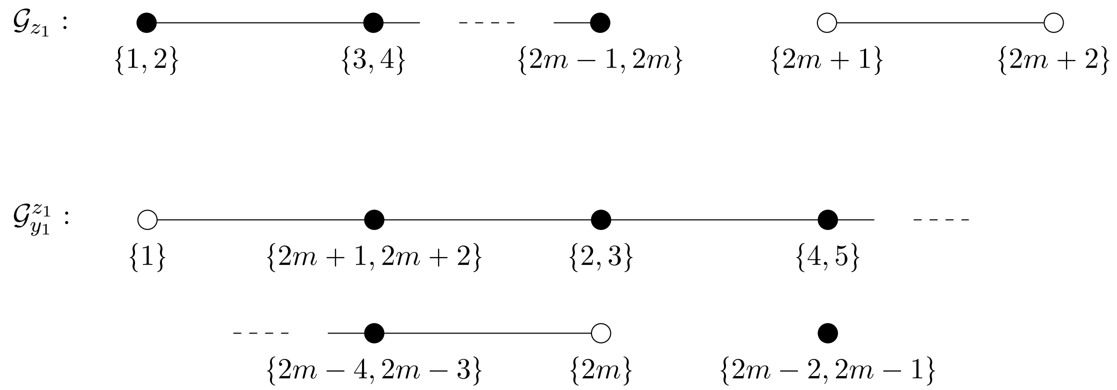


Figure B.69: The x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ from (4.3.18).

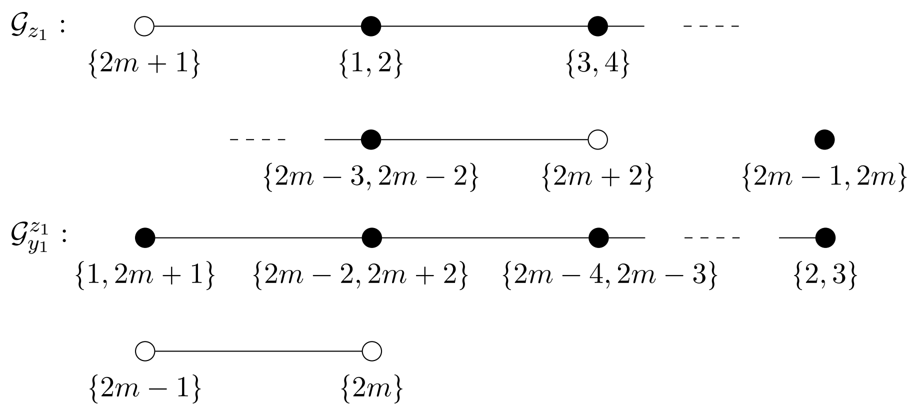


Figure B.70: The x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ from (4.3.19).

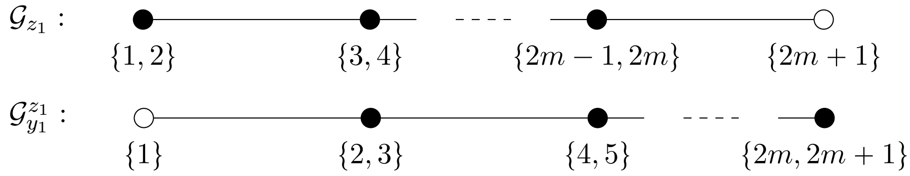


Figure B.71: The x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ from (4.3.21).

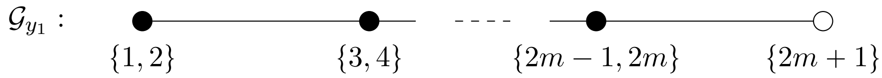


Figure B.72: The x-graph $\mathcal{G}_{y_1}$ from (4.3.22).

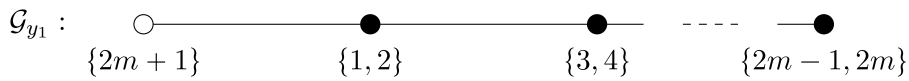


Figure B.73: The x-graph $\mathcal{G}_{y_1}$ from (4.3.23).

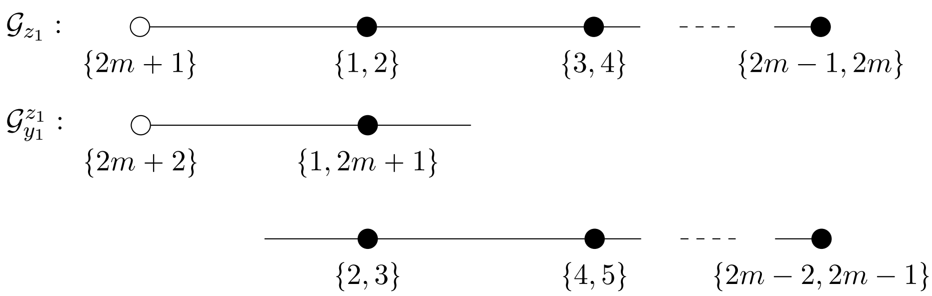


Figure B.74: The x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ from (4.3.24).

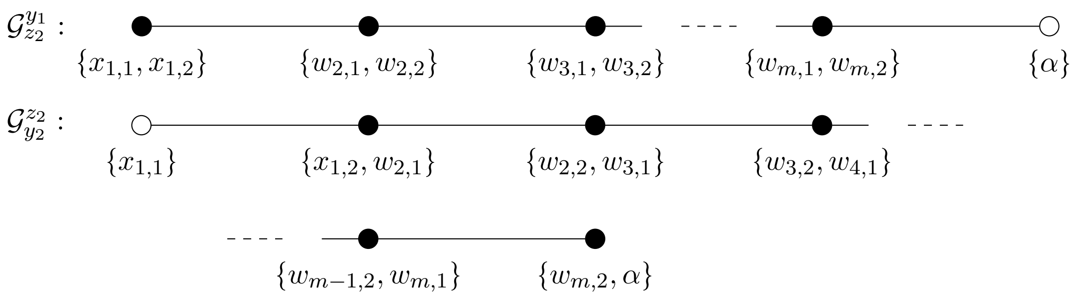


Figure B.75: The x-graphs $\mathcal{G}_{z_2}^{y_1}$ and $\mathcal{G}_{y_2}^{z_2}$ from (4.3.25).

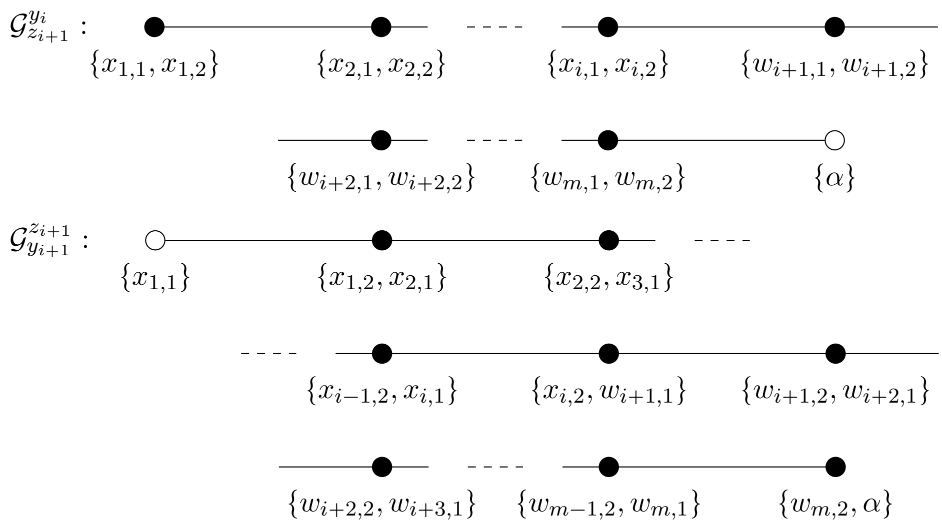


Figure B.76: The x-graphs $\mathcal{G}^{y_i}_{z_{i+1}}$ and $\mathcal{G}^{z_{i+1}}_{y_{i+1}}$ from (4.3.25).

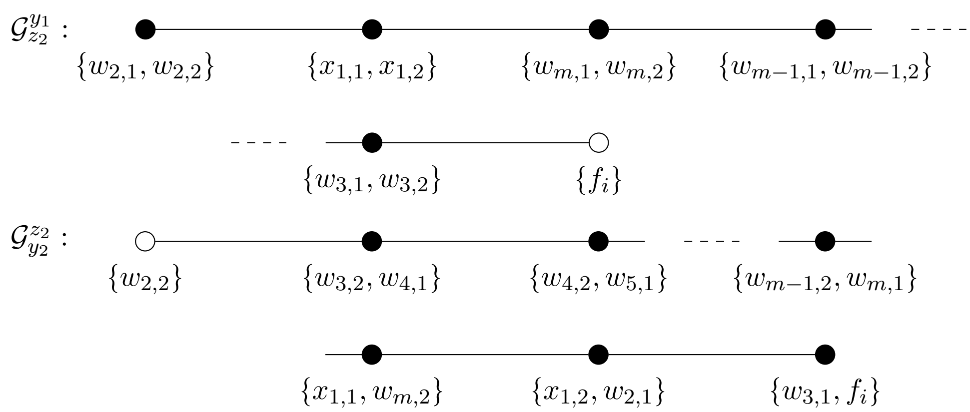


Figure B.77: The x-graphs $\mathcal{G}^{y_1}_{z_2}$ and $\mathcal{G}^{z_2}_{y_2}$ from (4.3.26).

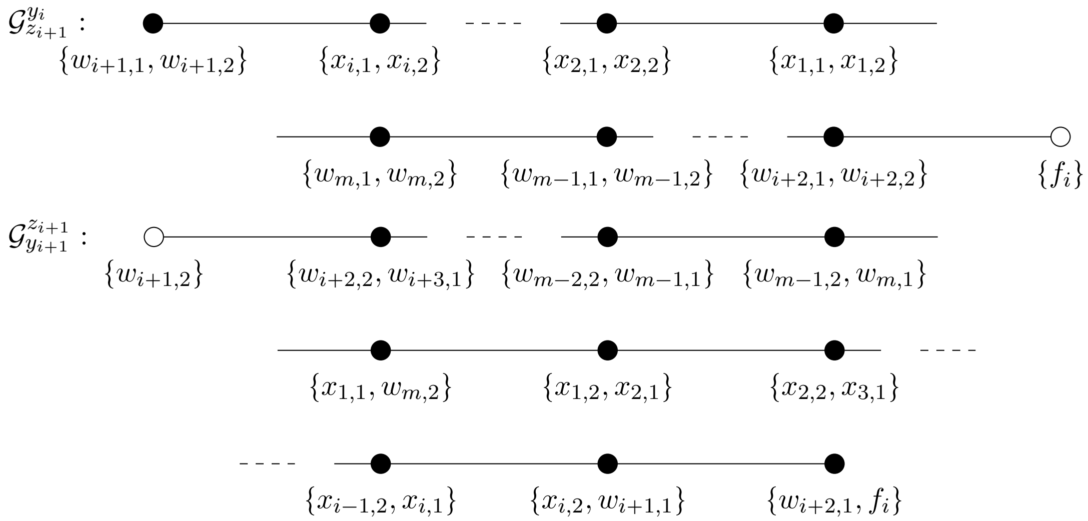


Figure B.78: The x-graphs $\mathcal{G}^{y_i}_{z_{i+1}}$ and $\mathcal{G}^{z_{i+1}}_{y_{i+1}}$ from (4.3.26).

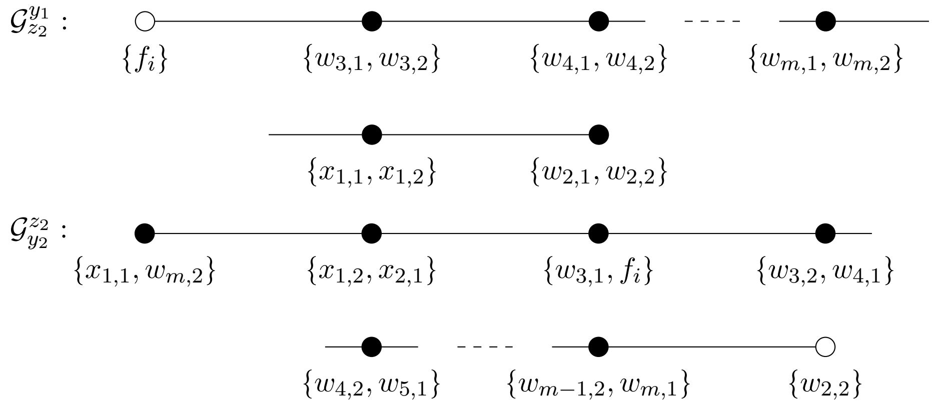


Figure B.79: The x-graphs $\mathcal{G}^{y_1}_{z_2}$ and $\mathcal{G}^{z_2}_{y_2}$ from (4.3.27).

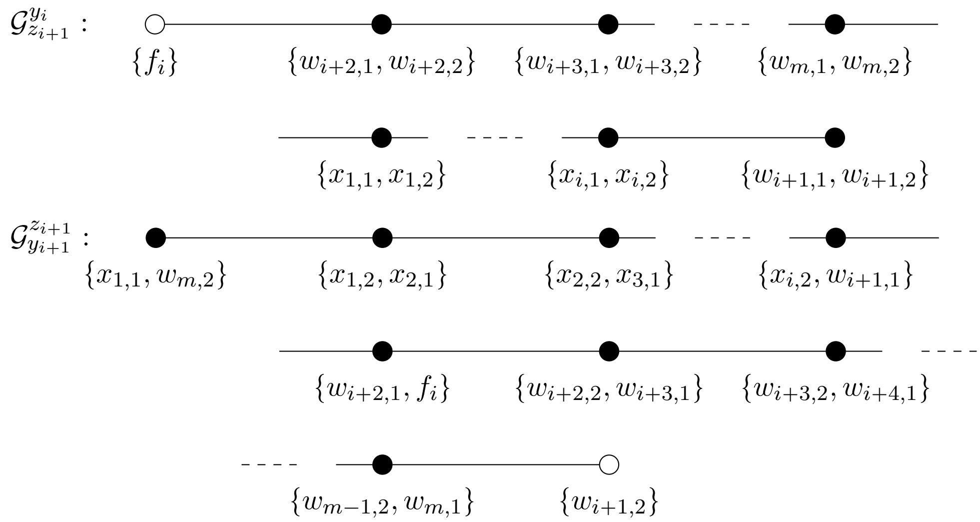


Figure B.80: The x-graphs $\mathcal{G}^{y_i}_{z_{i+1}}$ and $\mathcal{G}^{z_{i+1}}_{y_{i+1}}$ from (4.3.27).

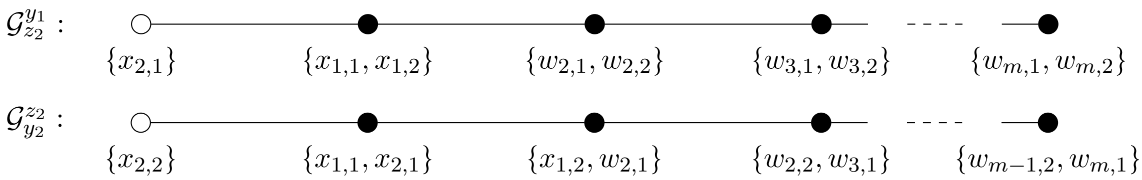


Figure B.81: The x-graphs $\mathcal{G}^{y_1}_{z_2}$ and $\mathcal{G}^{z_2}_{y_2}$ from (4.3.28).

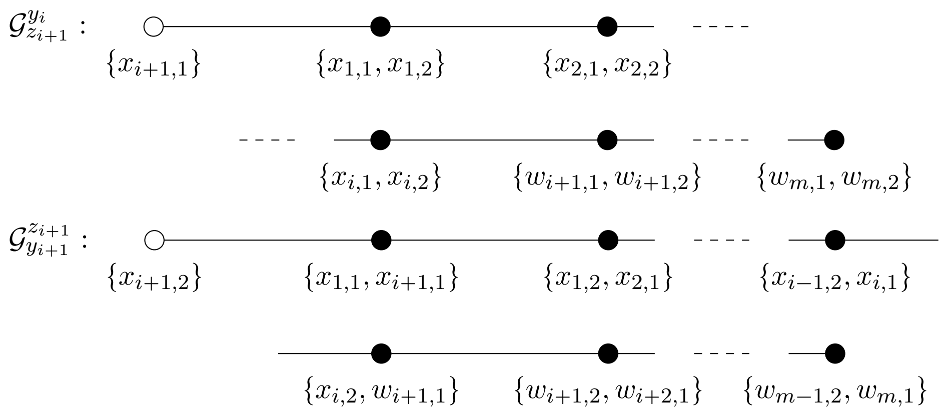


Figure B.82: The x-graphs $\mathcal{G}_{z_{i+1}}^{y_i}$ and $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ from (4.3.28).

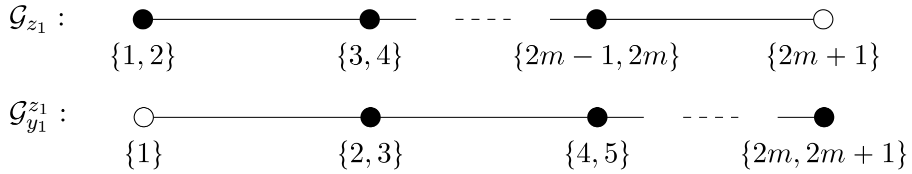


Figure B.83: The x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ from (4.3.29).

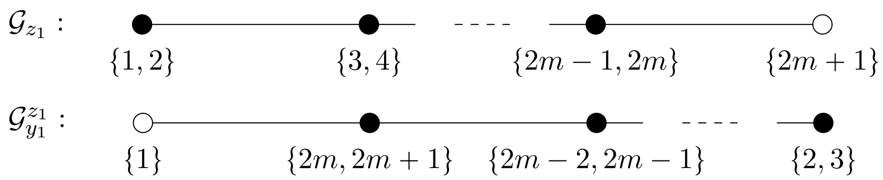


Figure B.84: The x-graphs $\mathcal{G}_{z_1}$ and $\mathcal{G}_{y_1}^{z_1}$ from (4.3.30).

Appendix C

MAGMA Code

The following code may be used in the computer algebra system MAGMA ([BC06], [CPB08] and [CP08]) to generate the matrix A_x for conjugate p-elements a and x in $\mathrm{Sym}(n)$ having full support (see Chapter 5 for full details). Here p is a prime number.

```
function CreateMatrix(a,x)
Deca:=CycleDecomposition(a);
n:=#Deca-#Fix(a);
p:=Order(a);
Ax:=Matrix(RationalField(),n,n,
     [<i,j,#(Deca[i]^x meet Deca[j])>:i,j in [1..n]]);
return Ax;
end function;
```

For a given matrix A, we can check if it is circulant using the code

```
function IsCirculant(A)
n:=#Rows(A);
I:=Matrix(RationalField(),n,n,[<i,i+1,1>:i in [1..n-1]])
     +Matrix(RationalField(),n,n,[<n,1,1>]);
B:=ZeroMatrix(RationalField(),n,n);
for i in {1..n} do
B:=B+A[1,i]*I^(i-1);
end for;
if A eq B then return true; else return false; end if;
end function;
```

For given p-elements $a, x \in \mathrm{Sym}(n)$, we may check if $\langle a, x\rangle \cong C_p \wr C_p$ using the code

```
function IsIsomorphicWreath(a,x)
p:=Order(a);
if Order(a) ne Order(x) then return false; else
Cp:=CyclicGroup(p);
```

```
Wp:=WreathProduct(Cp,Cp);
if IsIsomorphic(sub<Parent(a)|a,x>,Wp) then return true;
else return false; end if;
end if;
end function;
```

We may also create the circulant matrix $\text{circ}(y_0, \ldots, y_{p-1})$ from the vector $y = (y_0, \ldots, y_{p-1})$ over the finite field $GF(p)$

```
function CreateCirculant(y)
p:=#y;
P:=Matrix(GF(p),p,p, [<i,i+1,1>: i in [1..p-1]])
     +Matrix(GF(p),p,p,[<p,1,1>]);
C:=y[1]*P^p;
for i in {2..p} do
C:=C+y[i]*P^(i-1);
end for;
return C;
end function;
```

or over the real numbers.

```
function CreateRealCirculant(y)
p:=#y;
P:=Matrix(RealField(),p,p, [<i,i+1,1>: i in [1..p-1]])
     +Matrix(RealField(),p,p,[<p,1,1>]);
C:=y[1]*P^p;
for i in {2..p} do
C:=C+y[i]*P^(i-1);
end for;
return C;
end function;
```

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Table of Notation

$BS(M)$ the block sum matrix of the matrix M 108

$\mathcal{C}(G,X)$ the commuting graph of the group G with vertex set X 83

$C_G(g)$ the centralizer in G of g.......... 27

$\operatorname{char} k$ the characteristic of the field k.......... 27

$C^i(\Delta,\mathscr{F})$ the i-th co-chain space of the presheaf $\mathscr{F}$ defined on Δ.......... 51

$C^i(\mathscr{F})$ the i-th co-chain space of the presheaf $\mathscr{F}$.......... 52

$\operatorname{circ}(c_0,c_1,\ldots,c_{n-1})$ the $n\times n$ circulant matrix having first row $(c_0,c_1,\ldots,c_{n-1})$ 111

C_r the cyclic group of order r.......... 27

$C_r(\Delta,\mathscr{F})$ the r-th chain space of the presheaf $\mathscr{F}$ defined on Δ.......... 49

$C_r(\mathscr{F})$ the r-th chain space of the presheaf $\mathscr{F}$.......... 50

$d(\cdot,\cdot)$ the standard distance metric on a graph.......... 84

$\det(B)$ the determinant of the matrix B 110

$\mathrm{Dih}(2n)$ the dihedral group of order $2n$ 29

$\mathcal{F}(G,X)$ the local fusion graph of the group G having vertex set X 84

$f_C(X)$ the representer polynomial of the circulant matrix C.......... 111

$\operatorname{fix}(g)$ the fixed-point set of the element $g\in\mathrm{Sym}(n)$.......... 86

$\mathrm{Fun}(\Gamma,K)$ the set of all functions from Γ to K.......... 32

$\mathcal{F}_V$ the fixed-point presheaf of V.......... 48

$GF(p^n)$ the Galois field of order p^n.......... 27

$g_M(X)$ the reduced representer polynomial of the matrix M 128

$\mathcal{G}_x$ the x-graph corresponding to x.......... 86

$\mathcal{G}_x^y$ the x-graph corresponding to x with respect to y.......... 86

$H^i(\Delta,\mathscr{F})$ the i-th cohomology group of the presheaf $\mathscr{F}$ defined on Δ.......... 52

$H^i(\mathscr{F})$ the i-th cohomology group of the presheaf $\mathscr{F}$.......... 52

$H_r(\Delta,\mathscr{F})$ the r-th homology group of the presheaf $\mathscr{F}$ defined on Δ.......... 50

$H_r(\mathscr{F})$ the r-th homology group of the presheaf $\mathscr{F}$.......... 50

$i(V,W)$ the intertwining number of the modules V and W.......... 144

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